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Error Analysis for a Static Convergent Beam Triple LIDAR

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Abstract

In this paper we consider the problem of uncertainty propagation and quantification for the *converging* triple-beam LIDAR technology, proposed for measuring wind velocity passing through a fixed point in space. Converging triple-beam LIDAR employs the use of three non-parallel, non-coplanar, laser beams which are directed from a fixed platform, typically at ground level, that extend to meet at the point at which measurement of velocity is sought. Coordinate values of the velocity are ascertained with respect to unit vectors along the lines of sight of the laser beams (Doppler vectors), which are then resolved in order determine the velocity in terms of Cartesian coordinates (i.e. with respect to the standard basis). However, if there is any discrepancy between the recorded values of the coordinates with respect to the Doppler unit vectors and/or the perceived angle settings for such vectors with what they really should be, however small, then this will lead to errors in the reconstructed Cartesian coordinates. The aim of this paper to quantify the potential size of this error by consideration of the variance-covariance matrix of the reconstructed Cartesian coordinates.

KEYWORDS: WIND TURBINES;WIND VELOCITY FIELD; DOPPLER LIDAR; DOPPLER VECTORS; DIVERGING BEAM LIDAR; WIND MAPPING LIDAR; CONVERGING BEAM LIDAR; VELOCITY RECONSTRUCTION; MEASUREMENT ERROR

1 Introduction

Efficient deployment of wind power provides economic benefits as well as reducing carbon dioxide emissions and other pollution in accordance with UK and international government targets.

The onshore and offshore wind industry requires wind measurement at the pre-construction site planning stage in order to determine whether a given site has favourable wind conditions and to estimate the likely energy production from a wind farm located at the given site. Favourable wind conditions imply firstly that the average wind speed through the year is high enough, and secondly that the wind conditions are not too damaging due to excessive gusts, excessive turbulence intensity, extreme non-horizontal flow, or due to other extreme or abnormal flow conditions.

It is noted that site wind conditions vary considerably due to site weather conditions, as well as due to local terrain complexity and terrain roughness features such as forestry. Measurement data may be collected over some years in order to characterise a prospective wind farm site. Statistical and stochastic effects imply that there is uncertainty associated with extrapolating the measured

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time series data to a future expected wind regime which may be used to predict wind farm output. Reducing measurement uncertainty is beneficial in reducing uncertainty in predicted energy output as well as fatigue loading conditions. Therefore reduced measurement uncertainty allows for reduction in financial uncertainty and enables more efficient deployment of capital and resources.

Also, wind turbines continue to grow in size with rotor diameters as large as 180 metres and top tip height as high as 225 metres. This means that the variation in wind velocity field across the entire rotor area has a significant effect both on energy production, as well as fatigue loading throughout the wind turbine components such as blades, rotating drive train, tower and onshore/offshore foundations. Therefore wind measurement across a large area or volume of space can be beneficial towards optimization of wind farms, both at the pre-construction planning stage, as well as during the operational lifetime of wind farms and their wind turbines.

Traditionally the wind industry has employed mast mounted single point instruments such as spinning cup anemometers, wind vanes and ultrasonic equivalents. A mast may provide a number of measurements at different heights but typically ignores the lateral variation in wind field.

Often the wind flow has been assumed to be horizontal only, or the vertical component neglected, which is not always correct.

LIDAR (laser-based Light Detection And Ranging) wind velocity measurement works on the basis of the Doppler effect whereby laser radiation along a given laser line of sight is reflected back along that line of sight from microscopic aerosol particles within the air carried by the wind. The Doppler effect is well known and causes a frequency change in the reflected laser radiation. The frequency change may be measured by suitable signal processing and this provides a measured velocity component along the laser line of sight.

It is possible to select a measurement range by using timing gates, or "range gates", in the signal processing of pulsed LIDARs essentially fixing the measurement range by knowledge of the duration of time of flight of radiation there and back, or alternatively by controlling the focus range of focused CW (continuous wave) LIDARs relying on the fact that the integrated Doppler signal will be dominated by reflected radiation from the optically focused region.

Usually the LIDAR is placed on the ground and in the typical land-based deployment the chosen measurement range of interest corresponds to the typical wind turbine hub height such as 100 metres.

The wind industry has already been employing LIDAR technology. However, the usual deployment of wind LIDAR employs a diverging beam approach. For instance the conical scan approach employs a beam at a fixed angle to a rotation axis and then takes many measurements as the beam rotates around that axis, describing a cone. For instance one hundred measurements might be taken per revolution or conical scan. These measurements along different lines of sight may be combined into a single wind velocity measurement. However, by combining many line of sight measurements in this way there is an implicit assumption of simple wind flow. Effectively the diverging beam LIDAR averages the wind velocity around a circular probe region at a chosen range on the conical scan.

Apart from conical scan another commonly used diverging beam design is the "beam swinging" approach where typically four beam directions are employed by switching or swinging the beams. This configuration is commonly referred to as VAD scan (Velocity Azimuth Display). The beam swinging approach or VAD scan suffers the same problem as the conical scan in that it combines independent line of sight measurements from probe regions which are separated in space, typically by distances of approximately 100 metres.

Therefore the diverging beam LIDAR approach suffers from the assumption of uniform or simple linearly varying flow. The lack of general validity of this assumption gives rise to increased measurement error.

We know from simply observing the motion of leaves and branches of a tree in the wind that the wind is spatially varying and it is wrong to assume uniform or linearly varying flow. Considering that a large wind turbine (of rotor diameter 180 metres) may be ten times greater than the height of a large tree then it is noted that the wind velocity may vary considerably from top to bottom and from left to right across large rotors. Therefore the diverging beam LIDAR approach suffers from measurement ambiguity and measurement error uncertainties arising from the assumption of uniform flow.

In order to properly reconstruct a three dimensional wind velocity vector it is required to measure three independent non-parallel wind components. If the wind velocity vector field is varying non-linearly in space then to measure the wind velocity at a chosen point one should converge three LIDAR beams at the chosen measurement point.

A single point wind velocity measurement can be obtained by use of three or more fixed converging beams from three LIDARs. By employing an angle-scanning (or beam switching) LIDAR system with three separate LIDARs scanning (or switching) in cooperation to a succession of measurement points, it is then possible to measure a two-dimensional (planar) or three-dimensional (volumetric) wind velocity field map.

The convergence of three beams accounts for the fact that wind velocity is a three-dimensional vector quantity. The scanning or variation through a succession of measurement points accounts for the fact that the velocity vector field may vary three-dimensionally throughout a volume of space.

Proper measurement of non-horizontal flow, or measurement of yaw error in directional control of operational turbines, are examples of situations where it is beneficial to account for the three-dimensional nature of wind velocity.

Measurement of wind shear (changing wind speed with height) and wind veer (changing wind direction with height) are examples of where it is necessary to measure the three-dimensional variation of wind velocity through space. This may determine suitability of a site for wind turbine deployment.

In summary the measurement of the three-dimensional volumetrically varying wind velocity vector field offers numerous advantages to the wind industry, including that of better site assessment of damaging local conditions, better turbine site classifications (matching turbine strengths to site conditions), and better insurance and warranty conditions (ensuring turbines are operating within their design specifications). This may be achieved by a scanning converging beam triple LIDAR.

This paper discusses how to quantify the wind velocity measurement error when using a triple LIDAR converging beam approach.

2 Mathematical Preliminaries and Problem Formulation

Throughout we shall work with the index set $\mathcal{I} = \{1, 2, 3\}$. Let $\{\mathbf{u}_i : i \in \mathcal{I}\}$ be the unit vectors for the standard basis in $\mathbb{R}^3$, where $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$, correspond to the x, y, z directions in the right-handed Cartesian system, respectively.

Let $\{\hat{\mathbf{r}}_i : i \in \mathcal{I}\}$ be the unit vectors corresponding to the directions of each of the laser/LIDAR beams i.e. the Doppler LIDAR basis vectors. Thus, from the point of origin of each of the Doppler LIDAR beams, the point in space for which a velocity measurement is being sought, is represented by $\mathbf{r}_i = \widetilde{r}_i \hat{\mathbf{r}}_i$, for each lidar beam $i \in \mathcal{I}$, respectively.

Using converging triple beam LIDAR, a wind velocity, $\mathbf{v}$, relative to the aformentioned standard basis may represented as

$$\mathbf{v} = v_1\mathbf{u}_1 + v_2\mathbf{u}_2 + v_3\mathbf{u}_3. \tag{1}$$

Each Doppler LIDAR measurement obtains a component of wind velocity along the LIDAR line of sight. Therefore, employing the scalar product of two vectors (the unit line of sight vector and the

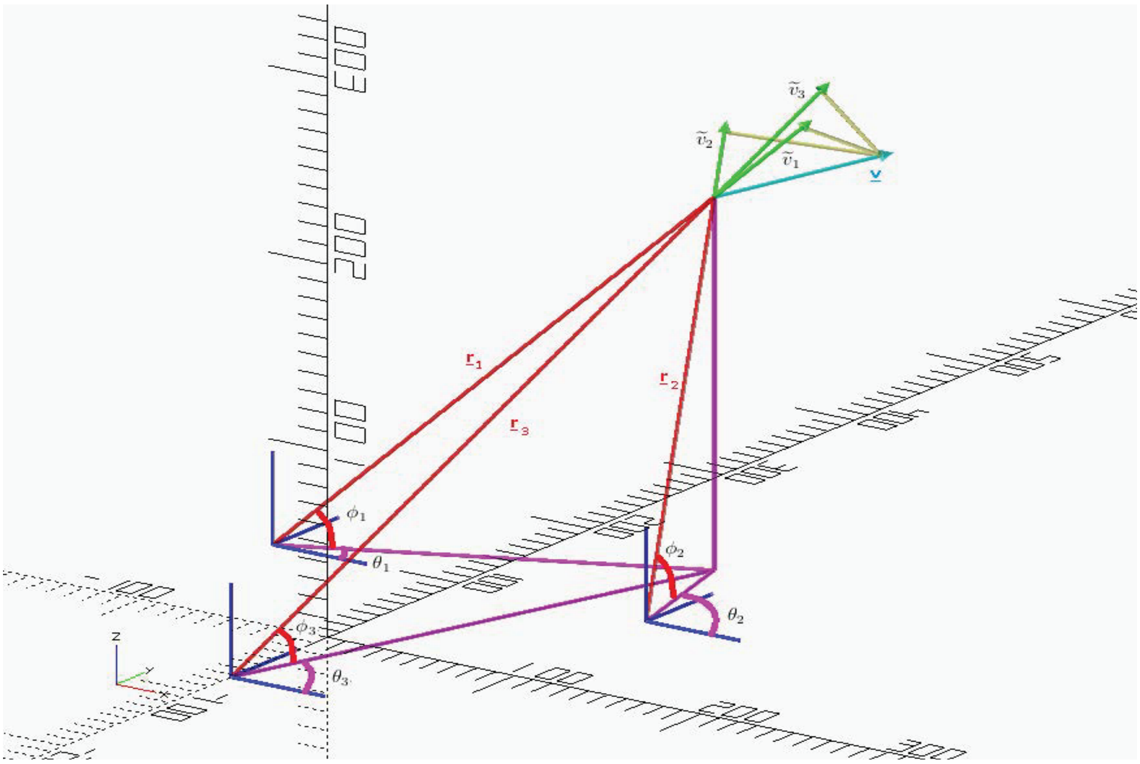

Figure 1: Geometric set-up of the convergent triple-beam LIDAR technology

true wind velocity vector relative to the standard basis) gives the Doppler line of sight component, given by

$$\widetilde{v}_i = \hat{\mathbf{r}}_i.\mathbf{v} = \sum_{j\in\mathcal{I}} r_{ij}v_j \text{ for all } i \in \mathcal{I}, \tag{2}$$

where $\hat{\mathbf{r}}_i = (r_{i1}, r_{i2}, r_{i3})'$, for $i \in \mathcal{I}$.

It is to be noted that the unit direction vectors do not depend on the position of the measurement point and in particular the above equation does not depend on the range or distance to the measurement point. Therefore range dependence of the velocity reconstruction exists only in the sense of increasing error on the three Doppler measurements which can be range dependent due to range limitations of LIDAR: however, such considerations will be ignored for the purposes of this discussion.

For each Doppler unit vector $\hat{\mathbf{r}}_i$, $i \in \mathcal{I}$, we may characterize its direction by:
$\theta_i \in [0, 2\pi)$, the azimuthal angle measured anti-clockwise from the x-axis; $\phi_i \in [-\pi/2, \pi/2]$, the elevation angle from the (x, y)-plane, taken to be positive when the z-coordinate is positive (as would normally be the case for a platform set on the ground measuring a point that is above ground). Then each of the Doppler unit vectors can be re-expressed in terms of the standard basis as follows:

$$\hat{\mathbf{r}}_i = (\cos\theta_i.\cos\phi_i)\mathbf{u_1} + (\sin\theta_i.\cos\phi_i)\mathbf{u_2} + (\sin\phi_i)\mathbf{u_3} \quad i \in \mathcal{I}. \tag{3}$$

Equations (2) and (3) can also be written in matrix form:

$$\widetilde{\mathbf{v}} = \mathbf{M}\mathbf{v} \tag{4}$$

where

$$M_{i1} = r_{i1} = \cos\theta_i.\cos\phi_i, \quad i \in \mathcal{I} \tag{5}$$

$$M_{i2} = r_{i2} = \sin\theta_i . \cos\phi_i, \quad i \in \mathcal{I} \tag{6}$$

$$M_{i3} = r_{i3} = \sin\phi_i, \quad i \in \mathcal{I} \tag{7}$$

and

$$\widetilde{v}_i = \sum_{j\in\mathcal{I}} [\mathbf{M}]_{ij} v_j = (\cos\theta_i . \cos\phi_i) v_1 + (\sin\theta_i . \cos\phi_i) v_2 + (\sin\phi_i) v_3, \quad i \in \mathcal{I}. \tag{8}$$

Thus, given the 6 Doppler angles and the 3 Doppler wind velocity coordinates, one can reconstruct the wind velocity vector in the standard basis:

$$\mathbf{v} = \mathbf{M}^{-1}\widetilde{\mathbf{v}} \tag{9}$$

or, equivalently,

$$v_j = \sum_{k\in\mathcal{I}} [\mathbf{M}^{-1}]_{jk} \widetilde{v}_k \quad j \in \mathcal{I}. \tag{10}$$

Therefore, with all of the elements of $\mathbf{M}$ in place, then this matrix can be inverted, provided that it has a non-zero determinant, in order to know the matrix $\mathbf{M}^{-1}$. Knowing the matrix allows us to reconstruct the true wind velocity (within the Cartesian reference frame) from the three Doppler LIDAR measured velocity components by use of equation (9). The objective in the remainder of this paper is to gauge the effect of uncertainty in the perceived polar angles and measured Doppler velocity components on the value of the reconstructed vector in Cartesian coordinates. We will do this by working with a metric commonly referred to as the error propagation formula and also by looking for an upper bound on this quantity that is relatively easier to compute than the exact metric (which also happens to have interesting qualitative interpretations).

3 Error Propagation Formulae

The determinant of the matrix $\mathbf{M}$ is given by $\Delta = \Delta(\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3)$, a function of the six angles defining three LIDAR beam directions:

$$\begin{aligned}\Delta = & \cos\theta_1 . \cos\phi_1 .(\sin\theta_2 . \cos\phi_2 . \sin\phi_3 - \sin\phi_2 . \sin\theta_3 . \cos\phi_3) \\ & - \sin\theta_1 . \cos\phi_1 .(\sin\phi_3 . \cos\theta_2 . \cos\phi_2 - \sin\phi_2 . \cos\theta_3 . \cos\phi_3) \\ & + \sin\phi_1 .(\cos\theta_2 . \cos\phi_2 . \sin\theta_3 . \cos\phi_3 - \sin\theta_2 . \cos\phi_2 . \cos\theta_3 . \cos\phi_3).\end{aligned} \tag{11}$$

Thus, the nine elements of $\mathbf{M}^{-1}$ are given as follows:

$$[\mathbf{M}^{-1}]_{11} = (r_{22}.r_{33} - r_{23}.r_{32})/\Delta \tag{12}$$

$$[\mathbf{M}^{-1}]_{12} = (r_{13}.r_{32} - r_{12}.r_{33})/\Delta \tag{13}$$

$$[\mathbf{M}^{-1}]_{13} = (r_{12}.r_{23} - r_{13}.r_{22})/\Delta \tag{14}$$

$$[\mathbf{M}^{-1}]_{21} = (r_{23}.r_{31} - r_{21}.r_{33})/\Delta \tag{15}$$

$$[\mathbf{M}^{-1}]_{22} = (r_{11}.r_{33} - r_{13}.r_{31})/\Delta \tag{16}$$

$$[\mathbf{M}^{-1}]_{23} = (r_{13}.r_{21} - r_{11}.r_{23})/\Delta \tag{17}$$

$$[\mathbf{M}^{-1}]_{31} = (r_{21}.r_{32} - r_{22}.r_{31})/\Delta \tag{18}$$

$$[\mathbf{M}^{-1}]_{32} = (r_{12}.r_{31} - r_{11}.r_{32})/\Delta \tag{19}$$

$$[\mathbf{M}^{-1}]_{33} = (r_{11}.r_{22} - r_{12}.r_{21})/\Delta. \tag{20}$$

It is noted from (5)-(20) that the entries of $\mathbf{M}^{-1}$ are not functionally dependent on the measured Doppler LIDAR components, i.e.:

$$[\mathbf{M}^{-1}]_{jk} = [\mathbf{M}^{-1}]_{ij}(\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3), \; j, k \in \mathcal{I}. \tag{21}$$

However, from (10) and (21) one notes that, in general, the six angles and the measured Doppler LIDAR components may contribute to the reconstructed vector components:

$$v_j = \sum_{k \in \mathcal{I}} [\mathbf{M}^{-1}]_{jk} \widetilde{v}_k = \sum_{k \in \mathcal{I}} [\mathbf{M}^{-1}]_{jk}(\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3) \widetilde{v}_k \tag{22}$$

i.e.

$$v_j = v_j(\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3, \widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3). \tag{23}$$

We will make the reasonable assumption that all 9 variables, $\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3, \widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3$, are mutually functionally independent of each other. As a result of this assumption, when measurement error is incorporated into these values and determined to be random, then it will also follow that the 9 variables are also statistically independent.

Set

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T \quad \boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T \quad \widetilde{\mathbf{v}} = (\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3)^T.$$

We express these variables in terms of constant vectors, $\boldsymbol{\theta}_0$, $\boldsymbol{\phi}_0$, $\widetilde{\mathbf{v}}_0$, representing the recorded azimuthal angles, elevation angles, and Doppler velocities, respectively; and also $\boldsymbol{\delta\theta}$, $\boldsymbol{\delta\phi}$ and $\boldsymbol{\delta\widetilde{v}}$, assumed to be random vectors, representing the corresponding measurement errors. Thus

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \boldsymbol{\delta\theta}; \quad \boldsymbol{\phi} = \boldsymbol{\phi}_0 + \boldsymbol{\delta\phi}; \quad \widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}_0 + \boldsymbol{\delta\widetilde{v}}.$$

The variance-covariance matrices of $\boldsymbol{\delta\theta}$, $\boldsymbol{\delta\phi}$ and $\boldsymbol{\delta\widetilde{v}}$ are given by

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}} = \text{diag}(\sigma^2_{\theta_1}, \sigma^2_{\theta_2}, \sigma^2_{\theta_3}), \quad \boldsymbol{\Sigma}_{\boldsymbol{\phi}} = \text{diag}(\sigma^2_{\phi_1}, \sigma^2_{\phi_2}, \sigma^2_{\phi_3}), \quad \boldsymbol{\Sigma}_{\widetilde{\mathbf{v}}} = \text{diag}(\sigma^2_{\widetilde{v}_1}, \sigma^2_{\widetilde{v}_2}, \sigma^2_{\widetilde{v}_3})$$

respectively.

Define

$$\nabla_{(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}})}(\cdot) = \left(\frac{\partial \cdot}{\partial \theta_1}, \frac{\partial \cdot}{\partial \theta_2}, \frac{\partial \cdot}{\partial \theta_3}, \frac{\partial \cdot}{\partial \phi_1}, \frac{\partial \cdot}{\partial \phi_2}, \frac{\partial \cdot}{\partial \phi_3}, \frac{\partial \cdot}{\partial \widetilde{v}_1}, \frac{\partial \cdot}{\partial \widetilde{v}_2}, \frac{\partial \cdot}{\partial \widetilde{v}_3} \right)^T.$$

Representing (23) more compactly as $v_j(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}})$, then define $\mathbf{g}_j$ as follows:

$$\mathbf{g}_j = \nabla_{(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}})}(v_j(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}}))\Big|_{(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)}.$$

Then, to first order, one can approximate $v_j(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}})$ as follows:

$$v_j(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}}) \approx v_j(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) + \mathbf{g}_j^T \begin{bmatrix} \boldsymbol{\delta\theta} \\ \boldsymbol{\delta\phi} \\ \boldsymbol{\delta\widetilde{v}} \end{bmatrix}.$$

This will be a "good" approximation provided that the measurement errors are small. Under such conditions, the covariance between $v_i(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}})$ and $v_j(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}})$ may be approximated as follows:

$$\text{cov}(v_i(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}}), v_j(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}})) \approx \mathbf{g}_i^T \text{diag}\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\phi}}, \boldsymbol{\Sigma}_{\widetilde{\mathbf{v}}}\right) \mathbf{g}_j$$

$$= \sum_{r \in \mathcal{I}} \left\{ \left(\frac{\partial v_i}{\partial \theta_r}\right) \left(\frac{\partial v_j}{\partial \theta_r}\right) \sigma_{\theta_r}^2 + \left(\frac{\partial v_i}{\partial \phi_r}\right) \left(\frac{\partial v_j}{\partial \phi_r}\right) \sigma_{\phi_r}^2 + \left(\frac{\partial v_i}{\partial \widetilde{v}_r}\right) \left(\frac{\partial v_j}{\partial \widetilde{v}_r}\right) \sigma_{\widetilde{v}_r}^2 \right\} \Bigg|_{(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} . \tag{24}$$

Setting $i = j$ yields an approximate expression for $\sigma_{v_j}^2 = \mathrm{var}\,(v_j(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{\mathbf{v}}))$, i.e.

$$\sigma_{v_j}^{*\,2} = \sum_{r \in \mathcal{I}} \left\{ \left(\frac{\partial v_j}{\partial \theta_r}\right)^2 \sigma_{\theta_r}^2 + \left(\frac{\partial v_j}{\partial \phi_r}\right)^2 \sigma_{\phi_r}^2 + \left(\frac{\partial v_j}{\partial \widetilde{v}_r}\right)^2 \sigma_{\widetilde{v}_r}^2 \right\} \Bigg|_{(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} , \tag{25}$$

the RHS being a form of the well-known error propagation formula.

Along with the values of the variances of the six angles and the three measured Doppler velocity components, it is also required to evaluate the partial derivatives that appear in (25).

The partial derivatives may be found by differentiating the expression of (10)/(22):

$$\frac{\partial v_j}{\partial \theta_i} = \sum_{k \in \mathcal{I}} \frac{\partial [\mathbf{M}^{-1}]_{jk} \widetilde{v}_k}{\partial \theta_i} = \sum_{k \in \mathcal{I}} \widetilde{v}_k \frac{\partial [\mathbf{M}^{-1}]_{jk}}{\partial \theta_i} \tag{26}$$

$$\frac{\partial v_j}{\partial \phi_i} = \sum_{k \in \mathcal{I}} \frac{\partial [\mathbf{M}^{-1}]_{jk} \widetilde{v}_k}{\partial \phi_i} = \sum_{k \in \mathcal{I}} \widetilde{v}_k \frac{\partial [\mathbf{M}^{-1}]_{jk}}{\partial \phi_i} \tag{27}$$

$$\frac{\partial v_j}{\partial \widetilde{v}_i} = \sum_{k \in \mathcal{I}} [\mathbf{M}^{-1}]_{jk} \frac{\partial \widetilde{v}_k}{\partial \widetilde{v}_i} = \sum_{k \in \mathcal{I}} [\mathbf{M}^{-1}]_{jk} \delta_{ik} \tag{28}$$

where for $m, n \in \mathcal{I}$, δ_{mn} is Kronecker delta.

To calculate the 27 terms given by $\frac{\partial [\mathbf{M}^{-1}]_{jk}}{\partial \theta_i}$, $i,j,k \in \mathcal{I}$ and 27 terms $\frac{\partial [\mathbf{M}^{-1}]_{jk}}{\partial \phi_i}$, $i,j,k \in \mathcal{I}$, it is required to compute the 6 terms that arise from differentiating Δ, with respect to each of the angles $\theta_1, \theta_2, \theta_3, \phi_1, \phi_2$ and ϕ_3:

$$\begin{aligned} \frac{\partial \Delta}{\partial \theta_1} &= [-\sin\theta_1].\cos\phi_1.(\sin\theta_2.\cos\phi_2.\sin\phi_3 - \sin\phi_2.\sin\theta_3.\cos\phi_3) \\ &\quad -[\cos\theta_1].\cos\phi_1.(\sin\phi_3.\cos\theta_2.\cos\phi_2 - \sin\phi_2.\cos\theta_3.\cos\phi_3) \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\partial \Delta}{\partial \theta_2} &= \cos\theta_1.\cos\phi_1.([\cos\theta_2].\cos\phi_2.\sin\phi_3) - \sin\theta_1.\cos\phi_1.(\sin\phi_3.[-\sin\theta_2].\cos\phi_2) \\ &\quad +\sin\phi_1.([-\sin\theta_2].\cos\phi_2.\sin\theta_3.\cos\phi_3 - [\cos\theta_2].\cos\phi_2.\cos\theta_3.\cos\phi_3)) \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\partial \Delta}{\partial \theta_3} &= \cos\theta_1.\cos\phi_1.(-\sin\phi_2.[\cos\theta_3].\cos\phi_3) - \sin\theta_1.\cos\phi_1.(-\sin\phi_2.[-\sin\theta_3].\cos\phi_3) \\ &\quad +\sin\phi_1.(\cos\theta_2.\cos\phi_2.[\cos\theta_3].\cos\phi_3 - \sin\theta_2.\cos\phi_2.[-\sin\theta_3].\cos\phi_3) \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial \Delta}{\partial \phi_1} &= \cos\theta_1.[-\sin\phi_1].(\sin\theta_2.\cos\phi_2.\sin\phi_3 - \sin\phi_2.\sin\theta_3.\cos\phi_3) \\ &\quad -\sin\theta_1.[-\sin\phi_1].(\sin\phi_3.\cos\theta_2.\cos\phi_2 - \sin\phi_2.\cos\theta_3.\cos\phi_3) \\ &\quad +[\cos\phi_1].(\cos\theta_2.\cos\phi_2.\sin\theta_3.\cos\phi_3 - \sin\theta_2.\cos\phi_2.\cos\theta_3.\cos\phi_3)) \end{aligned} \tag{32}$$

$$\frac{\partial\Delta}{\partial\phi_2} = \cos\theta_1.\cos\phi_1.(\sin\theta_2.[-\sin\phi_2].\sin\phi_3 - [\cos\phi_2].\sin\theta_3.\cos\phi_3)$$
$$-\sin\theta_1.\cos\phi_1.(\sin\phi_3.\cos\theta_2.[-\sin\phi_2] - [\cos\phi_2].\cos\theta_3.\cos\phi_3)$$
$$+\sin\phi_1.(\cos\theta_2.[-\sin\phi_2].\sin\theta_3.\cos\phi_3 - \sin\theta_2.[-\sin\phi_2].\cos\theta_3.\cos\phi_3)) \quad (33)$$

$$\frac{\partial\Delta}{\partial\phi_3} = \cos\theta_1.\cos\phi_1.(\sin\theta_2.\cos\phi_2.[\cos\phi_3] - \sin\phi_2.\sin\theta_3.[-\sin\phi_3])$$
$$-\sin\theta_1.\cos\phi_1.([\cos\phi_3].\cos\theta_2.\cos\phi_2 - \sin\phi_2.\cos\theta_3.[-\sin\phi_3])$$
$$+\sin\phi_1.(\cos\theta_2.\cos\phi_2.\sin\theta_3.[-\sin\phi_3] - \sin\theta_2.\cos\phi_2.\cos\theta_3.[-\sin\phi_3])). \quad (34)$$

The partial derivatives of $[\mathbf{M}^{-1}]_{jk},\ j,k \in \mathcal{I}$ with respect to each of the six angles, yielding 54 equations, can be found in the Appendix.

4 Main Results

4.1 Bounding the variance of the reconstructed velocity components

As should be clear from the previous section, it is possible to evaluate all 9 components of the 3×3 covariance matrix for the vector $(v_1(\boldsymbol{\theta},\boldsymbol{\phi},\widetilde{\mathbf{v}}), v_2(\boldsymbol{\theta},\boldsymbol{\phi},\widetilde{\mathbf{v}}), v_3(\boldsymbol{\theta},\boldsymbol{\phi},\widetilde{\mathbf{v}}))^T$ (to first order, which is adequate for our purposes), the behaviour of which can perhaps only be investigated by numerical computation across a suitable expanse of the product set in $\mathbb{R}^9$ arising from the ranges of values that could be inhabited by the variances of the 6 polar angles and 3 measured Doppler velocity components. In this section, we seek a more qualitative understanding of how potentially the errors may grow in the reconstructed velocity components. To this end, we first derive a bound on the error formula of (25).

Theorem. *Suppose that for each* $k \in \mathcal{I}$, $\widetilde{v}_k^{\min} \le \widetilde{v}_k \le \widetilde{v}_k^{\max}$ *and* $\widetilde{v}_k^{\text{bound}} = \max\left\{|\widetilde{v}_k^{\min}|, |\widetilde{v}_k^{\max}|\right\}$*. Then:*
(i) an upper bound on each ${\sigma^*_{v_j}}^2$, $j \in \mathcal{I}$, *is given by*

$$\sum_{r\in\mathcal{I}}\left\{\left(\sum_{k\in\mathcal{I}}\widetilde{v}_k^{\text{bound}}\left(\frac{2}{|\Delta|}+\frac{8}{\Delta^2}\right)\right)^2\sigma_{\theta_r}^2+\left(\sum_{k\in\mathcal{I}}\widetilde{v}_k^{\text{bound}}\left(\frac{2}{|\Delta|}+\frac{12}{\Delta^2}\right)\right)^2\sigma_{\phi_r}^2+\left(\sum_{k\in\mathcal{I}}\frac{2}{|\Delta|}\right)^2\sigma_{\widetilde{v}_r}^2\right\}; \quad (35)$$

(ii) the bound of (35) is of order $O(1/\Delta^4)$ *as* $\Delta \to 0$*;*
(iii) the bound of (35) is of order $O((\sum_{k\in\mathcal{I}}\widetilde{v}_k^{\text{bound}})^2)$ *as* $\sum_{k\in\mathcal{I}}\widetilde{v}_k^{\text{bound}} \to \infty$.

Proof
By exploiting the fact that the derivatives of Δ with respect to each of the six angles only involve sines and cosines (and thus taking values on the closed interval $[-1,1]$) and by applying the triangle inequality, then for each $r \in \mathcal{I}$,

$$\left|\frac{\partial\Delta}{\partial\theta_r}\right| \le 4, \quad \left|\frac{\partial\Delta}{\partial\phi_r}\right| \le 6.$$

By a similar rationale and combined with the above, we can deduce that

$$\left|\frac{\partial[\mathbf{M}^{-1}]_{jk}}{\partial\theta_r}\right| \le \frac{2}{|\Delta|} + \frac{2}{\Delta^2}\times 4 = \frac{2}{|\Delta|} + \frac{8}{\Delta^2} \quad (36)$$

$$\left|\frac{\partial[\mathbf{M}^{-1}]_{jk}}{\partial\phi_r}\right| \leq \frac{2}{|\Delta|} + \frac{2}{\Delta^2} \times 6 = \frac{2}{|\Delta|} + \frac{12}{\Delta^2} \tag{37}$$

$$\left|[\mathbf{M}^{-1}]_{jk}\right| \leq \frac{2}{|\Delta|}. \tag{38}$$

Applying the triangle inequality to the derivatives of the v_j and combining with (36)-(38), one obtains:

$$\left|\frac{\partial v_j}{\partial\theta_r}\right| \leq \sum_{k\in\mathcal{I}} |\widetilde{v}_k| \left|\frac{\partial[\mathbf{M}^{-1}]_{jk}}{\partial\theta_r}\right| \leq \sum_{k\in\mathcal{I}} \widetilde{v}_k^{\text{bound}} \left(\frac{2}{|\Delta|} + \frac{8}{\Delta^2}\right) \tag{39}$$

$$\left|\frac{\partial v_j}{\partial\phi_r}\right| \leq \sum_{k\in\mathcal{I}} |\widetilde{v}_k| \left|\frac{\partial[\mathbf{M}^{-1}]_{jk}}{\partial\theta_r}\right| \leq \sum_{k\in\mathcal{I}} \widetilde{v}_k^{\text{bound}} \left(\frac{2}{|\Delta|} + \frac{12}{\Delta^2}\right) \tag{40}$$

$$\left|\frac{\partial v_j}{\partial\widetilde{v}_i}\right| \leq \sum_{k\in\mathcal{I}} \left|[\mathbf{M}^{-1}]_{jk}\right| \leq \sum_{k\in\mathcal{I}} \frac{2}{|\Delta|}. \tag{41}$$

Combining (39)-(41) with (25) establishes (i).
The proofs for (ii) and (iii) are immediate from the form of (35). □

Although we do not know how tight the bound on ${\sigma^*_{v_j}}^2$, given by (35), actually is, we note that this bound increases quadratically with the overall potential size of the Doppler speeds (c.f. part (iii) above) and that the only controls available for offsetting such phenomena would be to increase the size of $|\Delta|$ (which has a maximum bound of 6, which can be seen by applying the triangle inequality to (11); indeed a tighter bound is given by 1 (see next section)) and/or the statistical precision (i.e. the inverse of the variance) in relation to each of the Doppler angles and the measured Doppler velocity components.

4.2 Geometric Interpretation of $|\Delta|$

It is noted that the absolute value of the determinant has a geometric interpretation in that it represents the volume of a parallelepiped whose edges are defined by the three unit vectors along the three LIDAR beam directions $\hat{\mathbf{r}}_1$, $\hat{\mathbf{r}}_2$ and $\hat{\mathbf{r}}_3$. If any two or more of these unit vectors approach each other, or if all three of these vectors become co-planar, then the parallelepiped volume tends to zero and thus the bound of (35) blows up toward infinity, making it more difficult to make the guarantee that the actual variance is constrained to stay below a certain value (in the absence of directly computing it from the formula of (25)). It is noted that when a measurement point approaches the plane of the three LIDARs then this parallelepiped becomes flattened and tends to zero volume, again implying that a guarantee that the true variance lies below a certain threshold will become increasingly difficult. On the other hand (and with all other quantities unchanged), the volume is at its maximum when the Doppler vectors are orthogonal to each other, resulting in a parallelepiped that is just a cube of unit length.

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Appendix

Recalling (11) and dealing with each element in turn:

$$\frac{\partial[\mathbf{M}^{-1}]_{11}}{\partial\theta_1} = \frac{\partial[(r_{22}r_{33} - r_{23}.r_{32})/\Delta]}{\partial\theta_1} = (1/\Delta)\frac{\partial[(r_{22}.r_{33} - r_{23}r_{32})]}{\partial\theta_1} - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32}).\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)\left[\frac{\partial r_{22}}{\partial\theta_1}.r_{33} + r_{22}.\frac{\partial r_{33}}{\partial\theta_1} - \frac{\partial r_{23}}{\partial\theta_1}.r_{32} - r_{23}.\frac{\partial r_{32}}{\partial\theta_1}\right] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)[(0).r_{33} + r_{22}.(0) - (0).r_{32} - r_{23}.(0)] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\theta_1}$$
$$= -(1/\Delta^2)(\sin\theta_2.\cos\phi_2.\sin\phi_3 - \sin\phi_2.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\theta_1} \tag{42}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{11}}{\partial\theta_2} = \frac{\partial[(r_{22}r_{33} - r_{23}.r_{32})/\Delta]}{\partial\theta_2} = (1/\Delta)\frac{\partial[(r_{22}.r_{33} - r_{23}r_{32})]}{\partial\theta_2} - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32}).\frac{\partial\Delta}{\partial\theta_2}$$
$$= (1/\Delta)\left[\frac{\partial r_{22}}{\partial\theta_2}.r_{33} + r_{22}.\frac{\partial r_{33}}{\partial\theta_2} - \frac{\partial r_{23}}{\partial\theta_2}.r_{32} - r_{23}.\frac{\partial r_{32}}{\partial\theta_2}\right] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\theta_2}$$
$$= (1/\Delta)[(\cos\theta_2.\cos\phi_2).r_{33} + r_{22}.(0) - (0).r_{32} - r_{23}.(0)] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\theta_2}$$
$$= (1/\Delta)[(\cos\theta_2.\cos\phi_2).\sin\phi_3] - (1/\Delta^2)(\sin\theta_2.\cos\phi_2.\sin\phi_3 - \sin\phi_2.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\theta_2} \tag{43}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{11}}{\partial\theta_3} = \frac{\partial[(r_{22}r_{33} - r_{23}.r_{32})/\Delta]}{\partial\theta_3} = (1/\Delta)\frac{\partial[(r_{22}.r_{33} - r_{23}r_{32})]}{\partial\theta_3} - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32}).\frac{\partial\Delta}{\partial\theta_3}$$
$$= (1/\Delta)\left[\frac{\partial r_{22}}{\partial\theta_3}.r_{33} + r_{22}.\frac{\partial r_{33}}{\partial\theta_3} - \frac{\partial r_{23}}{\partial\theta_3}.r_{32} - r_{23}.\frac{\partial r_{32}}{\partial\theta_3}\right] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\theta_3}$$
$$= (1/\Delta)[(0).r_{33} + r_{22}.(0) - (0).r_{32} - r_{23}.(\cos\theta_3.\cos\phi_3)] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\theta_3}$$
$$= (1/\Delta)[-\sin\phi_2.(\cos\theta_3.\cos\phi_3)] - (1/\Delta^2)(\sin\theta_2.\cos\phi_2.\sin\phi_3 - \sin\phi_2.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\theta_3} \tag{44}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{11}}{\partial\phi_1} = \frac{\partial[(r_{22}r_{33} - r_{23}.r_{32})/\Delta]}{\partial\phi_1} = (1/\Delta)\frac{\partial[(r_{22}.r_{33} - r_{23}r_{32})]}{\partial\phi_1} - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32}).\frac{\partial\Delta}{\partial\phi_1}$$
$$= (1/\Delta)\left[\frac{\partial r_{22}}{\partial\phi_1}.r_{33} + r_{22}.\frac{\partial r_{33}}{\partial\phi_1} - \frac{\partial r_{23}}{\partial\phi_1}.r_{32} - r_{23}.\frac{\partial r_{32}}{\partial\phi_1}\right] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\phi_1}$$
$$= (1/\Delta)[(0).r_{33} + r_{22}.(0) - (0).r_{32} - r_{23}.(0)] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\phi_1}$$

$$= -(1/\Delta^2)(\sin\theta_2.\cos\phi_2.\sin\phi_3 - \sin\phi_2.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_1} \tag{45}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{11}}{\partial\phi_2} = \frac{\partial[(r_{22}r_{33} - r_{23}.r_{32})/\Delta]}{\partial\phi_2} = (1/\Delta)\frac{\partial[(r_{22}.r_{33} - r_{23}r_{32})]}{\partial\phi_2} - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32}).\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)\left[\frac{\partial r_{22}}{\partial\phi_2}.r_{33} + r_{22}.\frac{\partial r_{33}}{\partial\phi_2} - \frac{\partial r_{23}}{\partial\phi_2}.r_{32} - r_{23}.\frac{\partial r_{32}}{\partial\phi_2}\right] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)[(-\sin\theta_2.\sin\phi_2).r_{33} + r_{22}.(0) - (\cos\phi_2).r_{32} - r_{23}.(0)] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)[(-\sin\theta_2.\sin\phi_2).\sin\phi_3 - (\cos\phi_2).\sin\theta_3.\cos\phi_3]$$
$$-(1/\Delta^2)(\sin\theta_2.\cos\phi_2.\sin\phi_3 - \sin\phi_2.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_2} \tag{46}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{11}}{\partial\phi_3} = \frac{\partial[(r_{22}r_{33} - r_{23}.r_{32})/\Delta]}{\partial\phi_3} = (1/\Delta)\frac{\partial[(r_{22}.r_{33} - r_{23}r_{32})]}{\partial\phi_3} - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32}).\frac{\partial\Delta}{\partial\phi_3}$$
$$= (1/\Delta)\left[\frac{\partial r_{22}}{\partial\phi_2}.r_{33} + r_{22}.\frac{\partial r_{33}}{\partial\phi_3} - \frac{\partial r_{23}}{\partial\phi_2}.r_{32} - r_{23}.\frac{\partial r_{32}}{\partial\phi_2}\right] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32})\frac{\partial\Delta}{\partial\phi_3}$$
$$= (1/\Delta)[(0).r_{33} + r_{22}.(\cos\phi_3) - (0).r_{32} - r_{23}.(\sin\theta_3.[-\sin\phi_3])] - (1/\Delta^2)(r_{22}.r_{33} - r_{23}.r_{32}).\frac{\partial\Delta}{\partial\phi_3}$$
$$= (1/\Delta)[(\sin\theta_2.\cos\phi_2).(\cos\phi_3) - (\sin\phi_2).(\sin\theta_3.[-\sin\phi_3])]$$
$$-(1/\Delta^2)(\sin\theta_2.\cos\phi_2.\sin\phi_3 - \sin\phi_2.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_3} \tag{47}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{12}}{\partial\theta_1} = \frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})/\Delta]}{\partial\theta_1} = (1/\Delta)\frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})]}{\partial\theta_1} - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33}).\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\theta_1}.r_{32} + r_{13}.\frac{\partial r_{32}}{\partial\theta_1} - \frac{\partial r_{12}}{\partial\theta_1}.r_{33} - r_{12}.\frac{\partial r_{33}}{\partial\theta_1}\right] - (1/\Delta^2)(r_{13}r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)[(0).r_{32} + r_{13}.(0) - ([\cos\theta_1].\cos\phi_1).r_{33} - r_{12}.(0)](1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33}).\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)[-\cos\theta_1.\cos\phi_1.\sin\phi_3] - (1/\Delta^2)(\sin\phi_1.\sin\theta_3.\cos\phi_3 - \sin\theta_1.\cos\phi_1.\sin\phi_3)\frac{\partial\Delta}{\partial\theta_1} \tag{48}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{12}}{\partial\theta_2} = \frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})/\Delta]}{\partial\theta_2} = (1/\Delta)\frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})]}{\partial\theta_2} - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33}).\frac{\partial\Delta}{\partial\theta_2}$$
$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\theta_2}.r_{32} + r_{13}.\frac{\partial r_{32}}{\partial\theta_2} - \frac{\partial r_{12}}{\partial\theta_2}.r_{33} - r_{12}\frac{\partial r_{33}}{\partial\theta_2}\right] - (1/\Delta^2)(r_{13}r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\theta_2}$$
$$= (1/\Delta)[(0).r_{32} + r_{13}.(0) - (0).r_{33} - r_{12}.(0)] - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\theta_2}$$
$$= -(1/\Delta^2)(\sin\phi_1.\sin\theta_3.\cos\phi_3 - \sin\theta_1.\cos\phi_1.\sin\phi_3)\frac{\partial\Delta}{\partial\theta_2} \tag{49}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{12}}{\partial\theta_3} = \frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})/\Delta]}{\partial\theta_3} = (1/\Delta)\frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})]}{\partial\theta_3} - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33}).\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\theta_3}.r_{32} + r_{13}.\frac{\partial r_{32}}{\partial\theta_3} - \frac{\partial r_{12}}{\partial\theta_3}.r_{33} - r_{12}\frac{\partial r_{33}}{\partial\theta_3}\right] - (1/\Delta^2)(r_{13}r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)[(0).r_{32} + r_{13}.([\cos\theta_3].\cos\phi_3) - (0).r_{33} - r_{12}.(0)] - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)[\sin\phi_1.(\cos\theta_3.\cos\phi_3)] - (1/\Delta^2)(\sin\phi_1.\sin\theta_3.\cos\phi_3 - \sin\theta_1.\cos\phi_1.\sin\phi_3)\frac{\partial\Delta}{\partial\theta_3} \quad (50)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{12}}{\partial\phi_1} = \frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})/\Delta]}{\partial\phi_1} = (1/\Delta)\frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})]}{\partial\phi_1} - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33}).\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\phi_1}.r_{32} + r_{13}.\frac{\partial r_{32}}{\partial\phi_1} - \frac{\partial r_{12}}{\partial\phi_1}.r_{33} - r_{12}\frac{\partial r_{33}}{\partial\phi_1}\right] - (1/\Delta^2)(r_{13}r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\cos\phi_1).r_{32} + r_{13}.(0) - (\sin\theta_1.[-\sin\phi_1]).r_{33} - r_{12}.(0)] - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\cos\phi_1).\sin\theta_3.\cos\phi_3 - (\sin\theta_1.[-\sin\phi_1]).\sin\phi_3]$$

$$-(1/\Delta^2)(\sin\phi_1.\sin\theta_3.\cos\phi_3 - \sin\theta_1.\cos\phi_1.\sin\phi_3)\frac{\partial\Delta}{\partial\phi_1} \quad (51)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{12}}{\partial\phi_2} = \frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})/\Delta]}{\partial\phi_2} = (1/\Delta)\frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})]}{\partial\phi_2} - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33}).\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\phi_2}.r_{32} + r_{13}.\frac{\partial r_{32}}{\partial\phi_2} - \frac{\partial r_{12}}{\partial\phi_2}.r_{33} - r_{12}\frac{\partial r_{33}}{\partial\phi_2}\right] - (1/\Delta^2)(r_{13}r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)[(0).r_{32} + r_{13}.(0) - (0).r_{33} - r_{12}.(0)] - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\phi_2}$$

$$= -(1/\Delta^2)(\sin\phi_1.\sin\theta_3.\cos\phi_3 - \sin\theta_1.\cos\phi_1.\sin\phi_3)\frac{\partial\Delta}{\partial\phi_2} \quad (52)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{12}}{\partial\phi_3} = \frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})/\Delta]}{\partial\phi_3} = (1/\Delta)\frac{\partial[(r_{13}r_{32} - r_{12}.r_{33})]}{\partial\phi_3} - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33}).\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\phi_3}.r_{32} + r_{13}.\frac{\partial r_{32}}{\partial\phi_3} - \frac{\partial r_{12}}{\partial\phi_3}.r_{33} - r_{12}.\frac{\partial r_{33}}{\partial\phi_3}\right] - (1/\Delta^2)(r_{13}r_{32} - r_{12}.r_{33})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)[(0).r_{32} + r_{13}.(\sin\theta_3.[-\sin\phi_3]) - (0).r_{33} - r_{12}.(\cos\phi_3)] - (1/\Delta^2)(r_{13}.r_{32} - r_{12}.r_{33}).\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)[(\sin\phi_1.(\sin\theta_3.[-\sin\phi_3]) - \sin\theta_1.\cos\phi_1.(\cos\phi_3)]$$

$$-(1/\Delta^2)(\sin\phi_1.\sin\theta_3.\cos\phi_3 - \sin\theta_1.\cos\phi_1.\sin\phi_3)\frac{\partial\Delta}{\partial\phi_3} \quad (53)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{13}}{\partial\theta_1} = \frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})/\Delta]}{\partial\theta_1} = (1/\Delta)\frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})]}{\partial\theta_1} - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22}).\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial\theta_1}.r_{23} + r_{12}.\frac{\partial r_{23}}{\partial\theta_1} - \frac{\partial r_{13}}{\partial\theta_1}.r_{22} - r_{13}.\frac{\partial r_{22}}{\partial\theta_1}\right] - (1/\Delta^2)(r_{12}r_{23} - r_{13}.r_{22})\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)[(\cos\theta_1.\cos\phi_1).r_{23} + r_{12}.(0) - (0).r_{22} - r_{13}.(0)] - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22})\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)[\cos\theta_1.\cos\phi_1.\sin\phi_2] - (1/\Delta^2)(\sin\theta_1.\cos\phi_1.\sin\phi_2 - \sin\phi_1.\sin\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\theta_1}. \tag{54}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{13}}{\partial\theta_2} = \frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})/\Delta]}{\partial\theta_2} = (1/\Delta)\frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})]}{\partial\theta_2} - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22}).\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial\theta_2}.r_{23} + r_{12}.\frac{\partial r_{23}}{\partial\theta_2} - \frac{\partial r_{13}}{\partial\theta_2}.r_{22} - r_{13}.\frac{\partial r_{22}}{\partial\theta_2}\right] - (1/\Delta^2)(r_{12}r_{23} - r_{13}.r_{22})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)[(0).r_{23} + r_{12}.(0) - (0).r_{22} - r_{13}.([\cos\theta_2].\cos\phi_2)] - (1/\Delta^2)(r_{12}r_{23} - r_{13}r_{22})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)[-\sin\phi_1.([\cos\theta_2].\cos\phi_2)] - (1/\Delta^2)(\sin\theta_1.\cos\phi_1.\sin\phi_2 - \sin\phi_1.\sin\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\theta_2} \tag{55}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{13}}{\partial\theta_3} = \frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})/\Delta]}{\partial\theta_3} = (1/\Delta)\frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})]}{\partial\theta_3} - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22}).\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial\theta_3}.r_{23} + r_{12}.\frac{\partial r_{23}}{\partial\theta_3} - \frac{\partial r_{13}}{\partial\theta_3}.r_{22} - r_{13}.\frac{\partial r_{22}}{\partial\theta_3}\right] - (1/\Delta^2)(r_{12}r_{23} - r_{13}.r_{22})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)[(0).r_{23} + r_{12}.(0) - (0).r_{22} - r_{13}.(0)] - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22})\frac{\partial\Delta}{\partial\theta_3}$$

$$= -(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\sin\phi_2 - \sin\phi_1.\sin\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\theta_3} \tag{56}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{13}}{\partial\phi_1} = \frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})/\Delta]}{\partial\phi_1} = (1/\Delta)\frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})]}{\partial\phi_1} - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22}).\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial\phi_1}.r_{23} + r_{12}.\frac{\partial r_{23}}{\partial\phi_1} - \frac{\partial r_{13}}{\partial\phi_1}.r_{22} - r_{13}.\frac{\partial r_{22}}{\partial\phi_1}\right] - (1/\Delta^2)(r_{12}r_{23} - r_{13}.r_{22})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\sin\theta_1.[-\sin\phi_1]).r_{23} + r_{12}.(0) - (\cos\phi_1).r_{22} - r_{13}.(0)] - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\sin\theta_1.[-\sin\phi_1]).\sin\phi_2 - (\cos\phi_1).\sin\theta_2.\cos\phi_2]$$
$$-(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\sin\phi_2 - \sin\phi_1.\sin\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\phi_1} \tag{57}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{13}}{\partial\phi_2} = \frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})/\Delta]}{\partial\phi_2} = (1/\Delta)\frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})]}{\partial\phi_2} - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22}).\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial \phi_2}.r_{23} + r_{12}.\frac{\partial r_{23}}{\partial \phi_2} - \frac{\partial r_{13}}{\partial \phi_2}.r_{22} - r_{13}.\frac{\partial r_{22}}{\partial \phi_2}\right] - (1/\Delta^2)(r_{12}r_{23} - r_{13}.r_{22})\frac{\partial \Delta}{\partial \phi_2}$$

$$= (1/\Delta)[(0).r_{23} + r_{12}.(\cos\phi_2) - (0).r_{22} - r_{13}.(\sin\theta_2.[-\sin\phi_2])] - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22})\frac{\partial \Delta}{\partial \phi_2}$$

$$= (1/\Delta)[\sin\theta_1.\cos\phi_1.(\cos\phi_2) - \sin\phi_1.(\sin\theta_2.[-\sin\phi_2])]$$
$$-(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\sin\phi_2 - \sin\phi_1.\sin\theta_2.\cos\phi_2)\frac{\partial \Delta}{\partial \phi_2} \quad (58)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{13}}{\partial \phi_3} = \frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})/\Delta]}{\partial \phi_3} = (1/\Delta)\frac{\partial[(r_{12}r_{23} - r_{13}.r_{22})]}{\partial \phi_3} - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22}).\frac{\partial \Delta}{\partial \phi_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial \phi_3}.r_{23} + r_{12}.\frac{\partial r_{23}}{\partial \phi_3} - \frac{\partial r_{13}}{\partial \phi_3}.r_{22} - r_{13}.\frac{\partial r_{22}}{\partial \phi_3}\right] - (1/\Delta^2)(r_{12}r_{23} - r_{13}.r_{22})\frac{\partial \Delta}{\partial \phi_3}$$

$$= (1/\Delta)[(0).r_{23} + r_{12}.(0) - (0).r_{22} - r_{13}.(0)] - (1/\Delta^2)(r_{12}.r_{23} - r_{13}.r_{22})\frac{\partial \Delta}{\partial \phi_3}$$

$$= -(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\sin\phi_2 - \sin\phi_1.\sin\theta_2.\cos\phi_2)\frac{\partial \Delta}{\partial \phi_3} \quad (59)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{21}}{\partial \theta_1} = \frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})/\Delta]}{\partial \theta_1} = (1/\Delta)\frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})]}{\partial \theta_1} - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33}).\frac{\partial \Delta}{\partial \theta_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{23}}{\partial \theta_1}.r_{31} + r_{23}.\frac{\partial r_{31}}{\partial \theta_1} - \frac{\partial r_{21}}{\partial \theta_1}.r_{33} - r_{21}.\frac{\partial r_{33}}{\partial \theta_1}\right] - (1/\Delta^2)(r_{23}r_{31} - r_{21}.r_{33})\frac{\partial \Delta}{\partial \theta_1}$$

$$= (1/\Delta)[(0).r_{23} + r_{31}.(0) - (0).r_{33} - r_{21}.(0)] - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33})\frac{\partial \Delta}{\partial \theta_1}$$

$$= -(1/\Delta^2)(\sin\phi_2.\cos\theta_3.\cos\phi_3 - \cos\theta_2.\cos\phi_2.\sin\phi_3)\frac{\partial \Delta}{\partial \theta_3} \quad (60)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{21}}{\partial \theta_2} = \frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})/\Delta]}{\partial \theta_2} = (1/\Delta)\frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})]}{\partial \theta_2} - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33}).\frac{\partial \Delta}{\partial \theta_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{23}}{\partial \theta_2}.r_{31} + r_{23}.\frac{\partial r_{31}}{\partial \theta_2} - \frac{\partial r_{21}}{\partial \theta_2}.r_{33} - r_{21}.\frac{\partial r_{33}}{\partial \theta_2}\right] - (1/\Delta^2)(r_{23}r_{31} - r_{21}.r_{33})\frac{\partial \Delta}{\partial \theta_2}$$

$$= (1/\Delta)[-([-\sin\theta_2].\cos\phi_2).\sin\phi_3]$$
$$-(1/\Delta^2)(\sin\phi_2.\cos\theta_3.\cos\phi_3 - \cos\theta_2.\cos\phi_2.\sin\phi_3)\frac{\partial \Delta}{\partial \theta_2} \quad (61)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{21}}{\partial \theta_3} = \frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})/\Delta]}{\partial \theta_3} = (1/\Delta)\frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})]}{\partial \theta_3} - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33}).\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{23}}{\partial \theta_3}.r_{31} + r_{23}.\frac{\partial r_{31}}{\partial \theta_3} - \frac{\partial r_{21}}{\partial \theta_3}.r_{33} - r_{21}.\frac{\partial r_{33}}{\partial \theta_3}\right] - (1/\Delta^2)(r_{23}r_{31} - r_{21}.r_{33})\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)[(0).r_{31} + r_{23}.([-\sin\theta_3].\cos\phi_3) - (0).r_{33} - r_{21}.(0)] - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33})\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)[(\sin\phi_2.([-\sin\theta_3].\cos\phi_3)]$$
$$-(1/\Delta^2)(\sin\phi_2.\cos\theta_3.\cos\phi_3 - \cos\theta_2.\cos\phi_2.\sin\phi_3)\frac{\partial\Delta}{\partial\theta_3} \quad (62)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{21}}{\partial\phi_1} = \frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})/\Delta]}{\partial\phi_1} = (1/\Delta)\frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})]}{\partial\phi_1} - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33}).\frac{\partial\Delta}{\partial\phi_1}$$
$$= (1/\Delta)\left[\frac{\partial r_{23}}{\partial\phi_1}.r_{31} + r_{23}.\frac{\partial r_{31}}{\partial\phi_1} - \frac{\partial r_{21}}{\partial\phi_1}.r_{33} - r_{21}.\frac{\partial r_{33}}{\partial\phi_1}\right] - (1/\Delta^2)(r_{23}r_{31} - r_{21}.r_{33})\frac{\partial\Delta}{\partial\phi_1}$$
$$= (1/\Delta)[(0).r_{31} + r_{23}.(0) - (0).r_{33} - r_{21}.(0)] - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33})\frac{\partial\Delta}{\partial\phi_1}$$
$$= -(1/\Delta^2)(\sin\phi_2.\cos\theta_3.\cos\phi_3 - \cos\theta_2.\cos\phi_2.\sin\phi_3)\frac{\partial\Delta}{\partial\phi_1} \quad (63)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{21}}{\partial\phi_2} = \frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})/\Delta]}{\partial\phi_2} = (1/\Delta)\frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})]}{\partial\phi_2} - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33}).\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)\left[\frac{\partial r_{23}}{\partial\phi_2}.r_{31} + r_{23}.\frac{\partial r_{31}}{\partial\phi_2} - \frac{\partial r_{21}}{\partial\phi_2}.r_{33} - r_{21}.\frac{\partial r_{33}}{\partial\phi_2}\right] - (1/\Delta^2)(r_{23}r_{31} - r_{21}.r_{33})\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)[(\cos\phi_2).r_{31} + r_{23}.(0) - (\cos\theta_2.[-\sin\phi_2]).r_{33} - r_{21}.(0)] - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33})\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)[(\cos\phi_2).\cos\theta_3.\cos\phi_3 - (\cos\theta_2.[-\sin\phi_2]).\sin\phi_3]$$
$$-(1/\Delta^2)(\sin\phi_2.\cos\theta_3.\cos\phi_3 - \cos\theta_2.\cos\phi_2.\sin\phi_3)\frac{\partial\Delta}{\partial\phi_2} \quad (64)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{21}}{\partial\phi_3} = \frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})/\Delta]}{\partial\phi_3} = (1/\Delta)\frac{\partial[(r_{23}r_{31} - r_{21}.r_{33})]}{\partial\phi_3} - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33}).\frac{\partial\Delta}{\partial\phi_3}$$
$$= (1/\Delta)\left[\frac{\partial r_{23}}{\partial\phi_3}.r_{31} + r_{23}.\frac{\partial r_{31}}{\partial\phi_3} - \frac{\partial r_{21}}{\partial\phi_3}.r_{33} - r_{21}.\frac{\partial r_{33}}{\partial\phi_3}\right] - (1/\Delta^2)(r_{23}r_{31} - r_{21}.r_{33})\frac{\partial\Delta}{\partial\phi_3}$$
$$= (1/\Delta)[(0).r_{31} + r_{23}.(\cos\theta_3.[-\sin\phi_3]) - (0).r_{33} - r_{21}.(\cos\phi_3)] - (1/\Delta^2)(r_{23}.r_{31} - r_{21}.r_{33})\frac{\partial\Delta}{\partial\phi_3}$$
$$= (1/\Delta)[\sin\phi_2.(\cos\theta_3.[-\sin\phi_3]) - \cos\theta_2.\cos\phi_2.(\cos\phi_3)]$$
$$-(1/\Delta^2)(\sin\phi_2.\cos\theta_3.\cos\phi_3 - \cos\theta_2.\cos\phi_2.\sin\phi_3)\frac{\partial\Delta}{\partial\phi_3} \quad (65)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{22}}{\partial\theta_1} = \frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})/\Delta]}{\partial\theta_1} = (1/\Delta)\frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})]}{\partial\theta_1} - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31}).\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\theta_1}.r_{33} + r_{11}.\frac{\partial r_{33}}{\partial\theta_1} - \frac{\partial r_{13}}{\partial\theta_1}.r_{31} - r_{13}.\frac{\partial r_{31}}{\partial\theta_1}\right] - (1/\Delta^2)(r_{11}r_{33} - r_{13}.r_{31})\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)[([-\sin\theta_1].\cos\phi_1).r_{33} + r_{11}.(0) - (0).r_{31} - r_{13}.(0)] - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31})\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)[([-\sin\theta_1].\cos\phi_1).\sin\phi_3] - (1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\phi_3 - \sin\phi_1.\cos\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\theta_1} \quad (66)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{22}}{\partial\theta_2} = \frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})/\Delta]}{\partial\theta_2} = (1/\Delta)\frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})]}{\partial\theta_2} - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31}).\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\theta_2}.r_{33} + r_{11}.\frac{\partial r_{33}}{\partial\theta_2} - \frac{\partial r_{13}}{\partial\theta_2}.r_{31} - r_{13}.\frac{\partial r_{31}}{\partial\theta_2}\right] - (1/\Delta^2)(r_{11}r_{33} - r_{13}.r_{31})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)[(0).r_{33} + r_{11}.(0) - (0).r_{31} - r_{13}.(0)] - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31})\frac{\partial\Delta}{\partial\theta_2}$$

$$= -(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\phi_3 - \sin\phi_1.\cos\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\theta_2} \quad (67)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{22}}{\partial\theta_3} = \frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})/\Delta]}{\partial\theta_3} = (1/\Delta)\frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})]}{\partial\theta_3} - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31}).\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\theta_3}.r_{33} + r_{11}.\frac{\partial r_{33}}{\partial\theta_3} - \frac{\partial r_{13}}{\partial\theta_3}.r_{31} - r_{13}.\frac{\partial r_{31}}{\partial\theta_3}\right] - (1/\Delta^2)(r_{11}r_{33} - r_{13}.r_{31})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)[(0).r_{33} + r_{11}.(0) - (0).r_{31} - r_{13}.([-\sin\theta_3].\cos\phi_3)] - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)[-\sin\phi_1.([-\sin\theta_3].\cos\phi_3)]$$

$$-(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\phi_3 - \sin\phi_1.\cos\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\theta_3} \quad (68)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{22}}{\partial\phi_1} = \frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})/\Delta]}{\partial\phi_1} = (1/\Delta)\frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})]}{\partial\phi_1} - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31}).\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\phi_1}.r_{33} + r_{11}.\frac{\partial r_{33}}{\partial\phi_1} - \frac{\partial r_{13}}{\partial\phi_1}.r_{31} - r_{13}.\frac{\partial r_{31}}{\partial\phi_1}\right] - (1/\Delta^2)(r_{11}r_{33} - r_{13}r_{31})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\cos\theta_1.[-\sin\phi_1]).r_{33} + r_{11}.(0) - (\cos\phi_1).r_{31} - r_{13}.(0)] - (1/\Delta^2)(r_{11}r_{33} - r_{13}r_{31})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\cos\theta_1.[-\sin\phi_1]).\sin\phi_3 - (\cos\phi_1).\cos\theta_3\cos\phi_3]$$

$$-(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\phi_3 - \sin\phi_1.\cos\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_1} \quad (69)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{22}}{\partial\phi_2} = \frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})/\Delta]}{\partial\phi_2} = (1/\Delta)\frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})]}{\partial\phi_2} - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31}).\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\phi_2}.r_{33} + r_{11}.\frac{\partial r_{33}}{\partial\phi_2} - \frac{\partial r_{13}}{\partial\phi_2}.r_{31} - r_{13}.\frac{\partial r_{31}}{\partial\phi_2}\right] - (1/\Delta^2)(r_{11}r_{33} - r_{13}r_{31})\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)[(0).r_{33} + r_{11}.(0) - (0).r_{31} - r_{13}.(0)] - (1/\Delta^2)(r_{11}r_{33} - r_{13}r_{31})\frac{\partial\Delta}{\partial\phi_2}$$

$$= -(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\phi_3 - \sin\phi_1.\cos\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_2} \quad (70)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{22}}{\partial\phi_3} = \frac{\partial[(r_{11}r_{33} - r_{13}.r_{31})/\Delta]}{\partial\phi_3} = (1/\Delta)\frac{\partial[(r_{11}r_{33} - r_{13}r_{31})]}{\partial\phi_3} - (1/\Delta^2)(r_{11}.r_{33} - r_{13}r_{31})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\phi_3}r_{33} + r_{11}\frac{\partial r_{33}}{\partial\phi_3} - \frac{\partial r_{13}}{\partial\phi_3}r_{31} - r_{13}\frac{\partial r_{31}}{\partial\phi_3}\right] - (1/\Delta^2)(r_{11}r_{33} - r_{13}r_{31})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)[(0).r_{33} + r_{11}.(\cos\phi_3) - (0).r_{31} - r_{13}.(\cos\theta_3.[-\sin\phi_3])] - (1/\Delta^2)(r_{11}.r_{33} - r_{13}.r_{31})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)[\cos\theta_1.\cos\phi_1.(\cos\phi_3) - \sin\phi_1.(\cos\theta_3.[-\sin\phi_3])]$$

$$-(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\phi_3 - \sin\phi_1.\cos\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_3} \quad (71)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{23}}{\partial\theta_1} = \frac{\partial[(r_{13}r_{21} - r_{11}r_{23})/\Delta]}{\partial\theta_1} = (1/\Delta)\frac{\partial[(r_{13}r_{21} - r_{11}r_{23})]}{\partial\theta_1} - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23}).\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\theta_1}r_{21} + r_{13}\frac{\partial r_{21}}{\partial\theta_1} - \frac{\partial r_{11}}{\partial\theta_1}r_{23} - r_{11}\frac{\partial r_{23}}{\partial\theta_1}\right] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)[(0).r_{21} + r_{13}.(0) - ([-\sin\theta_1].\cos\phi_1).r_{23} - r_{11}.(0)] - (1/\Delta^2)(r_{13}.r_{21} - r_{11}.r_{23})\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)[-([-\sin\theta_1].\cos\phi_1).\sin\phi_2]$$

$$-(1/\Delta^2)(\sin\phi_1.\cos\theta_2.\cos\phi_2 - \cos\theta_1.\cos\phi_1.\sin\phi_2)\frac{\partial\Delta}{\partial\theta_1} \quad (72)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{23}}{\partial\theta_2} = \frac{\partial[(r_{13}r_{21} - r_{11}r_{23})/\Delta]}{\partial\theta_2} = (1/\Delta)\frac{\partial[(r_{13}r_{21} - r_{11}r_{23})]}{\partial\theta_2} - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\theta_2}r_{21} + r_{13}\frac{\partial r_{21}}{\partial\theta_2} - \frac{\partial r_{11}}{\partial\theta_2}r_{23} - r_{11}\frac{\partial r_{23}}{\partial\theta_2}\right] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)[(0).r_{21} + r_{13}.([-\sin\theta_2].\cos\phi_2) - (0).r_{23} - r_{11}.(0)] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)[\sin\phi_1.([-\sin\theta_2].\cos\phi_2)] - (1/\Delta^2)(\sin\phi_1.\cos\theta_2.\cos\phi_2 - \cos\theta_1.\cos\phi_1.\sin\phi_2)\frac{\partial\Delta}{\partial\theta_2} \quad (73)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{23}}{\partial\theta_3} = \frac{\partial[(r_{13}r_{21} - r_{11}r_{23})/\Delta]}{\partial\theta_3} = (1/\Delta)\frac{\partial[(r_{13}r_{21} - r_{11}r_{23})]}{\partial\theta_3} - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\theta_3}r_{21} + r_{13}\frac{\partial r_{21}}{\partial\theta_3} - \frac{\partial r_{11}}{\partial\theta_3}r_{23} - r_{11}\frac{\partial r_{23}}{\partial\theta_3}\right] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)[(0).r_{21} + r_{13}.(0) - (0).r_{23} - r_{11}.(0)] - (1/\Delta^2)(r_{13}.r_{21} - r_{11}.r_{23})\frac{\partial\Delta}{\partial\theta_3}$$

$$= -(1/\Delta^2)(\sin\phi_1.\cos\theta_2.\cos\phi_2 - \cos\theta_1.\cos\phi_1.\sin\phi_2)\frac{\partial\Delta}{\partial\theta_3} \quad (74)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{23}}{\partial\phi_1} = \frac{\partial[(r_{13}r_{21} - r_{11}r_{23})/\Delta]}{\partial\phi_1} = (1/\Delta)\frac{\partial[(r_{13}r_{21} - r_{11}r_{23})]}{\partial\phi_1} - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\phi_1}$$
$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\phi_1}r_{21} + r_{13}\frac{\partial r_{21}}{\partial\phi_1} - \frac{\partial r_{11}}{\partial\phi_1}r_{23} - r_{11}\frac{\partial r_{23}}{\partial\phi_1}\right] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\phi_1}$$
$$= (1/\Delta)[(\cos\phi_1).r_{21} + r_{13}.(0) - (\cos\theta_1.[-\sin\phi_1]).r_{23} - r_{11}.(0)] - (1/\Delta^2)(r_{13}.r_{21} - r_{11}.r_{23})\frac{\partial\Delta}{\partial\phi_1}$$
$$= (1/\Delta)[(\cos\phi_1).\cos\theta_2.\cos\phi_2 - (\cos\theta_1.[-\sin\phi_1]).\sin\phi_2]$$
$$-(1/\Delta^2)(\sin\phi_1.\cos\theta_2.\cos\phi_2 - \cos\theta_1.\cos\phi_1.\sin\phi_2)\frac{\partial\Delta}{\partial\phi_1} \quad (75)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{23}}{\partial\phi_2} = \frac{\partial[(r_{13}r_{21} - r_{11}r_{23})/\Delta]}{\partial\phi_2} = (1/\Delta)\frac{\partial[(r_{13}r_{21} - r_{11}r_{23})]}{\partial\phi_2} - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\phi_2}r_{21} + r_{13}\frac{\partial r_{21}}{\partial\phi_2} - \frac{\partial r_{11}}{\partial\phi_2}r_{23} - r_{11}\frac{\partial r_{23}}{\partial\phi_2}\right] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)[(0).r_{21} + r_{13}.(\cos\theta_2.[-\sin\phi_2]) - (0).r_{23} - r_{11}.(\cos\phi_2)] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\phi_2}$$
$$= (1/\Delta)[\sin\phi_1.(\cos\theta_2.[-\sin\phi_2]) - \cos\theta_1.\cos\phi_1.(\cos\phi_2)]$$
$$-(1/\Delta^2)(\sin\phi_1.\cos\theta_2.\cos\phi_2 - \cos\theta_1.\cos\phi_1.\sin\phi_2) \quad (76)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{23}}{\partial\phi_3} = \frac{\partial[(r_{13}r_{21} - r_{11}r_{23})/\Delta]}{\partial\phi_3} = (1/\Delta)\frac{\partial[(r_{13}r_{21} - r_{11}r_{23})]}{\partial\phi_3} - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\phi_3}$$
$$= (1/\Delta)\left[\frac{\partial r_{13}}{\partial\phi_3}r_{21} + r_{13}\frac{\partial r_{21}}{\partial\phi_3} - \frac{\partial r_{11}}{\partial\phi_3}r_{23} - r_{11}\frac{\partial r_{23}}{\partial\phi_3}\right] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\phi_3}$$
$$= (1/\Delta)[(0).r_{21} + r_{13}.(0) - (0).r_{23} - r_{11}.(0)] - (1/\Delta^2)(r_{13}r_{21} - r_{11}r_{23})\frac{\partial\Delta}{\partial\phi_3}$$
$$= -(1/\Delta^2)(\sin\phi_1.\cos\theta_2.\cos\phi_2 - \cos\theta_1.\cos\phi_1.\sin\phi_2)\frac{\partial\Delta}{\partial\phi_3} \quad (77)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{31}}{\partial\theta_1} = \frac{\partial[(r_{21}r_{32} - r_{22}r_{31})/\Delta]}{\partial\theta_1} = (1/\Delta)\frac{\partial[(r_{21}r_{32} - r_{22}r_{31})]}{\partial\theta_1} - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)\left[\frac{\partial r_{21}}{\partial\theta_1}.r_{32} + r_{21}.\frac{\partial r_{32}}{\partial\theta_1} - \frac{\partial r_{22}}{\partial\theta_1}.r_{31} - r_{22}.\frac{\partial r_{31}}{\partial\theta_1}\right] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial\Delta}{\partial\theta_1}$$
$$= (1/\Delta)[(0).r_{32} + r_{21}.(0) - (0).r_{31} - r_{22}.(0)] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial\Delta}{\partial\theta_1}$$
$$= -(1/\Delta^2)(\cos\theta_2.\cos\phi_2.\sin\theta_3.\cos\phi_3 - \sin\theta_2.\cos\phi_2.\cos\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\theta_1} \quad (78)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{31}}{\partial\theta_2} = \frac{\partial[(r_{21}r_{32} - r_{22}r_{31})/\Delta]}{\partial\theta_2} = (1/\Delta)\frac{\partial[(r_{21}r_{32} - r_{22}r_{31})]}{\partial\theta_2} - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{21}}{\partial \theta_2}.r_{32} + r_{21}.\frac{\partial r_{32}}{\partial \theta_2} - \frac{\partial r_{22}}{\partial \theta_2}.r_{31} - r_{22}.\frac{\partial r_{31}}{\partial \theta_2}\right] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \theta_2}$$

$$= (1/\Delta)[([-\sin\theta_2].\cos\phi_2).r_{32} + r_{21}.(0) - (\cos\theta_2.\cos\phi_2).r_{31} - r_{22}.(0)] - (1/\Delta^2)(r_{21}.r_{32} - r_{22}.r_{31})\frac{\partial \Delta}{\partial \theta_2}$$

$$= (1/\Delta)[([-\sin\theta_2].\cos\phi_2).\sin\theta_3.\cos\phi_3 - (\cos\theta_2.\cos\phi_2).\cos\theta_3.\cos\phi_3]$$
$$-(1/\Delta^2)(\cos\theta_2.\cos\phi_2.\sin\theta_3.\cos\phi_3 - \sin\theta_2.\cos\phi_2.\cos\theta_3.\cos\phi_3)\frac{\partial \Delta}{\partial \theta_2} \quad (79)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{31}}{\partial \theta_3} = \frac{\partial[(r_{21}r_{32} - r_{22}r_{31})/\Delta]}{\partial \theta_3} = (1/\Delta)\frac{\partial[(r_{21}r_{32} - r_{22}r_{31})]}{\partial \theta_3} - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{21}}{\partial \theta_3}.r_{32} + r_{21}.\frac{\partial r_{32}}{\partial \theta_3} - \frac{\partial r_{22}}{\partial \theta_3}.r_{31} - r_{22}.\frac{\partial r_{31}}{\partial \theta_3}\right] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)[(0).r_{32} + r_{21}.([\cos\theta_3].\cos\phi_3) - (0).r_{31} - r_{22}.([-\sin\theta_3].\cos\phi_3)] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)[\cos\theta_2.\cos\phi_2.([\cos\theta_3].\cos\phi_3) - \sin\theta_2.\cos\phi_2.([-\sin\theta_3].\cos\phi_3)]$$
$$-(1/\Delta^2)(\cos\theta_2.\cos\phi_2.\sin\theta_3.\cos\phi_3 - \sin\theta_2.\cos\phi_2.\cos\theta_3.\cos\phi_3)\frac{\partial \Delta}{\partial \theta_3} \quad (80)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{31}}{\partial \phi_1} = \frac{\partial[(r_{21}r_{32} - r_{22}r_{31})/\Delta]}{\partial \phi_1} = (1/\Delta)\frac{\partial[(r_{21}r_{32} - r_{22}r_{31})]}{\partial \phi_1} - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \phi_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{21}}{\partial \phi_1}.r_{32} + r_{21}.\frac{\partial r_{32}}{\partial \phi_1} - \frac{\partial r_{22}}{\partial \phi_1}.r_{31} - r_{22}.\frac{\partial r_{31}}{\partial \phi_1}\right] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \phi_1}$$

$$= (1/\Delta)[(0).r_{32} + r_{21}.(0) - (0).r_{31} - r_{22}.(0)] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \phi_1}$$

$$= -(1/\Delta^2)(\cos\theta_2.\cos\phi_2.\sin\theta_3.\cos\phi_3 - \sin\theta_2.\cos\phi_2.\cos\theta_3.\cos\phi_3)\frac{\partial \Delta}{\partial \phi_1} \quad (81)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{31}}{\partial \phi_2} = \frac{\partial[(r_{21}r_{32} - r_{22}r_{31})/\Delta]}{\partial \phi_2} = (1/\Delta)\frac{\partial[(r_{21}r_{32} - r_{22}r_{31})]}{\partial \phi_2} - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \phi_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{21}}{\partial \phi_2}.r_{32} + r_{21}.\frac{\partial r_{32}}{\partial \phi_2} - \frac{\partial r_{22}}{\partial \phi_2}.r_{31} - r_{22}.\frac{\partial r_{31}}{\partial \phi_2}\right] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \phi_2}$$

$$= (1/\Delta)[(\cos\theta_2.[-\sin\phi_2]).r_{32} + r_{21}.(0) - (\sin\theta_2.[-\sin\phi_2]).r_{31} - r_{22}.(0)] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \phi_2}$$

$$= (1/\Delta)[(\cos\theta_2.[-\sin\phi_2]).\sin\theta_3.\cos\phi_3 - (\sin\theta_2.[-\sin\phi_2]).\cos\theta_3.\cos\phi_3]$$
$$-(1/\Delta^2)(\cos\theta_2.\cos\phi_2.\sin\theta_3.\cos\phi_3 - \sin\theta_2.\cos\phi_2.\cos\theta_3.\cos\phi_3)\frac{\partial \Delta}{\partial \phi_2} \quad (82)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{31}}{\partial \phi_3} = \frac{\partial[(r_{21}r_{32} - r_{22}r_{31})/\Delta]}{\partial \phi_3} = (1/\Delta)\frac{\partial[(r_{21}r_{32} - r_{22}r_{31})]}{\partial \phi_3} - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \phi_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{21}}{\partial \phi_3}.r_{32} + r_{21}.\frac{\partial r_{32}}{\partial \phi_3} - \frac{\partial r_{22}}{\partial \phi_3}.r_{31} - r_{22}.\frac{\partial r_{31}}{\partial \phi_3}\right] - (1/\Delta^2)(r_{21}r_{32} - r_{22}r_{31})\frac{\partial \Delta}{\partial \phi_3}$$

$$= (1/\Delta)[(0).r_{32} + r_{21}.(\sin\theta_3.[-\sin\phi_3]) - (0).r_{31} - r_{22}.(\cos\theta_3.[-\sin\phi_3])] - (1/\Delta^2)(r_{21}.r_{32} - r_{22}.r_{31})\frac{\partial \Delta}{\partial \phi_3}$$

$$= (1/\Delta)[\cos\theta_2.\cos\phi_2.(\sin\theta_3.[-\sin\phi_3]) - \sin\theta_2.\cos\phi_2.(\cos\theta_3.[-\sin\phi_3])]$$
$$-(1/\Delta^2)(\cos\theta_2.\cos\phi_2.\sin\theta_3.\cos\phi_3 - \sin\theta_2.\cos\phi_2.\cos\theta_3.\cos\phi_3)\frac{\partial \Delta}{\partial \phi_3} \tag{83}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{32}}{\partial \theta_1} = \frac{\partial[(r_{12}r_{31} - r_{11}r_{32})/\Delta]}{\partial \theta_1} = (1/\Delta)\frac{\partial[(r_{12}r_{31} - r_{11}r_{32})]}{\partial \theta_1} - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial \Delta}{\partial \theta_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial \theta_1}.r_{31} + r_{12}.\frac{\partial r_{31}}{\partial \theta_1} - \frac{\partial r_{11}}{\partial \theta_1}.r_{32} - r_{11}.\frac{\partial r_{32}}{\partial \theta_1}\right] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial \Delta}{\partial \theta_1}$$

$$= (1/\Delta)[(\cos\theta_1.\cos\phi_1).r_{31} + r_{12}.(0) - ([-\sin\theta_1].\cos\phi_1).r_{32} - r_{11}.(0)] - (1/\Delta^2)(r_{12}.r_{31} - r_{11}.r_{32})\frac{\partial \Delta}{\partial \theta_1}$$

$$= (1/\Delta)[(\cos\theta_1.\cos\phi_1).\cos\theta_3.\cos\phi_3 - ([-\sin\theta_1].\cos\phi_1).\sin\theta_3.\cos\phi_3]$$
$$-(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\cos\theta_3.\cos\phi_3 - \cos\theta_1.\cos\phi_1.\sin\theta_3.\cos\phi_3)\frac{\partial \Delta}{\partial \theta_1} \tag{84}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{32}}{\partial \theta_2} = \frac{\partial[(r_{12}r_{31} - r_{11}r_{32})/\Delta]}{\partial \theta_2} = (1/\Delta)\frac{\partial[(r_{12}r_{31} - r_{11}r_{32})]}{\partial \theta_2} - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial \Delta}{\partial \theta_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial \theta_2}.r_{31} + r_{12}.\frac{\partial r_{31}}{\partial \theta_2} - \frac{\partial r_{11}}{\partial \theta_2}.r_{32} - r_{11}.\frac{\partial r_{32}}{\partial \theta_2}\right] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial \Delta}{\partial \theta_2}$$

$$= (1/\Delta)[(0).r_{31} + r_{12}.(0) - (0).r_{32} - r_{11}.(0)] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial \Delta}{\partial \theta_2}$$

$$= -(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\cos\theta_3.\cos\phi_3 - \cos\theta_1.\cos\phi_1.\sin\theta_3.\cos\phi_3)\frac{\partial \Delta}{\partial \theta_2} \tag{85}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{32}}{\partial \theta_3} = \frac{\partial[(r_{12}r_{31} - r_{11}r_{32})/\Delta]}{\partial \theta_3} = (1/\Delta)\frac{\partial[(r_{12}r_{31} - r_{11}r_{32})]}{\partial \theta_3} - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial \theta_3}.r_{31} + r_{12}.\frac{\partial r_{31}}{\partial \theta_3} - \frac{\partial r_{11}}{\partial \theta_3}.r_{32} - r_{11}.\frac{\partial r_{32}}{\partial \theta_3}\right] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)[(0).r_{31} + r_{12}.([-\sin\theta_3].\cos\phi_3) - (0).r_{32} - r_{11}.([\cos\theta_3].\cos\phi_3)] - (1/\Delta^2)(r_{12}.r_{31} - r_{11}.r_{32})\frac{\partial \Delta}{\partial \theta_3}$$

$$= (1/\Delta)[\sin\theta_1.\cos\phi_1.([-\sin\theta_3].\cos\phi_3) - \cos\theta_1.\cos\phi_1.([\cos\theta_3].\cos\phi_3)]$$
$$-(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\cos\theta_3.\cos\phi_3 - \cos\theta_1.\cos\phi_1.\sin\theta_3.\cos\phi_3)\frac{\partial \Delta}{\partial \theta_3} \tag{86}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{32}}{\partial \phi_1} = \frac{\partial[(r_{12}r_{31} - r_{11}r_{32})/\Delta]}{\partial \phi_1} = (1/\Delta)\frac{\partial[(r_{12}r_{31} - r_{11}r_{32})]}{\partial \phi_1} - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial \Delta}{\partial \phi_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial\phi_1}.r_{31} + r_{12}.\frac{\partial r_{31}}{\partial\phi_1} - \frac{\partial r_{11}}{\partial\phi_1}.r_{32} - r_{11}.\frac{\partial r_{32}}{\partial\phi_1}\right] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\sin\theta_1.[-\sin\phi_1]).r_{31} + r_{12}.(0) - (\cos\theta_1.[-\sin\phi_1]).r_{32} - r_{11}.(0)] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\sin\theta_1.[-\sin\phi_1]).\cos\theta_3.\cos\phi_3 - (\cos\theta_1.[-\sin\phi_1]).\sin\theta_3.\cos\phi_3]$$
$$-(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\cos\theta_3.\cos\phi_3 - \cos\theta_1.\cos\phi_1.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_1} \quad (87)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{32}}{\partial\phi_2} = \frac{\partial[(r_{12}r_{31} - r_{11}r_{32})/\Delta]}{\partial\phi_2} = (1/\Delta)\frac{\partial[(r_{12}r_{31} - r_{11}r_{32})]}{\partial\phi_2} - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial\phi_2}.r_{31} + r_{12}.\frac{\partial r_{31}}{\partial\phi_2} - \frac{\partial r_{11}}{\partial\phi_2}.r_{32} - r_{11}.\frac{\partial r_{32}}{\partial\phi_2}\right] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)[(0).r_{31} + r_{12}.(0) - (0).r_{32} - r_{11}.(0)] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial\Delta}{\partial\phi_2}$$

$$= -(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\cos\theta_3.\cos\phi_3 - \cos\theta_1.\cos\phi_1.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_2} \quad (88)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{32}}{\partial\phi_3} = \frac{\partial[(r_{12}r_{31} - r_{11}r_{32})/\Delta]}{\partial\phi_3} = (1/\Delta)\frac{\partial[(r_{12}r_{31} - r_{11}r_{32})]}{\partial\phi_3} - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{12}}{\partial\phi_3}.r_{31} + r_{12}.\frac{\partial r_{31}}{\partial\phi_3} - \frac{\partial r_{11}}{\partial\phi_3}.r_{32} - r_{11}.\frac{\partial r_{32}}{\partial\phi_3}\right] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)[(0).r_{31} + r_{12}.(\cos\theta_3.[-\sin\phi_3]) - (0).r_{32} - r_{11}.(\sin\theta_3.[-\sin\phi_3])] - (1/\Delta^2)(r_{12}r_{31} - r_{11}r_{32})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)[\sin\theta_1.\cos\phi_1.(\cos\theta_3.[-\sin\phi_3]) - \cos\theta_1.\cos\phi_1.(\sin\theta_3.[-\sin\phi_3])]$$
$$-(1/\Delta^2)(\sin\theta_1.\cos\phi_1.\cos\theta_3.\cos\phi_3 - \cos\theta_1.\cos\phi_1.\sin\theta_3.\cos\phi_3)\frac{\partial\Delta}{\partial\phi_3} \quad (89)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{33}}{\partial\theta_1} = \frac{\partial[(r_{11}r_{22} - r_{12}r_{21})/\Delta]}{\partial\theta_1} = (1/\Delta)\frac{\partial[(r_{11}r_{22} - r_{12}r_{21})]}{\partial\theta_1} - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\theta_1}.r_{22} + r_{11}.\frac{\partial r_{22}}{\partial\theta_1} - \frac{\partial r_{12}}{\partial\theta_1}.r_{21} - r_{12}.\frac{\partial r_{21}}{\partial\theta_1}\right] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)[([-\sin\theta_1].\cos\phi_1).r_{22} + r_{11}.(0) - ([\cos\theta_1].\cos\phi_1).r_{21} - r_{12}.(0)] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_1}$$

$$= (1/\Delta)[([-\sin\theta_1].\cos\phi_1).\sin\theta_2.\cos\phi_2 - ([\cos\theta_1].\cos\phi_1).\cos\theta_2.\cos\phi_2]$$
$$-(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\theta_2.\cos\phi_2 - \sin\theta_1.\cos\phi_1.\cos\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\theta_1} \quad (90)$$

$$\frac{\partial[\mathbf{M}^{-1}]_{33}}{\partial\theta_2} = \frac{\partial[(r_{11}r_{22} - r_{12}r_{21})/\Delta]}{\partial\theta_2} = (1/\Delta)\frac{\partial[(r_{11}r_{22} - r_{12}r_{21})]}{\partial\theta_2} - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\theta_2}.r_{22} + r_{11}.\frac{\partial r_{22}}{\partial\theta_2} - \frac{\partial r_{12}}{\partial\theta_2}.r_{21} - r_{12}.\frac{\partial r_{21}}{\partial\theta_2}\right] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)[(0).r_{22} + r_{11}.([\cos\theta_2].\cos\phi_2) - (0).r_{21} - r_{12}.([-\sin\theta_2].\cos\phi_2)] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_2}$$

$$= (1/\Delta)[\cos\theta_1.\cos\phi_1.([\cos\theta_2].\cos\phi_2) - \sin\theta_1.\cos\phi_1.([-\sin\theta_2].\cos\phi_2)]$$

$$-(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\theta_2.\cos\phi_2 - \sin\theta_1.\cos\phi_1.\cos\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\theta_2} \tag{91}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{33}}{\partial\theta_3} = \frac{\partial[(r_{11}r_{22} - r_{12}r_{21})/\Delta]}{\partial\theta_3} = (1/\Delta)\frac{\partial[(r_{11}r_{22} - r_{12}r_{21})]}{\partial\theta_3} - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\theta_3}.r_{22} + r_{11}.\frac{\partial r_{22}}{\partial\theta_3} - \frac{\partial r_{12}}{\partial\theta_3}.r_{21} - r_{12}.\frac{\partial r_{21}}{\partial\theta_3}\right] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_3}$$

$$= (1/\Delta)[(0).r_{22} + r_{11}.(0) - (0).r_{21} - r_{12}.(0)] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\theta_3}$$

$$= -(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\theta_2.\cos\phi_2 - \sin\theta_1.\cos\phi_1.\cos\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\theta_3} \tag{92}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{33}}{\partial\phi_1} = \frac{\partial[(r_{11}r_{22} - r_{12}r_{21})/\Delta]}{\partial\phi_1} = (1/\Delta)\frac{\partial[(r_{11}r_{22} - r_{12}r_{21})]}{\partial\phi_1} - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\phi_1}.r_{22} + r_{11}.\frac{\partial r_{22}}{\partial\phi_1} - \frac{\partial r_{12}}{\partial\phi_1}.r_{21} - r_{12}.\frac{\partial r_{21}}{\partial\phi_1}\right] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\cos\theta_1.[-\sin\phi_1]).r_{22} + r_{11}.(0) - (\sin\theta_1.[-\sin\phi_1]).r_{21} - r_{12}.(0)] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_1}$$

$$= (1/\Delta)[(\cos\theta_1.[-\sin\phi_1]).\sin\theta_2.\cos\phi_2 - (\sin\theta_1.[-\sin\phi_1]).\cos\theta_2.\cos\phi_2]$$

$$-(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\theta_2.\cos\phi_2 - \sin\theta_1.\cos\phi_1.\cos\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\phi_1} \tag{93}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{33}}{\partial\phi_2} = \frac{\partial[(r_{11}r_{22} - r_{12}r_{21})/\Delta]}{\partial\phi_2} = (1/\Delta)\frac{\partial[(r_{11}r_{22} - r_{12}r_{21})]}{\partial\phi_2} - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\phi_2}.r_{22} + r_{11}.\frac{\partial r_{22}}{\partial\phi_2} - \frac{\partial r_{12}}{\partial\phi_2}.r_{21} - r_{12}.\frac{\partial r_{21}}{\partial\phi_2}\right] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)[(0).r_{22} + r_{11}.(\sin\theta_2.[-\sin\phi_2]) - (0).r_{21} - r_{12}.(\cos\theta_2.[-\sin\phi_2])] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_2}$$

$$= (1/\Delta)[\cos\theta_1.\cos\phi_1.(\sin\theta_2.[-\sin\phi_2]) - \sin\theta_1.\cos\phi_1.(\cos\theta_2.[-\sin\phi_2])]$$

$$-(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\theta_2.\cos\phi_2 - \sin\theta_1.\cos\phi_1.\cos\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\phi_2} \tag{94}$$

$$\frac{\partial[\mathbf{M}^{-1}]_{33}}{\partial\phi_3} = \frac{\partial[(r_{11}r_{22} - r_{12}r_{21})/\Delta]}{\partial\phi_3} = (1/\Delta)\frac{\partial[(r_{11}r_{22} - r_{12}r_{21})]}{\partial\phi_3} - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)\left[\frac{\partial r_{11}}{\partial\phi_3}.r_{22} + r_{11}.\frac{\partial r_{22}}{\partial\phi_3} - \frac{\partial r_{12}}{\partial\phi_3}.r_{21} - r_{12}.\frac{\partial r_{21}}{\partial\phi_3}\right] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_3}$$

$$= (1/\Delta)[(0).r_{22} + r_{11}.(0) - (0).r_{21} - r_{12}.(0)] - (1/\Delta^2)(r_{11}r_{22} - r_{12}r_{21})\frac{\partial\Delta}{\partial\phi_3}$$

$$= -(1/\Delta^2)(\cos\theta_1.\cos\phi_1.\sin\theta_2.\cos\phi_2 - \sin\theta_1.\cos\phi_1.\cos\theta_2.\cos\phi_2)\frac{\partial\Delta}{\partial\phi_3} \tag{95}$$