Birkbeck UNIVERSITY OF LONDON

Hypercubes as *dessins d'enfant*

By

Ben Fairbairn

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Abstract

We describe the action of the group $\mathrm{GL}_2(\mathbb{Z})$ on embeddings of hypercubes on compact orientable surfaces, specifically classifying the elements of finite order that can change the genus of the underlying surface by an arbitrarily large amount. In doing so we give an explicit illustration of the kind of computations encountered in the study of *dessins d'enfants* in the hope that those new to the area may find such an explicit example useful.

1 Introduction

We first recall the definition of a *dessin d'enfant.*

Definition 1. *A* ***dessin d'enfant****, or simply a* ***dessin****, is an ordered pair* $(\mathcal{S}, \mathcal{D})$ *where* $\mathcal{S}$ *is an oriented compact surface and* $\mathcal{D} \subset \mathcal{S}$ *is a finite graph such that*

(a) $\mathcal{D}$ *is connected;*

(b) $\mathcal{D}$ *is bipartite;*

(c) $X \setminus \mathcal{D}$ *is a union of finitely many topological spaces all of which are homeomorphic to the unit disc, which we call the* ***faces*** *of the dessin.*

When the underlying surface $\mathcal{S}$ *is clear we tend to write* $\mathcal{D}$ *instead of* $(\mathcal{S}, \mathcal{D})$*. We define the* ***genus*** *of the dessin* $(\mathcal{S}, \mathcal{D})$ *to be the genus of the surface* $\mathcal{S}$ *and write* $g(\mathcal{D})$ *for this.*

The prehistoric study of dessins goes at least as far back as Hamilton's 'icosian calculus' of the 1850s and the later work of Klein on his famous quartic curve in the 1870s [8]. In the 1970s, Jones and Singerman in [7] produced a unified framework for studying related ideas in geometry and combinatorics in the language of maps and hypermaps. These ideas reached the peak of their fame thanks to Grothendieck [4] stimulated by a theorem of Belyĭ in [1] that at the time was considered to be surprising. In particular, Grothendieck suggested that Belyĭ's theorem gave an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of dessins that would provide a means of studying, and thus understanding, this group. As an indication of how difficult understanding this group is, Hilbert's Inverse Galois Problem, arguably the hardest open problem in algebra today, may be thought of as determining if every finite group is a quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by a topologically closed normal subgroup. Indeed, the whole of algebraic number theory is in some sense encoded within this group.

Another approach to understanding $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is through its relationship to other groups. For example, some recent work of Guillot in [3] embeds this group inside the Grothendieck-Teichmüller group $\widehat{\mathcal{GT}}_0$ which is also of interest to mathematical physicists. Here we discuss another, less studied, action on dessins, namely the action of the general linear group $\mathrm{GL}_2(\mathbb{Z})$ that we shall describe in more detail in Section 2.

Some significant differences between $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are the following. The only non-trivial elements of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of finite order are conjugates of complex conjugation whilst in $\mathrm{GL}_2(\mathbb{Z})$ there are several such classes (we shall describe these explicitly in Section 2). Furthermore, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins preserves the underlying genus of a dessin whilst an element of $\mathrm{GL}_2(\mathbb{Z})$ can map a dessin to one with a different genus. We might naturally hope that elements of finite order in $\mathrm{GL}_2(\mathbb{Z})$ resemble elements of finite order in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and have no, or at least a bounded, effect on the genus of the underlying dessins. The following result dashes any such hope with the news that elements of finite order in $\mathrm{GL}_2(\mathbb{Z})$ can be arbitrarily badly behaved.

Theorem 2. *Given $n \in \mathbb{Z}^+$, there exist elements $M \in GL_2(\mathbb{Z})$ of finite order such that there exists a dessin $\mathcal{D}$ with the property that $|g(\mathcal{D}^M) - g(\mathcal{D})| > n$.*

In other words, elements of $\mathrm{GL}_2(\mathbb{Z})$ of finite order can change the genus of a dessin by an arbitrarily large amount.

In fact we prove something slightly stronger — we classify which of the elements of finite order have precisely the property described in the above Theorem. Indeed, the proof of the above gives an explicit and unusually easy to describe example of what is otherwise a somewhat difficult action to visualise and it is hoped that those new to the area may benefit from seeing such a 'down to Earth' example explicitly spelt out.

Several excellent general accounts of these and related matters have appeared in recent years, for example see those given by Girondo and González-Diez in [2, Chapter 4], by Guillot in [3] and by Širán in [10]. The reader is warned that owing to the somewhat disparate history of the subject, the terminology here is far from being fully standardised yet. For example, what Grothendieck considered to be a dessin [4], Jones and Singerman [7] call an 'oriented map' and is today known as a 'clean' dessin in which every vertex in one of the classes (the red/white vertices — see next section) has degree 2 (see [2, Remark 4.3]).

2 Preliminaries

Recall from Definition 1 that a dessin $\mathcal{D}$ is a bipartite graph. Conventionally, the vertices of $\mathcal{D}$ are referred to as being coloured black and white but in the interests of clarity in later diagrams we will consider the vertices here as being blue and red. Aside from this, unless otherwise stated, we follow the conventions adopted in [2].

With any dessin we can associate its 'permutation representation', sometimes called the 'monodromy group' of the dessin, defined as follows. First we fix an orientation of the underlying surface. To obtain our 'blue permutation' we consider a small disc around each of the blue vertices and permute the edges adjoined to that vertex moving around it in a way that is consistent with the chosen orientation. We call this σ_0. The 'red permutation', that we denote σ_1, is defined similarly around the red vertices. Just as the permutations σ_0 and σ_1 can be

Name	(A)	(B)	(C)	(D)
Representative	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Order	1	2	2	2
Name	(E)	(F)	(G)	
Representative	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	
Order	3	4	6	

Table 1: The conjugacy classes of elements of $\mathrm{GL}_2(\mathbb{Z})$ of finite order.

viewed as permutations of the edges around points of the surface located at the vertices of the graph, their composition $\sigma_2 = \sigma_1\sigma_0$ can be viewed as a permutation of the edges around points located at the centers of the faces.

The genus of the underlying surface of the dessin can be deduced from the cycle types of the permutations described in the previous paragraph thanks to the following result, a manifestation of the Euler-Poincaré formula. Given a permutation π we write $\#\pi$ for the number of cycles of π.

Proposition 3. *For a dessin $\mathcal{D}$ of genus g with N edges corresponding to the permutations σ_0, σ_1 and σ_2 we have that*

$$2 - 2g = \#\sigma_0 + \#\sigma_1 + \#\sigma_2 - N.$$

Proof. See [2, Proposition 4.10]. □

The bijective correspondence between the dessins and the 2-generated permutation groups $\langle \sigma_0, \sigma_1 \rangle$ gives us a natural action of $\mathrm{GL}_2(\mathbb{Z})$ on dessins as follows. Any finite 2-generated group is a quotient of the free group on two generators $F_2 := \langle X, Y \mid - \rangle$ by a normal subgroup $M \leq F_2$. Since the action of the inner automorphism group of F_2 is trivial on the set of all such normal subgroups it follows that we have an action of the outer automorphism group $Out(F_2) = Aut(F_2)/Inn(F_2)$ on these subgroups and thus on dessins. For any $n \geq 1$ there is an epimorphism

$$Out(F_n) \to Aut(F_n^{ab} = F_n/F_n' \cong \mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z}).$$

It may be shown that in the case $n = 2$ this epimorphism is in fact an isomorphism defining an action of $\mathrm{GL}_2(\mathbb{Z})$ on dessins. In [5] James showed that this action is faithful.

The elements of finite order in $\mathrm{GL}_2(\mathbb{Z})$ are well known — see the discussion given by Jones and Pinto in [6]. We list these in Table 1.

It will occasionally be useful to refer to what Jones and Singerman call a ‘dart’, that is, an ordered pair (v, e) where v is a vertex and e is an edge adjoined to it.

3 Effects of elements of finite order on genera

3.1 Elements preserving the genus

By Proposition 3 any element of $\mathrm{GL}_2(\mathbb{Z})$ that preserves the set of cycle types of σ_0, σ_1 and σ_2, and thus $\#\sigma_0 + \#\sigma_1 + \#\sigma_2$, will preserve the genus of the corresponding dessin. Now,

- type (A) elements preserve all three permutations and thus their cycle types;
- type (B) elements invert σ_0 and σ_1 whilst

$$\sigma_1^{-1}\sigma_0^{-1} = \sigma_1^{-1}(\sigma_1\sigma_0)^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}\sigma_1$$

  so the cycle types are all preserved;
- type (C) elements interchange σ_0 and σ_1 whilst

$$\sigma_0\sigma_1 = \sigma_0(\sigma_1\sigma_0)\sigma_0^{-1} = \sigma_0\sigma_2\sigma_0^{-1}$$

  so the set of cycle types is preserved;
- type (E) elements simply permute our elements

$$\sigma_0 \mapsto \sigma_1 \mapsto \sigma_2 \mapsto \sigma_0$$

  so the set of cycle types is again preserved;
- finally type (G) elements are simply the product of the type (B) element with a type (E) element and therefore has no effect on the genus by the above calculations.

The type (A), (C) and (E) permutations are sometimes referred to as the 'Machì operations' [9].

3.2 Elements not preserving the genus

It remains to determine the effect of type (D) and type (F) elements. We will first discuss the type (D) elements.

Let n be a positive integer. Recall that the *hypercube* Q_n (sometimes called an 'n-cube') is the graph whose vertices are precisely the 2^n strings of the symbols 0 and 1 of length n with two strings adjoined by an edge if and only if they differ in just one place, so for example in the 7-cube the vertex 0111001 is adjoined to both of 0110001 and 1111001 (among others) but none of 1111101, 0011011, 0100001 and 0000001 (among others). Since a dart is distinguished by a vertex and an edge adjoined to it, in a hypercube we can denote a dart by specifying the vertex with a string of 0s and 1s and distinguishing which entry is changed to determine the edge. For example, in the 7-cube, the dart determined by the vertex 0111001 and the edge whose end points are 0111001 and 0110001 will be denoted $011\underline{1}001$. Note that the darts $011\underline{1}001$ and $011\underline{0}001$ correspond to the same edge.

It is not too difficult to see that we can define a permutation of the edges that is consistent with a choice of orientation of the underlying surface as follows. At a vertex we simply move from left to right along the sequence defining that vertex changing the entries one at a time. Performing this operation at every vertex defined by a string with an even number of entries equal to 1 gives us the 'blue' permutation of the dessin — see Figure 1. Performing the same operation at all the other vertices, that is, the vertices defined by a sequence with an odd number of 1s, defines a red permutation that is consistent with the opposite orientation. To see this, recall that either by definition, if we view a Riemann surface as being a complex 1-manifold, or by the Riemann existence theorem, if we view a Riemann surface as being a complex algebraic curve, we can focus on a simply connected neighborhood of a vertex of the dessin where this all becomes clear — see Figure 2. (This is sometimes called a 'Petrie walk'.)

For example, the permutation defined by the vertices of the hypercube Q_4 corresponding to sequences with an even number of 1s in them using the above procedure is the blue permutation

$$(\underline{0}000, 0\underline{0}00, 00\underline{0}0, 000\underline{0})(\underline{1}100, 1\underline{1}00, 11\underline{0}0, 110\underline{0})(\underline{1}010, 1\underline{0}10, 10\underline{1}0, 101\underline{0})$$

$$(\underline{1}001, 1\underline{0}01, 10\underline{0}1, 100\underline{1})(\underline{0}110, 0\underline{1}10, 01\underline{1}0, 011\underline{0})(\underline{0}101, 0\underline{1}01, 01\underline{0}1, 010\underline{1})$$

$$(\underline{0}011, 0\underline{0}11, 00\underline{1}1, 001\underline{1})(\underline{1}111, 1\underline{1}11, 11\underline{1}1, 111\underline{1}).$$

Applying the same procedure to the other vertices gives a permutation defined by the opposite orientation, but implementing the procedure by moving along the sequence not from left to right but from right to left instead soon remedies this. For example, applying this procedure to the vertices of the 4-cube corresponding to sequences with an odd number of 1s in them gives the red permutation

$$(100\underline{0}, 10\underline{0}0, 1\underline{0}00, \underline{1}000)(010\underline{0}, 01\underline{0}0, 0\underline{1}00, \underline{0}100)(001\underline{0}, 00\underline{1}0, 0\underline{0}10, \underline{0}010)$$

$$(000\underline{1}, 00\underline{0}1, 0\underline{0}01, \underline{0}001)(111\underline{0}, 11\underline{1}0, 1\underline{1}110, \underline{1}110)(110\underline{1}, 11\underline{0}1, 1\underline{1}01, \underline{1}101)$$

$$(101\underline{1}, 10\underline{1}1, 1\underline{0}11, \underline{1}011)(011\underline{1}, 01\underline{1}1, 0\underline{1}11, \underline{0}111).$$

It is clear that this is a degree $n2^{n-1}$ action and that every cycle of the permutations constructed by the above procedures has length n and so the two permutations σ_0 and σ_1 both have 2^{n-1} cycles, that is $\#\sigma_0 + \#\sigma_1 = 2^n$. The only ambiguity in determining the genus of the dessin is the number of cycles of the product of the blue and red permutations by Proposition 3.

Recall that each edge has two names in terms its relationship to its vertices, for example $\underline{0}00\cdots$ and $\underline{1}00\cdots$ are the same edge. Direct calculation now gives us the following.

$$\begin{aligned} \underline{0}000\cdots/\underline{1}000\cdots &\xrightarrow{\sigma_0} 0\underline{0}00\cdots/0\underline{1}00\cdots \\ &\xrightarrow{\sigma_1} \underline{0}100\cdots/\underline{1}100\cdots \\ &\xrightarrow{\sigma_0} 1\underline{0}000\cdots/1\underline{1}00\cdots \\ &\xrightarrow{\sigma_1} \underline{0}000\cdots/\underline{1}000\cdots \end{aligned}$$

Figure 1: The orientation of the blue permutation σ_0.

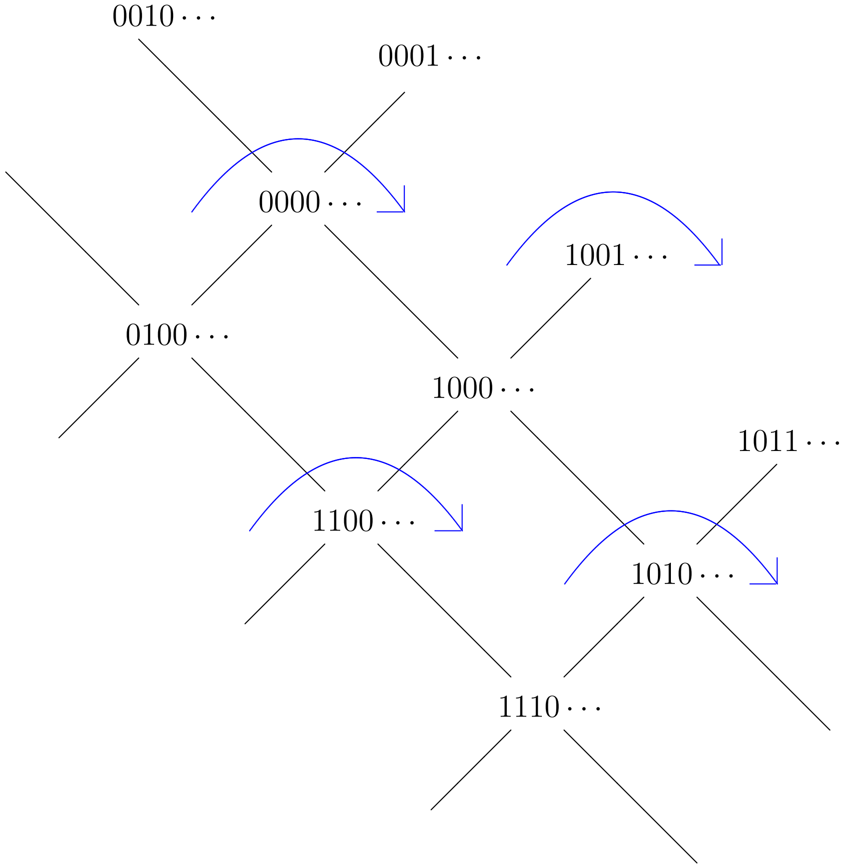
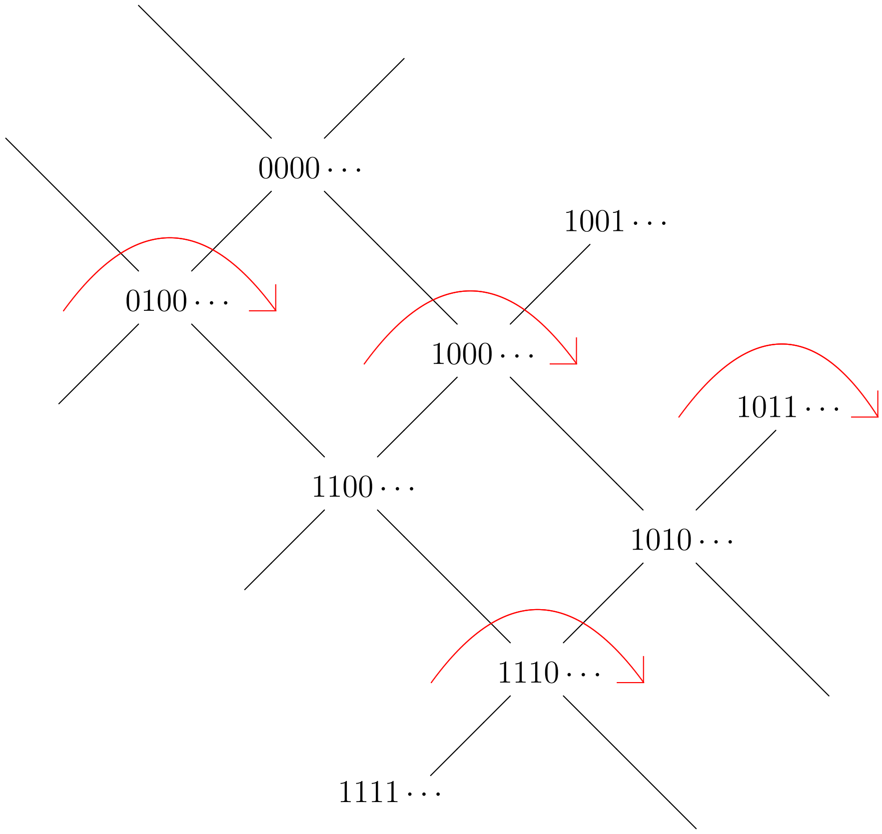


Figure 2: The orientation of the red permutation, σ_1.



It is easy to see from the above that calculations performed on other edges will give similar results and so $o(\sigma_0\sigma_1) = 2$. It follows that σ_2 has $n2^{n-2}$ cycles and so from Proposition 3 the genus in this case for $n > 1$ will be $1 + 2^{n-3}(n-4)$.

If we apply the type (D) element given in Table 1 to this dessin, then this has the effect of inverting one of the two permutations. If we invert σ_1, then to determine the genus of the image of our dessin we need to calculate $\sigma_1^{-1}\sigma_0$. Direct calculation now gives us the following.

$$\underline{0}000\cdots/\underline{1}000\cdots \xrightarrow{\sigma_0} 0\underline{0}00\cdots/0\underline{1}00\cdots \xrightarrow{\sigma_1^{-1}} 01\underline{1}0\cdots/01\underline{0}0\cdots$$

The permutation $\sigma_1^{-1}\sigma_0$ has the effect of 'changing 0/1s in the sequences in pairs'. In particular, we have that $o(\sigma_1^{-1}\sigma_0) = n$ and so this permutation has 2^{n-1} cycles. It follows from Proposition 3 that the genus in this case is $1 + 2^{n-2}(n-3) \geq 1 + 2^{n-3}(n-4)$. In particular, the increase in the genus of our dessin under the action of our element is $2^{n-3}(n-2)$ and this can be made arbitrarily large by taking n to be large enough. This proves Theorem 2.

For completeness, we describe the action of type (F) elements. The element given in Table 1 has the effect of replacing σ_0 with σ_1 and replacing σ_1 with σ_0^{-1}. Since $\sigma_0^{-1}\sigma_1 = (\sigma_1^{-1}\sigma_0)^{-1}$ the effect on the cycle types of these permutations and thus the genus of the underlying surface will be the same as that of the type (D) elements that we described above.

4 Concluding remarks

Question 4. *Does there exist a family of graphs $\mathcal{D}_n$ and a finite order element M such that $g(\mathcal{D}_n^M)/g(\mathcal{D}_n) \to \infty$ as $n \to \infty$.*

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