# BIROn - Birkbeck Institutional Research Online

# Human Dynamics with Limited Complexity

Mark Levene mark@dcs.bbk.ac.uk, Trevor Fenner trevor@dcs.bbk.ac.uk, and George Loizou george@dcs.bbk.ac.uk

Department of Computer Science and Information Systems,
Birkbeck, University of London, London WC1E 7HX, U.K.

**Abstract**

Human dynamics suggest statistical models that may explain and provide us with better insight into human behaviour in various social contexts. Here, we provide a critical overview of human dynamics in the context of complex systems and sociophysics. One of the principal ideas in sociophysics is that, in a similar framework to that of statistical physics, individual humans can be thought of as "social atoms", each exhibiting simple individual behaviour and possessing limited intelligence, but nevertheless collectively giving rise to complex social patterns. In this context, we propose a generative multiplicative decrease process having an *attrition function* that controls the rate of decrease of the population at each stage of the process. The discrete solution to the model takes the form of a product, and a continuous approximation of this solution is derived via the renewal equation that describes age-structured population dynamics. We also investigate some limited complexity variations of the attrition function within this model.

*Keywords:* human dynamics, generative model, multiplicative process, attrition function, survival analysis, rank-order distribution

## 1 Introduction

Social and technological networks are examples of complex social systems [Bar07], which give rise to human dynamics that may be explained by generative stochastic processes. The availability of large data sets, such as mobile phone records, has widened the applicability of human dynamics investigations. On example is the attempt by Schneider et al. [SBCG13] to uncover the characteristics of daily mobility patterns, and another is the research of Brockmann and Thies [BT08] to discover regularities underlying multiscale human mobility, which employed the geographic circulation of individual bank notes as a proxy for human traffic. Such applications for detecting long-range human patterns may also inform about the geographic spread of human diseases.

Human dynamics is not limited to the study of behaviour within communication networks, as can be seen, for example, by the proposal of Mitnitski et al. [MSR13], who apply a simple stochastic queueing model to the complex phenomenon of ageing in order to illustrate how health deficits accumulate with age. In fact, human dynamics has a broader remit, similar to the aims of *sociophysics* [Gal08, SC14], where notions from statistical physics are used to examine social phenomena in a comparable fashion to the investigation of economics phenomena in *econophysics*. (Previously, the terminology *social physics* was in use [Ste50].) One of the principal ideas arising from statistical physics is that individuals can be thought of as "social atoms", each displaying simple behaviour and possessing limited reasoning capacity, however, in aggregate, giving rise to complex social patterns [Buc07, Bal12]. For example, as described in [PS10], the popularity of movies emerges as collective choice behaviour arising from individual, possibly independent, choices. The employed methodology often involves the empirical investigation of collective choice dynamics, resulting in the postulation of statistical laws governing, for example, the universal properties of election results [CMF13]. A fundamental issue in the process of statistical model building is that the model must be tested against experimental data, and superseded by a newer model that better explains the data when such a model is found [Gal08]. Pentland [Pen14] views social physics as bringing together big data about human behaviour and social science theory to create a practical science that can be applied to real-world settings. For example, Pentland looks at how the flow of ideas within a social network can bring about changes in behaviour that result in social actions.

Social physics has a long history going back to the polymath Quetelet in the 19th century, who applied statistical laws to the study of human characteristics and social aptitudes. For example, in deriving the body mass index, he discovered from data that body weight is approximately proportional to the square of the body height [Ekn08]. His main endeavour was to define characteristics of an "average man" by fitting the normal distribution to the data he collected. An early paper by the enigmatic physicist Majorana [Man05], published posthumously in 1942, suggested the possibility of using statistical physics in modelling social phenomena. In particular, he postulated that by treating individuals as "social atoms" it is possible to formulate social statistical laws through empirical observation.

The foundations of 20th century social physics can be attributed to the Princeton physicist Stewart [Ste50], whose research was linked to applying gravitational potential theory to the geographic distribution of populations. Social physics describes social processes using methods from the natural sciences, such as physics, in order to propose theories and laws by analysing empirical data. In particular, any measurable phenomenon involving people or time, or both, is within the scope of social physics. Iberall [Ibe84] attempted to apply the physics of systems, such as statistical mechanics and thermodynamics, to the study of civilisation, starting from the smallest time-scale and building up to larger scales through aggregation. In this view, society is an ensemble of primitive individuals, and their interaction is tracked by identifying the quantities that are conserved upon these interactions. Iberall used the term *homeokinetics* to describe such study of complex systems at all levels from atomistic to social and biological. Weidlich [Wei05] conceived an operational enterprise, termed *sociodynamics*, that constructs master equations that use state variables and the transitions between them, resulting in equations describing the evolution of a probabilistic system. This method has been shown to be applicable to describing several scenarios within the social sciences, such as political opinion formation and migration between geographical regions.

In the context of human dynamics, we have been particularly interested in formulating *generative models* in the form of stochastic processes by which complex systems evolve and give rise to power laws or other distributions [FLL07, FLL12, FLL15]. This type of research builds on the early work of Simon [Sim55], and the more recent work of Barabási's group [AB02] and other researchers [Est11]. In the bigger picture, one can view the goal of such research as being similar to that of *social mechanisms* [HS98], which looks into the processes, or mechanisms, that can explain observed social phenomena. Using an example given in

[Sch98], the growth in the sales of a book can be explained by the well-known logistic growth model [TW02]. More recently, we have shown that the process of conference registration with an early bird deadline can be modelled by bi-logistic growth [FLL13].

In this paper we employ the generative model presented in [FKLL17, FLL18], which defines a stochastic multiplicative decrease process [Mit04, Zan08] that generates a rank-order distribution [SKKV96]; the model is formally introduced in Section 2. A related approach leading to the same underlying equations was presented in [FLL15]; this defines an urn-based stochastic process that captures the essential dynamics of survival analysis applications [KK12]. In both approaches the continuous approximation of the model leads to the well-known transport equation from fluid dynamics [Lax06], which is equivalent to the renewal equation in age-structured models of population dynamics [Cha94]. A fundamental feature of our generative model is the *attrition function*, which controls the rate of decrease of the stochastic process; in Section 3 we will introduce several attrition functions that are of limited complexity.

In [FLL15] we applied the model to a longitudinal data set of popular search engine queries covering 114 months, and showed that the survival function of these queries is closely matched by the solution of our model with power-law attrition (see Subsection 3.5). More recently, we showed that a similar result can be obtained by utilising an exponential mixture model [FLL16]. In addition, in [FLL18] we applied the model to the UK parliamentary election results in 2005, 2010 and 2015, showing that the rank-order distribution produced by our model, using a mixture of power-law attrition functions, closely matches the empirical data (see Subsection 3.5). Moreover, in [FKLL17] we applied the model to the UK 2016 EU referendum results, showing that the rank-order distribution produced by our model, now using a beta-like attrition function, closely matches the data (see Subsection 3.8).

The attrition function is a key feature in capturing the shape of the distribution generated by our model. There is, of course, a large variety of attrition functions to choose from. However, of particular interest to us, are those of limited complexity that, nevertheless, lead to rich dynamics.

Bentley et al. [BHS04, BO11] have studied situations where agents have a limited amount of "intelligence", in the form of copying the behaviour of others, together with possible random drift (which can be viewed as a form of innovation). Both copying and innovation assume very little in terms of what an agent might know when making a decision, provided the decision-making is carried out using only local information based on the agent's active interactions with others agents. Copying can be viewed as a form of preferential attachment, as was shown in [Eva07] using local decision-making in a social network model, and more generally as a mechanism for network evolution [Sim55, Bar07, Per14].

The rest of the paper is organised as follows. In Section 2, we present a generative model in the form of a *multiplicative process* [Mit04, Zan08], which was initially introduced in [FLL15] as an urn-based transfer model; this can also be viewed as a survival model. In Section 3, we describe several limited complexity attrition functions, which we believe may be of practical importance for models of human dynamics. Finally, in Section 4, we give our concluding remarks.

## 2 A multiplicative process for generating a rank-order distribution

In this section we present a generative model in the form of a multiplicative process, which has applications in human dynamics. In its simplest form, a multiplicative process generates a log-normal distribution [JKB94, LSA01], and has applications in many fields, such as economics, biology and ecology [Mit04]. The solution to the multiplicative process we propose generates a rank-ordered distribution [SKKV96], which can be constructed by ranking data objects according to some numerically described feature. This feature is then plotted against rank, and finally the resulting distribution is analysed. Examples of rank-order distributions are: the distribution of large earthquakes [SKKV96], the distribution of oil reserve sizes [LS98], Zipf's rank-frequency distribution [MH99], the size distribution of cities [BGVV99], and the distribution of historical extreme events [CTT$^+$12].

We assume a countable number of indices, where the $i$th index represents the $i$th *object* ranked in descending order according to some numerical feature, which we call the *size* of the object. If the object is, for example, a node in a network, then the numerical feature could be the number of links the object possesses. Similarly, if the object is a candidate in an election district, then the numerical feature could be the number of votes the candidate attained in the district. We will use the term *vote* as a generic numerical feature, so that the size of an object is taken to be the number of votes it has attained.

The stochastic process will proceed over a number of stages. For any stage $s$, $s \geq 0$, we let $\mu(i,s)$, $0 \leq \mu(i,s) \leq 1$, be the probability that the $i$th object "loses" a potential vote at that stage. In the context of survival models $\mu(i,s)$ is often referred to as a *hazard function* [KK12], but here we prefer to call it an *attrition function*, which is more descriptive in the context of human dynamics. We always require that $\mu(0,s) = 0$ for all $s$.

We now let $F(i,s)$, $0 \leq F(i,s) \leq 1$, be a discrete function representing the expected proportion of votes potentially attainable for object $i$ at stage $s$. We postulate a dummy object 0 that satisfies

$$F(0,s) = 1 \quad \text{for} \quad s \geq 0.$$

The dynamics of the multiplicative process are captured by the following two equations:

$$F(i,0) = 0 \quad \text{for} \quad i > 0, \tag{1}$$

and

$$F(i+1,s+1) = \big(1 - \mu(i,s)\big)\, F(i,s) \quad \text{for} \quad 0 \leq i \leq s. \tag{2}$$

Equations (1) and (2) define the expected behaviour of a stochastic process [Ros96] describing how, as $i$ increases, the numbers of votes for successive objects decrease, considering these objects as being less popular. For any particular vote, the attrition function is the probabilistic mechanism that decides whether the vote will be "lost" or not. The process obeys Gibrat's Law [Eec04], which in its original form states that the proportional rate of growth of a firm is independent of its absolute size. In our context, Gibrat's Law states that the proportional rate of decrease in the "popular" vote for successive objects is independent of the actual number of votes cast.

As in [FLL15], we approximate the discrete function $F(i,s)$ by a continuous function $f(i,s)$, and $\mu(i,s)$ is now also a continuous function; $f(i,s)$ is known as the *survival function.*

The boundary conditions are that $f(0,s) = 1$ for $s \geq 0$ and $f(i,0) = 0$ for $i > 0$. The dynamics of the model are now captured by the first-order hyperbolic partial differential equation [Lax06],

$$\frac{\partial f(i,s)}{\partial s} + \frac{\partial f(i,s)}{\partial i} + \mu(i,s) f(i,s) = 0, \tag{3}$$

which is the same as the *renewal equation* encountered in age-structured models of population dynamics [Cha94]. It is also the well-known *transport equation* in fluid dynamics [Lax06].

From Eq. (1.22) in [Cha94], the solution of Eq. (3), when $i \leq s$, is given by

$$f(i,s) = \exp\left(-\int_0^i \mu\left(i-t, s-t\right) dt\right). \tag{4}$$

As noted above, $f(i,s)$ is well-defined provided $i \leq s$. In practice, $s$ is bounded above by the number of available objects, say $n$, and therefore only $n$ stages of Eq.(2) are necessary.

# 3 Limited complexity attrition functions

In this section, we demonstrate the utility of the model by describing some carefully chosen, limited complexity attrition functions that we believe have practical significance. We do not claim to have included all possible limited complexity attrition functions, and our choice of what we consider to be "limited complexity" is guided, not only by ease of computation, but also whether the solution corresponds to some well-known function.

## 3.1 Constant attrition

In the simplest case, we let

$$\mu(i,s) = C,$$

for some positive constant $C$.

Substituting this into (4), we obtain

$$f(i,s) = \exp\left(-Ci\right),$$

which is the survival function of the exponential distribution, with rate parameter $C$.

We have applied the constant attrition function in modelling the popularity of search engine queries with an exponential mixture, where each exponential component has a constant attrition function [FLL16].

## 3.2 Rank-dependent attrition

In this case, we let

$$\mu(i,s) = \frac{\alpha}{i+\kappa},$$

for some positive constants $\alpha$ and $\kappa$.

Substituting this into (4), we obtain

$$f(i,s) = \left(\frac{\kappa}{i+\kappa}\right)^{\alpha},$$

which is a shifted power law function.

It can thus be seen that the decay due to rank-dependent attrition is slower than with constant attrition, since $\mu(i,t)$ decreases with $i$.

### 3.3 Rank-independent attrition

In this case, we let

$$\mu(i,s) = \frac{\kappa}{s},$$

for some positive constant $\kappa$.

Substituting this into (4), we obtain

$$f(i,s) = \left(1 - \frac{i}{s}\right)^{\kappa},$$

which is a polynomial decay function that was investigated in [FLL14] in the context of survival analysis data.

### 3.4 Preferential attrition

In this case, we let

$$\mu(i,s) = \frac{\kappa i}{s^2},$$

for some positive constant $\kappa$.

Substituting this into (4), we obtain

$$f(i,s) = \left(1 - \frac{i}{s}\right)^{\kappa} \exp\left(\frac{\kappa i}{s}\right),$$

which is an exponential function with a polynomial decay, as in [FLL15].

Preferential attrition can be interpreted as having two components, $i/s$ and $\kappa/s$, the first proportional to the size of the object and the second rank-independent.

### 3.5 Power-law attrition

In this case, we let

$$\mu(i,s) = \lambda(1+\rho)i^{\rho},$$

for some shape parameter $\rho$, $-1 \leq \rho \leq 1$, and positive scale parameter $\lambda$.

Substituting this into (4), we obtain

$$f(i,s) = \exp\left(-\lambda i^{1+\rho}\right),$$

which is a Weibull survival function that was investigated in [FLL15] in the context of modelling the popularity of queries, and in [FLL18] in the context of the UK parliamentary elections.

### 3.6 Normal-like attrition

This is a special case of power-law attrition, with $\rho = 1$, i.e.,

$$\mu(i, s) = Ci,$$

for some positive constant $C$.

We thus obtain

$$f(i, s) = \exp\left(-\frac{Ci^2}{2}\right),$$

which is a normal-like survival function.

It is interesting to compare constant attrition to normal-like attrition, where the survival function becomes Gaussian when the attrition increases in proportion to the rank.

### 3.7 Gamma-like attrition

In this case, we let

$$\mu(i, s) = \frac{\alpha}{i + \kappa} + \beta,$$

for some positive constants $\kappa, \alpha$ and $\beta$, which is the sum of rank-dependent and constant attrition functions.

This gives

$$f(i, s) = \left(\frac{\kappa}{i + \kappa}\right)^{\alpha} \exp\left(-\beta i\right),$$

which is a gamma-like survival function.

### 3.8 Beta-like attrition

In this case, we let

$$\mu(i, s) = \frac{\alpha}{i + \kappa} + \frac{\beta}{s},$$

for some positive constants $\kappa, \alpha$ and $\beta$, which is the sum of rank-dependent and rank-independent attrition functions.

This gives

$$f(i, s) = \left(\frac{\kappa}{i + \kappa}\right)^{\alpha} \left(1 - \frac{i}{s}\right)^{\beta},$$

which is a beta-like survival function, investigated in [FKLL17] in the context of the UK 2016 EU referendum.

Figure 1, taken from [FKLL17], shows the Remain and Leave votes for the UK as a whole and for Scotland on its it own, together with the fitted beta-like survival functions. In [FKLL17] we presented the fitted parameters and the $R^2$ values, which indicate very good fits to the empirical data. This can also be seen visually in Figure 1.

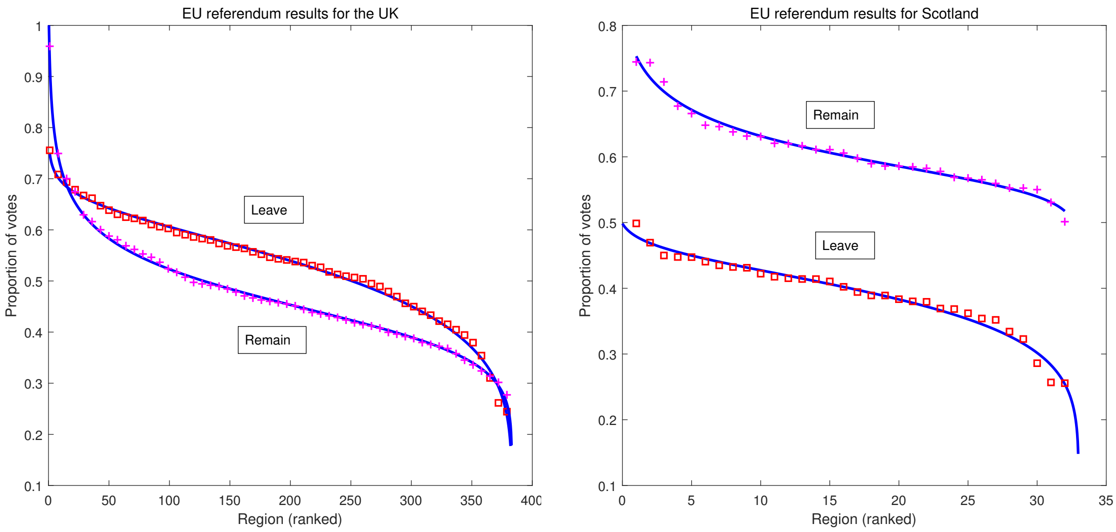


Figure 1: Beta-like regression curves and regional data points for the UK (left) and for Scotland (right).

## 4 Concluding remarks

Human dynamics arise in the context of complex social systems, the goal being to build statistical models that can help us understand and analyse social phenomena concealed in large data sets. Its aims are very similar to those of social physics, which has a long history and is experiencing a revival due to the availability of vast quantities of social data and an abundance of computational power [BW14].

We have concentrated on a particular generative model in the form of a multiplicative decrease process that gives rise to the renewal equation (4) encountered in modelling age-structured population dynamics. One attraction of this model is its flexibility in being able to accommodate the specific attrition function that may be suitable for the application in hand. In particular, we have cherry-picked some particular limited complexity attrition functions that demonstrate the potential of this process in modelling complex social behaviour using relatively simple attrition functions, which may correspond to agents' decision-making.

Although the goals of human dynamics are far from being solved, we believe that such models, akin to those used in statistical physics, will allow significant progress to be made in this field.

## References

[AB02] R. Albert and A.-L. Barabási. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 74:47–97, 2002.

[Bal12] P. Ball. *Why Society is a Complex Matter: Meeting Twenty-first Century Challenges with a New Kind of Science*. Springer-Verlag, Berlin, 2012.

[Bar07] A.-L. Barabási. The architecture of complexity: From network structure to human dynamics. *IEEE Control Systems Magazine*, 27:33–42, 2007.

[BGVV99] S. Brakman, H. Garretsen, C. Van Marrewijk, and M. Van den Burg. The return of Zipf: Towards a further understanding of the rank-size distribution. *Journal of Regional Science*, 39:183–213, 1999.

[BHS04] R.A. Bentley, M.W. Hahn, and S.J. Shennan. Random drift and culture change. *Proceedings of the Royal Society of London, Series B*, 271:1443–1450, 2004.

[BO11] A. Bentley and P. Ormerod. Agents, intelligence, and social atoms. In E. Slingerland and M. Collard, editors, *Creating Consilience: Integrating the Sciences and the Humanities*, pages 205–222. Oxford University Press, New York, NY, 2011.

[BT08] D. Brockmann and F. Theis. Money circulation, trackable items, and the emergence of universal human mobility patterns. *IEEE Pervasive Computing*, 7:28–35, 2008.

[Buc07] M. Buchanan. *The Social Atom: Why the rich get richer, cheats get caught and you neighbor usually looks like you.* Cyan Books and Marshall Cavendish, London, 2007.

[BW14] T.J. Barnes and M.W. Wilson. Big data, social physics, and spatial analysis: The early years. *Big Data & Society*, 1:1–14, 2014.

[Cha94] B. Charlesworth. *Evolution in age-structured populations.* Cambridge Studies in Mathematical Biology: 13. Cambridge University Press, Cambridge, UK, second edition, 1994.

[CMF13] A. Chatterjee, M. Mitrović, and S. Fortunato. Universality in voting behavior: an empirical analysis. *Nature Scientific Reports*, 3:1049, 2013.

[CTT+12] C.-C. Chen, C.-Y. Tseng, L. Telesca, S.-C. Chi, and L.-C. Sun. Collective Weibull behavior of social atoms: Application of the rank-ordering statistics to historical extreme events. *Europhysics Letters*, 97:48010–1–48010–6, 2012.

[Eec04] J. Eeckhout. Gibrat's Law for (all) cities. *The American Economic Review*, 94:1429–1451, 2004.

[Ekn08] G. Eknoyan. Adolphe Quetelet (1796 – 1874) – the average man and indices of obesity. *Nephrology Dialysis Transplantation*, 23:47–51, 2008.

[Est11] E. Estrada. *The Structure of Complex Networks: Theory and Applications.* Oxford University Press, Oxford, 2011.

[Eva07] T.S. Evans. Exact solutions for network rewiring models. *European Physical Journal B*, 56:65–69, 2007.

[FKLL17] T. Fenner, E. Kaufmann, M. Levene, and G. Loizou. A multiplicative process for generating a beta-like survival function with application to the UK 2016 EU referendum results. *International Journal of Modern Physics C*, 28:1750132, 2017. 14 pages.

[FLL07] T. Fenner, M. Levene, and G. Loizou. A model for collaboration networks giving rise to a power-law distribution with an exponential cutoff. *Social Networks*, 29:70–80, 2007.

[FLL12] T. Fenner, M. Levene, and G. Loizou. A discrete evolutionary model for chess players ratings. *IEEE Transactions on Computational Intelligence and AI in Games*, 4:84–93, 2012.

[FLL13] T. Fenner, M. Levene, and G. Loizou. A bi-logistic growth model for conference registration with an early bird deadline. *Central European Journal of Physics*, 11:904–909, 2013.

[FLL14] T. Fenner, M. Levene, and G. Loizou. A stochastic evolutionary model for survival dynamics. *Physica A*, 410:595–600, 2014.

[FLL15] T. Fenner, M. Levene, and G. Loizou. A stochastic evolutionary model for capturing human dynamics. *Journal of Statistical Mechanics: Theory and Experiment*, 2015:P08015, August 2015.

[FLL16] T. Fenner, M. Levene, and G. Loizou. A stochastic evolutionary model generating a mixture of exponential distributions. *European Physical Journal B*, 89:1–7, 2016.

[FLL18] T. Fenner, M. Levene, and G. Loizou. A multiplicative process for generating the rank-order distribution of UK election results. *Quality & Quantity*, 52:1069–1079, 2018.

[Gal08] S. Galam. Sociophysics: A review of Galam models. *Journal of Modern Physics C*, 19:409–440, 2008.

[HS98] P. Hedström and R. Swedberg. Social mechanisms: An introductory essay. In P. Hedström and R. Swedberg, editors, *Social Mechanisms: An Analytical Approach to Social Theory*, pages 1–31. Cambridge University Press, Cambridge, UK, 1998.

[Ibe84] A. Iberall. Contributions to a physical science for the study of civilization. *Journal of Social and Biological Structures*, 7:259–283, 1984.

[JKB94] N.L. Johnson, S. Kotz, and N. Balkrishnan. *Continuous Univariate Distributions, Volume 1*, chapter 13 Normal distributions, pages 80–206. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York, NY, second edition, 1994.

[KK12] D.G. Kleinbaum and M. Klein. *Survival Analysis, A Self-Learning Text.* Springer Science+Business Media, LLC, New York, NY, third edition, 2012.

[Lax06] P.D. Lax. *Hyperbolic Partial Differential Equations.* Courant Lecture Notes. American Mathematical Society, Providence, RI, 2006.

[LS98] J. Laherrère and D. Sornette. Stretched exponential distributions in nature and economy: fat tails with characteristic scales. *European Physical Journal B*, 2:525–539, 1998.

[LSA01] E. Limpert, W.A. Stahel, and M. Abbt. Log-normal distributions across the sciences: Keys and clues. *BioScience*, 51:341–352, 2001.

[Man05] N.R. Mantegna. Presentation of the English translation of Ettore Majorana's paper: The value of statistical laws in physics and social sciences. *Quantitative Finance*, 5:133–140, 2005.

[MH99] C.D. Manning and H.Schütze. *Foundations of Statistical Natural Language Processing: Section 1.4.3*. MIT Press, Cambridge, MA., 1999.

[Mit04] M. Mitzenmacher. A brief history of generative models for power law and lognormal distributions. *Internet Mathematics*, 1:226–251, 2004.

[MSR13] A. Mitnitski, X. Song, and K. Rockwood. Assessing biological aging: the origin of deficit accumulation. *Biogerontology*, 14:709–717, 2013.

[Pen14] A. Pentland. *Social Physics: How Good Ideas Spread–The Lessons from a New Science*. Penguin Press, New York, NY, 2014.

[Per14] M. Perc. The Matthew effect in empirical data. *Journal of the Royal Society Interface*, 11:20140378, 2014.

[PS10] R.K. Pan and S. Sinha. The statistical laws of popularity: universal properties of the box-office dynamics of motion pictures. *New Journal of Physics*, 12:115004 (23pp), 2010.

[Ros96] S.M. Ross. *Stochastic Processes*. John Wiley & Sons, New York, NY, second edition, 1996.

[SBCG13] C.M. Schneider, V. Belik, T. Couronné, and M.C. González. Unravelling daily human mobility motifs. *Journal of the Royal Society Interface*, 10:20130246, 2013.

[SC14] P. Sen and B.K. Chakrabarti. *Sociophysics: An Introduction*. Oxford University Press, Oxford, 2014.

[Sch98] T.C. Schelling. Social mechanisms and social dynamics. In P. Hedström and R. Swedberg, editors, *Social Mechanisms: An Analytical Approach to Social Theory*, pages 32–44. Cambridge University Press, Cambridge, UK, 1998.

[Sim55] H.A. Simon. On a class of skew distribution functions. *Biometrika*, 42:425–440, 1955.

[SKKV96] D. Sornette, L. Knopoff, Y.Y. Kagan, and C. Vanneste. Rank-ordering statistics of extreme events: Application to the distribution of large earthquakes. *Journal of Geophysical Research*, 101:13–883–13–893, 1996.

[Ste50] J.Q. Stewart. The development of social physics. *American Journal of Physics*, 18:239–253, 1950.

[TW02] A. Tsoularis and J. Wallace. Analysis of logistic growth models. *Mathematical Biosciences*, 179:21–55, 2002.

[Wei05] W Weidlich. Thirty years of sociodynamics. An integrated strategy of modelling in the social sciences: applications to migration and urban evolution. *Chaos, Solitons and Fractals*, 24:45–56, 2005.

[Zan08] D.H. Zanette. Multiplicative processes and city sizes. In S. Albeverio, D. Andrey, P. Giordano, and A. Vancheri, editors, *The Dynamics of Complex Urban Systems An Interdisciplinary Approach*, pages 457–472. Physica-Verlag, Heidelberg, 2008.