BIROn - Birkbeck Institutional Research Online

On the size-Ramsey number of cycles

R. Javadi[a], F. Khoeini[a], G.R. Omidi[a,c,1], A. Pokrovskiy[b,2]

[a]Department of Mathematical Sciences, Isfahan University of Technology,
Isfahan, 84156-83111, Iran

[b]Department of Mathematics, ETH Zürich, Switzerland

[c]School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran

rjavadi@cc.iut.ac.ir, romidi@cc.iut.ac.ir,
f.khoeini@math.iut.ac.ir, dr.alexey.pokrovskiy@gmail.com

Abstract

For given graphs $G_1, \ldots, G_k$, the size-Ramsey number $\hat{R}(G_1, \ldots, G_k)$ is the smallest integer m for which there exists a graph H on m edges such that in every k-edge coloring of H with colors $1, \ldots, k$, H contains a monochromatic copy of G_i of color i for some $1 \leq i \leq k$. We denote $\hat{R}(G_1, \ldots, G_k)$ by $\hat{R}_k(G)$ when $G_1 = \cdots = G_k = G$.

Haxell, Kohayakawa and Łuczak showed that the size-Ramsey number of a cycle C_n is linear in n i.e. $\hat{R}_k(C_n) \leq c_k n$ for some constant c_k. Their proof, however, is based on the regularity lemma of Szemerédi and so no specific constant c_k is known.

In this paper, we give various upper bounds for the size-Ramsey numbers of cycles. We provide an alternative proof of $\hat{R}_k(C_n) \leq c_k n$, avoiding the use of the regularity lemma, where c_k is exponential and doubly-exponential in k, when n is even and odd, respectively. In particular, we show that for sufficiently large n we have $\hat{R}(C_n, C_n) \leq 10^5 \times cn$, where $c = 6.5$ if n is even and $c = 1989$ otherwise.

Keywords: Ramsey number, Size Ramsey number, Random graphs, Cycles.
AMS subject classification: 05C55, 05D10

1 Introduction

For given graphs $G_1, \ldots, G_k$ and a graph H, we say that H is *Ramsey* for $(G_1, \ldots, G_k)$ and we write $H \longrightarrow (G_1, \ldots, G_k)$, if no matter how one colors the edges of H with k colors $1, \ldots, k$, there exists a monochromatic copy of G_i of color i in H, for some $1 \leq i \leq k$. Ramsey's theorem [16] states that for given graphs $G_1, \ldots, G_k$, there exists a graph H that is Ramsey for $(G_1, \ldots, G_k)$. Note that, if a graph H is Ramsey for $(G_1, \ldots, G_k)$ and H is a subgraph of H', then H' is also Ramsey for $(G_1, \ldots, G_k)$. In this view, in order to study the collection of graphs which are Ramsey for $(G_1, \ldots, G_k)$, it suffices to study the collection $\mathcal{F}(G_1, \ldots, G_k)$ of graphs which are minimal subject to being Ramsey for $(G_1, \ldots, G_k)$. These graphs are called *Ramsey minimal* for $(G_1, \ldots, G_k)$.

Many interesting problems in graph theory concern the study of various parameters related to Ramsey minimal graphs for $(G_1, \ldots, G_k)$. The most well-known and well-studied

---

one, is the smallest number of vertices of a graph in $\mathcal{F}(G_1, \ldots, G_k)$ which is referred to as the *Ramsey number* of $(G_1, \ldots, G_k)$ and is denoted by $R(G_1, \ldots, G_k)$. In diagonal case, where $G = G_1 = \cdots = G_k$, we may write $R_k(G)$ for $R(G_1, \ldots, G_k)$. Estimating $R(K_n) = R_2(K_n)$ is one of the main open problems in Ramsey theory. Erdős [8] and Erdős and Szekeres [10] showed that $2^{n/2} \leq R(K_n) \leq 2^{2n}$, and despite a lot of efforts, there have not been much improvements to the exponents of the bounds. For further results about the Ramsey numbers of graphs, see [5, 15] and the references therein.

In this paper, we consider another well-studied parameter called the *size-Ramsey number* $\hat{R}(G_1, \ldots, G_k)$ of the given graphs $G_1, \ldots, G_k$, which is defined as the minimum number of edges of a graph in $\mathcal{F}(G_1, \ldots, G_k)$. The investigation of the size-Ramsey numbers of graphs was initiated by Erdős, Faudree, Rousseau, and Schelp [9] in 1978. Since then, the size-Ramsey numbers of graphs have been studied with particular focus on the case of trees, bounded degree graphs and sparse graphs. The survey paper due to Faudree and Schelp [11] collects some results about size-Ramsey numbers.

One of the most studied directions in this area is the size-Ramsey number of paths. In 1983 Beck [4], showed that $\hat{R}(P_n) = \hat{R}(P_n, P_n) < 900\,n$ for sufficiently large n, where P_n is a path on n vertices. This verifies the linearity of the size-Ramsey number of paths in terms of the number of vertices and from then, different approaches were attempted by several authors to reduce the constant coefficient in the upper bound, see [3, 6, 14]. Most of these approaches are based on the classic models of random graphs. Currently, the best known upper bound is due to Dudek and Prałat. [7] which proved that $\hat{R}(P_n) \leq 74\,n$, for sufficiently large n.

In this paper, we investigate the size-Ramsey number of cycles. The linearity of $\hat{R}_k(C_n)$ (in terms of n) follows from the earlier result by Haxell, Kohayakawa and Łuczak [13]. Nevertheless, their proof is based on the regularity lemma and therefore is unable to determine a specific constant coefficient. The standard techniques for proving a linear bounds for paths, avoiding the use of the regularity lemma, seems to be insufficient to prove a linear bound for cycles. Here, we give such a proof for the following theorem.

Theorem 1.1. *Let $n_1, n_2, \ldots, n_t$ be a sequence of positive integers with t_e even numbers and t_o odd numbers. Let $c = 4.6 \times 10^{2^{t_o}-1} \times 15^{t_e}$, $n = \max(n_1, \ldots, n_t)$ and suppose that for all i, we have $n_i \geq 2\lceil \log(nc) \rceil + 2$. Then*

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leq (\ln c + 1)\,c^2\,n.$$

The above theorem is proved by showing that an Erdős-Renyi random graph with suitable edge probability is almost surely a Ramsey graph for a collection of cycles. By considering binomial random bipartite graph model we will give further improvement on the bound in Theorem 1.1 (see Theorems 3.2 and 3.4).

Throughout the paper, the notations $\log x$ and $\ln x$ refer to the logarithms to the bases 2 and Euler's number e, respectively. Also, for a graph G and a subset $S \subseteq V(G)$, $N_G(S)$ stands for the set of all vertices of G which have at least one neighbor in S.

2   Cycles versus a complete bipartite graph

In this section, we prove some auxiliary results which will later be used to bound the size-Ramsey numbers of cycles. Specifically, we prove some linear upper bounds (in terms

of the number of vertices) for the Ramsey and bipartite Ramsey numbers of cycles versus a complete bipartite graph. First, we give some definitions and lemmas.

A rooted tree with at most two children for each vertex is called a *binary tree*. The *depth* of a vertex in a binary tree T is the distance from the vertex to the root of T and the maximum depth of any vertex is called the *height* of T. If a tree has only one vertex (the root), the height is zero. A *perfect binary tree* is a binary tree with all leaves at the same depth where every internal vertex (non-leaf vertex) has exactly two children. Now we begin with the following lemma.

Lemma 2.1. *For every positive integer $n \geq 2$, there is a binary tree of height $\lceil \log n \rceil$ and at most $2n + \lceil \log n \rceil - 2$ vertices which has exactly n leaves, all of the same depth.*

Proof. We give a proof by induction on n. The case $n = 2$ is trivial. Now assume that $k \geq 3$ and the statement of the lemma is true for $n < k$. We are going to prove the statement for $n = k$. If $k = 2^t$ for some t, then clearly the perfect binary tree of height t has exactly k leaves and $2k - 1$ vertices and we are done. Now, assume that $k = 2^{t_1} + \cdots + 2^{t_r}$, where $r \geq 2$ and $t_1 > \cdots > t_r \geq 0$. For each $1 \leq i \leq r$, let T_i be the perfect binary tree of height t_i with $2^{t_i+1} - 1$ vertices and 2^{t_i} leaves. Now, we construct a binary tree T as follows. Consider the vertex disjoint binary trees $T_1, \ldots, T_r$ with roots $x_1, \ldots, x_r$ and a new path $P = v_1 \ldots v_{t_1-t_r+1}$ and add an edge from $v_{t_1-t_i+1}$ to the root x_i of T_i for each $1 \leq i \leq r$. One can easily check that T is a binary tree rooted at v_1 of height $t_1 + 1$ with $k = 2^{t_1} + \cdots + 2^{t_r}$ leaves and

$$|V(T)| = \sum_{i=1}^{r} 2^{t_i+1} - r + t_1 - t_r + 1 \leq 2k + t_1 - 1$$

vertices. Clearly $\lceil \log k \rceil = t_1 + 1$ and so T is a binary tree with k leaves of depth $\lceil \log k \rceil$ and at most $2k + \lceil \log k \rceil - 2$ vertices. $\square$

We also need the following tree-universality result due to Haxell and Kohayakawa.

Theorem 2.2. [12] *Let $1 \leq d \leq t$ be fixed integers. Suppose that G is a bipartite graph with associated bipartition (V_1, V_2), such that for every subset $S \subseteq V_i$ ($i \in \{1, 2\}$) with $|S| \leq 2t/d$, we have $|N_G(S)| \geq 2d|S|$. Then, G contains as a subgraph every tree with maximum degree at most d whose each bipartition class has at most t vertices.*

The above theorem is used to prove the following lemma about finding red paths in a 2-coloured balanced complete bipartite graph. The proof technique of this lemma is similar to the techniques from [2].

Lemma 2.3. *Let n, m_1, m_2 be positive integers such that n is even and $\min\{m_1, m_2\} \geq n \geq (\lceil \log m_1 \rceil + \lceil \log m_2 \rceil + 1)$. Suppose that we have a 2-edge-colored $K_{7m_1+8m_2, 8m_1+7m_2}$ with colors red and blue and bipartition classes V_1 and V_2 which has no blue $K_{m,m}$. Then for each $i \in \{1, 2\}$, there is $V_i' \subseteq V_i$ with $|V_i'| = m_i$ such that for every $x \in V_1'$ and $y \in V_2'$ there is a red path of length $n - 1$ from x to y.*

Proof. Assume that the edges of $H = K_{N_1, N_2}$ are colored by red and blue and (V_1, V_2) is the bipartition of H with $|V_1| = N_1 = 7m_1 + 8m_2$ and $|V_2| = N_2 = 8m_1 + 7m_2$. Let H_r

and H_b be the subgraphs of H induced on the red and blue edges, respectively. By our assumption, H_b is $K_{m,m}$-free. Thus, we have

$$\forall\, i,j \in \{1,2\},\ |N_{H_r}(S)| > N_{i+1} - m_{j+1}, \quad \text{for every } S \subseteq V_i \text{ with} \quad |S| \geq m_j, \quad (2.1)$$

(reading $i+1$ modulo 2).

Claim 1. There is an induced subgraph $G \subseteq H_r$ with parts $\hat{V}_i \subseteq V_i$, $i = 1, 2$, which satisfies

$$\forall\, i \in \{1,2\}, \quad |N_G(S)| \geq 6|S|, \quad \text{for every } S \subseteq \hat{V}_i \text{ with } |S| \leq m_1 + m_2. \quad (2.2)$$

Proof of the claim. Define

$$\mathcal{E} = \{(X_0, X_1) \mid X_i \subseteq V_i, |X_i| \leq m_1 + m_2, |N_{H_r \setminus X_{i+1}}(X_i)| \leq 6|X_i|, \text{ for each } i \in \{1,2\}\},$$

(reading $i+1$ modulo 2) and let (A_1, A_2) be a pair in $\mathcal{E}$ with largest $|A_1| + |A_2|$. We claim that $|A_i| < m_{i+1}$ ($i \in \{1,2\}$), since otherwise, by (2.1), we have

$$N_{i+1} - m_i - |A_{i+1}| < |N_{H_r \setminus A_{i+1}}(A_i)| \leq 6|A_i|,$$

which implies that $N_{i+1} < 6|A_i| + |A_{i+1}| + m_i \leq 7m_1 + 7m_2 + m_i = N_{i+1}$, which is a contradiction. Therefore, $|A_i| < m_{i+1}$ ($i \in \{1,2\}$).

Now, for each $i \in \{1,2\}$, let $\hat{V}_i = V_i \setminus A_i$ and G be the induced subgraph of H_r on $\hat{V}_1 \cup \hat{V}_2$. To see (2.2), for some $i \in \{1,2\}$, let $S \subseteq \hat{V}_i$ be a subset with $|S| \leq m_1 + m_2$. For the contrary, suppose that $|N_G(S)| < 6|S|$. Then

$$|N_{H_r \setminus A_{i+1}}(S \cup A_i)| \leq |N_G(S)| + |N_{H_r \setminus A_{i+1}}(A_i)| < 6|S| + 6|A_i| = 6|S \cup A_i|.$$

Also, $|N_{H_r \setminus (S \cup A_i)}(A_{i+1})| \leq |N_{H_r \setminus A_i}(A_{i+1})| \leq 6|A_{i+1}|$. Thus, by maximality of (A_1, A_2), we have $|S \cup A_i| > m_1 + m_2$. On the other hand, we have $|A_i| < m_{i+1}$. Therefore, $|S| > m_i$. Hence, using (2.1), we have

$$N_{i+1} - m_{i+1} < |N_{H_r}(S)| \leq |N_G(S)| + |A_{i+1}| < 6|S| + m_i \leq 6m_1 + 6m_2 + m_i,$$

which implies that $N_{i+1} < 7m_1 + 7m_2$, again a contradiction. This proves the claim. $\square$

Now, let G be the subgraph of H_r which satisfies (2.2). In the light of Theorem 2.2, G (and so H_r) contains a copy of any tree T with maximum degree 3 which size of its bipartition classes is at most $3/2(m_1 + m_2)$. Now, for $i \in \{1,2\}$, let T_i be a binary tree with m_i leaves of depth $\lceil \log m_i \rceil$ and at most $2m_i + \lceil \log m_i \rceil - 2$ vertices (which exists due to Lemma 2.1). Also, let T be a tree on at most $n + 2m_1 + 2m_2$ vertices formed by attaching the roots of T_1 and T_2 by a path of length $n - 1 - \lceil \log m_1 \rceil - \lceil \log m_2 \rceil$. Note that T has maximum degree 3 with leaves $x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}$, where there is a path of length $n - 1$ from x_i to y_j for every $1 \leq i \leq m_1$ and $1 \leq j \leq m_2$. Also note that since n is even, $\{x_1, \ldots, x_{m_1}\}$ and $\{y_1, \ldots, y_{m_2}\}$ are contained in different parts of the bipartition of T. Without loss of generality, we can assume that $V_1' = \{x_1, \ldots, x_{m_1}\} \subseteq V_1$ and $V_2' = \{y_1, \ldots, y_{m_2}\} \subseteq V_2$. Thus, it is clear that the size of the bipartition class of T contained in V_i ($i \in \{1,2\}$) is at most $n/2 + 3/2\, m_i + m_{i+1} \leq 3/2(m_1 + m_2)$. Hence, by Theorem 2.2, H_r contains a copy of T. This completes the proof. $\square$

4

Given bipartite graphs $G_1, \ldots, G_k$, the *bipartite Ramsey number $BR(G_1, \ldots, G_k)$* is defined as the smallest integer b such that for any edge coloring of the complete bipartite graph $K_{b,b}$ with k colors $1, \ldots, k$, there exists a monochromatic copy of G_i of color i in $K_{b,b}$, for some $1 \leq i \leq k$. In other words, it is the smallest integer b such that $K_{b,b} \to (G_1, \ldots, G_k)$. The above lemma can be used to give an upper bound for the bipartite Ramsey number of an even cycle versus a complete bipartite graph.

Lemma 2.4. *Let n, m be positive integers such that n is even with $m \geq n \geq 2\lceil \log m \rceil + 1$. Then, $BR(C_n, K_{m,m}) \leq 15m$.*

Proof. Let H be a 2-edge-colored complete bipartite graph with bipartition (V_1, V_2) such that $|V_1| = |V_2| = 15m$. Suppose that H contains no blue $K_{m,m}$. To prove the lemma, it is enough to show that H contains a red C_n.

By Lemma 2.3, there are $V_i' \subseteq V_i$ with $|V_i'| = m$ ($i \in \{1, 2\}$) such that for every $x \in V_1'$ and $y \in V_2'$ there is a red path of length $n - 1$ from x to y. Since H has no blue $K_{m,m}$, there is a red edge xy for some $x \in V_1'$ and $y \in V_2'$. Adding this edge to the red path of length $n - 1$ from x to y gives a red cycle of length n as required. $\square$

The following corollary is an immediate consequence of Lemma 2.4.

Corollary 2.5. *Let m and $n_1, \ldots, n_t$ be positive integers such that for every $1 \leq i \leq t$, n_i is even and $m \geq n_i \geq 2\lceil \log(15^{t-1}m) \rceil + 1$. Then, $BR(C_{n_1}, \ldots, C_{n_t}, K_{m,m}) \leq 15^t m$.*

Proof. We give a proof by induction on t. The case $t = 1$ follows from Lemma 2.4. Now, assuming the assertion holds for $t < t_0$, we are going to prove it for $t = t_0$. To see this, consider the graph $H = K_{15^{t_0}m, 15^{t_0}m}$ whose edges are colored by the colors $1, 2, \ldots, t_0 + 1$ and suppose that there is no copy of C_{n_i} of color i in H for all $1 \leq i \leq t_0$. We show that there is a copy of $K_{m,m}$ of color $t_0 + 1$ in H. By the induction hypothesis, we have $BR(C_{n_1}, \ldots, C_{n_{t_0-1}}, K_{15m,15m}) \leq 15^{t_0}m$ and so there is a copy of $K_{15m,15m}$ in H whose edges are colored by colors t_0 and $t_0 + 1$. Now using Lemma 2.4, this copy contains either a copy of $C_{n_{t_0}}$ of color t_0, or a copy of $K_{m,m}$ of color $t_0 + 1$. By the assumption, the earlier case does not hold. Hence, there is a copy of $K_{m,m}$ of color $t_0 + 1$ in H and we are done. $\square$

For the case of odd cycles, we need a variant of Lemma 2.4 for 3-partite graphs which is stated as follows.

Lemma 2.6. *Let n, m_1, m_2 be positive integers, where $\min\{m_1, m_2\} \geq n \geq (\lceil \log m_1 \rceil + \lceil \log m_2 \rceil + 2)$. Then*

$$K_{X,Y,Z} \longrightarrow (C_n, K_{m_1,m_2}),$$

where $K_{X,Y,Z}$ is a complete 3-partite graph with color classes X, Y, Z of sizes $|X| = 7m_1 + 8m_2$, $|Y| = 8m_1 + 7m_2$ and $|Z| = m_1 + m_2 - 1$.

Proof. The case when n is even follows from Lemma 2.3 (it just suffices to consider the subgraph $K_{X,Y}$ of $K_{X,Y,Z}$ and apply Lemma 2.3). Now, let n be odd. Consider a $2-$edge coloring of $K_{X,Y,Z}$ and suppose that there is no blue K_{m_1,m_2}.

By Lemma 2.3, there are sets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = m_1$ and $|Y'| = m_2$ such that for every $x \in X'$ and $y \in Y'$ there is a red path of length $n - 2$ from x to y contained in $X \cup Y$.

5

Now, since there is no blue K_{m_1,m_2} in the 2-edge-colored $K_{X,Z}$, we have $|N_Z^r(X')| \geq m_1$, where $N_Z^r(S)$ is the set of all vertices in Z which have a neighbour in S in the red subgraph of $K_{X,Y,Z}$. Similarly, since there is no blue K_{m_1,m_2} in the 2-edge-colored $K_{Y,Z}$, we have $|N_Z^r(Y')| \geq m_2$. Therefore, since $|Z| = m_1 + m_2 - 1$, we have $N_Z^r(X') \cap N_Z^r(Y') \neq \emptyset$. Hence, there are some vertices $x \in X'$, $y \in Y'$, and $z \in N_Z^r(x) \cap N_Z^r(y)$. Now, the concatenation of the edges yz and zx and the path of length $n-2$ from x to y in $X \cup Y$ comprises a red C_n, as required. $\qquad\square$

Lemma 2.6 along with the fact that the graph $K_{X,Y,Z}$ in Lemma 2.6 is a subgraph of $K_{16m_1+16m_2-1}$ immediately imply the following result.

Corollary 2.7. *Let n, m_1, m_2 be positive integers, where $\min\{m_1, m_2\} \geq n \geq (\lceil \log m_1 \rceil + \lceil \log m_2 \rceil + 2)$. Then, $R(C_n, K_{m_1,m_2}) \leq 16m_1 + 16m_2 - 1$.*

Let $f_1(m_1, m_2) = 16m_1 + 16m_2 - 1$ and for every $t \geq 2$, define

$$f_t(m_1, m_2) = f_{t-1}(f_{t-1}(m_1 + m_2 - 1, 8m_1 + 7m_2), 7m_1 + 8m_2). \qquad (2.3)$$

In the following, we show that $f_t(m_1, m_2)$ is an upper bound for the Ramsey number of t cycles (with some restrictions on their sizes) versus the graph K_{m_1,m_2}.

Theorem 2.8. *Let m_1, m_2 and $n_1, \ldots, n_t$ be positive integers such that $\min\{m_1, m_2\} \geq n_i \geq 2\lceil \log(f_t(m_1, m_2)) \rceil + 2$ for each $1 \leq i \leq t$. Then,*

$$R(C_{n_1}, \ldots, C_{n_t}, K_{m_1,m_2}) \leq f_t(m_1, m_2).$$

Proof. We give a proof by induction on t. The case $t = 1$ follows from Corollary 2.7. Now, assuming correctness of the assertion for $t < t_0$, we are going to prove it for $t = t_0$. Consider the $(t_0 + 1)$-edge-colored graph $H = K_N$ with colors $1, 2, \ldots, t_0 + 1$, where $N = f_{t_0}(m_1, m_2)$. We assume that H contains no copy of C_{n_i} of color i for each $1 \leq i \leq t_0$ and we show that there is a copy of K_{m_1,m_2} of color $t_0 + 1$. By the induction hypothesis, we have $R(C_{n_1}, \ldots, C_{n_{t_0-1}}, K_{N_1,N_2}) \leq N$ for $N_1 = f_{t_0-1}(m_1 + m_2 - 1, 8m_1 + 7m_2)$ and $N_2 = 7m_1 + 8m_2$. Thus, there is a copy of 2-edge-colored K_{N_1,N_2} with parts X and Y by colors t_0 and $t_0 + 1$ in K_N. Now, again by the induction hypothesis, we have

$$|X| = N_1 = f_{t_0-1}(m_1 + m_2 - 1, 8m_1 + 7m_2) \geq R(C_{n_1}, \ldots, C_{n_{t_0-1}}, K_{m_1+m_2-1,8m_1+7m_2}).$$

Therefore, there is a copy of a 2-edge-colored $K_{m_1+m_2-1,8m_1+7m_2}$ by colors t_0 and $t_0 + 1$ with parts X' and X'' in the induced subgraph of K_N on X. Thus, the edges of the complete 3-partite graph with the color classes Y, X' and X'' are colored by colors t_0 and $t_0 + 1$ and so by Lemma 2.6, there is a copy of K_{m_1,m_2} of color $t_0 + 1$ in H and we are done. $\qquad\square$

The following corollary follows from Theorem 2.8 and the fact that $f_2(m_1, m_2) = 2416m_1 + 2176m_2 - 273$.

Corollary 2.9. *Let m_1, m_2, n_1, n_2 be positive integers such that $\min\{m_1, m_2\} \geq n_1, n_2 \geq 2\lceil \log(2416m_1 + 2176m_2 - 273) \rceil + 2$. Then, we have*

$$R(C_{n_1}, C_{n_2}, K_{m_1,m_2}) \leq 2416m_1 + 2176m_2 - 273.$$

6

By calculating the function $f_t(m_1, m_2)$ and using Theorem 2.8, we can prove the following theorem.

Theorem 2.10. *Let $t \geq 3$ and m_1, m_2 and $n_1, \ldots, n_t$ be positive integers such that $\min\{m_1, m_2\} \geq n_i \geq 2\lceil \log(10^{2^{t-1}}(m_1 + m_2)) \rceil + 2$ for each $1 \leq i \leq t$. Then,*

$$R(C_{n_1}, \ldots, C_{n_t}, K_{m_1, m_2}) \leq 10^{2^{t-1}}(m_1 + m_2).$$

Proof. Using Theorem 2.8, it just suffices to show that $f_t(m_1, m_2) \leq 10^{2^{t-1}}(m_1 + m_2)$. To see this, let $f_t(m_1, m_2) = a_t m_1 + b_t m_2 + c_t$, where a_t, b_t and c_t are three functions in terms of t. From (2.3) one can easily see that for each $t \geq 2$,

$$a_t = a_{t-1}^2 + 8a_{t-1}b_{t-1} + 7b_{t-1},$$

$$b_t = a_{t-1}^2 + 7a_{t-1}b_{t-1} + 8b_{t-1},$$

and

$$c_t = -a_{t-1}^2 + a_{t-1}c_{t-1} + c_{t-1}.$$

Clearly for every $i \geq 2$ we have $b_i < a_i$ and so for every $t \geq 3$,

$$a_t < a_{t-1}^2 + 8a_{t-1}^2 + 7a_{t-1} \leq 10a_{t-1}^2.$$

Therefore, by induction on t we can see that for every $t \geq 3$,

$$b_t \leq a_t \leq 10^{2^{t-1}}.$$

On the other hand, again by induction on t, we have $c_t \leq 0$. Therefore

$$f_t(m_1, m_2) \leq a_t m_1 + b_t m_2 \leq 10^{2^{t-1}}(m_1 + m_2).$$

$\square$

With all these results in hand, we can prove the main result of this section, as follows.

Theorem 2.11. *Let t_e and t_o be respectively the number of even and odd integers in the sequence $(n_1, \ldots, n_t)$ and suppose that $m \geq n_i \geq 2(\lceil \log N \rceil + 1)$ for each $1 \leq i \leq t$, where $N = 4.6 \times 10^{2^{t_o}-1} \times 15^{t_e}m$. Then*

$$R(C_{n_1}, \ldots, C_{n_t}, K_{m,m}) \leq N.$$

Proof. The case $t_o = 0$ follows from Corollary 2.5 (note that in this case the complete graph K_N has a complete bipartite graph $K_{15^{t_e}m, 15^{t_e}m}$ as a subgraph). So, let $t_o \geq 1$. Also, without loss of generality, assume that n_i is odd for all $1 \leq i \leq t_o$. Consider a $(t+1)$-edge-colored K_N with colors $1, 2, \ldots, t+1$. Assume that there is no copy of C_{n_i} of color i for each $1 \leq i \leq t$. Our goal is to show that there is a copy of $K_{m,m}$ of color $t+1$. Using Corollary 2.7 and Corollary 2.9 when $t_o = 1, 2$ and Theorem 2.10 when $t_o \geq 3$, we have $R(C_{n_1}, \ldots, C_{n_{t_o}}, K_{15^{t_e}m, 15^{t_e}m}) \leq N$ and so there is a copy of $K_{15^{t_e}m, 15^{t_e}m}$ in K_N whose edges are colored by $t_e + 1$ colors $t_o + 1, \ldots, t+1$. Now, Corollary 2.5 implies that $BR(C_{n_{t_o+1}}, \ldots, C_{n_t}, K_{m,m}) \leq 15^{t_e}m$ and so there is a copy of $K_{m,m}$ of color $t+1$ in the $(t_e + 1)$-edge-colored $K_{15^{t_e}m, 15^{t_e}m}$, as desired. $\square$

3 Random graphs and upper bounds

In this section, we will apply the obtained results in Section 2 on random graphs to give some linear upper bounds in terms of the number of vertices for the size-Ramsey number of large cycles. For this purpose, we deploy two random structure models namely binomial random graphs and binomial random bipartite graphs.

The first model called *binomial random graph* $\mathcal{G}(n, p)$ is the random graph G with the vertex set $[n] := \{1, 2, \ldots, n\}$ in which every pair $\{i, j\} \subseteq [n]$ appears independently as an edge in G with probability p. Note that an event in a probability space is called to holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. To see more about random graphs we refer the reader to see [1, 3]. We will state some results that hold a.a.s. and we always assume that n is large enough.

The first lemma asserts that there is a graph on N vertices whose number of edges is linear in terms of N, while it has no large hole (a pair of disjoint subsets of vertices with no edge between them). It should be noted that a very similar fact has been proved in [6] to prove a linear upper bound for the size-Ramsey number of paths. Here, we give a proof for completeness. For two subsets of vertices S, T, the number of edges with one end in S and one end in T is denoted by $e(S, T)$.

Lemma 3.1. *Let $c \in \mathbb{R}_+$ and let $d = d(c)$ be such that*

$$(1 - 2c) \ln(1 - 2c) + 2c \ln(c) + c^2 d \geq 0. \tag{3.1}$$

Then, in the graph $G \in \mathcal{G}(N, d/N)$, a.a.s. for every two disjoint sets of vertices S and T with $|S| = |T| = cN$, we have $e(S, T) \geq 1$.

Proof. Let S and T with $|S| = |T| = cN$ be fixed and let $X = X_{S,T} = e(S, T)$. Clearly, $\mathbb{E}X = c^2 dN > 0$ and by the Chernoff bound

$$\Pr(X = 0) = \Pr(X \leq 0) \quad \leq \quad \exp\left(-\mathbb{E}X\right) = \exp\left(-c^2 dN\right).$$

Thus, by the union bound over all choices of S and T we have

$$\Pr\left(\bigcup_{S,T}(X_{S,T} = 0)\right) \leq \binom{N}{cN}\binom{(1-c)N}{cN}\exp\left(-c^2 dN\right)$$

$$= \frac{N!}{(cN)!(cN)!((1-2c)N)!}\exp\left(-c^2 dN\right).$$

Using Stirling's formula ($x! \sim \sqrt{2\pi x}(x/e)^x$) we get

$$\Pr\left(\bigcup_{S,T}(X_{S,T} = 0)\right) \leq \frac{1}{2\pi c\sqrt{1 - 2c}N} \cdot \left(\frac{(1-2c)^{2c-1}\exp\left(-c^2 d\right)}{c^{2c}}\right)^N$$

$$\leq \frac{1}{2\pi c\sqrt{1 - 2c}N} = o(1),$$

where the last inequality is due to (3.1). This completes the proof. $\qquad \square$

Combining Theorem 2.11 and Lemma 3.1, gives some information on the size-Ramsey numbers of cycles. Roughly speaking, these two facts imply that $\mathcal{G}(N, d/N) \longrightarrow (C_{n_1}, \ldots, C_{n_t})$ for sufficiently large N when we have some restrictions on the parameters. In the following result, which is the main result of this paper, we use this fact to give a linear upper bound for the size-Ramsey number of large cycles.

Theorem 3.2. *Let $f = 4.6 \times 10^{2^{t_o}-1} \times 15^{t_e}$, where t_e and t_o are respectively the number of even and odd integers in the sequence $(n_1, \ldots, n_t)$. Also let $c = \min\{19773, f\}$ if $t = 2$ and $c = f$, otherwise. Suppose that $n = \max\{n_1, \ldots, n_t\}$ and for each $1 \leq i \leq t$, we have $n_i \geq 2\lceil \log(nc) \rceil + 2$. Then, for sufficiently large n, we have*

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leq (\ln c + 1)\, c^2\, n.$$

Proof. Let $N = nc$, $d = ((2c^{-1} - 1)\ln(1 - 2c^{-1}) - 2c^{-1}\ln(c^{-1}))/c^{-2}$ and $G = \mathcal{G}(N, d/N)$. By Lemma 3.1, a.a.s. for every two disjoint sets of vertices S and T in $V(G)$ with $|S| = |T| = n$, we have $e(S, T) \geq 1$. Therefore, a.a.s. the complement of G does not contain $K_{n,n}$ as a subgraph. On the other hand, the expected number of edges of G is $\frac{d}{N}\binom{N}{2} \leq Nd/2$ and the concentration around the expectation follows immediately from the Chernoff bound. Hence, for sufficiently large N, there exists a graph H on N vertices with at most $Nd/2$ edges whose complement does not contain $K_{n,n}$ as a subgraph. Hence, by Corollary 2.9 and Theorem 2.11, we have $H \longrightarrow (C_{n_1}, \ldots, C_{n_t})$. This means that for sufficiently large N we have

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leq \frac{Nd}{2} \leq \frac{c\ln c - (c-2)\ln(c-2)}{2}\, c^2\, n \leq (\ln c + 1)\, c^2\, n,$$

where the last inequality follows by applying the mean value theorem to the function $x \ln x$. $\qquad \square$

Based on Theorem 3.2, for sufficiently large n, we have

$$\hat{R}(C_n, C_n) \leq \begin{cases} 1989 \times 10^5\, n & \text{if } n \text{ is odd,} \\ 86 \times 10^5\, n & \text{if } n \text{ is even.} \end{cases}$$

It should be noted that other random graph models can be applied in the above method to improve the obtained bounds. One of these models is *random regular graphs* which gives slightly better results, however we omit its computations because it does not give much more improvement. See [7] for an application of this method to size Ramsey numbers. Another model is the binomial random bipartite graphs, described in below, which gives better upper bound when all the cycles are even.

The *binomial random bipartite graph* $\mathcal{G}(n, n, p)$ (where p may be a function of n) is the random bipartite graph $G = (V_1 \cup V_2, E)$ whose partite sets V_1, V_2 are of order n and each pair $(i, j) \in V_1 \times V_2$ appears independently as an edge in G with probability p. The following is the counterpart of Lemma 3.1 for the random bipartite graphs. Once again, it is well known and we include its proof for completeness.

Lemma 3.3. *Let $0 < c < 1$ and let $d = d(c)$ be such that*

$$2(1 - c)\ln(1 - c) + 2c\ln c + c^2 d \geq 0.$$

Then, a.a.s. for every two sets of vertices S and T in different color classes of $G \in \mathcal{G}(N, N, d/N)$ with $|S| = |T| = cN$, we have $e(S, T) \geq 1$.

9

Proof. Let S and T with $|S| = |T| = cN$ be fixed and let $X = X_{S,T} = e(S,T)$. Clearly, $\mathbb{E}X = c^2 dN > 0$ and by the Chernoff bound

$$\Pr(X = 0) = \Pr(X \leq 0) \leq \exp(-\mathbb{E}X) = \exp\left(-c^2 dN\right).$$

Thus, by the union bound over all choices of S and T we have

$$\Pr\left(\bigcup_{S,T}(X_{S,T} = 0)\right) \leq \binom{N}{cN}^2 \exp\left(-c^2 dN\right)$$

$$= \left(\frac{N!}{(cN)!((1-c)N)!}\right)^2 \exp\left(-c^2 dN\right).$$

Using Stirling's formula we get

$$\Pr\left(\bigcup_{S,T}(X_{S,T} = 0)\right) \leq \frac{1}{2\pi c(1-c)N} \cdot \left(\frac{\exp\left(-c^2 d/2\right)}{c^c(1-c)^{1-c}}\right)^{2N}$$

$$\leq \frac{1}{2\pi c(1-c)N} = o(1),$$

as desired. □

The following theorem gives an improvement of Theorem 3.2 when the lengths of all cycles are even.

Theorem 3.4. *Assume that $n_1, \ldots, n_t$ are even positive integers and $n = \max\{n_1, \ldots, n_t\}$. Also, suppose that for each $1 \leq i \leq t$ we have $n_i \geq 2\lceil \log(15^t n)\rceil + 2$. Then for sufficiently large n, we have*

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leq 2 \times 15^{2t}(t \ln 15 + 1)\, n.$$

Proof. Let $c = 15^{-t}$, $N = n/c$, $d = (-2(1-c)\ln(1-c) - 2c\ln c)/c^2$ and $G = \mathcal{G}(N, N, d/N)$. By Lemma 3.3, a.a.s. for every two sets of vertices S and T in different color classes of G with $|S| = |T| = n$, we have $e(S,T) \geq 1$. Therefore a.a.s. the complement of G with respect to $K_{N,N}$ does not contain $K_{n,n}$ as a subgraph and so by Corollary 2.5, we have $G \longrightarrow (C_{n_1}, \ldots, C_{n_t})$. On the other hand, the expected number of edges of G is Nd and concentration around the expectation follows immediately from the Chernoff bound. This means that for sufficiently large n we have

$$\hat{R}(C_{n_1}, \ldots, C_{n_t}) \leq Nd = 2c^{-2}(c^{-1}\ln c^{-1} - (c^{-1} - 1)\ln(c^{-1} - 1))\,n \leq 2c^{-2}(\ln c^{-1} + 1)\,n,$$

where the last inequality is due to the mean value theorem. □

As a consequence of Theorem 3.4, for sufficiently large even n, we have

$$\hat{R}(C_n, C_n) \leq 65 \times 10^4 n.$$

References

[1] N. Alon and J. H. Spencer, The Probabilistic Method, 4th ed., Wiley, New York, 2016.

[2] I. Balla, A. Pokrovskiy, and B. Sudakov. Ramsey goodness of bounded degree trees, *Combin. Probab. Comput.* 27 (2018), no. 3, 289–309.

[3] B. Bollobás, Random Graphs, Cambridge University Press, Cambridge, 2001.

[4] J. Beck, On size Ramsey number of paths, trees, and circuits I, *J. Graph Theory* **7** (1983), 115–129.

[5] D. Conlon, J. Fox, B. Sudakov, Recent developments in graph Ramsey theory, in: Surveys in Combinatorics 2015, Cambridge University press, 2015, 49–118.

[6] A. Dudek and P. Prałat, An alternative proof of the linearity of the size-Ramsey number of paths, *Combin. Probab. Comput.* 24 (2015), 551–555.

[7] A. Dudek and P. Prałat, On some multicolour Ramsey properties of random graphs, *SIAM J. Discrete Math.* 31 (2017), no. 3, 2079–2092.

[8] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* 53 (1947), 292–294.

[9] P. Erdős, R. Faudree, C. Rousseau and R. Schelp, The size Ramsey number, *Period. Math. Hungar.* 9 (1978), no. 1–2, 145–161.

[10] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* 2 (1935), 463–470.

[11] R. Faudree and R. Schelp, A survey of results on the size Ramsey number, Paul Erdos and his mathematics, II (Budapest, 1999), *Bolyai Soc. Math. Stud.*, **11**, *Janos Bolyai Math. Soc.*, Budapest, 2002, 291–309.

[12] P. E. Haxell and Y. Kohayakawa, The size-Ramsey number of trees, *Israel J. Math.* 89 (1995), no. 1-3, 261-274.

[13] P. Haxell, Y. Kohayakawa, and T. Luczak, The induced size-Ramsey number of cycles, *Combin. Probab. Comput.* 4 (1995), no. 3, 217–239.

[14] S. Letzter, Path Ramsey number for random graphs, *Combin. Probab. Comput.* 25 (2016), 612–622.

[15] S. P. Radziszowski, Small Ramsey numbers. *Electron. J. Combin.* 1 (1994), Dynamic Surveys, DS1.14 (Jan 9, 2014).

[16] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* 30 (1930), 264–286.