Derivation of an Analytical Expression for the Power Coupling Coefficient for Offset Launch Into Multimode Fiber

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***Abstract*—The demand for higher bandwidth in Local Area Networks (LANs) has fuelled considerable research in techniques for mitigating modal dispersion in multimode fiber (MMF). These techniques include selective mode excitation, offset launching, angular multiplexing and electronic dispersion compensation, all of which strive to optimize the channel impulse response of a MMF. To obtain the optimal bandwidth-enhancement results from these techniques, knowledge of the distribution of power coupling coefficients given an arbitrary offset launch in a MMF is important. In this paper, an analytical expression for the power coupling coefficient for an incident Gaussian beam launched with a radial offset into a MMF having an infinite parabolic refractive index profile is derived. This expression is useful in understanding the parameters which may affect the power coupling coefficient and how they may enhance the MMF bandwidth. The power coupling coefficients obtained from the derived analytical expression are compared with numerical results and are in excellent agreement. The analytical expression may be extended to manufactured MMF.**

***Index Terms*—LAN, laser coupling, multimode fiber, offset launch.**

I. Introduction

GRADED-INDEX multimode fiber (MMF) has become the predominant fiber in Local Area Networks (LANs) due to its low cost and ease of installation, as well as the availability of cost-effective light transceivers [2]–[5]. In the last two decades, there has been a large increase in the volume of multimedia data carried on LANs. This has stimulated the need for techniques to support the gigabit rate demands over existing multimode fiber backbones which were initially designed for 10-Mb/s and 100-Mb/s systems. Although alternatives such as single mode fiber and ribbon link fiber exist [6], the cost for replacing the current infrastructure would be high.

The bandwidth of a MMF is limited mainly by modal dispersion, caused by the propagation delay differences between the modes. The demand for higher bandwidth in LANs has attracted wide interest in techniques to mitigate modal dispersion, including variants of selective mode excitation [7]–[12], offset launching [13], angular multiplexing [14] and electronic dispersion compensation (EDC) [15]–[18]. In selective mode excitation, only a subset of propagating modes is excited to minimize the differences in the propagation delays between the modes [7]–[12]. For offset launching, a laser beam is positioned radially offset from the centre of the fiber core to excite higher-order modes [13]. For angular multiplexing, only a small portion of the total angular span is used per channel to reduce the pulse spread [14]. Electronic dispersion compensation (EDC) is carried out by equalizing the modal amplitudes of the propagating modes by an adaptive algorithm, either at the transmitter or the receiver [15], [16]. The common goal of these off-centre launch techniques [7]–[18] is to achieve a high bandwidth without the stringent mechanical alignment required in centre launch schemes. This is accomplished by exciting a large number of adjacent mode groups, without aiming the optical beam solely at the centre of the MMF where imperfections in the refractive index profile are dominant. In a typical LAN where MMF lengths are mostly shorter than 300 meters [19], exciting a large number of adjacent mode groups in a MMF link limits the negative effects of modal dispersion and power modal coupling on the channel bandwidth. In addition to the relaxed mechanical alignment requirements, off-centre techniques also give rise to the potential for a multi-channel system within a single MMF [17]. Due to these advantages, off-centre launch techniques have garnered significant interest in the quest for higher LAN bandwidths in recent years.

In order to optimize the bandwidth achieved by these off-centre launches, it is necessary to position the beam in the best manner possible to excite desired modes and to suppress others whenever viable to minimize modal dispersion. To ascertain the optimal bandwidth enhancement from these techniques, it is important to know, among other parameters, the distribution of power coupling coefficients for a typical MMF, for centre launch as well as for offset launch. In this paper, we derive the exact analytical expression for the power coupling coefficient for an offset launch in a MMF with an infinite parabolic refractive index profile, assuming weak guidance. Our analysis should prove useful in understanding the parameters which affect the power coupling coefficient and how it may enhance the bandwidth.

Sections II lays the foundation for our derivation of an analytical expression for the power coupling coefficient for an offset launch in a parabolic-index MMF. An overview of the refractive index profile of a graded-index MMF is given in Section II.A. The overlap integral for the power coupling coefficient, used as the basis for the derivation, is introduced in Section II.B. The

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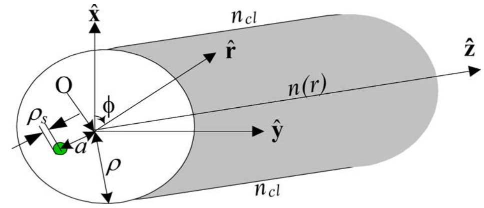

Fig. 1. Launch condition at the endface of a weakly guiding fiber [1].

computation of the power coupling coefficient requires expressions for both the modal electric field in an infinite parabolic refractive index MMF and the incident electric field distribution of the offset beam. The expressions for both of these electric fields are derived briefly in Sectionss II.C and II.D respectively. The derivation of the analytical expression for the power coupling coefficient for an offset launch in an infinite parabolic refractive index MMF is presented in Section III. This is followed by a general proof of power conservation from the derived analytical expression in Section IV. A comparison of the power coupling coefficients from the derived analytical expression with those computed numerically is presented in Section V. It is shown in Section VI that the applicability of the derived analytical expression for the power coupling coefficient for offset launch for an infinite parabolic refractive index profile may be extended to manufactured MMF characterized by refractive index profiles with profile parameters within the range of $1.8 < \alpha < 2.2$.

II. Preliminary Concepts

A. Refractive Index Profile of Multimode Fiber

In this paper, we assume that the refractive index of the multimode fiber is modeled by an infinite parabolic profile described by [1]

$$n(R) = n_{\rm co}(1 - \Delta R^2) \tag{1}$$

where $n_{\rm co}$ is the maximum refractive index of the core. The normalized radius R is defined by $R = r/\rho$ where ρ is the core radius. The profile height parameter is given by $\Delta = (n_{\rm co}^2 - n_{\rm cl}^2)/(2n_{\rm co}^2)$ where $n_{\rm cl}$ is the refractive index of the cladding at $R = 1$. For the derivations, weak guidance $(n_{\rm cl} \approx n_{\rm co})$ is assumed. Hence, $\Delta \underline{\equiv} (n_{\rm co} - n_{\rm cl})/n_{\rm co}$ [1]

B. Overlap Integral for Power Coupling Coefficient

The launch condition of an optical beam into a weakly guiding graded index MMF is shown in Fig. 1. The optical beam, assumed to have a Gaussian distribution, is launched to a position radially offset from the centre of the fiber. The relative power coupled into a fiber mode is given by the power coupling coefficient

$$\eta = \frac{\left|\iint\limits_{A_{\rm core}} \mathbf{E}_{\rm inc}(x,y) \cdot \mathbf{e}_t * (x,y)dxdy\right|^2}{\iint\limits_{A_{\rm core}} |\mathbf{E}_{\rm inc}(x,y)|^2 dxdy \iint\limits_{A_{\rm core}} |\mathbf{e}_t(x,y)|^2 dxdy} \tag{2}$$

where $\mathbf{e}_t$ is the transverse field of the relevant fiber mode, $\mathbf{E}_{\rm inc}$ is the incident electric field of the offset beam and $A_{\rm core}$ is the cross-sectional area of the fiber core.

C. Modal Electric Fields of an Infinite Parabolic Refractive Index MMF

Under the weakly guiding approximation, the transverse modal electric field of a MMF with an infinite parabolic refractive index profile may be found analytically by solving the scalar wave equation. The detailed analysis is given in [1] and is summarized here.

The spatial dependence of the transverse electric field is governed by the scalar wave equation

$$\left[\nabla_{\mathbf{t}}^2 + k^2 n^2(x,y) - \tilde{\beta}_{lm}^2\right] e_{lm} = 0 \tag{3}$$

where $\nabla_{\mathbf{t}}^2 \varphi = (1/r)(\partial/\partial r)(r(\partial\varphi)/(\partial r)) + (1/r^2)(\partial^2\varphi)/(\partial\phi^2)$ is the transverse Laplacian operator, $\mathrm{k} = 2\pi/\lambda$ is the free-space propagation constant, λ is the free-space wavelength, e_{lm} is the transverse modal electric field with azimuthal mode number l and radial mode number m. $\tilde{\beta}_{lm}$ is the scalar propagation constant, given by

$$\tilde{\beta}_{lm} = \frac{V}{\rho\,(2\Delta)^{1/2}}\left[1 - \frac{4\Delta}{V}(2m + l - 1)\right]^{1/2}. \tag{4}$$

The modes are commonly referred to as LP_{lm} modes. Modes which have the same value of $2m + l$ are characterized by the same scalar propagation constant, $\tilde{\beta}_{lm}$ and share the same mode group order. The total electric field, $\mathbf{E}$ of a weakly guiding multimode fiber may be expressed as the sum of individual LP_{lm} modes

$$\mathbf{E}(x,y,z) = \sum_l \sum_m c_{lm} \cdot \mathbf{e}_{lm}(x,y)\exp(i\tilde{\beta}_{lm}z) \tag{5}$$

where c_{lm} are constants.

For each value of $\tilde{\beta}_{lm}$, there are generally four solutions for the LP_{lm} modes, which can be expressed as superpositions of the function Ψ defined by

$$\Psi = F_l(r)\cos l\phi; \quad \Psi = F_l(r)\sin l\phi. \tag{6}$$

The radial wavefunction F_l is given by

$$F_l = R^l L_{m-1}^{(l)}(VR^2)\exp(-0.5VR^2). \tag{7}$$

$L_{m-1}^{(l)}$ is the generalized Laguerre polynomial, given in [20]. The Laguerre polynomial is an orthogonal function on the interval $[0, \infty]$, thus the modes are orthogonal. V is the normalized frequency, given by $V = (2\pi \mathrm{NA}\rho)/\lambda$, where NA is the numerical aperture of the particular MMF and λ is the free-space wavelength of the incident optical source into the fiber.

The solution of the scalar wave equation does not give any information about the polarization of the vector field. Assuming the transverse modal electric field must be polarized along the optical axes of the fiber and that any pair of orthogonal x- and y-axes may be chosen as optical axes in the fiber cross-section, then there are four possible polarizations for e_{lm}. The complete

TABLE I
MODAL TRANSVERSE FIELDS OF WEAKLY GUIDING FIBERS [1]

Degenerate mode number, *i*	*Mode*	$\mathbf{e}_{lm}$
Fundamental* HE$_{11}$ *and* HE$_{1m}$ *(l = 0) modes		
1	*Even* HE$_{1m}$	$\hat{\mathbf{x}}F_o$
3	*Odd* HE$_{1m}$	$\hat{\mathbf{y}}F_o$
Higher-order modes (l ≥ 0)		
1	*Even* HE$_{l+1,\,m}$	$\{\hat{\mathbf{x}}\cos l\phi - \hat{\mathbf{y}}\sin l\phi\}F_l$
2	TM$_{0m}$*(l=1)*	$\{\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi\}F_l$
2	*Even* EH$_{l-1,\,m}$ *(l>1)*	$\{\hat{\mathbf{x}}\cos l\phi + \hat{\mathbf{y}}\sin l\phi\}F_l$
3	*Odd* HE$_{l+1,\,m}$	$\{\hat{\mathbf{x}}\sin l\phi + \hat{\mathbf{y}}\cos l\phi\}F_l$
4	TE$_{0m}$*(l=1)*	$\{\hat{\mathbf{x}}\sin\phi - \hat{\mathbf{y}}\cos\phi\}F_l$
4	*Odd* EH$_{l-1,\,m}$*(l>1)*	$\{\hat{\mathbf{x}}\sin l\phi - \hat{\mathbf{y}}\cos l\phi\}F_l$

The fields F_o and F_l in the table are defined in Eq. (7); $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors parallel to the *x*-axis and *y*-axis respectively.

modal fields from both the solutions of the scalar wave equation and their polarization properties are summarized in Table I, taken from [1], together with the corresponding vector solutions of the wave equation. The degeneration of the LP_{lm} into the traditional HE, EH, TE and TM modes is shown in Table I, taken from [1], where i denotes the degenerate mode number. The exact propagation constant for the i-th degenerate mode of an LP_{lm} mode may be calculated from the scalar propagation constant $\tilde{\beta}_{lm}$ [1].

D. Electric Field of Radially Offset Gaussian Beam

In deriving the analytical expression for the power coupling coefficient, an expression for the amplitude of incident radially offset Gaussian beam is required. Here, a brief derivation of the electric field of the radially offset beam is given.

Without loss of generality, the amplitude of a Gaussian beam with spot size ρ_s and offset launch at $x = a$ can be written as

$$\mathbf{E}_{\text{inc}} = C\exp\left[-\frac{((x-a)^2+y^2)}{2\rho_s^2}\right]\hat{\mathbf{x}} \tag{8}$$

where C is a normalization coefficient. This can be rewritten in cylindrical coordinates as

$$\mathbf{E}_{\text{inc}} = C\exp\left[-\frac{(R^2+A^2-2AR\cos\phi)}{2\Omega^2}\right]\hat{x} \tag{9}$$

where ϕ is the azimuthal angle, $A = a/\rho$ is the normalized radial offset, $\Omega = \rho_s/\rho$ is the normalized spot size and $R = r/\rho$, where ρ is the radius of the MMF core.

III. Derivation of an Analytical Expression for the Power Coupling Coefficient for Offset Launch

Having given the expressions for the modal electric field of an infinite parabolic refractive index MMF and the electric field of a radially offset Gaussian beam, we now proceed to derive the analytical expression for the power coupling coefficient for an offset launch. This result was originally derived by Grau *et al.* [23]. However, as their original paper only quotes the final result with very little detail of the lengthy analysis, we have given full details here. Our analysis is valid for the weak guiding approximation which applies to the majority of communications grade optical fibers [1].

From Section II, let the polarized modal transverse electric field

$$\mathbf{e}_{lm} = BR^l L_{m-1}^l(VR^2)\exp(-0.5VR^2)\cos l\phi\hat{\mathbf{x}} \tag{10}$$

where B is a normalization constant to be determined.

Substituting $\mathbf{e}_{lm}$ from (10), $\mathbf{E}_{\text{inc}}$ from (9) into (2), the overlap integral for the power coupling coefficient for mode LP_{lm}, η_{lm} may be written as

$$\eta_{lm} = \frac{\left|\iint\limits_{A_{\text{core}}} \mathbf{E}_{\text{inc}}(x,y)\cdot\mathbf{e}_{lm}*(x,y)dxdy\right|^2}{\iint\limits_{A_{\text{core}}} |\mathbf{E}_{\text{inc}}(x,y)|^2 dxdy \iint\limits_{A_{\text{core}}} |\mathbf{e}_{lm}(x,y)|^2 dxdy}. \tag{11}$$

To solve the power coupling integral of (11) analytically, the denominator is first normalized. Setting $\iint\limits_{A_{\text{core}}} |\mathbf{E}_{\text{inc}}(x,y)|^2 dx\,dy = 1$, we find, $C = 1/\Omega\sqrt{\pi}$. Now, setting $\iint\limits_{A_{\text{core}}} |\mathbf{e}_{lm}(x,y)|^2 dx\,dy = 1$, we find that the normalization constant B is given by

$$B^2\int\limits_{R=0}^{\infty}\int\limits_{\phi=0}^{2\pi}\exp\left(-\frac{R^2}{2\Omega^2}\right)R^{2l}\left[L_{m-1}^l(VR^2)\right]^2 \times\exp(-VR^2)\cos^2 l\phi d\phi R\,dR = 1. \tag{12}$$

Solving the ϕ -component integral

$$\int\limits_{\phi=0}^{2\pi}\cos^2 l\phi d\phi = (1+\delta_{0l})\pi \tag{13}$$

where δ_{0l} is the Kronecker delta function.

Substituting the solution for the ϕ-component integral in (13) back into (12)

$$(1+\delta_{0l})\pi B^2\int\limits_{R=0}^{\infty}R^{2l}\left[L_{m-1}^l(VR^2)\right]^2\exp(-VR^2)R\,dR = 1. \tag{14}$$

Letting $x = VR^2$

$$\frac{(1+\delta_{0l})\pi B^2}{2V^{l+1}}\int\limits_{x=0}^{\infty}x^l\left[L_{m-1}^l(x)\right]^2\exp(-x)dx = 1. \tag{15}$$

From the orthogonality relation of the Laguerre polynomials [21], we have

$$\int\limits_{x=0}^{\infty}x^l\exp(-x)\,L_k^l(x)L_n^l(x)dx = \frac{\Gamma(1+l+n)}{\Gamma(n+1)}\delta_{nk}. \tag{16}$$

Let $k = n = m - 1$ and substituting (16) into (15), we find

$$\frac{\pi B^2}{2V^{l+1}} \frac{\Gamma(l+m)}{\Gamma(m)} (\delta_{0l} + 1) = 1. \qquad (17)$$

This gives B as

$$B = \left(\frac{2V^{l+1}\Gamma(m)}{\pi(\delta_{0l} + 1)\Gamma(l+m)} \right)^{1/2}. \qquad (18)$$

Next, substituting the expressions for B and C back into the numerator of (11), we have

$$\eta_{lm} = \frac{1}{\Omega^2 \pi^2} \left(\frac{2V^{l+1}\Gamma(m)}{\delta_{0l} + 1)\Gamma(l+m)} \right) \times \left| \begin{array}{l} \int_{R=0}^{\infty} \int_{\phi=0}^{2\pi} \exp\left(-\frac{(R^2+A^2-2AR\cos\phi)}{2\Omega^2}\right) \times \\ R^l L_{m-1}^l (VR^2) \exp\left(-\frac{VR^2}{2}\right) \cos l\phi R \, dR \, d\phi \end{array} \right|^2. \qquad (19)$$

Letting $\phi \to \phi - \pi$ and rewriting the ϕ-integral

$$\begin{aligned} \int_0^{2\pi} \exp\left(\frac{AR\cos\phi}{\Omega^2}\right) \cos\, l\phi \, d\phi &= \int_0^{\pi} \exp\left(\frac{AR\cos\phi}{\Omega^2}\right) \cos\, l\phi \, d\phi \\ &\quad + (-1)^l \int_0^{\pi} \exp\left(-\frac{AR\cos\phi}{\Omega^2}\right) \cos l\phi \, d\phi. \end{aligned} \qquad (20)$$

The Modified Bessel function is given in [22]

$$I_n(z) = \frac{1}{\pi} \int_0^{\pi} \exp(z\cos\theta) \, \cos(n\,\theta) d\theta. \qquad (21)$$

Letting $n = l$ and $z = (AR)/\Omega^2$in (21) and substituting into (20)

$$\begin{aligned} &\int_0^{\pi} \exp(z\cos\phi) \cos l\phi \, d\phi \\ &\quad + (-1)^l \int_0^{\pi} \exp(-z\cos\phi) \cos\, l\phi \, d\phi \\ &\quad = \pi \left[I_l(z) + (-1)^l I_l(-z) \right] \\ &\quad = \pi \left[I_l\left(\frac{AR}{\Omega^2}\right) + (-1)^l I_l\left(-\frac{AR}{\Omega^2}\right) \right]. \end{aligned} \qquad (22)$$

Using the property $I_l(-z) = (-1)^l I_l(z)$, the ϕ-integral becomes

$$\int_0^{2\pi} \exp\left(\frac{AR\cos\phi}{\Omega^2}\right) \cos\, l\phi \, d\phi = 2\pi I_l\left(\frac{AR}{\Omega^2}\right). \qquad (23)$$

Substituting the ϕ-integral back into (19), we get

$$\begin{aligned} \eta_{lm} = \frac{8}{\Omega^2} &\left(\frac{V^{l+1}\Gamma(m)}{\delta_{0l} + 1)\Gamma(l+m)} \right) \exp\left(-\frac{A^2}{\Omega^2}\right) \\ &\times \left| \int_0^{\infty} I_l\left(\frac{AR}{\Omega^2}\right) L_{m-1}^l (VR^2) \right. \\ &\left. \times \exp\left(-\frac{R^2}{2}\left(\frac{1}{\Omega^2} + V\right)\right) R^{l+1} dR \right|^2. \end{aligned} \qquad (24)$$

Using the table of integrals in [20, Eq. 7.421, No. 4, p. 847]

$$\begin{aligned} \int_{x=0}^{\infty} & x^{\nu+1} \exp(-\beta x^2) L_n^{\nu}(\alpha x^2) \, J_{\nu}(xy) dx \\ &= 2^{-\nu-1} \frac{(\beta-\alpha)^n}{\beta^{\nu+n+1}} y^{\nu} \\ &\quad \times \exp\left(-\frac{y^2}{4\beta}\right) L_n^{\nu}\left(\frac{\alpha y^2}{4\beta(\alpha-\beta)}\right) \end{aligned} \qquad (25)$$

letting $\nu = l,\ \beta = 2^{-1}(\Omega^{-2} + V), \quad n = m-1, \alpha = V, y = iA/\Omega^2$, and using $I_l(z) = (-i)^l J_l(iz)$, (24) becomes

$$\eta_{lm} = \frac{8}{\Omega^2} \left(\frac{V^{l+1}\Gamma(m)}{\delta_{0l} + 1)\Gamma(l+m)} \right) \exp\left(-\frac{A^2}{\Omega^2}\right) \times \left| \begin{array}{l} 2^{-l-1} \frac{\left[\frac{1}{2}(\Omega^{-2}-V)\right]^{m-1}}{\left[\frac{1}{2}(\Omega^{-2}+V)\right]^{l+m}} \left(\frac{A}{\Omega^2}\right)^l \\ \times \exp\left(\frac{(A\Omega^{-2})^2}{2(\Omega^{-2}+V)}\right) L_{m-1}^l \left(\frac{-V(A\Omega^{-2})^2}{(V+\Omega^{-2})(V-\Omega^{-2})}\right) \end{array} \right|^2. \qquad (26)$$

Simplifying

$$\begin{aligned} \eta_{lm} = \frac{8A^{2l}}{\Omega^{4l+2}} &\left(\frac{V^{l+1}\Gamma(m)}{\delta_{0l} + 1)\Gamma(l+m)} \right) \left(\frac{1}{\Omega^2} - V \right)^{2m-2} \\ &\times \left(\frac{1}{\Omega^2} + V \right)^{-2l-2m} \exp\left[-\left(\frac{A^2 V}{1 + V\Omega^2} \right) \right] \\ &\times \left[L_{m-1}^l \left(\frac{V(A\Omega^{-2})^2}{\Omega^{-4} - V^2} \right) \right]^2. \end{aligned} \qquad (27)$$

Equation (27) is our main result and is the exact analytical solution for the power coupling coefficient of the LP_{lm} mode for an offset launch in an infinite parabolic refractive index MMF. This is the same expression derived by Grau, Leminger and Sauter [23] for the case where no tilt is present ((17) of Grau's paper [23]). Setting $A = 0$ in (27), the power coupling coefficient can be reduced after some more algebra to that for the special case of centre launch.

We also point out that in [24], Saijonmaa *et al.* used a different approach for deriving an analytical expression for the power coupling coefficients of an offset launch, based on Gaussian-Hermite modes factorised into x and y components.

IV. General Proof of Power Conservation

In this section, we show that (27) satisfies power conservation

$$\sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \eta_{lm} = 1. \qquad (28)$$

The first summation in (28) is the summation over the radial mode number, m. Replacing m by $m+1$ in (27) and (28), the summation over m can be written as

$$S = \sum_{m=0}^{\infty} \frac{m!}{\Gamma(m+l+1)} \left(\frac{\frac{1}{\Omega^2} - V}{\frac{1}{\Omega^2} + V} \right)^{2m} \times \left[L_m^l \left(\frac{V\left(\frac{A}{\Omega^2}\right)^2}{\frac{1}{\Omega^4} - V^2} \right) \right]^2. \qquad (29)$$

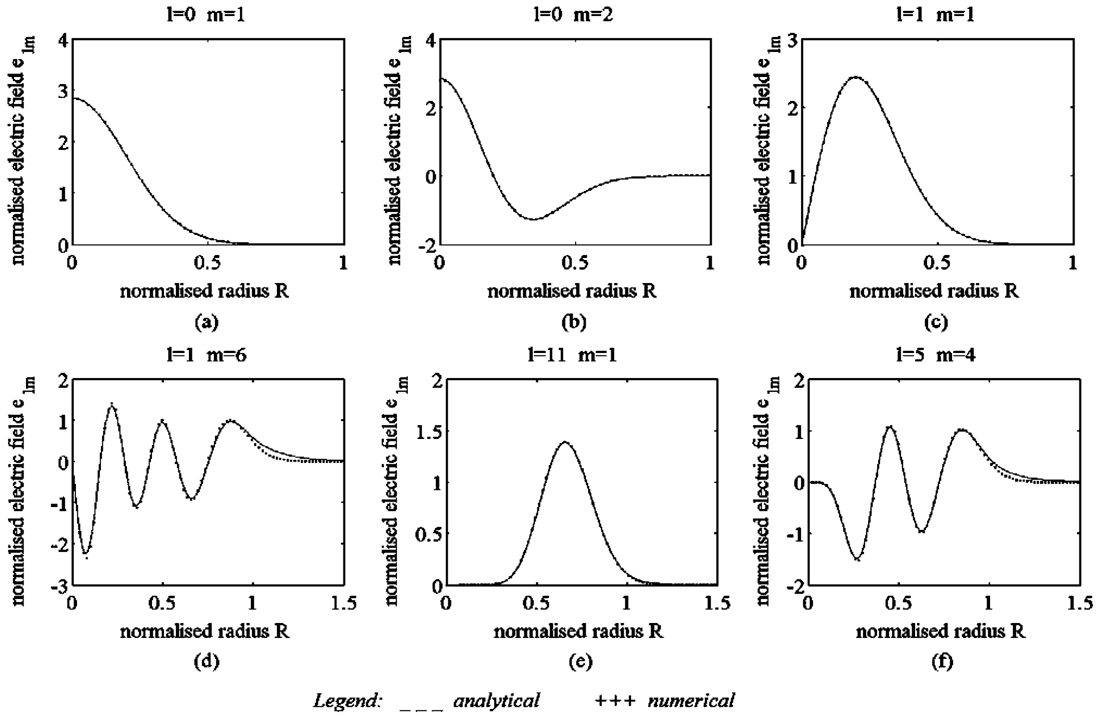

Fig. 2. Comparison of analytical and numerical values of electric field for a graded-index MMF for profile parameter $\alpha = 2.0$.

We use the following result from [20] to calculate S :

$$\sum_{n=0}^{\infty} n! \frac{L_n^{\alpha}(x) L_n^{\alpha}(y) z^n}{\Gamma(n+\alpha+1)} = \frac{(xyz)^{-\alpha/2}}{1-z} \times \exp\left(\frac{-z(x+y)}{1-z}\right) I_{\alpha}\left(\frac{2\sqrt{xyz}}{1-z}\right). \quad (30)$$

Comparing (29) to (30), let

$$z = \left(\frac{1-\Omega^2 V}{1+\Omega^2 V}\right)^2, \quad (31)$$

$$x = y = \frac{VA^2}{1-\Omega^4 V^2} \quad (32)$$

and

$$\alpha = l. \quad (33)$$

Thus, substituting (31), (32), and (33) into (30), we get

$$S = \frac{V^{-(l+1)} A^{-2l}}{4\Omega^2} (1+\Omega^2 V)^{2l+2} \times \exp\left[-\frac{A^2}{2\Omega^2}\left(\frac{1-\Omega^2 V}{1+\Omega^2 V}\right)\right] I_l\left(\frac{A^2}{2\Omega^2}\right). \quad (34)$$

Having calculated the summation over m, we now combine S with the original expression for $\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \eta_{lm}$ in (28), yielding $\sum_{l,m} \eta_{lm} = \sum_l \eta_l$ where

$$\eta_l = \frac{8A^{2l}}{\Omega^{4l+2}} \frac{V^{l+1}}{\delta_{0l}} \times \exp\left[-\frac{A^2}{\Omega^2}\left(\frac{V\Omega^2}{1+V\Omega^2}\right)\right]\left(\frac{1}{\Omega^2}+V\right)^{-2l-2} \times \frac{V^{-(l+1)}}{4\Omega^2} A^{-2l} (1+\Omega^2 V)^{2l+2} \times \exp\left[-\frac{A^2}{2\Omega^2}\left(\frac{1-\Omega^2 V}{1+\Omega^2 V}\right)\right] \times I_l\left(\frac{A^2}{2\Omega^2}\right). \quad (35)$$

Simplifying (35)

$$\eta_l = \frac{2}{\delta_{0l}+1} \exp\left(-\frac{A^2}{2\Omega^2}\right) I_l\left(\frac{A^2}{2\Omega^2}\right). \quad (36)$$

Now, to sum over l, we use the generating function of the Bessel function of the first kind [20]:

$$\sum_{l=-\infty}^{\infty} t^l I_l(z) = \exp\left[\frac{1}{2}z\left(t+\frac{1}{t}\right)\right] \quad (37)$$

Setting $t = l$ gives

$$\sum_{l=-\infty}^{\infty} I_l(z) = \exp(z). \quad (38)$$

Using the property $I_{-l}(z) = I_l(z)$

$$\sum_{l=-\infty}^{\infty} I_l(z) = I_o(z) + 2\sum_{l=1}^{\infty} I_l(z) = \exp(z). \quad (39)$$

Thus, summing η_l over l

$$\sum_{l=0}^{\infty} \eta_l = 2\exp\left(-\frac{A^2}{2\Omega^2}\right)\sum_{l=0}^{\infty}\left[I_l\left(\frac{A^2}{2\Omega^2}\right)/(\delta_{0l}+1)\right]$$

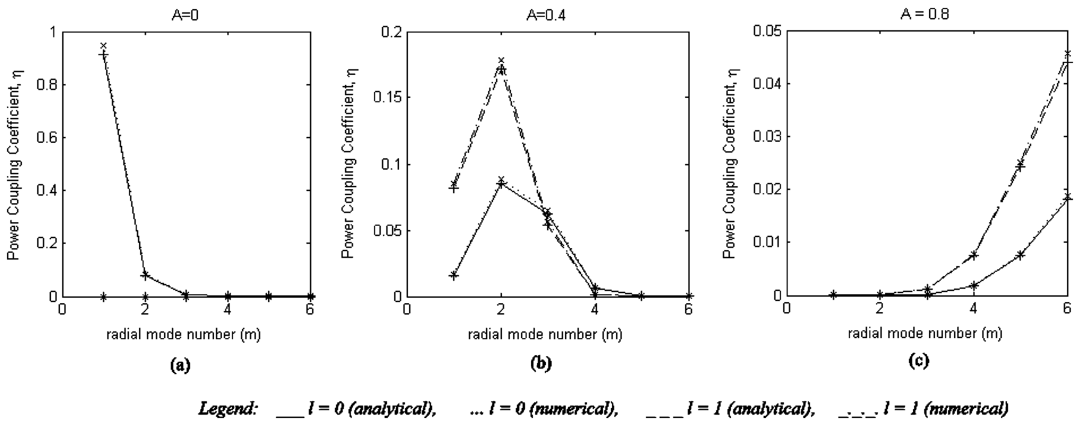

Fig. 3. Comparison of power coupling coefficients of a perfect parabolic graded-index MMF ($\alpha = 2.0$) obtained using analytical and numerical models, for various normalized radial offsets, A.

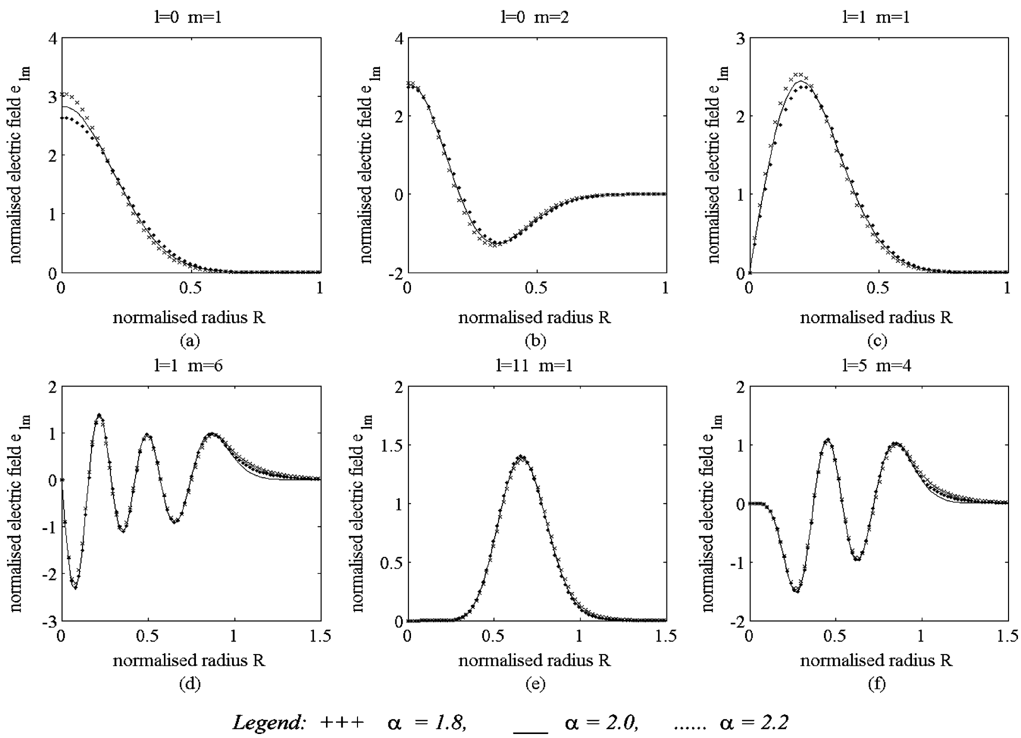

Fig. 4. Comparison of electric field variation along the core of a graded index MMF for different values of refractive index profile parameter, α.

$$
\begin{aligned}
&= \exp\left(-\frac{A^2}{2\Omega^2}\right) \\
&\quad \times \left[I_o\left(\frac{A^2}{2\Omega^2}\right) + 2\sum_{l=1}^{\infty} I_l\left(\frac{A^2}{2\Omega^2}\right) \right] \\
&= \exp\left(-\frac{A^2}{2\Omega^2}\right) \exp\left(\frac{A^2}{2\Omega^2}\right) = 1
\end{aligned} \tag{40}
$$

as required.

V. Comparison With Numerical Results for Refractive Index With $\alpha = 2.0$

To confirm the validity of the derived analytical expression ((27)) for the power coupling coefficient for an offset launch, a finite difference method [25] which solves the scalar wave equation for each mode group order in a graded index MMF was used. Using this method, the electric field of a graded index MMF with a refractive index of a given profile parameter, α for any specified LP mode was computed. As a reference, the electric field values in a MMF with a refractive index profile with $\alpha = 2.0$ were calculated for several modes using the finite difference method to solve the scalar wave equation. It was confirmed that the electric field computed numerically using the finite difference method was in good agreement with the electric field for the infinite parabolic profile used in the derivation of the analytical expression for the power coupling coefficient, presented in (27). This is illustrated in Fig. 2, for a number of different LP modes. In Fig. 2 the finite difference equation was solved for a fiber with a core diameter of 62.5 μm, a cladding diameter of 125 μm, a core refractive index of 1.45 and cladding

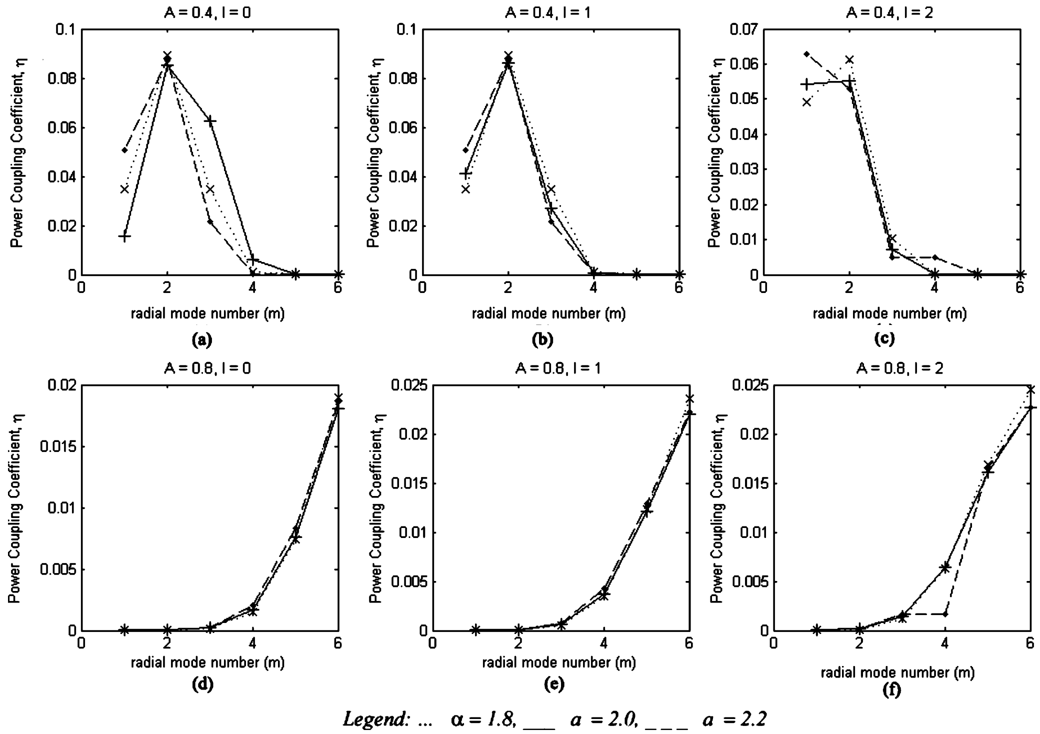

Fig. 5. Comparison of power coupling coefficients of a graded index MMF obtained using different values of refractive index profile parameter, α for various radial offsets, A.

index of 1.436, a numerical aperture of 0.2 and a wavelength of 1.55 μm. This give a normalized frequency $V = 25.3$ and from (4) the condition on m and l for guided modes to have an effective index larger than 1.436 is $2m + l \leq 13$. In Fig. 2(a)–(c) we compare the computed fields with those of an infinite parabolic profile for several low order modes, and in Fig. 2(d)–(f) for modes just at cutoff where $2m + l = 13$. In all cases the two results are in extremely close agreement confirming that the analytical solutions for an infinite parabolic profile are very accurate.

The power coupling coefficients for different offset launches in a MMF were calculated numerically using (11), with the electric field values calculated earlier by solving the scalar wave equation from the finite difference method. The power coupling coefficients were then compared with those calculated from the derived analytical expression in (27). The examples shown in Fig. 3 confirm that the analytical and numerical results agree well both for low order modes and modes close to cutoff.

VI. Comparison With Numerical Results for Refractive Index With $1.8 < \alpha < 2.2$

Current commercial manufacturing processes are unable to produce optical fibers with a constant, accurate refractive index profile [26]. One of the manufacturing defects is the imperfection in the parabolic index profile [26]. Instead of a perfect parabolic profile with a profile parameter, $\alpha = 2.0$, manufactured optical fibers usually have profile parameters ranging from $1.8 < \alpha < 2.2$. In this section, we show that the modal fields are insensitive to α in this range and consequently that the power coupling coefficients are also insensitive to α.

The electric fields of a MMF with parameters given above were calculated numerically using the finite difference method in [25], for profile parameters $\alpha = 1.8$ and $\alpha = 2.2$. The results in Fig. 4 show that the electric fields for refractive index profiles with profile parameters $1.8 < \alpha < 2.2$ closely agree with the electric fields of an infinite parabolic profile fiber with $\alpha = 2.0$ computed using (10) for both low order modes and modes close to cutoff.

Power coupling coefficients for various modes were calculated for refractive index profiles with $\alpha = 1.8$ and $\alpha = 2.2$ using the electric fields computed numerically in Fig. 5. Plots of power coupling coefficients over several radial and azimuthal mode numbers for various radial offsets in Fig. 6 demonstrate that the power coupling coefficients for an offset launch in a MMF within the profile parameter $1.8 < \alpha < 2.2$ are in good agreement with the power coupling coefficients for a perfect parabolic index profile with $\alpha = 2.0$.

Importantly, this implies that the derived analytical expression for the power coupling coefficient for offset launch in a MMF in Section III may be extended for use in imperfect parabolic refractive indices with profile parameters within the range of $1.8 < \alpha < 2.2$.

VII. Conclusion

An analytical expression for the power coupling coefficient for an offset launch in a graded index MMF with a refractive index of profile parameter $\alpha = 2.0$ has been derived. It is shown that the derived analytical expression satisfies power conservation.

The electric field in a MMF with a refractive index of profile parameters $\alpha = 1.8$ and $\alpha = 2.2$ were found to be in good agreement with the electric field for a MMF with a perfect parabolic refractive index profile $(\alpha = 2.0)$. In light of this, the validity of the derived analytical expression for power coupling coefficient may be extended to manufactured MMF with profile parameters within the range of $1.8 < \alpha < 2.2$.

The derived expression should prove useful in analyzing parameters which affect power coupling coefficient in commercial fibers. Analytical expressions for modal time delays in MMF [27] for near-parabolic refractive index profile parameters exist. An appropriate analytical expression for modal time delay in near-parabolic MMF, used alongside the derived analytical expression for power coupling coefficient allows an analytical expression for the channel impulse response for a commercial MMF to be constructed. The complete analytical expression for the channel impulse response should prove valuable in ascertaining optimal launch conditions for experiments such as offset launching, selective mode excitation and EDC.

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