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# NEW L-STABLE MODIFIED TRAPEZOIDAL METHODS FOR THE INITIAL VALUE PROBLEMS

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New L-stable trapezoidal formulas obtained by modifying the nonlinear trapezoidal formulas are presented. Numerical results are presented to confirm the theoretical stability analysis.

*Keywords*: L-Stability; Trapezoidal; Nonlinear methods

*CR Categories*: G.1.7

## 1. INTRODUCTION

In Sanugi [1], it was shown that the geometric mean (GM) trapezoidal formula is L-stable. In recent work, Chawla and Al-Zanaidi [2] proposed that by modifying the classical arithmetic mean (AM), geometric mean (GM) and harmonic mean ($H_aM$) trapezoidal formulas, a new class of modified trapezoidal formulas with L-stability is obtained.

The classical trapezoidal formula is given by

$$y_{n+1} = y_n + \frac{h}{2}[f_n + f_{n+1}] \tag{1.1}$$

for the numerical integration of the initial value problem

$$y' = f(x, y), \quad y(a) = y_0. \tag{1.2}$$

Alternative formulas to (1.1) have been proposed by using various other types of means for $f_n$ and $f_{n+1}$. A contraharmonic mean ($C_oM$) trapezoidal formula:

$$y_{n+1} = y_n + h\left[\frac{f_n^2 + f_{n+1}^2}{f_n + f_{n+1}}\right] \tag{1.3}$$

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was discussed by Evans and Yaakub [3] and a centroidal mean ($C_eM$) was proposed by Evans and Yaakub [4]:

$$y_{n+1} = y_n + h\left[\frac{2(f_n^2 + f_n f_{n+1} + f_{n+1}^2)}{3(f_n + f_{n+1})}\right] \tag{1.4}$$

and a root-mean square (RMS) was also proposed by Evans and Yaakub [5]:

$$y_{n+1} = y_n + h\sqrt{\left(\frac{f_n^2 + f_{n+1}^2}{2}\right)}. \tag{1.5}$$

In this paper, we present a new class of trapezoidal formulas by modifying the contraharmonic mean ($C_oM$), the centroidal mean ($C_eM$) and root-mean square (RMS). By using a standard test equation

$$f(x, y) = \lambda y, \quad \lambda \in C, \quad \mathrm{Re}(z) < 0 \tag{1.6}$$

we show that each of these modified trapezoidal formulas is second order and L-stable for autonomous problems.

The stability theory for ordinary differential equations including zero stability, weak stability and absolute stability or A-stability are discussed in Lambert [6]. In our discussion the term L-stability was defined as:

DEFINITION 1 *A one-step method is said to be L-stable if it is A-stable, and when applied to the equation* $y' = \lambda y$ *with* $\mathrm{Re}(\lambda) < 0$, *it gives* $y_{n+1} = Q(h\lambda)y_n$, *where* $|Q(h\lambda)| \to 0$ *as* $\mathrm{Re}(h\lambda) \to -\infty$.

## 2. L-STABLE MODIFIED TRAPEZOIDAL FORMULAS

The classical Euler method

$$y_{n+1} = y_n + hf_n \tag{2.1}$$

applied backwards at $x_{n+1}$ in the negative $x$-direction gives

$$y_n = y_{n+1} - hf_{n+1}. \tag{2.2}$$

Formula (2.2) was called the explicit backward Euler formula [2] and defined by

$$\hat{y}_n = y_{n+1} - hf_{n+1} \tag{2.3}$$

and we set

$$\hat{f}_n = f(x_n, \hat{y}_n).$$

## 2.1 Modified Arithmetic Mean (MAM) Trapezoidal Formula

The classical AM trapezoidal formula (1.2) was modified with Eq. (2.3) and written as

$$y_{n+1} = y_n + \frac{h}{2}[\hat{f}_n + f_{n+1}] \tag{2.4}$$

where we can obtain the local truncation error in Eq. (2.4) since

$$\hat{y}_n = y_n - \frac{h^2}{2}y''_n + 0(h^3)$$

and

$$\hat{f}_n = f_n - \frac{h^2}{2}y''_n g_n + 0(h^3) \tag{2.5}$$

where $g_n = \partial f_n / \partial y_n$. By substituting Eq. (2.5) into (2.4) we obtain

$$\begin{aligned} y_{n+1} &= y_n + \left(\frac{h}{2}y'_n + \frac{h}{2}y'_n\right) + \frac{h^2}{2}y''_n + \frac{h^3}{4}(y'''_n - y''g_n) + 0(h^4) \\ &= y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{12}(3y'''_n - 3y''_n g_n) + 0(h^4) \end{aligned} \tag{2.6}$$

and by Taylor series as

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + 0(h^4). \tag{2.7}$$

By subtracting (2.6) from (2.7), it can be shown that the local truncation error for the MAM trapezoidal formula is

$$\mathrm{LTE}_{\mathrm{MAM}} = \frac{h^3}{12}[-y'''_n + 3y''_n g_n] + 0(h^4). \tag{2.8}$$

THEOREM 1. *The MAM trapezoidal formula* (2.4) *is L-stable.*

*Proof* By applying the test Eq. (1.6) in (2.3) we obtain

$$\hat{y}_n = (1 - z)y_{n+1} \tag{2.9}$$

and

$$\hat{f}_n = \lambda(1 - z)y_{n+1}. \tag{2.10}$$

By substituting Eqs. (2.9) and (2.10) in (2.4), we obtain

$$y_{n+1} = y_n + \frac{z}{2}[(2 - z)y_{n+1}]. \tag{2.11}$$

Following Evans and Sanugi [7], we write

$$\frac{y_{n+1}}{y_n} = Q$$

in Eq. (2.11), to obtain

$$Q(z) = \frac{2}{2 - 2z + z^2}. \tag{2.12}$$

Now, we set $w = 2 - 2z + z^2$ and $\mathrm{Re}(z) = -x - iy$ in Eq. (2.12) to give

$$w = (2 + 2x + x^2 - y^2) + i2xy \quad \text{and} \quad |w|^2 = (2 + 2x + x^2 - y^2)^2 + (2xy)^2.$$

For $x > 0$, since $|z| \geq x$, it follows that

$$|w|^2 \geq (2 + 2x + x^2)^2 \tag{2.13}$$

and from (2.12) we obtain

$$|Q(z)| < \frac{2}{2 + 2x + x^2}. \tag{2.14}$$

From Eq. (2.14), $|Q(z)| < 1$ and for $\mathrm{Re}(z) < 0$ we can see that $|Q(z)| \to 0$ as $x \to \infty$. This proves that the MAM trapezoidal formula (2.4) is L-stable.

### 2.2 Modified Contraharmonic Mean ($MC_oM$) Trapezoidal Formula

With (2.3), we extend the modification to $C_oM$ method in (1.3) to obtain

$$y_{n+1} = y_n + h\left[\frac{\hat{f}_n^2 + f_{n+1}^2}{\hat{f}_n + f_{n+1}}\right]. \tag{2.15}$$

The local truncation error is obtained by using (2.5) to give

$$\begin{aligned}\frac{\hat{f}_n^2 + f_{n+1}^2}{\hat{f}_n + f_{n+1}} &= f_n + \frac{h}{2}y_n'' + \frac{h^2}{4}\left(y_n''' - y_n''g_n + \frac{(y'')^2}{f_n}\right) \\ &\quad + \frac{h^3}{12}\left[y_n^{iv} + 3\frac{y_n''y_n'''}{f_n} - \frac{3(y_n'')^2}{2f_n} + 3\frac{(y_n'')^2 g_n}{f_n}\right] + 0(h^4).\end{aligned} \tag{2.16}$$

By substituting (2.16) into (2.15) we obtain

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2}y_n'' + \frac{h^3}{4}\left[y_n''' + \frac{(y_n'')^2}{f_n} - y_n''g_n\right] + 0(h^4). \tag{2.17}$$

From (2.7) and (2.17) it can be shown that the local truncation error for the $MC_oM$ trapezoidal formula is given by

$$\mathrm{LTE}_{MC_oM} = \frac{h^3}{12}\left[3y_n''g_n - 3\frac{(y_n'')^2}{f_n} - y_n'''\right]. \tag{2.18}$$

THEOREM 2. *The $MC_oM$ trapezoidal formula* (2.15) *is L-stable.*

*Proof* By applying the $MC_oM$ trapezoidal formula (2.15) to the test equation in (1.6), we obtain

$$y_{n+1} = y_n + z\left[\frac{(2-z)^2}{(2-z)}y_{n+1}\right]. \tag{2.19}$$

Again we write

$$\frac{y_{n+1}}{y_n} = Q$$

in the Eq. (2.19), to obtain

$$Q(z) = \frac{2-z}{2-5z+4z^2-z^3}. \tag{2.20}$$

Now, we set $w = 2 - 5z + 4z^2 - z^3$ and $\mathrm{Re}(z) = -x - iy$ in Eq. (2.20) to give

$$w = (2 + 5x + 4x^2 - 4y^2 + xy^2 + 2xy^2 + x^3) + i(8xy + 5y - y^3 + 3x^2y) \quad \text{and}$$
$$|w|^2 = (2 + 5x + x^3)^2 + (9 + 52x + 32x^2 + 16x^3)y^2 + (26 + 8x + 3x^2)y^4 + y^6.$$

For $x > 0$, since $|z| \geq x$, it follows that

$$|w|^2 \geq (2 + 5x + 4x^2 + x^3)^2 \tag{2.21}$$

and from (2.20) we obtain

$$|Q(z)| < \frac{2+x}{2+5x+4x^2+x^3}. \tag{2.22}$$

From Eq. (2.22), $|Q(z)| < 1$ and for $\mathrm{Re}(z) < 0$ we can see that $|Q(z)| \to 0$ as $x \to \infty$. Thus, we show that our $MC_oM$ trapezoidal formula (2.15) is L-stable.

### 2.3 Modified Centroidal Mean ($MC_eM$) Trapezoidal Formula

With (2.3), we modify the $C_eM$ trapezoidal formula (1.3) to obtain

$$y_{n+1} = y_n + h\left[\frac{2(\hat{f}_n^2 + \hat{f}_n f_{n+1} + f_{n+1}^2)}{3(\hat{f}_n + f_{n+1})}\right]. \tag{2.23}$$

By using Mathematica and with (2.5), we obtain

$$\frac{2(\hat{f}_n^2 + \hat{f}_n f_{n+1} + f_{n+1}^2)}{3(\hat{f}_n + f_{n+1})} = f_n + \frac{h}{2} y_n'' + \frac{h^2}{12}\left[3y_n''' + \frac{(y_n'')^2}{f_n} - 3y_n'' g_n\right]$$
$$+ \frac{h^3}{24}\left[2f_n^2 y_n^{iv} + 2f_n y_n'' y_n''' - (y_n'')^2 + 2f_n (y_n'')^2 g_n\right] + 0(h^4). \qquad (2.24)$$

By substituting (2.24) into (2.23) we obtain

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} y_n'' + \frac{h^3}{12}\left[3y_n''' \frac{(y_n'')^2}{f_n} - 3y_n'' g_n\right] + 0(h^4). \qquad (2.25)$$

From (2.7) and (2.25) it can be shown that the local truncation error for the $MC_eM$ trapezoidal formula is given by

$$\mathrm{LTE}_{MC_eM} = \frac{h^3}{12}\left[3y_n'' g_n - \frac{(y_n'')^2}{f_n} - y_n'''\right]. \qquad (2.26)$$

THEOREM 3. *The $MC_eM$ trapezoidal formula* (2.23) *is L-stable.*

*Proof* By applying the $MC_eM$ trapezoidal formula (2.23) to the test equation in (1.6), we obtain

$$y_{n+1} = y_n + \frac{2z}{3}\left[\frac{(3 - 3z + z^2)}{(2 - z)} y_{n+1}\right] \qquad (2.27)$$

Following Evans and Sanugi [7], we write

$$\frac{y_{n+1}}{y_n} = Q$$

in the Eq. (2.27), to obtain

$$Q(z) = \frac{6 - 3z}{6 - 9z + 6z^2 - 2z^3}. \qquad (2.28)$$

Now, we set $w = 6 - 9z + 6z^2 - 2z^3$ and $\mathrm{Re}(z) = -x - iy$ in Eq. (2.28) to give

$$w = (6 + 9x + 6x^2 + 2x^3 - 6y^2 - 6xy^2) + i(9y + 12xy - y^3 + 6x^2 y) \quad \text{and}$$
$$|w|^2 = (6 + 9x + 6x^2 + 2x^3)^2 + (9 + 36x + 72x^2 + 48x^3 + 12x^4)y^2 + (24x + 12x^2)y^4 + 4y^6.$$

For $x > 0$, since $|z| \geq x$, it follows that

$$|w|^2 \geq (6 + 9x + 6x^2 + 2x^3)^2 \qquad (2.29)$$

and from (2.28) we obtain

$$|Q(z)| < \frac{6+3x}{6+9x+6x^2+2x^3}. \tag{2.30}$$

From Eq. (2.30), $|Q(z)| < 1$ and for $\mathrm{Re}(z) < 0$ we can see that $|Q(z)| \to 0$ as $x \to \infty$. Thus, the $MC_eM$ trapezoidal formula (2.23) is L-stable.

### 2.4 Modified Root-mean-square (MRMS) Trapezoidal Formula

With (2.3), we modify the RMS trapezoidal formula (1.3) to obtain

$$y_{n+1} = y_n + h\sqrt{\left(\frac{\hat{f}_n^2 + f_{n+1}^2}{2}\right)}. \tag{2.31}$$

By using Mathematica and with (2.5), we obtain

$$\sqrt{\left(\frac{\hat{f}_n^2 + f_{n+1}^2}{2}\right)} = f_n + \frac{h}{2}y_n'' + \frac{h^2}{12}\left[3y_n''' + \frac{3(y_n'')^2}{2f_n} - 3y_n''g_n\right] + \frac{h^3}{48}\left[4y_n^{iv} + 6\frac{y_n''y_n'''}{f_n} - 3\frac{(y_n'')^3}{f_n^2} + 6\frac{(y_n'')^2}{f_n}\right] + 0(h^4). \tag{2.32}$$

By substituting (2.32) into (2.31) we obtain

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2}y_n'' + \frac{h^3}{12}\left[3y_n''' + \frac{3(y_n'')^2}{2f_n} - 3y_n''g_n\right] + 0(h^4). \tag{2.33}$$

From (2.7) and (2.33) it can be shown that the local truncation error for the MRMS trapezoidal formula is given by

$$\mathrm{LTE}_{\mathrm{MRMS}} = \frac{h^3}{12}\left[3y_n''g_n - \frac{3(y_n'')^2}{2f_n} - y_n'''\right]. \tag{2.34}$$

THEOREM 4. *The MRMS trapezoidal formula* (2.31) *is L-stable.*

*Proof* By applying the MRMS trapezoidal formula (2.31) to test the equation in (1.6), we obtain

$$y_{n+1} = y_n + \frac{z}{\sqrt{2}}\left[\left(\sqrt{2-2z+z^2}\right)y_{n+1}\right]. \tag{2.35}$$

As previously, we write

$$\frac{y_{n+1}}{y_n} = Q$$

in the Eq. (2.35), to obtain

$$Q(z) = \frac{\sqrt{2}}{\sqrt{2} - z(2 - 2z + z^2)^{1/2}}. \tag{2.36}$$

Now, we set $w = \sqrt{2} - z(2 - 2z + z^2)^{1/2}$ and $\mathrm{Re}(z) = -x - iy$ in Eq. (2.36) and by setting

$$\sinh(2r) = \frac{(2 + 2x)y}{2 + 2x + x^2} \tag{2.37}$$

we obtain

$$(2 - 2z + z^2)^{1/2} = [(2 + 2x + x^2 - y^2) + i(2 + 2x)y]^{1/2} \tag{2.38}$$

or Eq. (2.38) can also be written as

$$\begin{aligned} (2 - 2z + z^2)^{1/2} &= [(2 + 2x + x^2) + i(2 + 2x)y]^{1/2} \\ &= (2 + 2x + x^2)^{1/2}[1 + i\sinh(2r)]^{1/2} \\ &= (2 + 2x + x^2)^{1/2}[\cosh(r) + i\sinh(r)]. \end{aligned} \tag{2.39}$$

From Eq. (2.36), we set

$$\begin{aligned} w &= \sqrt{2} - z(2 - 2z + z^2)^{1/2} \\ &= \sqrt{2} + (x + iy)(2 + 2x + x^2)^{1/2}[\cosh(r) + i\sinh(r)] \\ &= \left(\sqrt{2} + x\sqrt{2 + 2x + x^2}\cosh(r) - y\sqrt{2 + 2x + x^2}\sinh(r)\right) \\ &\quad + i\left(x\sqrt{2 + 2x + x^2}\sinh(r) + y\sqrt{2 + 2x + x^2}\cosh(r)\right) \end{aligned}$$

and we can show that

$$\begin{aligned} |w|^2 &= 2 + x\sqrt{2 + 2x + x^2}\left[x\sqrt{2 + 2x + x^2}\cosh(2r) + 2^{1/2}\sinh(r)\right] \\ &\quad + \sqrt{2}y\sqrt{2 + 2x + x^2}\left[\frac{1}{\sqrt{2}}y\sqrt{2 + 2x + x^2}\cosh(2r) - 2\sinh(r)\right]. \end{aligned} \tag{2.40}$$

By substituting $y$ from (2.37) into (2.40), we obtain

$$\begin{aligned} |w|^2 &= 2 + x\sqrt{2 + 2x + x^2}\left[x\sqrt{2 + 2x + x^2}\cosh(2r) + 2^{1/2}\sinh(r)\right] \\ &\quad + 2^{1/2}\frac{(2 + 2x + x^2)^{1/2}}{(2 + 2x)}\sinh^2(2r)\cosh(r) \\ &\quad \times \left[\frac{(2 + 2x + x^2)^{1/2}}{\sqrt{2}(2 + 2x)}\cosh(r)\cosh(2r) - 1\right]. \end{aligned} \tag{2.41}$$

From Eq. (2.41), we can see that for $x > 0$

$$\frac{(2+2x+x^2)^{1/2}}{\sqrt{2}(2+2x)}\cosh(r)\cosh(2r) > 1$$

and since $|z| \geq x$, it follows that

$$|w| > 2 + x(2+2x+x^2)^{1/2} \tag{2.42}$$

and from (2.36) we obtain

$$|Q(z)| < \frac{2}{2+x(2+2x+x^2)^{1/2}}. \tag{2.43}$$

Thus from Eq. (2.43), $|Q(z)| < 1$ and for $\mathrm{Re}(z) < 0$ we can see that $|Q(z)| \to 0$ as $x \to \infty$. Thus, the MRMS trapezoidal formula (2.31) is L-stable.

## 3. NUMERICAL EXAMPLE

*Example 1* We consider the initial value problem

$$y' = -2y, \qquad y(0) = 1, \quad 0 \leq x \leq 1, \tag{3.1}$$

where the exact solution is $y(x) = \exp(-2x)$. The absolute error in the numerical solution using this new class of modified trapezoidal formula obtained with step-size $h = 0.01$ after every ten steps are shown in Table I.

*Example 2* The second-order differential equation IVP given by

$$y'' + 101y' + 100y = 0, \quad y(0) = 1.01, \; y'(0) = -2 \tag{3.2}$$

over the range $0 \leq x \leq 1$.

The general solutions for (3.2) is $y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ and the theoretical solution of problem (3.2) in the specified range is $y(x) = 0.01\exp(-100x) + \exp(-x)$.

TABLE I The Absolute Error for Solving Equation (3.1) Using the Various Modified Trapezoidal Formulas.

| $x$ | *MAM* | *MGM* | $MH_aM$ | $MC_oM$ | $MC_eM$ |
|---|---|---|---|---|---|
| 0.1 | 0.44493E − 04 | 0.52598E − 04 | 0.60703E − 04 | 0.28284E − 04 | 0.39085E − 04 |
| 0.2 | 0.72858E − 04 | 0.86131E − 04 | 0.99403E − 04 | 0.46314E − 04 | 0.64002E − 04 |
| 0.3 | 0.89479E − 04 | 0.10578E − 03 | 0.12208E − 03 | 0.56879E − 04 | 0.78602E − 04 |
| 0.4 | 0.97682E − 04 | 0.11548E − 03 | 0.13327E − 03 | 0.62093E − 04 | 0.85808E − 04 |
| 0.5 | 0.99972E − 04 | 0.11819E − 03 | 0.13639E − 03 | 0.63547E − 04 | 0.87819E − 04 |
| 0.6 | 0.98223E − 04 | 0.11612E − 03 | 0.13401E − 03 | 0.62435E − 04 | 0.86282E − 04 |
| 0.7 | 0.93823E − 04 | 0.11092E − 03 | 0.12801E − 03 | 0.59638E − 04 | 0.82417E − 04 |
| 0.8 | 0.87792E − 04 | 0.10379E − 03 | 0.11978E − 03 | 0.55804E − 04 | 0.77119E − 04 |
| 0.9 | 0.80865E − 04 | 0.95599E − 04 | 0.11033E − 03 | 0.51400E − 04 | 0.71034E − 04 |
| 1.0 | 0.73565E − 04 | 0.86975E − 04 | 0.10038E − 03 | 0.46759E − 04 | 0.64621E − 04 |

TABLE II The Absolute Error for Solving Equation (3.3) by the Various Modified Trapezoidal Formulas.

| $x$ | $MAM$ | $MGM$ | $MH_aM$ | $MC_oM$ | $MC_eM$ |
|---|---|---|---|---|---|
| 0.1 | 0.93712E − 07 | 0.60644E − 05 | 0.12227E − 04 | 0.12359E − 04 | 0.41858E − 05 |
| 0.2 | 0.10856E − 06 | 0.54558E − 05 | 0.11024E − 04 | 0.11192E − 04 | 0.38037E − 05 |
| 0.3 | 0.14766E − 06 | 0.48779E − 05 | 0.99074E − 05 | 0.10158E − 04 | 0.34827E − 05 |
| 0.4 | 0.17833E − 06 | 0.43606E − 05 | 0.89031E − 05 | 0.92192E − 05 | 0.31885E − 05 |
| 0.5 | 0.20184E − 06 | 0.38976E − 05 | 0.80002E − 05 | 0.83672E − 05 | 0.29186E − 05 |
| 0.6 | 0.21925E − 06 | 0.34832E − 05 | 0.71886E − 05 | 0.75938E − 05 | 0.26713E − 05 |
| 0.7 | 0.23152E − 06 | 0.31124E − 05 | 0.64589E − 05 | 0.68919E − 05 | 0.24446E − 05 |
| 0.8 | 0.23947E − 06 | 0.27806E − 05 | 0.58031E − 05 | 0.62548E − 05 | 0.22369E − 05 |
| 0.9 | 0.24382E − 06 | 0.24838E − 05 | 0.52135E − 05 | 0.56766E − 05 | 0.20466E − 05 |
| 1.0 | 0.24516E − 06 | 0.22183E − 05 | 0.46837E − 05 | 0.51517E − 05 | 0.18722E − 05 |

The problem (3.2) can also be written as a system, *i.e.*

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -100 & -101 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1.01 \\ -2 \end{pmatrix}. \tag{3.3}$$

The matrix of the system given by Eq. (3.3) has the eigenvalues $\lambda_1 = -100$ and $\lambda_2 = -1$ and the solutions of system (3.3) are $y_1(x) = y(x) = 0.01\exp(-100x) + \exp(-x)$ and $y_2(x) = z(x) = -\exp(-100x) - \exp(-x)$.

According to the analysis of the trapezoidal method for absolute stability we need $-2.0 < h\lambda < 0$ or $-2.0 < -h < 0$ and $-2.0 < -100h < 0$. These inequalities are both satisfied only if $h < 2/100 = 0.02$ in order to get a decreasing solution. The absolute errors in the numerical solutions for solving Eq. (3.3) using the modified trapezoidal formulas MAM, MGM, $MH_aM$, $MC_oM$ and $MC_eM$ obtained with $h = 0.001$ are shown in Table II after every hundred steps.

From Table I, we can see that the modified formulas based on the contraharmonic mean $MC_oM$ gives greater accuracy. While the results in Table II as expected show that the linear modified trapezoidal formula based on the arithmetic mean (AM) for solving system of stiff equation gives more accuracy. However in the same class of nonlinear modified trapezoidal formulas, we see that the centroidal mean $MC_eM$ method performs better.

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