Auslander-Reiten theory in functorially finite resolving subcategories

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Abstract

We analyze the Auslander-Reiten quiver Γ_Ω of a functorially finite resolving subcategory Ω. Chapter 1 gives a short introduction into the basic definitions and theorems of Auslander-Reiten theory in A-mod. We generalize these definitions and theorems in Chapter 2 and find a constant p such that $l(X) \leq pl(Y)$ if there is an Ω-irreducible morphism from X to Y. This constant enables us to prove the Brauer-Thrall 1.5 conjecture for Ω. Moreover, we find a connection between sectional paths in A-mod and irreducible morphisms in Ω.

In Chapter 3 we introduce degrees of irreducible morphisms and use this notion to prove the generalization of the Happel-Preiser-Ringel theorem for Ω. Finally, in Chapter 4, we analyze left stable components of Γ_Ω and find out that their left subgraph types are given by Dynkin diagrams if and only if Ω is finite. In the preparation of the proof we discover connected components with certain properties and name them helical components due to their shape. It turns out later that these components are the same as coray tubes. In the final section we discuss under which conditions the length of modules tends to infinity if we knit to the left in a component and give a complete description of all connected components in which this is not the case.

Acknowledgements

Contents

Introduction

When the notion of almost split sequences was first introduced in the early 1970s its significance and impact on representation theory of Artin algebras was not immediately recognized. These short exact sequences contain information about how certain morphisms split and their existence was first proved in [AR74]. More intensive studies of almost split sequences began after Ringel embedded them into Auslander-Reiten quivers [R78]. These quivers give a complete description how any morphism between finitely generated modules splits into irreducible morphisms.

It has been proved that almost split sequences do not only exist in A-mod, the category of all finitely generated left A-modules over an Artin algebra A, but also in certain subcategories [AS81], [R91], [K97]. This implies that there are also Auslander-Reiten quivers in these categories. One of the most important example of these categories is $\mathfrak{F}(\Delta)$, the category of standard-filtered modules of a quasi-hereditary algebra.

Since the vertices of Auslander-Reiten quivers are given by the isomorphism classes of indecomposable modules in A-mod or a subcategory χ, these quivers are finite if and only if A-mod or χ is finite respectively, i.e. there are up to isomorphism only finitely many indecomposable modules in A-mod or χ respectively. Naturally the question arose if there are criteria on the Auslander-Reiten quiver of A for the algebra to be representation finite. A partial answer to this question was given by Riedtmann by introducing the notion of tree type of stable translation quivers [R80], which allowed her to classify all selfinjective algebras of finite representation type by their Auslander-Reiten quivers [R80'], [R83], [BLR81]. She proved that if an algebra is representation finite, then the tree type of each stable component of its Auslander-Reiten quiver is given by a Dynkin diagram.

We generalize this statement and prove this generalization for Auslander-Reiten quivers of any algebra or certain subcategories that have Auslander-Reiten quivers. In order to establish this, we introduce a concept similar to the tree type which does not only work for stable components of Auslander-Reiten quivers.

In the whole dissertation A denotes an associative Artin algebra with a multiplicative identity over an algebraically closed field K. For convenience, we suppose that A is indecomposable as this simplifies the notation significantly. Nevertheless, all results can be applied to decomposable algebras as all indecomposable direct summands of a decomposable algebra have disjoint Auslander-Reiten quivers. X, Y and Z usually denote modules in A-mod, which is the category of all finitely generated left A-modules. Simple, projective and injective modules are denoted by S, P and I respectively. If for a morphism $f : Z \to Y$ there is a module X and morphisms $g : Z \to X$, $h : X \to Y$, we say f factors through g and h and f factors over X. The Jordan-Hölder length of a module X is denoted by $l(X)$. Most definitions are taken from [ARS95]. Note that for some theorems we also provide the dual statements, but only give one proof and leave the dual to the reader.

When we speak about Dynkin diagrams we always mean undirected Dynkin diagrams, which are classified below. In the infinite series A_n and D_n the index n equals the number of vertices in the diagram.
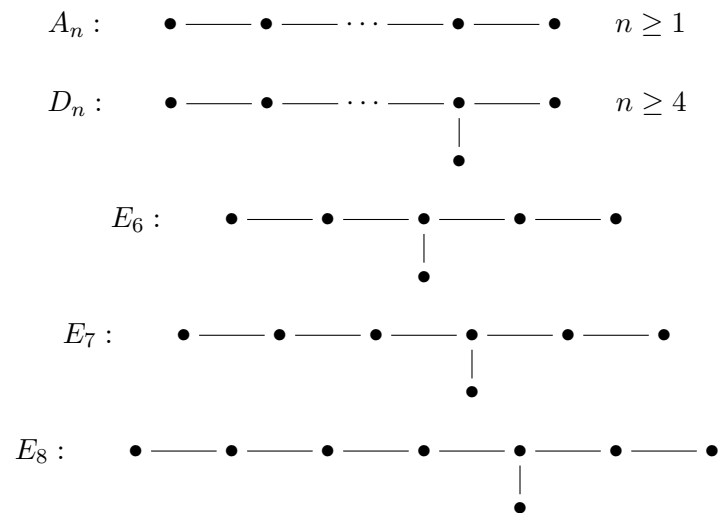
Moreover, the following are the Euclidean diagrams, which are obtained by adding one vertex to a Dynkin diagram such that the new diagram is not Dynkin anymore and every finite diagram without loops that is not a Dynkin diagram contains a Euclidean diagram. In the infinite series A_n and D_n the number of vertices in the diagram is $n + 1$.

$$\widetilde{E}_8 : \quad \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet$$
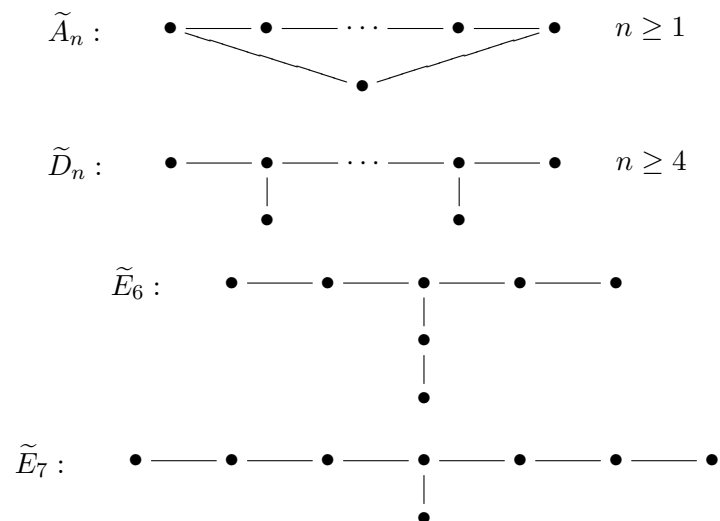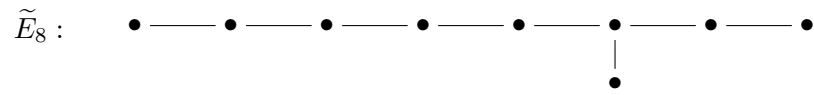
Consequently, the only diagrams without loops which are not Dynkin but do not contain a Euclidean diagram are the following infinite diagrams.

$$A_\infty : \quad \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\! \cdots$$

$$D_\infty : \quad \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\! \cdots$$

$$A_\infty^\infty : \quad \cdots \!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\! \bullet \!\!-\!\! \cdots$$
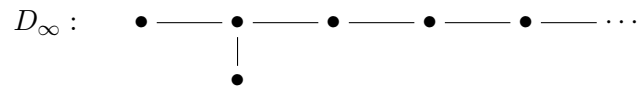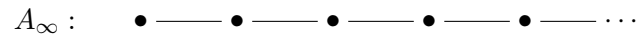
Chapter 1

Introduction to Auslander-Reiten theory

This chapter provides a short introduction to the basic definitions and most important theorems of Auslander-Reiten theory alongside with a few examples of Auslander-Reiten quivers. For further reading we recommend [ARS95].

1.1 Almost split sequences

Definition 1.1.1

(a) *We say a morphism $f : X \to Y$ is a **split epimorphism** if the identity on Y factors through f, that is there is a morphism $g : Y \to X$ such that $fg = \mathrm{Id}_Y$.*

$$
\begin{array}{ccc}
 & X & \\
\scriptstyle g \nearrow & & \searrow \scriptstyle f \\
Y & \xrightarrow{\ \ \mathrm{Id}\ \ } & Y
\end{array}
$$

(b) *Dually, a morphism $g : Y \to X$ is called a **split monomorphism** if there is a morphism $f : X \to Y$ such that $fg = Id_Y$.*

$$
\begin{array}{ccc}
 & X & \\
\scriptstyle g \nearrow & & \searrow \scriptstyle f \\
Y & \xrightarrow{\ \ \mathrm{Id}\ \ } & Y
\end{array}
$$

Definition 1.1.2

(a) *A morphism $f : X \to Y$ is called **right almost split** if it is not a split epimorphism and any morphism $Z \to Y$ that is not a split epimorphism factors through f.*

$$
\begin{array}{ccc}
Z & & \\
\downarrow & \searrow & \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

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*(b) We dually define a morphism $g : Y \to X$ to be **left almost split** if it is not a split monomorphism and any morphism $Y \to Z$ that is not a split monomorphism factors through g.*

$$Y \xrightarrow{g} X$$
$$\searrow \quad \vdots$$
$$Z$$

For convenience, we say that a right almost split morphism that maps to a module Y is a right almost split morphism for Y. Dually, we call a left almost split morphism $g : Y \to X$ a left almost split morphism for Y. An example for a right almost split morphism is the embedding $i : \mathrm{rad}P \to P$ for an indecomposable projective module P. Dually, the morphism $\Pi : I \to I/\mathrm{soc}I$ is left almost split for an indecomposable injective module I.

Lemma 1.1.3 *[ARS95, V. Lemma 1.7] Let $f : X \to Y$ be a morphism.*

(a) If f is right almost split, then Y is an indecomposable module.

(b) If f is left almost split, then X is an indecomposable module.

Definition 1.1.4 *A short exact sequence*

$$0 \longrightarrow X \xrightarrow{g} Y \xrightarrow{f} Z \longrightarrow 0$$

*is called an **almost split sequence** if g is left almost split and f is right almost split.*

Moreover, in an almost split sequence g and f are **left** and **right minimal morphisms** respectively, i.e. for all $h : Y \to Y$ such that $hg = g$ or $fh = f$, h is an isomorphism. Morphisms that are both right minimal and right almost split are called **minimal right almost split morphisms**, dually, a **minimal left almost split morphism** is both left minimal and left almost split. Minimal right almost split and minimal left almost split morphisms exist for every indecomposable module and their domains and codomains are unique up to isomorphism [ARS95, V.1]. This is important for the definition of Auslander-Reiten quivers in the next section.

It has been shown in [ARS95, V.1] that in an almost split sequence the modules X and Z are determined by each other and can be computed using certain functors. We consider a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \longrightarrow Z \longrightarrow 0$$

of Z. Let T denote the contravariant functor $\mathrm{Hom}_A(\ , A) : A\text{-mod} \to A^{op}\text{-mod}$. Then $\mathrm{Coker}(T(f))$ is an A^{op}-module, which we call the transpose of Z or $Tr(Z)$. Note that $Tr(Z)$ is zero if and only if Z is projective. We then call $\tau = DTr$ the **Auslander-Reiten translation**, where D denotes the usual duality $\mathrm{Hom}_A(\ , K)$.

Lemma 1.1.5 *[ARS95, IV. Proposition 1.10] $DTr : A\text{-mod} \to A\text{-mod}$ induces a bijection between the set of isomorphism classes of non-projective indecomposable modules and the set of isomorphism classes of non-injective indecomposable modules with TrD as inverse.*

Consequently, we write τ^{-1} for TrD. By definition we have $\tau(P) = \tau^{-1}(I) = 0$ for a projective module P and an injective module I respectively. Nevertheless, for an indecomposable module X and an integer n we say $\tau^n(X)$ exists only if $\tau^n(X) \neq 0$.

Theorem 1.1.6 *[ARS95, V. Theorems 1.15, 1.16]*

(a) *Let Z be an indecomposable non-projective A-module. Then there is an up to isomorphism unique almost split sequence*

$$0 \longrightarrow \tau(Z) \overset{g}{\longrightarrow} Y \overset{f}{\longrightarrow} Z \longrightarrow 0$$

ending in Z.

(b) *Let X be an indecomposable non-injective A-module. Then there is an up to isomorphism unique almost split sequence*

$$0 \longrightarrow X \overset{g}{\longrightarrow} Y \overset{f}{\longrightarrow} \tau^{-1}(X) \longrightarrow 0$$

starting in X.

1.2 Auslander-Reiten quivers

The Auslander-Reiten quiver of an algebra is a **biquiver**, i.e. a quiver with two disjoint sets of arrows, containing information on all almost split sequences. For its definition we need to consider a different point of view on the morphisms in an almost split sequence.

Definition 1.2.1 *A morphism $f : X \to Y$ is called **irreducible** if f is neither a split monomorphism nor a split epimorphism and, if $f = gh$ for some morphisms $h : X \to M$ and $g : M \to Y$, then either h is a split monomorphism or g is a split epimorphism.*

For example every minimal left or right almost split morphism is also an irreducible morphism. The following theorem describes the close connection between the two concepts.

Theorem 1.2.2 *[ARS95, V. Theorem 5.3]*

(a) *Let Z be an indecomposable module and Y a non-zero module in A-mod. A morphism $g : Y \to Z$ is irreducible if and only if there exists a morphism $g' : Y' \to Z$ such that the induced morphism $(g, g') : Y \oplus Y' \to Z$ is a minimal right almost split morphism.*

(b) *Dually, if X is an indecomposable module, then a morphism $f : X \to Y$ is irreducible if and only if there is a morphism $f' : X \to Y'$ such that the induced morphism $(f, f')^T : X \to Y \oplus Y'$ is a minimal left almost split morphism.*

For an almost split sequence

$$0 \longrightarrow X \overset{g}{\longrightarrow} Y \overset{f}{\longrightarrow} Z \longrightarrow 0$$

let $Y = \oplus Y_i$ be a decomposition of Y into indecomposable modules and let Π_i denote the projection from Y to Y_i. Furthermore, we set $f_i = f\Pi_i$ and $g_i = \Pi_i g$ to be the induced

morphisms from X to Y_i and from Y_i to Z respectively. Note that not only f and g are irreducible morphisms but also all f_i and g_i. Moreover, for indecomposable modules M and N any irreducible morphism $h : M \to N$ can be extended to a minimal left almost split morphism and to a minimal right almost split morphism by 1.2.2.

Theorem 1.2.3 *[ARS95, V. Proposition 5.9] Let*

$$0 \longrightarrow X \xrightarrow{g} Y \xrightarrow{f} Z \longrightarrow 0$$

be an exact sequence. It is an almost split sequence if and only if f and g are both irreducible.

If only either f or g is irreducible, then

$$0 \longrightarrow X \xrightarrow{g} Y \xrightarrow{f} Z \longrightarrow 0$$

is not necessarily an almost split sequence. For example, consider a minimal right almost split morphism $f : Y \to Z$ such that Y admits a non-trivial decomposition $Y_1 \oplus Y_2$ of Y such that $f_1 : Y_1 \to Z$ is surjective. Then there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(f_1) \longrightarrow Y_1 \xrightarrow{f_1} Z \longrightarrow 0$$

which is not an almost split sequence, but f_1 is clearly an irreducible morphism by 1.2.2. Alongside with irreducible morphisms we introduce the **radical** $\mathrm{rad}(X,Y)$, which is the set of all $f \in \mathrm{Hom}_A(X,Y)$ such that gfh is not an isomorphism for any $h : Z \to X$ and $g : Y \to Z$ with Z indecomposable. **Powers of the radical** are defined inductively, that is an $f \in \mathrm{Hom}_A(X,Y)$ is in $\mathrm{rad}^n(X,Y)$ if there is a module Z such that $f = gh$ for some $h \in \mathrm{rad}(X,Z)$ and $g \in \mathrm{rad}^{n-1}(Z,Y)$. This gives rise to a sequence of submodules

$$\cdots \subset \mathrm{rad}^n(X,Y) \subset \mathrm{rad}^{n-1}(X,Y) \subset \cdots \subset \mathrm{rad}(X,Y) \subset \mathrm{Hom}_A(X,Y)$$

which motivates to define

$$\mathrm{rad}^\infty(X,Y) = \bigcap_{n \in \mathbb{N}} \mathrm{rad}^n(X,Y).$$

For convenience, a morphism $f : X \to Y$ is called a **radical morphism** if $f \in \mathrm{rad}(X,Y)$. There is a crucial connection between the radical and irreducible morphisms.

Lemma 1.2.4 *[ARS95, V. Proposition 7.3] Let $f : X \to Y$ be a morphism between indecomposable modules X, Y. Then f is irreducible if and only if $f \in \mathrm{rad}(X,Y) \backslash \mathrm{rad}^2(X,Y)$.*

The proof is also given in a more general version in Lemma 2.2.3. So for X and Y indecomposable the elements of $\mathrm{rad}^n(X,Y) \backslash \mathrm{rad}^{n+1}(X,Y)$ are the non-zero morphisms that can be written as a sum of compositions of irreducible morphisms such that the shortest of these compositions has length n and cannot be written as a composition of more than n radical morphisms.

Definition 1.2.5 *The **Auslander-Reiten quiver** Γ_A of an algebra A and its module category A-mod is a biquiver, i.e. a quiver consisting of vertices Γ_0 and two disjoint sets of arrows between them, 1-arrows Γ_1 and 2-arrows Γ_2. There is a vertex for every isomorphism class of indecomposable modules of the algebra. Given two indecomposable modules X and Y there is a 1-arrow from the vertex corresponding to the isomorphism class of X to the vertex corresponding to the isomorphism class of Y if there is an irreducible morphism from X to Y. For each almost split sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

there is a 2-arrow from the vertex corresponding to the isomorphism class of Z to the vertex corresponding to the isomorphism class of X. In order to distinguish between different types of arrows, 2-arrows are drawn as dotted arrows.

Definition 1.2.6 *Let $h : M \to N$ be an irreducible morphism between indecomposable modules and $f : X \to N$ and $g : M \to Y$ extensions to minimal right almost split and minimal left almost split morphisms respectively. We denote the number of M in a sum decomposition of X and the number of N in a sum decomposition of Y by m and n respectively. We then say the 1-arrow from M to N has **valuation** (m, n).*

Throughout the dissertation we usually do not distinguish between an indecomposable module, its isomorphism class or the corresponding vertices in Γ_A. There are squared brackets to mark injective and projective modules in Γ_A. For convenience, we simply say arrows for 1-arrows in Γ_A and, if the valuation of an arrow is $(1, 1)$, we say it is **trivially valuated**. Furthermore, for an indecomposable module Y we call each module X such that there is an arrow from X to Y in Γ_A an **immediate predecessor of Y**. Dually, each module Z such that there is an arrow from Y to Z in Γ_A is called an **immediate successor of Y**.

If a subquiver Γ of Γ_A contains two modules X and Y such that there is a 1-arrow or a 2-arrow between them in Γ_A, then we define that Γ automatically also contains this 1-arrow or 2-arrow respectively and the 1-arrow in Γ has the same valuation as in the whole Auslander-Reiten quiver. We call a non-empty subquiver Γ of Γ_A a **connected component** if for each X in Γ the modules in Γ are precisely all modules Y such that there is a walk between X and Y in Γ_A, that is X and Y are connected by arrows irrespective of their direction. Clearly, each indecomposable module X is contained in the unique connected component of Γ_A consisting of all modules that are connected to X by a walk in the Auslander-Reiten quiver. We call this component the connected component of X.

For a different point of view on arrow valuation we introduce some more notation. Recall that X and Y are indecomposable since they are modules in Γ_A. We denote the factor module $\mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y)$ by $\mathrm{Irr}(X, Y)$ and the division algebra $\mathrm{End}_A(X)/\mathrm{rad}(\mathrm{End}_A(X))$ by T_X, where in this case $\mathrm{rad}(\mathrm{End}_A(X))$ is the Jacobson radical of the algebra $\mathrm{End}_A(X)$. $\mathrm{Irr}(X, Y)$ then becomes a T_Y-T_X^{op}-bimodule and, if X and Y are indecomposable, a T_Y-T_X^{op}-vector space [ARS95, VII.1].

Theorem 1.2.7 *[ARS95, VII. Proposition 1.3] Let X and Y denote indecomposable A-modules and assume there is an irreducible morphism $f : X \to Y$. Let the valuation of the corresponding arrow be (m, n). Then m equals the dimension of $\operatorname{Irr}(X, Y)$ as a T_X^{op}-vector space while n equals the dimension of $\operatorname{Irr}(X, Y)$ as a T_Y-vector space.*

Lemma 1.2.8 *[ARS95, VII. Proposition 1.5] Suppose there is an arrow in Γ_A from X to Y with valuation (m, n). If Y is non-projective, then the valuation of the arrow from $\tau(Y)$ to X is (n, m).*

Corollary 1.2.9 *[ARS95, VII. Corollary 2.3] Since K is assumed to be algebraically closed, m equals n in all valuations of arrows in Γ_A.*

Let X and Y be indecomposable and assume there is an irreducible morphism from X to Y. If the corresponding arrow is valued (n, n), we say that there are **n arrows** from X to Y and we draw n arrows from X to Y in Γ_A. In particular, if there is only one arrow from X to Y, the arrow has trivial valuation. Consequently, if there are n arrows from X to Y in Γ_A, then there are n arrows from $\tau^k(X)$ to $\tau^k(Y)$ and from $\tau^k(Y)$ to $\tau^{k-1}(X)$ for all $k \in \mathbb{Z}$ such that these modules exists. Moreover, we say there are multiple arrows from X to Y if $n \geq 2$ and we say a connected component contains multiple arrows if and only if there are multiple arrows from X to Y for any modules X and Y in that connected component. In order to show that the module categories in the following examples are finite, we state another well-known theorem.

Theorem 1.2.10 *[ARS95, VI. Theorem 1.4] If there is a connected component Γ in Γ_A such that the length of all modules in Γ is bounded, then it is the only connected component and A is representation finite.*

Example 1.2.11 *Let A be the path algebra given by the quiver*

$$e_1 \longrightarrow e_2 \longrightarrow e_3$$

We compute the Auslander-Reiten quiver of A. The indecomposable modules are $S_3 = P_3, S_2, S_1 = I_1, P_1 = I_3, P_2$ and I_2. Since the canonical morphisms $\operatorname{rad}(P) \to P$ for projective modules and $I \to I/\operatorname{soc}(I)$ for injective modules are irreducible, we already have given irreducible morphisms $P_3 \to P_2$, $P_2 \to P_1$, $I_3 \to I_2$ and $I_2 \to I_1$. Considering P_2 we notice that it is the only indecomposable module that is domain of a non-zero non-isomorphism to S_2. Consequently, we have the following Auslander-Reiten quiver.
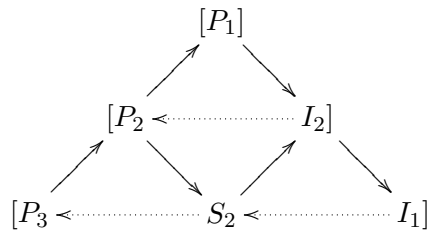
A slightly more complicated example shows that in some cases the Auslander-Reiten quiver contains an isomorphism class of an indecomposable module more than once. In these cases we consider the corresponding vertices to be different. This differentiation is important in Chapter 4. Nevertheless, we say a component Γ of the Auslander-Reiten quiver is finite if its vertices correspond to only finitely many isomorphism classes of indecomposable modules. So when we draw the Auslander-Reiten quiver of an algebra of finite representation type, we make sure that every different 1-arrow and 2-arrow occurs at least once, i.e. there is an illustration for every almost split sequence and every irreducible morphism.

Example 1.2.12 *Let A be the path algebra*

$$e_1 \xrightarrow{\ \alpha\ } e_2 \, \circlearrowleft \beta$$

with the relation $\beta^2 = 0$. Then some isomorphism classes of indecomposable modules occur more than once in the Auslander-Reiten quiver of A-mod.

Since the shape of the Auslander-Reiten quiver is crucial for this example, we calculate it in more detail then usual. First of all we name all occurring modules and write down their Jordan-Hölder composition series. We have simple modules S_1 and S_2, where S_1 is also the injective module I_1. Besides that there are projective and injective modules

$$
P_1 = \begin{matrix} S_1 \\ S_2 \\ S_2 \end{matrix}
\qquad
P_2 = \begin{matrix} S_2 \\ S_2 \end{matrix}
\qquad
I_2 = \begin{matrix} & S_1 & \\ S_1 & & S_2 \\ & S_2 & \end{matrix}
$$

and two other modules

$$
X = \begin{matrix} S_1 \\ S_2 \end{matrix}
\qquad
Y = \begin{matrix} S_1 & S_2 \\ & S_2 \end{matrix} .
$$

We start by computing $\tau(I_2)$, which is P_2. Clearly, P_1 is a direct summand of the middle term of the related almost split sequence, because the embedding $\mathrm{rad}(P_1) \to P_1$ is always irreducible. Consequently, Y must be the other summand in order to have a short exact sequence. The projection $I_2 \to I_2/\mathrm{soc}(I_2) = S_1 \oplus X$ is also irreducible, so the two direct summands must be $\tau^{-1}(P_1)$ and $\tau^{-1}(Y)$ respectively. But there is no morphism from P_1 to X that factors over a simple module, so we have $\tau^{-1}(P_1) = S_1$ by the Jordan-Hölder multiplicities of P_1, I_2 and X. It is easy to verify that $\tau(Y) = S_2$. On the other hand, by Jordan-Hölder multiplicities S_2 is also a direct summand of the almost split sequence from Y to X. These calculations give rise to the Auslander-Reiten quiver, which is shown on the next page.

$$
\begin{array}{ccccccc}
 & & [P_1 \xleftarrow{\quad\quad} I_1] & & \\
 & [P_2 \xleftarrow{\quad} I_2] & & \\
S_2 \xleftarrow{\quad} Y \xleftarrow{\quad} X \xleftarrow{\quad} S_2 & \\
X \xleftarrow{\quad} S_2 \xleftarrow{\quad} Y \xleftarrow{\quad} X & \\
 & [P_2 \xleftarrow{\quad} I_2] & \\
 & & [P_1 \xleftarrow{\quad} I_1] &
\end{array}
$$
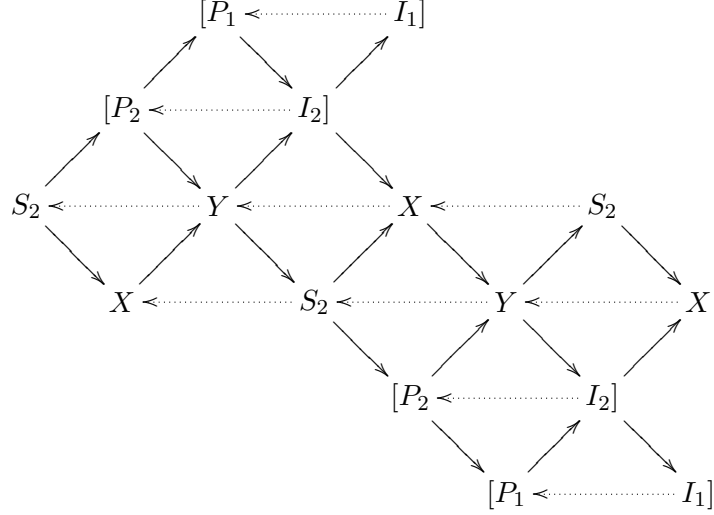
In order to show the necessity of certain assumptions in the following chapters, we compute the Auslander-Reiten quiver of another algebra, which we refer to as the **standard example**. Its significance for this dissertation is due to the following.

If A is a quasi-hereditary algebra, then there is a natural ordering of the indecomposable projective modules $P_1, \ldots, P_n$. We then define Δ_i to be the largest factor module of P_i such that for all simple blocks $S_j = P_j/(\mathrm{rad}(P_j))$ in the Jordan-Hölder series of Δ_i we have $j \leq i$. This gives a collection of modules $\Delta_1, \ldots, \Delta_n$, which are called the **standard modules** of A. Then a module M in A-mod such that there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ with $M_j/M_{j-1} \cong \Delta_{i_j}$ for all $j = 1, \ldots, k$ and some $1 \leq i_j \leq k$ is called **standard-filtered** and $\mathfrak{F}(\Delta)$ consists of all modules with this property. It has been shown that $\mathfrak{F}(\Delta)$ is a functorially finite resolving subcategory [R91]. Dually, we define the **costandard module** ∇_i as the largest submodule of I_i such that for all simple blocks S_j in ∇_i we have $j \leq i$.

Example 1.2.13 *Let A be the quasi-hereditary path algebra given by the quiver*

$$
e_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} e_2 \underset{g_2}{\overset{f_2}{\rightleftarrows}} e_3 \underset{g_3}{\overset{f_3}{\rightleftarrows}} e_4
$$

with relations $f_1 g_1 = g_2 f_2$, $f_2 g_2 = g_3 f_3$ and $f_3 g_3 = f_2 f_1 = f_3 f_2 = g_1 g_2 = g_2 g_3 = 0$.

The structure of $_A A$, which is A considered as a left A-module, is given by

$$
\begin{array}{cccc}
S_1 & S_2 & S_3 & S_4 \\
S_2 \oplus S_1 & S_3 \oplus S_2 & S_4 \oplus & S_3 \\
S_1 & S_2 & S_3 &
\end{array}.
$$

For convenience, we give all modules proper names. Later Theorem 1.2.10 is used to see that the modules mentioned are in fact all indecomposable modules up to isomorphism. S_1, S_2, S_3 and S_4 naturally denote the simple modules. The projective and injective modules are given by

$$P_1 = I_1 = \begin{matrix} S_1 \\ S_2 \\ S_1 \end{matrix} \qquad P_2 = I_2 = \begin{matrix} S_2 \\ S_1 \;\; S_3 \\ S_2 \end{matrix}$$

$$P_3 = I_3 = \begin{matrix} S_3 \\ S_2 \;\; S_4 \\ S_3 \end{matrix} \qquad P_4 = \begin{matrix} S_4 \\ S_3 \end{matrix} \qquad I_4 = \begin{matrix} S_3 \\ S_4 \end{matrix}.$$

The standard modules and costandard modules are

$$\Delta_1 = \nabla_1 = S_1 \qquad \Delta_2 = \begin{matrix} S_2 \\ S_1 \end{matrix} \qquad \Delta_3 = \begin{matrix} S_3 \\ S_2 \end{matrix}$$

$$\nabla_2 = \begin{matrix} S_1 \\ S_2 \end{matrix} \qquad \nabla_3 = \begin{matrix} S_2 \\ S_3 \end{matrix},$$

where $\Delta_4 = P_4$ and $\nabla_4 = I_4$. Finally we name all other modules.

$$X = \begin{matrix} S_2 \;\; S_4 \\ S_1 \;\; S_3 \end{matrix} \qquad Y = \begin{matrix} S_1 \;\; S_3 \\ S_2 \;\; S_4 \end{matrix}$$

$$M_2 = \begin{matrix} S_2 \\ S_1 \;\; S_3 \end{matrix} \qquad M_3 = \begin{matrix} S_3 \\ S_2 \;\; S_4 \end{matrix}$$

$$N_2 = \begin{matrix} S_1 \;\; S_3 \\ S_2 \end{matrix} \qquad N_3 = \begin{matrix} S_2 \;\; S_4 \\ S_3 \end{matrix}$$

We start the computation of the Auslander-Reiten quiver of A-mod by observing that P_1 and P_3 are also injective. It follows that there are almost split sequences

$$0 \longrightarrow \Delta_2 \longrightarrow P_1 \oplus S_2 \longrightarrow \nabla_2 \longrightarrow 0$$

$$0 \longrightarrow N_3 \longrightarrow S_2 \oplus S_4 \oplus P_3 \longrightarrow M_3 \longrightarrow 0$$

by [ARS95, V. Proposition 5.5]. We connect these almost split sequences at the module S_2 and obtain the whole Auslander-Reiten quiver by knitting to either left or right.
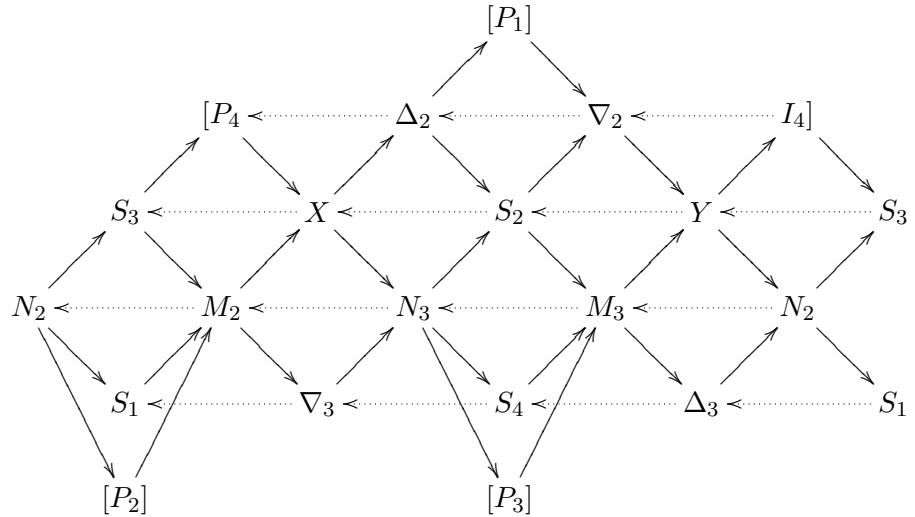
Chapter 2

Irreducible morphisms in functorially finite resolving subcategories

2.1 Functorially finite resolving subcategories

As mentioned in the introduction, almost split sequences do not only exist in A-mod but also occur in subcategories of A-mod. Let χ be a subcategory of A-mod. By **subcategory** we always mean a full subcategory closed under isomorphisms, direct sums and summands.

Definition 2.1.1

(a) *For modules X and Y in χ a morphism $f : X \to Y$ is called **right almost split in** χ if it is not a split epimorphism and any morphism $Z \to Y$ with Z in χ that is not a split epimorphism factors through f.*

$$\begin{array}{ccc} Z & & \\ \downarrow & \searrow & \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

(b) *We dually define a morphism $g : Y \to X$ to be **left almost split in** χ if it is not a split monomorphism and any morphism $Y \to Z$ with Z in χ that is not a split monomorphism factors through g.*

$$\begin{array}{ccc} Y & \xrightarrow{\ g\ } & X \\ & \searrow & \downarrow \\ & & Z \end{array}$$

Similarly to A-mod, we say that a right almost split morphism $f : X \to Y$ in χ is a right almost split morphism for Y in χ. Dually, we call a left almost split morphism $g : Y \to X$ in χ a left almost split morphism for Y in χ .

Definition 2.1.2 *For modules X, Y, Z in χ a short exact sequence*

$$0 \longrightarrow X \xrightarrow{\ g\ } Y \xrightarrow{\ f\ } Z \longrightarrow 0$$

*is called an **almost split sequence in χ** if g is minimal left almost split in χ and f is minimal right almost split in χ.*

If there are up to isomorphism unique almost split sequences in χ ending in every indecomposable non-projective module in χ, then we clearly also have an Auslander-Reiten quiver for χ. One type of subcategory that we are particularly interested in with this property are functorially finite resolving subcategories. The motivating example for these categories is the category of all modules with a standard filtration of a quasi-hereditary algebra or in general categories obtained by a generalized cotilting module. For an arbitrary indecomposable non-projective module X in χ, the module $\tau(X)$ is not necessarily in χ. Therefore, we introduce approximations, which give rise to an operation on the indecomposable modules in χ whose properties in χ are similar to those of the Auslander-Reiten translation in A-mod. We start by recalling the definitions.

Definition 2.1.3

(a) *A **right χ-approximation** of a module Y is a morphism $f_Y : X_Y \to Y$, where X_Y is in χ, such that for all Z in χ every morphism $g : Z \to Y$ factors through f_Y.*

$$
\begin{array}{ccc}
Z & & \\
\big\downarrow & \searrow{\scriptstyle g} & \\
X_Y & \xrightarrow{\ f_Y\ } & Y
\end{array}
$$

(b) *Dually, a **left χ-approximation** of a module Y is a morphism $f^Y : Y \to X^Y$, where X^Y is in χ, such that for all Z in χ every morphism $g : Y \to Z$ factors through f^Y.*

$$
\begin{array}{ccc}
 & & Z \\
 & \nearrow{\scriptstyle g} & \big\uparrow \\
Y & \xrightarrow{\ f^Y\ } & X^Y
\end{array}
$$

Definition 2.1.4 *A subcategory χ of A-mod is called*

(a) **contravariantly finite** *if right χ-approximations exist for all modules in A-mod,*

(b) **covariantly finite** *if left χ-approximations exist for all modules in A-mod,*

(c) **functorially finite** *if it is both contravariantly and covariantly finite.*

If a right χ-approximation is also a right minimal morphism, we call it a **minimal right χ-approximation**. Dually, if a left χ-approximation is also a left minimal morphism, we call it a **minimal left χ-approximation**. Unless otherwise specified we assume all right and left χ-approximations to be right minimal and left minimal respectively. By definition the domain X_Y of a minimal right χ-approximation and the codomain of a minimal left χ-approximation are unique up to isomorphism.

Definition 2.1.5 *Let Y be an arbitrary module and let X be a module in χ.*

(a) *A $\boldsymbol{\chi}$-section is a morphism $f : X \to Y$ such that for all modules Z in χ and morphisms $h : X \to Z$ and $g : Z \to Y$ with $f = gh$, h must be a split monomorphism.*

(b) *A $\boldsymbol{\chi}$-contraction is a morphism $f : Y \to X$ such that for all modules Z in χ and morphisms $h : Z \to X$ and $g : Y \to Z$ with $f = hg$, h must be a split epimorphism.*

The relation between χ-sections and minimal right χ-approximations corresponds to the relation between irreducible and minimal right almost split morphisms. This means a morphism $f : X \to Y$ is a χ-section if and only if there are a module X' in χ and a morphism $f' : X' \to Y$ such that $(f, f') : X \oplus X' \to Y$ is a minimal right χ-approximation [KP03, Proposition 2.2]. We call the morphism (f, f') an **extension of $\boldsymbol{f}$** to a minimal right χ-approximation.

Definition 2.1.6

(a) *A module I in χ is called **Ext-injective in $\boldsymbol{\chi}$** if $\operatorname{Ext}_A^i(M, I) = 0$ for all M in χ and all $i \geq 1$.*

(b) *Dually, a module P in χ is called **Ext-projective in $\boldsymbol{\chi}$** if $\operatorname{Ext}_A^i(P, M) = 0$ for all M in χ and all $i \geq 1$.*

If we deal with only one subcategory, we simply say I is Ext-injective or P is Ext-projective.

Definition 2.1.7 *A subcategory χ of A-mod is called **closed under extensions** if for all X, Z in χ the existence of a short exact sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

implies that Y is in χ.

Definition 2.1.8 *Let Ω be a subcategory of A-mod. We call Ω **resolving** if*

(a) *it is closed under extensions,*

(b) *$_AA$ is in Ω and*

(c) *it is closed under kernels of epimorphisms.*

Definition 2.1.9 *An A-module T is called a **generalized cotilting module** if*

(a) *its injective dimension is finite, that is there is an exact sequence*

$$0 \longrightarrow T \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0,$$

where all I_j are injective A-modules,

(b) *$\operatorname{Ext}_A^i(T, T) = 0$ for all $i \geq 1$ and*

(c) there is a finite resolution

$$0 \longrightarrow T_n \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow I \longrightarrow 0$$

for each injective module I where T_i in $\mathrm{add}(T)$, i.e. each T_i is a sum of direct summands of T.

It has been shown by Auslander and Reiten in [AR91] that for a generalized cotilting module T the category $^{\perp}T = \{\mathrm{Ext}_A^i(-, T) = 0\}$ is contravariantly finite and resolving. Moreover, if A has finite global dimension, then there is a bijection between contravariantly finite resolving subcategories $^{\perp}T$ and isomorphism classes of basic cotilting modules T [R07, Corollary 2.4], where basic means that no indecomposable summand of T occurs more than once in T.

Let χ denote a functorially finite subcategory of A-mod that is closed under extensions. Let Y be an indecomposable module in χ that is not Ext-projective, we then can decompose the domain of a right χ-approximation $X_{\tau(Y)} = M \oplus I$ where $M \neq 0$ is indecomposable and not Ext-injective [K97, Theorem 2.3] whereas I is Ext-injective [KP03, Theorem 5.3]. The module M is then denoted by $\tau_\chi(Y)$. For left χ-approximations the dual statement holds, i.e. if Y is not Ext-injective, we can decompose the codomain of a left χ-approximation $X^{\tau^{-1}(Y)} = N \oplus P$ where $N \neq 0$ is indecomposable and not Ext-projective while P is Ext-projective. Analogously, the module N is denoted by $\tau_\chi^{-1}(Y)$. We call τ_χ the **relative Auslander-Reiten translation of χ** and $\tau_\chi(Y)$ the relative Auslander-Reiten translate of a module Y in χ. It follows from these results that almost split sequences in χ have similar properties as they do in A-mod.

Lemma 2.1.10 *[K97, Proposition 2.7]*

(a) Let

$$0 \longrightarrow \mathrm{Ker}(f) \overset{g}{\longrightarrow} Y \overset{f}{\longrightarrow} Z \longrightarrow 0$$

be an exact sequence in χ. If f is a minimal right almost split morphism in χ, then $\mathrm{Ker}(f) \cong \tau_\chi(Z)$ and g is a minimal left almost split morphism in χ.

(b) Let

$$0 \longrightarrow X \overset{g}{\longrightarrow} Y \overset{f}{\longrightarrow} \mathrm{Coker}(g) \longrightarrow 0$$

be an exact sequence in χ. If g is a minimal left almost split morphism in χ, then $\mathrm{Coker}(g) \cong \tau_\chi^{-1}(X)$ and f is a minimal right almost split morphism in χ.

Throughout the dissertation χ denotes a functorially finite subcategory of A-mod that is closed under extensions while $\Omega = {}^{\perp}T$ denotes a functorially finite resolving subcategory of χ that is generated by a generalized cotilting module T. In particular, χ contains all projective modules since they are contained in Ω. Most of the upcoming results still hold if we just demand χ and Ω to be contravariantly finite, but all examples that are discussed are functorially finite subcategories. Moreover, we want to deal with Auslander-Reiten quivers in which we are able to knit both to the left and the right.

By definition resolving subcategories contain all projective modules and these are precisely the Ext-projective modules. Hence if χ contains an Ext-projective module P which is not projective, then P is not in Ω. An arbitrary resolving subcategory usually does not contain all injective modules. In the case of $^{\perp}T$ the Ext-injective modules are all modules in $\mathrm{add}(T)$ by construction. In particular, in Ω the number of indecomposable non-isomorphic projective modules coincides with the number of indecomposable non-isomorphic Ext-injective modules by [R07, Proposition 1.4]. Note that A-mod itself is clearly a functorially finite resolving subcategory.

Lemma 2.1.11 *Let*

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence and let Z be an indecomposable module in a functorially finite resolving subcategory Ω. Then there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X_N & \xrightarrow{i} & X_M & \xrightarrow{gf_M} & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_N} & & \downarrow{\scriptstyle f_M} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & Z & \longrightarrow & 0
\end{array}
$$

such that the top row splits if and only if the bottom row splits.

Proof:
Let f_M be a right Ω-approximation of M, then the composition gf_M is surjective as a composition of surjective maps. We denote its kernel by X_N and prove that the natural morphism from X_N to N is a right Ω-approximation. As $0 = gf_M i$, we know that $\mathrm{Im}(f_M i) \subset \mathrm{Ker}(g) \cong N$ embeds naturally into N. We call this morphism f_N and show that it is a right Ω-approximation of N. Let X be a module in Ω and $h : X \to N$ a morphism, then the composition fh factors through f_M, so we have $fh = f_M h'$ for some $h' : X \to X_M$. Moreover, $gf_M h' = gfh = 0$, so h' maps to the Kernel of gf_M, which equals X_N. In other words, there is a $h'' : X \to X_N$ such that $h' = ih''$ and, in particular, $fh = f_M i h'' = f f_N h''$. As f is injective, this implies $h = f_N h''$, which proves that f_N is a right Ω-approximation.

Suppose now that the top row splits, then Z is a direct summand of X_M and gf_M is a split epimorphism. In particular, g is a split epimorphism and the bottom row also splits. Lastly, we assume that the bottom row splits, then Z is a direct summand of M and a minimal right Ω-approximation is given by the identity on Z. Hence the top row also splits, which completes the proof. $\qquad\square$

Note that if the bottom row is an almost split sequence in A-mod, then the diagram can be reduced to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{i|_X} & Y & \xrightarrow{gf_M|_Y} & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_N|_X} & & \downarrow{\scriptstyle f_M|_Y} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & Z & \longrightarrow & 0
\end{array}
$$

where $X = \tau_{\Omega}(Z)$ is the unique direct summand of $X_N = X \oplus I$ that is not Ext-injective and $Y \cong X_M/i(I)$.

We now recall a result on contravariantly finite subcategories that we apply to Ω frequently throughout the dissertation.

Lemma 2.1.12 *[AR91, Proposition 3.3]*

(a) *Let $f_Y : X_Y \to Y$ be a minimal right Ω-approximation of Y. Then f_Y induces an isomorphism*

$$\mathrm{Ext}_A^i(X, X_Y) \cong \mathrm{Ext}_A^i(X, Y)$$

for all $i \geq 1$ and X in Ω.

(b) *Dually, if $f^X : X \to X^X$ is a minimal left Ω-approximation of X, then f^X induces an isomorphism*

$$\mathrm{Ext}_A^i(X^X, Y) \cong \mathrm{Ext}_A^i(X, Y)$$

for all $i \geq 1$ and Y in Ω.

Clearly, the short exact sequences corresponding to the isomorphism in (a) for $i = 1$ are given by the diagram of Lemma 2.1.11. In particular, the middle term of the sequence corresponding to $\mathrm{Ext}_A^1(X, X_Y)$ is the domain of a minimal right Ω-approximation of the middle term of the sequence corresponding to $\mathrm{Ext}_A^1(X, Y)$. More generally, the following holds for a functorially finite subcategory that is closed under extensions.

Lemma 2.1.13 *[K97, Lemma 2.1] Let N be a module and let $f_N : X_N \to N$ and $f^N : N \to X^N$ denote minimal right and left χ-approximations respectively. Then we have the following.*

(a) *$\mathrm{Ext}_A^1(\ , f_N)|_\chi : \mathrm{Ext}_A^1(\ , X_N)|_\chi \to \mathrm{Ext}_A^1(\ , N)|_\chi$ is a monomorphism of functors.*

(b) *$\mathrm{Ext}_A^1(f^N, \ )|_\chi : \mathrm{Ext}_A^1(X^N, \ )|_\chi \to \mathrm{Ext}_A^1(N, \ )|_\chi$ is a monomorphism of functors.*

Lemma 2.1.14 *[KP03, Lemma 4.3] Let X be a module in χ, N an arbitrary module and $f : X \to N$ a morphism that induces a monomorphism of contravariant functors*

$$\mathrm{Ext}_A^1(\ , f)|_\chi : \mathrm{Ext}_A^1(\ , X)|_\chi \to \mathrm{Ext}_A^1(\ , N)|_\chi.$$

(a) *If $\mathrm{Ext}_A^1(\tau^{-1}(N), X) = 0$, then X is Ext-injective in χ.*

(b) *If $\tau^{-1}(N)$ is in χ, then $\mathrm{Ext}_A^1(\tau^{-1}(N), X) = 0$ if and only if X is Ext-injective in χ.*

Dually, let $f : N \to X$ be a morphism that induces a monomorphism of covariant functors

$$\mathrm{Ext}_A^1(f, \ )|_\chi : \mathrm{Ext}_A^1(X, \ )|_\chi \to \mathrm{Ext}_A^1(N, \ )|_\chi.$$

(c) *If $\mathrm{Ext}_A^1(X, \tau(N)) = 0$, then X is Ext-projective in χ.*

(d) *If $\tau(N)$ is in χ, then $\mathrm{Ext}_A^1(X, \tau(N)) = 0$ if and only if X is Ext-projective in χ.*

In order to show that the relative Auslander-Reiten translation in Ω can be defined analogously via the relative Auslander-Reiten translation in χ, we prove the following statements.

Theorem 2.1.15 *Let Z be a module in Ω.*

(a) *If Z is not projective, then the domain of a minimal right Ω-approximation $f_{\tau_\chi(Z)}$: $X_{\tau_\chi(Z)} \to \tau_\chi(Z)$ can be decomposed $X_{\tau_\chi(Z)} = I \oplus \tau_\Omega(Z)$ where I is Ext-injective in Ω.*

(b) *If Z is not Ext-injective in Ω, then the codomain of a minimal left Ω-approximation $f^{\tau_\chi^{-1}(Z)} : X^{\tau_\chi^{-1}(Z)} \to \tau_\chi^{-1}(Z)$ can be decomposed $X^{\tau_\chi^{-1}(Z)} = P \oplus \tau_\Omega^{-1}(Z)$ where P is projective.*

Proof:
Firstly, note that Z is not projective implies it is not Ext-projective in Ω and hence cannot be Ext-projective in χ, so $\tau_\chi(Z)$ exists. We prove statement (a) and statement (b) follows by duality.

Let $f_{\tau(Z)} : X_{\tau(Z)} \to \tau(Z)$ denote a minimal right χ-approximation of $\tau(Z)$ and consider a minimal right Ω-approximation $g_{\tau(Z)} : Y_{\tau(Z)} \to \tau(Z)$, which is also an Ω-section. Then there is a factorization $g_{\tau(Z)} = f_{\tau(Z)}g'$ for some morphism $g' : Y_{\tau(Z)} \to X_{\tau(Z)}$ as $Y_{\tau(Z)}$ is in χ. On the other hand, there is also a factorization $g' = f_{X_{\tau(Z)}}h$ for some morphism $h : Y_{\tau(Z)} \to X_{X_{\tau(Z)}}$, where $f_{X_{\tau(Z)}} : X_{X_{\tau(Z)}} \to X_{\tau(Z)}$ is a minimal right Ω-approximation. Consequently, as $g_{\tau(Z)} = f_{\tau(Z)}f_{X_{\tau(Z)}}h$ is an Ω-section, h must be a split monomorphism and $Y_{\tau(Z)}$ is a direct summand of $X_{X_{\tau(Z)}}$.

$$
\begin{array}{ccc}
& Y_{\tau(Z)} & \\
h \nearrow & \downarrow g' & \searrow g_{\tau(Z)} \\
X_{X_{\tau(Z)}} \xrightarrow{f_{X_{\tau(Z)}}} X_{\tau(Z)} \xrightarrow{f_{\tau(Z)}} \tau(Z)
\end{array}
$$

Hence we get a chain of monomorphisms

$$\mathrm{Ext}_A^1(Z, Y_{\tau(Z)}) \hookrightarrow \mathrm{Ext}_A^1(Z, X_{X_{\tau(Z)}}) \hookrightarrow \mathrm{Ext}_A^1(Z, X_{\tau(Z)}) \hookrightarrow \mathrm{Ext}_A^1(Z, \tau(Z))$$

of $\mathrm{End}_A(Z)^{op}$-modules by Lemma 2.1.13, which is in fact a chain of isomorphisms by Lemma 2.1.12. In particular, $\mathrm{Ext}_A^1(Z, X_{X_{\tau(Z)}}) \cong \mathrm{Ext}_A^1(Z, Y_{\tau(Z)}) \cong \mathrm{Ext}_A^1(Z, \tau_\Omega(Z))$ as $\tau_\Omega(Z)$ is the only direct summand of $Y_\tau(Z)$ that is not Ext-injective. Hence we can decompose $X_{X_{\tau(Z)}} = I \oplus \tau_\Omega(Z)$ and obtain $\mathrm{Ext}_A^1(Z, I) = 0$. Note that the restriction of $f = f_{X_{\tau(Z)}}|_I : I \to X_{\tau(Z)}$ still induces a monomorphism of contravariant functors $\mathrm{Ext}_A^1(\ , f)|_\Omega : \mathrm{Ext}_A^1(\ , I)|_\Omega \to \mathrm{Ext}_A^1(\ , X_{\tau(Z)})|_\Omega$ and hence I is Ext-injective in Ω by Lemma 2.1.14.

It remains to show that $\tau_\Omega(Z)$ is actually a direct summand of $X_{\tau_\chi(Z)}$ if $X_{\tau(Z)} = J \oplus \tau_\chi(Z)$ denotes the decomposition of $X_{\tau(Z)}$ into its Ext-injective part and the indecomposable relative Auslander-Reiten translate of Z in χ. Suppose $\tau_\Omega(Z)$ is not a direct summand of $X_{\tau_\chi(Z)}$ then we have

$$\mathrm{Ext}_A^1(Z, \tau_\Omega(Z)) \cong \mathrm{Ext}_A^1(Z, X_J) \cong \mathrm{Ext}_A^1(Z, J) = 0,$$

which cannot be true. Consequently, $\tau_\Omega(Z)$ is a direct summand of the domain of $f_{\tau_\chi(Z)}$: $X_{\tau_\chi(Z)} \to \tau_\chi(Z)$ and all other direct summands of $X_{\tau_\chi(Z)}$ must be Ext-injective as they are also direct summands of I. $\qquad\square$

Lemma 2.1.16 *If M is module in A-mod and X_M denotes the domain of a minimal right Ω-approximation of M, then we have*

$$l(X_M) \leq l(M)s$$

where $s = \max\{l(X_S)|S \text{ a simple module in } A\text{-mod}\}$

Proof:
We prove the statement by induction on $l(M)$, but first we recall a result from [AR91]. Let

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\ f\ } Z \longrightarrow 0$$

be an exact sequence in A-mod. Moreover, let $f_X : X_X \to X$ and $f_Z : X_Z \to Z$ denote minimal right Ω-approximations of X and Z respectively and let $X_Z \times_Z Y$ denote the pullback of f_Z and f. We then obtain a short exact sequence

$$0 \longrightarrow X \longrightarrow X_Z \times_Z Y \longrightarrow X_Z \longrightarrow 0$$

and by Lemma 2.1.12 we have $\mathrm{Ext}^1_A(X_Z, X) \cong \mathrm{Ext}^1_A(X_Z, X_X)$ from which we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X_X & \longrightarrow & Y_Y & \longrightarrow & X_Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle h} & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & X_Z \times_Z Y & \longrightarrow & X_Z & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_Z} & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{\ f\ } & Z & \longrightarrow & 0.
\end{array}
$$

By [AR91, Proposition 3.6] the composition $gh = g_Y : Y_Y \to Y$ is a right Ω-approximation of Y, which is not necessarily right minimal. If $l(M) = 1$, then M is a simple A-module and it follows that

$$l(X_M) \leq \max\{l(X_S)|S \text{ a simple module in } A\text{-mod}\} = l(M)s.$$

Suppose now the statement is true for modules of length $n-1$ and $l(M) = n$. Let S be a simple module that embeds into M, we then obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X_S & \longrightarrow & Y_M & \longrightarrow & X_{M/S} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_S} & & \downarrow{\scriptstyle g_M} & & \downarrow{\scriptstyle f_{M/S}} & & \\
0 & \longrightarrow & S & \longrightarrow & M & \xrightarrow{\ f\ } & M/S & \longrightarrow & 0.
\end{array}
$$

By induction we get

$$l(Y_M) = l(X_S) + l(X_{M/S}) \leq l(S)s + l(M/S)s = l(M)s.$$

In particular, if $f_M : X_Y \to Y$ denotes a minimal right Ω-approximation, then

$$l(X_M) \leq l(Y_M) \leq l(M)s.$$

$\square$

It follows from this proof that if f_M in the commutative diagram of Lemma 2.1.11 is a minimal right Ω-approximation of M, then f_N is also right minimal. Note that in general even if f_X and f_Z are minimal right Ω-approximations, the morphism f_Y we obtain by the construction in the previous proof is not necessarily a minimal right Ω-approximation as the standard example shows.

Example 2.1.17 *Consider the standard example 1.2.13 and let $\Omega = \mathfrak{F}(\Delta)$ be the category of standard-filtered modules. Then the almost split sequence*

$$0 \longrightarrow M_2 \longrightarrow X \oplus \nabla_3 \longrightarrow N_3 \longrightarrow 0.$$

gives rise to a commutative diagram with an approximation of $X \oplus \nabla_3$ that is not minimal.

Let us first recall the Auslander-Reiten quiver and mark all standard-filtered modules red.
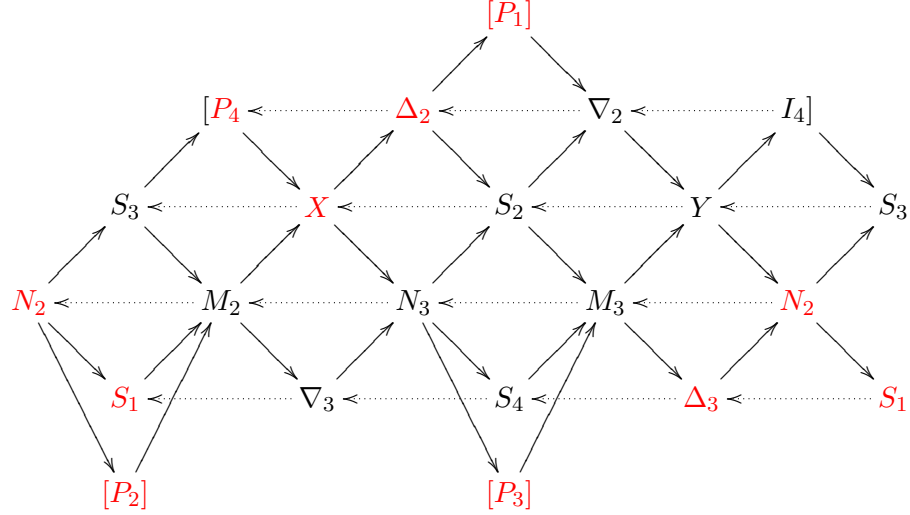
It is now easy to see that $f_{M_2} : P_2 \oplus S_1 \to M_2$, $f_{X \oplus \nabla_3} : X \oplus P_2 \to X \oplus \nabla_3$ and $f_{N_3} : X \to N_3$ are minimal right Ω-approximations of the modules in the given almost split sequence. On the other hand, the commutative diagram we obtain by the construction in the proof Lemma 2.1.16 is the following.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_2 \oplus S_1 & \longrightarrow & X \oplus P_2 \oplus S_1 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{M_2}} & & \downarrow & & \downarrow{\scriptstyle f_{N_3}} & & \\
0 & \longrightarrow & M_2 & \longrightarrow & X \oplus \nabla_3 & \longrightarrow & N_3 & \longrightarrow & 0.
\end{array}
$$

Clearly the right Ω-approximation of $X \oplus \nabla_3$ is not minimal as the direct summand S_1 is redundant.

2.2 Irreducible morphisms in subcategories

Definition 2.2.1 *Let X, Y be modules in χ. We call a morphism $f : X \to Y$ **irreducible in χ** or **χ-irreducible**, if f is neither a split monomorphism nor a split epimorphism and, whenever we have a Z in χ such that there are morphisms $h : X \to Z$ and $g : Z \to Y$ that satisfy $f = gh$, then either h is a split monomorphism or g a split epimorphism.*

The connection between irreducible and minimal right almost split morphisms is the same as for A-mod.

Theorem 2.2.2

(a) *Let Z be an indecomposable module in χ and Y a non-zero module in χ. Then a morphism $g : Y \to Z$ is irreducible in χ if and only if there exists a morphism $g' : Y' \to Z$ such that the induced morphism $(g, g') : Y \oplus Y' \to Z$ is a minimal right almost split morphism in χ.*

(b) *Dually, if X is an indecomposable module in χ, then a morphism $f : X \to Y$ is χ-irreducible if and only if there is a morphism $f' : X \to Y'$ such that the induced morphism $(f, f')^T : X \to Y \oplus Y'$ is a minimal left almost split morphism in χ.*

Proof:
Assume first that $g : Y \to Z$ is irreducible in χ and let $h : M \to Z$ be a minimal right almost split morphism in χ. Since g is not a split epimorphism, it factors through h, i.e. $g = h\varphi$ where φ is a split monomorphism by irreducibility of g. We set $Y' = \text{Coker}(\varphi)$ and hence obtain $M \cong Y \oplus Y'$. Moreover, $g' = h|_{Y'} : Y' \to Z$ is a morphism such that $(g, g') : Y \oplus Y' \to Z$ is a minimal right almost split morphism.

Suppose now that $h : M \to Z$ is a minimal right almost split morphism in χ. Furthermore, let $M = Y \oplus Y'$ with Y non-zero and let $g = h|_Y : Y \to Z$. Assume that $g = st$ for some morphisms $t : Y \to N$ and $s : N \to Z$ such that s is not a split epimorphism. Since h is a right almost split morphism, there exist a morphism $(u, v)^T : N \to Y \oplus Y'$ such that $s = (g, g')(u, v)^T$. We obtain the following commutative diagram

$$
Y \oplus Y' \xrightarrow{\left(\begin{smallmatrix} t & 0 \\ 0 & \text{Id}_{Y'} \end{smallmatrix}\right)} N \oplus Y' \xrightarrow{\left(\begin{smallmatrix} u & 0 \\ v & \text{Id}_{Y'} \end{smallmatrix}\right)} Y \oplus Y'
$$

with (g, g'), (s, g'), (g, g') all mapping to Z.

Hence the composition $\left(\begin{smallmatrix} ut & 0 \\ vt & \text{Id}_{Y'} \end{smallmatrix}\right)$ is an isomorphism as (g, g') is right minimal. Thus $ut : Y \to Y$ is an isomorphism, showing that t is a split monomorphism. It follows that g is χ-irreducible. $\qquad\square$

Analogously to A-mod, one can define the **χ-radical** of $\text{Hom}_A(X, Y)$ for modules X, Y in χ, i.e. $\text{rad}_\chi(X, Y)$ is the set of all $f \in \text{Hom}_A(X, Y)$ such that gfh is not an isomorphism for any $h : Z \to X$ and $g : Y \to Z$ with Z indecomposable in χ. Again, we define **powers of the χ-radical** inductively, so an $f \in \text{Hom}_A(X, Y)$ is in $\text{rad}_\chi^n(X, Y)$ if there is a Z in

χ such that $f = gh$ for some $h \in \text{rad}_\chi(X, Z)$, $g \in \text{rad}_\chi^{n-1}(Z, Y)$. Moreover, we also set $\text{rad}_\chi^\infty(X, Y) = \bigcap_{n \in \mathbb{N}} \text{rad}_\chi^n(X, Y)$ and $\text{Irr}_\chi(X, Y) = \text{rad}_\chi(X, Y)/\text{rad}_\chi^2(X, Y)$. Note that we can swap the order of g and h in the definition of power of the χ-radical, that is $\text{rad}_\chi^n(X, Y)$ is also the set of all $f \in \text{Hom}_A(X, Y)$ such that there is a Z in χ such that $f = gh$ for some $h \in \text{rad}_\chi^{n-1}(X, Z)$, $g \in \text{rad}_\chi(Z, Y)$.

Lemma 2.2.3 *Let $f : X \to Y$ be a morphism between indecomposable modules X, Y in χ. Then f is an irreducible morphism in χ if and only if $f \in \text{rad}_\chi(X, Y) \backslash \text{rad}_\chi^2(X, Y)$.*

Proof:
We first show for X an indecomposable and M an arbitrary module in χ that $\text{rad}_\chi(X, M)$ is the set of morphisms from X to M that are not split monomorphisms whereas $\text{rad}_\chi(M, X)$ is the set of morphisms from M to X that are not split epimorphisms.

Let $\phi : X \to M$ be a morphism. If ϕ is a split monomorphism, we have $\phi'\phi = \text{Id}_X$ and trivially get an isomorphism $\phi'\phi\text{Id}_X$. If there are morphisms $h : Z \to X$ and $g : M \to Z$ with Z indecomposable in χ and $g\phi h$ an isomorphism, it is clear that $hg\phi$ is an isomorphism on $\text{Im}(h)$ and hence X decomposes into $\text{Im}(h) \oplus \text{Ker}(g\phi)$. Since X is indecomposable, we then know that h must be an isomorphism and ϕ a split monomorphism.

Now we consider $\psi : M \to X$. If ψ is a split epimorphism, we obtain an isomorphism $\text{Id}_X\psi\psi'$. If we have morphisms $h : Z \to M$ and $g : X \to Z$ with Z indecomposable in χ and $g\psi h$ an isomorphism, we know that ψhg is an isomorphism on $\text{Im}(\psi h)$ and X decomposes into $\text{Im}(\psi h) \oplus \text{Ker}(g)$, i.e. $\text{Ker}(g) = 0$, g is an isomorphism and ψ is a split epimorphism.

Let f be irreducible, then it is neither a split monomorphism nor a split epimorphism and, therefore, in $\text{rad}_\chi(X, Y)$. If $f \in \text{rad}_\chi^2(X, Y)$, we have a decomposition $f = gh$ with $g \in \text{rad}_\chi(X, Z)$ and $h \in \text{rad}_\chi(Z, Y)$, thus neither g nor h splits, contradicting f to be irreducible. Hence f is not in $\text{rad}_\chi^2(X, Y)$.

On the other hand, let $f \in \text{rad}\chi(X, Y) \backslash \text{rad}_\chi^2(X, Y)$ and let $f = gh$ be a factorization of f in χ. Then either g or h is not in some χ-radical and, therefore, a split monomorphism or split epimorphism respectively as X and Y are indecomposable. □

Note that we have $\text{rad}_\chi(X, Y) = \text{rad}(X, Y)$ for X, Y in χ, but usually $\text{rad}_\chi^n(X, Y) \subsetneq \text{rad}^n(X, Y)$ for $n > 1$.

Definition 2.2.4 *The **Auslander-Reiten quiver** of χ is a biquiver, i.e. a quiver consisting of vertices Γ_0 and two disjoint sets of arrows between them, 1-arrows Γ_1 and 2-arrows Γ_2. There is a vertex for every isomorphism class of indecomposable modules of χ. Given two indecomposable modules X and Y in χ there is a 1-arrow from the vertex corresponding to the isomorphism class of X to the vertex corresponding to the isomorphism class of Y if there is an irreducible morphism from X to Y. For each almost split sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

there is a 2-arrow from the vertex corresponding to the isomorphism class of Z to the vertex corresponding to the isomorphism class of X. To distinguish between different types of arrows, 2-arrows are drawn as dotted arrows.

Definition 2.2.5 *Let $h : M \to N$ be an irreducible morphism in χ between indecomposable modules and $f : X \to N$ and $g : M \to Y$ extensions to minimal right almost split and minimal left almost split morphisms in χ respectively. We denote the number of copies of M in a sum decomposition of X and the number of copies of N in a sum decomposition of Y by n and m respectively. As in A-mod, the arrow from X to Y in the Auslander-Reiten quiver of χ has* **valuation** *(n, m).*

Throughout the dissertation we denote the Auslander-Reiten quiver of χ by Γ_χ. Just as for Auslander-Reiten quivers of A, we mostly do not distinguish between an indecomposable module, its isomorphism class or a corresponding vertex in Γ_χ. There are squared brackets to mark Ext-injective and Ext-projective modules in Γ_χ. For convenience, we say arrows for 1-arrows and, if the valuation of an arrow is $(1, 1)$, we say it is **trivially valuated**.

If a subquiver Γ of Γ_χ contains two modules X and Y such that there is a 1-arrow or a 2-arrow between them, then Γ also contains this 1-arrow or 2-arrow respectively and the 1-arrow in Γ has the same valuation as in the whole Auslander-Reiten quiver. We call a non-empty subquiver Γ of Γ_χ a **connected component** of χ if for each X in Γ the modules in Γ are precisely all modules Y such that there is a walk between X and Y in Γ_χ. Clearly, each indecomposable module X in χ is contained in a unique connected component of Γ_χ, which we call the connected component of X in Γ_χ. Analogously to A-mod, we say a connected component is finite if it contains only finitely many indecomposable non-isomorphic modules.

Note that also in the Auslander-Reiten quiver of χ it frequently happens that the same indecomposable module occurs more than once. In this case we again differentiate between the corresponding vertices of this module in Γ_χ. Having defined valuation for arrows in χ, we can generalize a fact about valuation of arrows in A-mod.

Theorem 2.2.6 *Let X and Y be indecomposable modules in χ such that there is an irreducible morphism from X to Y. Suppose the valuation of the 1-arrow from X to Y is (n, m), i.e. there is a module M such that X is not a direct summand of M and a minimal right almost split morphism $f : nX \oplus M \to Y$ in χ. Let $f_1, \ldots, f_n$ name the morphisms obtained by restricting f to the different copies of X in $nX \oplus M$. Then $\{\overline{f_1}, \ldots, \overline{f_n}\}$ is a basis for $\mathrm{Irr}_\chi(X, Y)$ as a T_X^{op} and T_Y-vector space. In particular, $n = m$.*

Proof:
The induced morphism $(f_1, \ldots, f_n) : nX \to Y$ is irreducible as it can be extended to a minimal right almost split morphism in χ. Suppose $\{\overline{f_1}, \ldots, \overline{f_n}\}$ is linearly dependent, i.e. there are endomorphisms $\alpha_i : X \to X$ such that $\overline{f_1 \alpha_1} + \overline{f_2 \alpha_2} + \cdots + \overline{f_n \alpha_n} = 0$ with at least one $\overline{\alpha_i}$ being non-zero. This relation implies that $f_1 \alpha_1 + f_2 \alpha_2 + \cdots + f_n \alpha_n$ is in $\mathrm{rad}_\chi^2(X, Y)$.

However, since at least one $\overline{\alpha_i}$ is non-zero, α_i must be an isomorphism and the morphism $\alpha = (\alpha_i)^T : X \to nX$ is a split monomorphism. Therefore, there is a morphism $\beta : nX \to X$ such that $\beta\alpha = \mathrm{Id}_X$ and $nX = (n-1)X \oplus X = \mathrm{Ker}(\beta) \oplus \mathrm{Im}(\alpha)$. It follows that we can extend α to an isomorphism $\varphi = (\alpha, \alpha') : X \oplus (n-1)X \to nX$. This gives rise to a new minimal right almost split morphism $f\varphi$. The diagram below shows how an arbitrary morphism in χ mapping to Y factors over $f\varphi$ using that f is a minimal right almost split morphism.

$$
\begin{array}{ccc}
& Z & \\
\varphi^{-1}g \nearrow & \downarrow g & \searrow \\
nX \xrightarrow{\ \varphi\ } & nX \xrightarrow{\ f\ } & Y
\end{array}
$$

Since we have extended the composition $f\alpha = f_1\alpha_1 + f_2\alpha_2 + \cdots + f_n\alpha_n$ to a minimal right almost split morphism, it must be irreducible in χ by 2.2.2, contradicting the fact that $f_1\alpha_1 + f_2\alpha_2 + \cdots + f_n\alpha_n$ is in $\mathrm{rad}_\chi^2(X, Y)$. Consequently, $\{\overline{f_1}, \ldots, \overline{f_n}\}$ is linearly independent.

Now we show that $\{\overline{f_1}, \ldots, \overline{f_n}\}$ span $\mathrm{Irr}_\chi(X, Y)$. Let $g : X \to Y$ be irreducible in χ and $\overline{g}$ its coset in $\mathrm{Irr}_\chi(X, Y)$. Since g is not a split epimorphism, there is a morphism $h : X \to nX \oplus M$ such that $g = fh$. We can decompose this morphism into $g = f|_{nX}ph + f|_Mqh$ where p and q name the projections from $nX \oplus M$ to nX and M respectively. Clearly, qh is in $\mathrm{rad}_\chi(X, M)$ as M does not contain X as a direct summand. Thus we get $f|_Mqh \in \mathrm{rad}_\chi^2(X, Y)$ and $\overline{g} = \overline{f|_{nX}ph}$. Using that ph maps from X to nX we decompose it into a set of endomorphisms $\{\alpha_i\} \subset \mathrm{End}_A(X)$ such that $\overline{g} = \sum \overline{f_i\alpha_i}$.

Since A is a K-algebra over an algebraically closed field, we have $T_X^{op} \cong T_Y \cong K$, hence the statement follows immediately for T_Y. As a trivial consequence of that, we obtain $n = m$. $\qquad\square$

This theorem enables us to generalize another well-established result.

Corollary 2.2.7 *Let X and Y be indecomposable modules in χ such that there is an arrow from X to Y in the Auslander-Reiten quiver of χ with valuation (n, n). If Y is not Ext-projective, then the arrow from $\tau_\chi(Y)$ to X is also valued (n, n).*

Proof:
By our assumptions n is the number of direct summand isomorphic to X in the domain of a minimal right almost split morphism $f : M \to Y$ in χ. Consequently, by Lemma 2.1.10, n is also the number of direct summands isomorphic to X in the codomain of a minimal left almost split morphism $g : \tau_\chi(Y) \to M$. It follows from Theorem 2.2.6 that the valuation of the corresponding arrow is indeed (n, n). $\qquad\square$

Since any arrow from X to Y in the Auslander-Reiten quiver of χ is valued (n, n) for some $n \in \mathbb{N}$, it again makes sense to say there are **n arrows** from X to Y. Moreover, we draw n arrows from X to Y in the Auslander-Reiten quiver of χ. We say there are multiple arrows from X to Y if $n \geq 2$ and a connected component contains multiple arrows if and only if there are multiple arrows from X to Y for some modules X and Y in that connected component.

Let us recall the well-known lemma by Harada and Sai, which clearly can be applied to irreducible morphisms in χ. Therefore, we can generalize a criterion for the Auslander-Reiten quiver of an Artin algebra to consist of only one component to a criterion for the Auslander-Reiten quiver of χ to have this property.

Lemma 2.2.8 (Harada-Sai Lemma) *[ARS95, VI. Lemma 1.2] Let $X_0, \ldots, X_{2^n-1}$ be indecomposable modules such that $l(X_i) \leq n$ for $i = 0, \ldots, 2^n - 1$. Suppose $f_i : X_i \to X_{i+1}$ are non-isomorphisms, then $f_{2^n-1} \cdots f_1 = 0$.*

Theorem 2.2.9 *Let A be an indecomposable algebra. If there is a component Γ in the Auslander-Reiten quiver of χ such that the length of all modules in Γ is bounded, then it is the only component.*

Proof:
Let n be the positive integer such that $l(X) \leq n$ for all X in Γ. Let X be in Γ and Y be an arbitrary indecomposable module in χ such that $\operatorname{rad}_\chi^{2^n}(X, Y) \neq 0$. We then know there must be a chain of at least 2^n non-isomorphisms from X to Y. This contradicts Lemma 2.2.8, so we have $\operatorname{rad}_\chi^{2^n}(X, -) = \operatorname{rad}_\chi^{2^n}(-, X) = 0$. In particular, we have $\operatorname{Hom}_A(X, Y) = \operatorname{Hom}_A(Y, X) = 0$ for Y not in Γ.

Since χ contains a functorially finite resolving subcategory Ω, it follows that it, in particular, contains all projective modules. But there is a projective module P such that $\operatorname{Hom}_A(P, X) \neq 0$ and hence P is in Γ. As projective modules of an indecomposable algebra are always connected, we conclude that all projective modules are in Γ. Moreover, for every indecomposable module Y there is a non-zero homomorphism from a projective module to Y, so every indecomposable module must be contained in Γ. $\square$

Note that this theorem does not hold for arbitrary functorially finite subcategories that are closed under extensions, as the standard example shows.

Example 2.2.10 *Let A be the algebra of the form*

$$
\begin{array}{cccc}
S_1 & S_2 & S_3 & S_4 \\
S_2 \oplus S_1 & S_3 \oplus S_2 & S_4 \oplus S_3 & \\
S_1 & S_2 & S_3 &
\end{array}
$$

which we have analyzed in Example 1.2.13. Then there is a functorially finite subcategory χ of A-mod that is finite and closed under extensions, but its Auslander-Reiten quiver has 2 connected components.

We have already seen that A is representation finite in Example 1.2.13. Hence every subcategory of A-mod is functorially finite. We consider the subcategory $\chi = \operatorname{add}(P_1 \oplus I_4)$ that consists of all modules whose indecomposable direct summands are isomorphic to P_1 or I_4. Their Jordan-Hölder composition series

$$
P_1 = I_1 = \begin{array}{c} S_1 \\ S_2 \\ S_1 \end{array} \qquad I_4 = \begin{array}{c} S_3 \\ S_4 \end{array}.
$$

show that $\mathrm{Hom}_A(P_1, I_4) = \mathrm{Hom}_A(I_4, P_1) = 0$. Moreover, since P_1 is also injective, χ is closed under extensions, but its Auslander-Reiten quiver consists of two components that contain one indecomposable module each.

Moreover, with the help of the lemma of Harada and Sai we can also prove a generalization of the first Brauer-Thrall conjecture.

Corollary 2.2.11 *The subcategory χ is finite if and only if the Jordan-Hölder length of its indecomposable modules is bounded.*

Proof:
Let n be a positive integer such $l(X) \leq n$ for all indecomposable modules X in χ. It follows that $\mathrm{rad}_\chi^{2^n}(X, Y) = \mathrm{rad}_\chi^\infty(X, Y) = 0$ for all indecomposable modules X, Y in χ by Lemma 2.2.8. Moreover, for every module X there is a projective module P and a path in Γ_χ from P to X of length shorter than 2^n. Since there are only finitely many projective modules and finally many paths of length shorter than 2^n, there can only be finitely many indecomposable modules in χ up to isomorphism. The converse is trivial. □

The second Brauer-Thrall conjecture, that is an Artin algebra A is representation infinite if and only if there are infinitely many positive integers $n_1, n_2, \ldots$ such that for each $i \in \mathbb{N}$ there are infinitely many non-isomorphic indecomposable modules of Jordan-Hölder length n_i, has been proved for Artin algebras over an algebraically closed field. The original proof can be found in [NR75], a sketch of the proof in English is given in [R80*]. The problem is still open for algebras over arbitrary fields or rings, but it has been established in [S80] that if there is one $n \in \mathbb{N}$ such that there are infinitely many non-isomorphic indecomposable modules of length n, then there are infinitely many $n_i \in \mathbb{N}$ with that property. This is nothing but the induction step for a proof of the second Brauer-Thrall conjecture and is often referred to as the one and a half Brauer-Thrall conjecture. We generalize the aforementioned result using the same techniques, but we need to find a boundary for the length of an indecomposable module X in terms of the length of an indecomposable module Y if there is an irreducible morphism from X to Y in Ω. The first lemma is the dual of [S80, Lemma 3.1], which fits better into our context.

Lemma 2.2.12 *Let X and Y be indecomposable modules and assume $f : X \to Y$ is an irreducible morphism. Then*

$$|l(X) - l(Y)| \leq l(Y) \cdot m^2$$

where $m = \max\{l({}_A A), l(A_A)\}$.

Proof:
If Y is projective, then f is a monomorphism and $l(X) < l(Y)$, so obviously the stated inequation holds in this case. Suppose now that Y is not projective, then there is an almost split sequence

$$0 \longrightarrow DTr(Y) \longrightarrow X \oplus X' \longrightarrow Y \longrightarrow 0.$$

This gives us $l(DTr(Y)) + l(Y) = l(X) + l(X')$ and, therefore, we obtain $|l(X) - l(Y)| \leq$ $\max\{l(Y), l(DTr(Y))\}$. Moreover, we have $l(DTr(Y)) = l(Tr(Y))$ since D is a duality. So let $P_1 \to P_0 \to Y$ be a minimal projective resolution, then we obtain a morphism $\mathrm{Hom}_A(f, A) : \mathrm{Hom}_A(P_0, A) \to \mathrm{Hom}_A(P_1, A)$ such that $Tr(Y) = \mathrm{Coker}(\mathrm{Hom}_A(f, A))$. Therefore, we have

$$l(DTr(Y)) = l(Tr(Y)) \leq l(\mathrm{Hom}_A(P_1, A)).$$

Since $\mathrm{Hom}_A(P_1, A)$ is a projective A^{op}-module, we know that $l(\mathrm{Hom}_A(P_1, A)) \leq l(A_A)n$ where n denotes the number of indecomposable direct summands of P_1. Clearly, $n \leq l(P_0)$ and the number of direct summands of P_0 is at most $l(Y)$ and hence $n \leq l(P_0) \leq l(Y)l(_AA)$ as the Jordan-Hölder length of each direct summand of P_0 is less than or equal to $l(_AA)$. Overall we obtain $l(DTr(Y)) \leq l(Y)m^2$ and

$$|l(X) - l(Y)| \leq l(Y)m^2$$

□

Lemma 2.2.13 *Let X and Y be indecomposable modules in Ω and suppose there is an irreducible morphism $f : X \to Y$ in Ω. Then*

$$|l(X) - l(Y)| \leq l(Y)p$$

where $p = (s(1 + m^2) - 1)$, $m = \max\{l(_AA), l(A_A)\}$ and $s = \max\{l(X_{S_i})\}$ for all simple modules S_i.

Proof:
Since a minimal right almost split morphism in Ω is given by a minimal right Ω-approximation of the domain of a minimal right almost split morphism in A-mod, there is a module M in A-mod such that there is an irreducible morphism $g : M \to Y$ and X is a direct summand of the domain X_M of a right Ω-approximation of M. We then know that

$$l(X) \leq l(M)s$$

by Lemma 2.1.16, where s is the maximal Jordan-Hölder length of a minimal right Ω-approximation of a simple A-module. Note that s is always finite as A is an Artin algebra.

$$|l(X) - l(Y)| \leq l(Y)(s(1 + m^2) - 1)$$

clearly holds if $l(X) \leq l(Y)$. So assume now $l(X) > l(Y)$, then

$$|l(X) - l(Y)| \leq |l(M)s - l(Y)| = |l(M)s - l(Y)s + l(Y)s - l(Y)|$$

$$\leq |l(M) - l(Y)|s + |l(Y)s - l(Y)| \leq l(Y)sm^2 + l(Y)(s - 1) = l(Y)p,$$

where we use the inequation obtained in Lemma 2.2.12. □

This inequation allows us to prove the one and a half Brauer-Thrall conjecture for a functorially finite resolving subcategory Ω. The proof closely follows [S80, Theorem 3.2].

Theorem 2.2.14 *Let Ω be a functorially finite resolving subcategory of A-mod such that there exist $\mathfrak{N} \geq \mathfrak{N}_0$ non-isomorphic indecomposable modules of length n, where $\mathfrak{N}_0$ is the cardinality of a countable set. Then there are infinitely many positive integers n_i with $\mathfrak{N}$ non-isomorphic indecomposable modules of length n_i.*

Proof:
Let $\{M_i\}_{i \in \mathfrak{N}}$ be the set of non-isomorphic, indecomposable modules of length n. We prove the statement by showing that for each integer $n' > n$ there is an integer $k \geq n'$ such that there are $\mathfrak{N}$ indecomposable, non-isomorphic modules of length k.

Suppose there is an integer $n' > n$ such that for all $m \geq n'$ the number of modules of length m is strictly less than $\mathfrak{N}$ and fix such an m. We know that there is a simple module S such that S is a direct summand of $\mathrm{soc}(M_i)$ for at least $\mathfrak{N}$ modules M_i and, therefore, there are non-zero morphisms $f_i : S \to M_i$ for these modules. It follows that there exist $\mathfrak{N}$ non-zero morphisms $g_i : X_S \to M_i$.

Let us first assume that $\mathfrak{N}$ of the morphisms g_i are not sums of compositions of irreducible morphisms in Ω, that is there are $\mathfrak{N}$ morphisms g_i in $\mathrm{rad}_\Omega^\infty(X_S, M_i)$. We then get $\mathfrak{N}$ chains

$$\cdots \longrightarrow M_{i,2^m+1} \longrightarrow M_{i,2^m} \longrightarrow \cdots \longrightarrow M_{i,1} \longrightarrow M_{i,0} \longrightarrow M_i$$

of arbitrary length of indecomposable modules $M_{i,j}$ and Ω-irreducible morphisms. Then by Lemma 2.2.8 there is an $m' \in \mathbb{N}$ such that in each of these chain in the last m' steps there is a module of length greater or equal to m. Let M be such a module, then there is only a finite number of modules M_i such that M occurs in the last m' steps of the chains associated to the modules M_i as Γ_Ω is a locally finite Auslander-Reiten quiver. Furthermore, the length of modules in the last m' steps of each chain associated to an M_i is bounded by $(np)^{m'}$ where p is the constant defined in Lemma 2.2.13. It follows that there are at least $\mathfrak{N}$ non-isomorphic indecomposable modules with length in an interval $[m, (np)^{m'}]$. So there exists a $k \in \mathbb{N}$, $m \leq k \leq (np)^{m'}$ such that there are $\mathfrak{N}$ non-isomorphic indecomposable modules of length k, which is a contradiction.

Suppose now that for $\mathfrak{N}$ of the M_i there is a chain of Ω-irreducible morphisms with non-zero composition starting in X_S and ending in M_i. Since there are only finitely many arrows ending in any module in an Auslander-Reiten quiver, we know that only finitely many of the chains can be shorter than m' and the same argument as in the first case can be applied to the other chains.

So in both cases we get that there is no bound on the integers k such that there exist $\mathfrak{N}$ non-isomorphic indecomposable modules of length k, which completes the proof of the theorem. $\qquad\square$

2.3 Decompositions of morphisms

The purpose of this section is to create the general setup for the subsequent section. When an arbitrary morphism is decomposed into a sum of compositions of irreducible morphisms between indecomposable modules, this decomposition is not unique in general. In particular, not even the aforementioned indecomposable modules are determined by the morphism. In order to give a description when at least these modules are unique, we need to establish a setting in which it makes sense to consider decompositions. First of all we generalize a result from [ARS95, V 7.4].

Lemma 2.3.1 *Let X and Y be indecomposable modules in χ and let $f \in \operatorname{rad}_\chi^n(X, Y)$ with $n \geq 2$. Then we have the following.*

(a) *There exist an integer $s \geq 1$, indecomposable modules $Z_1, \ldots, Z_s$, morphisms $f_i \in \operatorname{rad}_\chi(X, Z_i)$ and morphisms $g_i : Z_i \to Y$ with each g_i a sum of compositions of $n-1$ irreducible morphisms between indecomposable modules such that $f = \sum_{i=1}^s g_i f_i$.*

(b) *If $f \in \operatorname{rad}_\chi^n(X, Y) \backslash \operatorname{rad}_\chi^{n+1}(X, Y)$ then at least one of the f_i in (a) is irreducible and $f = u + v$, where u is a non-zero sum of compositions of n irreducible morphisms between indecomposable modules and $v \in \operatorname{rad}_\chi^{n+1}(X, Y)$.*

(c) *Dually, there exist an integer $t \geq 1$, indecomposable modules $W_1, \ldots, W_t$, morphisms $f_i : X \to W_i$ and morphisms $g_i \in \operatorname{rad}_\chi(W_i, Y)$ with each f_i a sum of compositions of $n-1$ irreducible morphisms between indecomposable modules such that $f = \sum_{i=1}^t g_i f_i$.*

(d) *If $f \in \operatorname{rad}_\chi^n(X, Y) \backslash \operatorname{rad}_\chi^{n+1}(X, Y)$ then at least one of the g_i in (c) is irreducible and $f = u + v$, where u is a non-zero sum of compositions of n irreducible morphisms between indecomposable modules and $v \in \operatorname{rad}_\chi^{n+1}(X, Y)$.*

Proof:
We proof all statements by induction on n. For $n = 2$ let $g : Z \to Y$ be a minimal right almost split morphism in χ. The decomposition of Z into indecomposable modules $Z \cong \bigoplus_{i=1}^s Z_i$ induces morphisms $g_i : Z_i \to Y$. Since $f \in \operatorname{rad}_\chi^2(X, Y)$, it is not a split epimorphism and there is a morphism $f' : X \to Z$ such that $f = gf'$. Let $f_i : X \to Z_i$ be the morphisms induced by f' and the decomposition $Z \cong \bigoplus_{i=1}^s Z_i$. Then we get that $f_i \in \operatorname{rad}_\chi(X, Z_i)$, the morphisms g_i are irreducible and $f = \sum_{i=1}^s g_i f_i$. Moreover, if $f \notin \operatorname{rad}_\chi^3(X, Y)$, then not all f_i are in $\operatorname{rad}_\chi^2(X, Z_i)$. Hence for at least one $i \in \{1, \ldots, s\}$ we have that f_i is irreducible. We obtain the decomposition $f = u + v$ by setting $u = \sum g_i f_i$, where we sum over all i such that f_i is irreducible. This establishes the claim for $n = 2$.

Assume now that $f \in \operatorname{rad}_\chi^n(X, Y)$ for $n \geq 3$. Let $g' : Z \to Y$ be a minimal right almost split morphism in χ with $Z \cong \bigoplus_{i=1}^t Z_i$ and induced morphisms $g_i' : Z_i \to Y$. As f is in $\operatorname{rad}_\chi^n(X, Y)$, there is a morphism $f' \in \operatorname{rad}_\chi^{n-1}(X, Z)$ such that $f = g'f'$. Furthermore, we consider the induced morphisms $f_i' : X \to Z_i$ and rewrite $f = \sum_{i=1}^t g_i' f_i'$. Each f_i' is in $\operatorname{rad}_\chi^{n-1}(X, Z_i)$ as it maps to a distinct direct summand of Z. By induction there are indecomposable modules Z_{ij} for $j = 1, \ldots, s_i$ and morphisms $f_{ij} \in \operatorname{rad}_\chi(X, Z_{ij})$ and $h_{ij} : Z_{ij} \to Z_i$ which are sums of compositions of $n-2$ irreducible morphisms between indecomposable modules such that $f_i' = \sum_{j=1}^{s_i} h_{ij} f_{ij}$. Then the composed morphism $g_{ij} = g_i' h_{ij}$

is clearly a sum of compositions of $n-1$ irreducible morphisms between indecomposable modules and we have that

$$f = \sum_{i=1}^{t} \sum_{j=1}^{s_i} g_{ij} f_{ij}.$$

In order to prove (b), observe that when $f \notin \operatorname{rad}_\chi^{n+1}$ then not all f_{ij} can be in $\operatorname{rad}_\chi^2(X, Z_{ij})$. This shows that at least one f_{ij} is irreducible. We obtain the decomposition $f = u + v$ by setting $u = \sum g_{ij} f_{ij}$, where we sum over all ij such that f_{ij} is irreducible. $\qquad \square$

Let X and Y be indecomposable modules in χ such that there is a χ-irreducible morphism from X to Y. Suppose the corresponding arrow in Γ_χ is valued (n, n), i.e. there are n arrows from X to Y. By Theorem 2.2.6 there are irreducible morphisms $f_1, \ldots, f_n \in \operatorname{rad}_\chi(X, Y) \backslash \operatorname{rad}_\chi^2(X, Y)$ such that $\{\overline{f_1}, \ldots, \overline{f_n}\}$ is a basis of $\operatorname{Irr}_\chi(X, Y)$. We label each of the n arrows with one morphism f_i and fix this labeling for the chapter.

By doing this we have not only chosen a morphism for each arrow in Γ_χ, but also a minimal right almost split morphism and a minimal left almost split morphism for each module. After making this choice it is possible to consider a path in Γ_χ as a morphism that is a composition of irreducible morphisms between indecomposable modules; therefore, we do not distinguish between a path γ and a composition of irreducible morphisms between indecomposable modules g if $\gamma = g\varphi$ for an isomorphism φ. In particular, we can consider the morphisms g_i appearing in the first part of Lemma 2.3.1 as a sum of paths, because they are all obtained from minimal right almost split morphisms by construction. Applying this labeling and the choice of minimal right almost split morphisms to 2.3.1 we can prove the existence of decompositions given by our fixed labeling.

Lemma 2.3.2 *Let X and Y be indecomposable modules in χ and let f be a morphism in $\operatorname{rad}_\chi^n(X, Y) \backslash \operatorname{rad}_\chi^{n+1}(X, Y)$. Then for each $m \geq n$ f can be written as $f = v + \sum_{i=n}^{m-1} u_i$, where u_i is a sum of paths of length i and $v \in \operatorname{rad}_\chi^m(X, Y)$.*

Proof:
We have already seen that $f = \sum_{i=1}^{s} g_i f_i$, where each g_i is a sum of paths of length $n-1$ and $f_i \in \operatorname{rad}_\chi(X, Z_i)$. Without loss of generality, let $f_1, \ldots, f_k$ be the only irreducible morphisms of the f_i. So each f_i for $i = 1, \ldots, k$ can be written as $f_i = h_i \varphi_i$, where h_i is the minimal right almost split morphism mapping to Z_i given by the labeling of the Auslander-Reiten quiver and φ_i is a split monomorphism. We go into more detail and write

$$f_i = \sum_{j=1}^{k_i} h_{ij} \varphi_{ij} + \sum_{j=k_i+1}^{n_i} h_{ij} \varphi_{ij}$$

where $\varphi_{ij} : X \to X$ are isomorphisms for $j = 1, \ldots, k_i$ and φ_{ij} radical morphisms for $j = k_i + 1, \ldots, n_i$ which might be zero. Note that h_{ij} are the irreducible morphisms between indecomposable modules forming h_i, i.e. $h_i = (h_{i1}, \ldots, h_{in_i})$. By construction $g_i h_{ij} \varphi_{ij}$ is a sum of paths from X to Y for all $j = 1, \ldots, k_i$, where $g_i h_{ij} \varphi_{ij} \in \operatorname{rad}_\chi^{n+1}(X, Y)$ if $j \geq k_i + 1$. Consequently, we set $u_n = \sum_{i=1}^{k} \sum_{j=1}^{k_i} g_i h_{ij} \varphi_{ij}$. We continue inductively with $f - u_n \in \operatorname{rad}_\chi^{n+1}(X, Y)$ until we have the designated decomposition. $\qquad \square$

To make sure that for an arbitrary morphism this inductive method terminates at some point we need to prove another Lemma. The proof of the first statement closely follows [ARS95, V 7.2].

Lemma 2.3.3 *Let X and Y be indecomposable modules in χ. Then there is an $m \in \mathbb{N}$ such that $\mathrm{rad}_\chi^m(X, Y) = \mathrm{rad}_\chi^\infty(X, Y)$. In particular, there are only finitely many non-zero paths from X to Y in the Auslander-Reiten quiver of χ.*

Proof:
Both X and Y are finitely generated modules, so $\mathrm{Hom}_A(X, Y)$ is a finitely generated K-module and hence of finite length. Therefore, the descending chain

$$\mathrm{Hom}_A(X, Y) \supset \mathrm{rad}_\chi(X, Y) \supset \cdots \supset \mathrm{rad}_\chi^m(X, Y) \supset \cdots$$

becomes stable and there is an $m \in \mathbb{N}$ such that

$$\mathrm{rad}_\chi^\infty(X, Y) = \bigcap_{n \in \mathbb{N}} \mathrm{rad}_\chi^n(X, Y) = \mathrm{rad}_\chi^m(X, Y).$$

This implies that there cannot be non-zero paths of length greater or equal to m from X to Y in the Auslander-Reiten quiver of χ. Since for each module there is only a finite number of immediate predecessors in an Auslander-Reiten quiver, there are clearly at most finitely many paths of each length from X to Y and overall there are only finitely many paths from X to Y. $\qquad\square$

The entire setup suggests the following definition.

Definition 2.3.4 *Let $f : X \to Y$ be a morphism between indecomposable modules in χ that is in $\mathrm{rad}_\chi^n(X, Y) \backslash \mathrm{rad}_\chi^{n+1}(X, Y)$. A **decomposition** of f is a sum of morphisms*

$$f = v + \sum_{i=n}^m u_i$$

such that $v \in \mathrm{rad}_\chi^\infty(X, Y)$ and u_i is a sum of paths of length i, where the single paths are given by the construction in Lemma 2.3.1 and, therefore, equal the labeling of the Auslander-Reiten quiver up to an isomorphism of X.

Without loss of generality, we assume that any sum of paths occurring in a decomposition is non-zero.

Definition 2.3.5 *Let γ be a path of length k from X to Y in the Auslander-Reiten quiver of χ. We then say that a decomposition $f = v + \sum_{i=n}^m u_i$ **contains** γ if γ occurs as a summand of u_k, i.e. γ equals a path in u_k up to an isomorphism of X.*

2.4   Uniqueness of sectional paths

Recall that a **sectional path**

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

in the Auslander-Reiten quiver of A-mod contains no module X_i such that $\tau(X_i) = X_{i-2}$.

Theorem 2.4.1 *Let*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

be a sequence of irreducible morphisms between indecomposable A-modules, i.e. a path in the Auslander-Reiten quiver of A-mod. Let $f = f_n \cdots f_1 + f_n \phi$, where ϕ is a morphism from X_0 to X_{n-1} such that there exists a decomposition of ϕ that does not contain the path $\gamma = f_{n-1} \cdots f_1$. Suppose that f factors through a morphism $g : Y \to X_n$ for some module Y such that $(f_n, g) : X_{n-1} \oplus Y \to X_n$ is irreducible. Then the sequence is not sectional and f factors over $\tau(X_n)$.

Proof:

Note that $f = 0$ becomes a special case of this theorem by setting $Y = 0$ and $g = 0$, because $(f_n, 0) : X_{n-1} \oplus 0 \to X_n$ is clearly irreducible. The proof is roughly based on [ARS95, VII. Lemma 2.5] and is done by induction on n. Let $n = 2$ and $f = f_2(f_1 + \phi) = gh$. We consider

$$X_0 \xrightarrow{(f_1+\phi,-h)^T} X_1 \oplus Y \xrightarrow{(f_2,g)} X_2$$

and conclude $(f_2, g)(f_1 + \phi, -h)^T = 0$. If (f_2, g) is a monomorphism, we have $f_1 + \phi = 0$ and, in particular, $\overline{f_1} = \overline{\phi}$ in $\mathrm{rad}(X_0, X_1)/\mathrm{rad}^2(X_0, X_1)$, which is a contradiction to our assumption that there is a decomposition of ϕ not containing the arrow labeled f_1. Therefore, (f_2, g) is a non-injective irreducible morphism and hence an epimorphism. Consequently, X_2 is not projective and there is an almost split sequence

$$0 \longrightarrow \tau(X_2) \xrightarrow{(f_2',g',t')^T} X_1 \oplus Y \oplus Z \xrightarrow{(f_2,g,t)} X_2 \longrightarrow 0.$$

Due to $\mathrm{Im}(f_1 + \phi, -h) \subset \mathrm{Ker}(f_2, g) \subset \mathrm{Ker}(f_2, g, t) = \mathrm{Im}(f_2', g', t') \cong \tau(X_2)$ there is a natural morphism $h' : X_0 \to \tau(X_2)$ such that the diagram

$$
\begin{array}{ccc}
& & X_0 \\
& \overset{h'}{\swarrow} & \big\downarrow {\scriptstyle (f_1+\phi,-h,0)^T} \\
0 \longrightarrow \tau(X_2) \xrightarrow{(f_2',g',t')^T} X_1 \oplus Y \oplus Z \xrightarrow{(f_2,g,t)} X_2 \longrightarrow 0
\end{array}
$$

commutes. This shows the second statement; moreover, we have $f_1 + \phi = f_2' h'$. Suppose now that $f_1 + \phi$ is not irreducible, i.e. $f_1 + \phi \in \mathrm{rad}^2(X_0, X_1)$ and $\overline{f_1} = \overline{\phi}$ in $\mathrm{rad}(X_0, X_1)/\mathrm{rad}^2(X_0, X_1)$. It follows that every decomposition of ϕ contains the arrow labeled f_1, which contradicts our assumptions. Thus both $f_1 + \phi$ and f_2' are irreducible, so h' is a split monomorphism and hence an isomorphism, because $\tau(X_2)$ is indecomposable.

We now assume the claim holds for $n-1$. Let $\gamma = f_{n-1} \cdots f_1$ and $f_n(\gamma + \phi) = gh$ for some $h : X_0 \to Y$ and $g : Y \to X_n$ such that $(f_n, g) : X_{n-1} \oplus Y \to X_n$ is irreducible. Then the composition $(f_n, g)(\gamma + \phi, -h)^T$ is zero. If (f_n, g) is an epimorphism, then X_n is not projective. There is an irreducible morphism $f_n' : \tau(X_n) \to X_{n-1}$ and we obtain a morphism $h' : X_0 \to \tau(X_n)$, in the same way as for $n = 2$, such that $\gamma + \phi = f_n' h'$. Hence we have shown that f factors over $\tau(X_n)$.

$$
\begin{array}{ccccccccc}
 & & & & X_0 & & & & \\
 & & {}^{h'}\nearrow & & \downarrow {}^{(\gamma+\phi,-h,0)^T} & & & & \\
0 \longrightarrow & \tau(X_n) & \xrightarrow{(f_n',g',t')^T} & X_{n-1} \oplus Y \oplus Z & \xrightarrow{(f_n,g,t)} & X_n & \longrightarrow & 0
\end{array}
$$

If $X_{n-2} \cong \tau(X_n)$, we are done, if $X_{n-2} \not\cong \tau(X_n)$, we take a closer look at ϕ. By our assumption there is a decomposition of $\phi = v + \sum_{i=k}^m u_i$ that does not contain γ, where k and m denote the minimal and maximal length of paths in this decomposition respectively. We arbitrarily choose v_1, v_2, v_3 such that $v = (f_{n-1}, \phi', f_n')(v_1, v_2, v_3)^T$, where $\phi' : M \to X_{n-1}$ is irreducible such that (f_{n-1}, ϕ', f_n') is the minimal right almost split morphism of X_{n-1} given by the labeling of the Auslander-Reiten quiver of A-mod. Furthermore, we divide the u_i in the same way, namely $\sum_{i=k}^m u_i = (f_{n-1}, \phi', f_n')(\phi_1, \phi_2, \phi_3)^T$. Note that ϕ_1 clearly has a decomposition not containing $f_{n-2} \cdots f_1$ by construction. This implies that $\phi_1 + v_1$ has a decomposition not containing $f_{n-2} \cdots f_1$ since $v_1 \in \mathrm{rad}_\chi^\infty(X_0, X_{n-2})$. Moreover, we have $\phi = f_{n-1}(\phi_1 + v_1) + \phi'(\phi_2 + v_2) + f_n'(\phi_3 + v_3)$.

Inserting this equation into $\gamma + \phi = f_n' h'$ we obtain $\gamma + f_{n-1}(\phi_1 + v_1) = f_n'(h' - \phi_3 - v_3) - \phi'(\phi_2 + v_2)$. Since $(f_{n-1}, \phi', f_n') : X_{n-2} \oplus M \oplus \tau X_n \to X_{n-1}$ is irreducible as a minimal right almost split morphism, we can apply the induction hypothesis on $\gamma + f_{n-1}(\phi_1 + v_1)$.
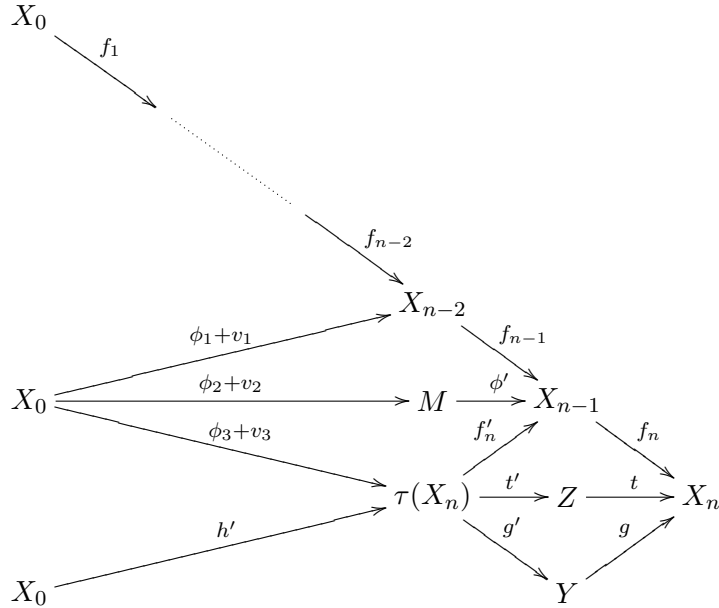
On the other hand, if (f_n, g) is injective, then $\gamma + \phi$ is zero and we consider a decomposition of $\phi = v + \sum_{i=k}^m u_i$ that does not contain γ. We can write $v = (f_{n-1}, \phi')(v_1, v_2)^T$, where $(f_{n-1}, \phi') : X_{n-2} \oplus M \to X_{n-1}$ is the minimal right almost split morphism induced by the labeling of the Auslander-Reiten quiver. Furthermore, we split up the the given sum

of paths in the same way as before, $\sum_{i=k}^{m} u_i = (f_{n-1}, \phi')(\phi_1, \phi_2)^T$. By construction there is a decomposition of $\phi_1 + v_1$ not containing $f_{n-2} \cdots f_1$ and we can apply the induction hypothesis on $\gamma + f_{n-1}(\phi_1 + v_1)$ as it factors through ϕ'. This completes the proof. $\square$

The same proof holds, with a minor restriction, for functorially finite subcategories. Therefore, we first need to generalize the definition of sectional paths in the natural way.

Definition 2.4.2 *Let*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

*be a path in the Auslander-Reiten quiver of χ. The path is called **sectional** if there is no i such that $X_{i+2} = \tau_\chi(X_i)$.*

Theorem 2.4.3 *Let*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

be a sequence of irreducible morphisms in χ between indecomposable modules such that X_i is not Ext-projective for $i \geq 1$. Let $f = f_n \cdots f_1 + f_n \phi$, where ϕ is a morphism from X_0 to X_{n-1} such that there exist a decomposition of ϕ that does not contain the path $\gamma = f_{n-1} \cdots f_1$. Suppose f factors through a morphism $g : Y \to X_n$ for some module Y in χ such that $(f_n, g) : X_{n-1} \oplus Y \to X_n$ is irreducible χ. Then the sequence is not sectional in χ and f factors over $\tau_\chi(X_n)$.

Proof:
The proof works just as for A-mod except that we do not have to show that the X_i are not Ext-projective. $\square$

In the following, there are several statements for a functorially finite subcategory χ and a sectional path

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

in its Auslander-Reiten quiver such that X_i is not Ext-projective for $i \geq 1$. Note that if χ is A-mod, the restriction to non-projective modules is not necessary, because the proofs of these statements are always based on the previous theorems.

Theorem 2.4.4 *Let*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

be a sectional path in χ such that X_i is not Ext-projective for $i \geq 1$. Let f be a morphism that has a decomposition containing $\gamma = f_n \cdots f_1$. Then every decomposition of f contains γ.

Proof:
Suppose that there is a decomposition of f not containing γ. We rewrite this decomposition as $f = g_1 + g_2$, where g_1 factors through f_n and g_2 factors through a morphism $g : Y \to X_n$

such that $(f_n, g) : X_{n-1} \oplus Y \to X_n$ is the minimal right almost split morphism in χ given by the labeling of the Auslander-Reiten quiver. Let $f' = f - \gamma\varphi$, where φ is the isomorphism of the path γ in the decomposition of f containing γ. Consequently, there is a decomposition of f' not containing γ. Moreover, we decompose this decomposition of f' in the same way as before, i.e. $f' = h_1 + h_2$, where h_1 factors through f_n and h_2 factors through g. So we have $\gamma\varphi + h_1 - g_1 = g_2 - h_2$ and, consequently, $\gamma + (h_1 - g_1)\varphi^{-1} = (g_2 - h_2)\varphi^{-1}$. By construction γ is not one of the paths contained in $(h_1 - g_1)\varphi^{-1}$, so γ is not sectional by Theorem 2.4.3, which contradicts the assumptions. Hence every decomposition of f must contain γ. $\qquad\square$

Note that different non-sectional paths can occur in distinct decompositions, even the number of paths in a decomposition may depend on the choice of an isomorphism in another path as the following example shows.

Example 2.4.5 *Let A be the path algebra of the quiver*

$$e_1 \quad \overset{f}{\underset{g}{\rightleftarrows}} \quad e_2$$

with the relation $gfg = 0$. We set $P_1 = Ae_1$ and $P_2 = Ae_2$ and note that $P_2 = \mathrm{rad}(P_1)$. Then the decomposition of the embedding $i : P_2 \to P_1$ depends on the choice of isomorphisms in P_2.

The Jordan-Hölder composition series of the non-simple indecomposable modules are

$$
P_1 = I_2 = \begin{matrix} S_1 \\ S_2 \\ S_1 \\ S_2 \end{matrix}
\qquad
P_2 = \begin{matrix} S_2 \\ S_1 \\ S_2 \end{matrix}
\qquad
I_1 = \begin{matrix} S_1 \\ S_2 \\ S_1 \end{matrix}
$$

$$
I_1/S_1 = \begin{matrix} S_1 \\ S_2 \end{matrix}
\qquad
P_2/S_2 = \begin{matrix} S_2 \\ S_1 \end{matrix}.
$$

We label the Auslander-Reiten quiver of A with the natural embedding and projection morphisms. Then there is a decomposition of i just containing the arrow from P_2 to P_1. Since this arrow is also a sectional path, it is contained in any decomposition. If we choose the isomorphism corresponding to this path to be $e_2 \mapsto e_2 + fg$ instead of the identity, then the decomposition of i contains the arrow from P_2 to P_1 and the path from P_2 to P_1 that factors over S_2.
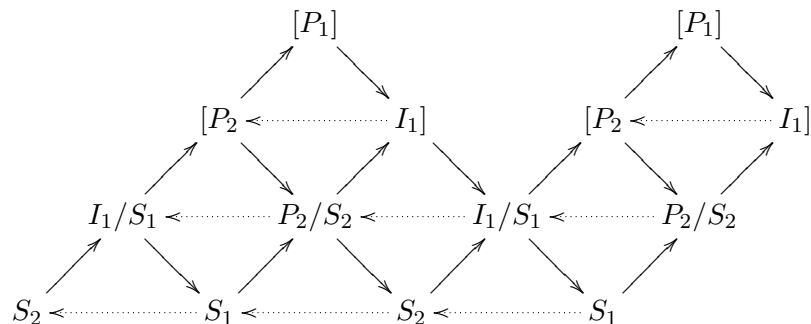
Corollary 2.4.6 *Let*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

be a sectional path in the Auslander-Reiten quiver of χ such that X_i is not Ext-projective for $i \geq 1$. If a morphism $f : X_0 \to X_n$ has a decomposition containing $f_n \cdots f_1$ and f factors over some module Y in χ, then X_i is a direct summand of Y for some $i \in \{0, \ldots, n\}$.

Proof:
Assume X_i is not a direct summand of Y, then there is a decomposition of f not containing $f_n \cdots f_1$, which clearly contradicts Theorem 2.4.4. $\qquad\square$

Corollary 2.4.7 *Let $f : X_0 \to X_n$ be a morphism between indecomposable modules in χ that has a decomposition containing a sectional path*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

such that X_i is not Ext-projective for $i \geq 1$. Then f is not in $\mathrm{rad}_\chi^{n+1}(X_0, X_n)$; in particular, f is non-zero.

Proof:
By Theorem 2.4.4 every decomposition of f contains $f_n \cdots f_1$. This path can clearly only be decomposed into at most n radical morphisms and, therefore, $f \notin \mathrm{rad}_\chi^{n+1}(X_0, X_n)$.
$\square$

Note that f can be any sum of distinct paths such that at least one path is sectional.

Corollary 2.4.8 *If there is a sectional cycle in the Auslander-Reiten quiver of χ, then there is at least one Ext-projective and one Ext-injective module on it.*

Proof:
As a reminder, a sectional cycle in an Auslander-Reiten quiver is a sectional path

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = X_0$$

from some indecomposable X_0 in χ to itself such that the composition of the path with itself is again sectional. Let $f = f_n \cdots f_1$, then by the lemma of Harada and Sai there is an integer k such that $f^k = 0$. Suppose none of the X_i is Ext-projective, then $f^k \neq 0$ by Corollary 2.4.7, which is a contradiction. If we assume that none of the X_i is Ext-injective, we can apply the same arguments to

$$\tau_\chi^{-1}(X_0) \xrightarrow{f_1} \tau_\chi^{-1}(X_1) \xrightarrow{f_2} \cdots \longrightarrow \tau_\chi^{-1}(X_{n-1}) \xrightarrow{f_n} \tau_\chi^{-1}(X_n) = \tau_\chi^{-1}(X_0),$$

which completes the proof. $\qquad\square$

2.5 Sectional paths and subcategories

Starting in this section we consider how compositions of irreducible morphisms in χ behave in a functorially finite resolving subcategory of χ. Recall that Ω denotes such a subcategory of χ in the whole dissertation. In order to be able to find relations between χ and Ω, we have to require that they have the same Ext-projective modules. But clearly in Ω every Ext-projective module is projective as Ω contains all projective A-modules and is closed under kernels of epimorphisms. From now on let χ be a functorially finite subcategory such that all Ext-projective modules in χ are projective. In particular, for all results in this chapter χ can be specialized to equal A-mod. Firstly, we observe another easy consequence from the previous section.

Corollary 2.5.1 *Let M be an indecomposable module in χ. If there is an indecomposable module X in Ω and there is a sectional path γ from X to M in Γ_χ such that no other module besides X on this path is in Ω, then γ is an Ω-section and can be extended to a minimal right Ω-approximation of M.*

Proof:

Let γ be given by

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = M.$$

Moreover, let Y be a module in Ω and $g : X \to Y$ and $f : Y \to M$ morphisms such that $\gamma = fg$. Since no module on the path is in Ω, the modules X_i are, in particular, not projective for $i \geq 1$. Therefore, every module that occurs in a factorization of γ contains a direct summand X_i by Corollary 2.4.6. Consequently, $X_0 = X$ is a direct summand of Y and, as every decomposition of fg contains γ, g is a split monomorphism. $\qquad\square$

Theorem 2.5.2 *Let X, Y be modules in Ω such that Y is not projective and let*

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n = Y$$

be a sectional path in χ such that X_i is not in Ω for $i = 1, \ldots, n-1$. Then every morphism f whose decompositions contain $\gamma = f_n \cdots f_0$ is irreducible in Ω. Moreover, if we denote all sectional paths from X to Y in χ such that all modules along these paths are not in Ω by $\gamma_1, \ldots, \gamma_n$, then their cosets $\{\overline{\gamma_1}, \ldots, \overline{\gamma_n}\}$ in $\mathrm{Irr}_\Omega(X, Y)$ are linearly independent.

Proof:

Let Z be a module in Ω and $g : X \to Z$ and $h : Z \to Y$ morphisms such that $f = hg$. Since no module on the path is in Ω, the modules X_i are, in particular, not projective for $i \geq 1$. Therefore, every module that occurs in a factorization of f contains a direct summand X_i by Corollary 2.4.6. Hence X or Y is a direct summand of Z and either g is a split monomorphism or h is a split epimorphism.

Suppose $\{\overline{\gamma_1}, \ldots, \overline{\gamma_n}\}$ is a linearly dependent set in $\mathrm{Irr}_\Omega(X, Y)$. This means we can rewrite $\overline{\gamma_1} = \sum_{i=2}^{n} \overline{\gamma_i h_i}$ for some $h_i \in \mathrm{End}_A(X)$, i.e. $\gamma_1 - \sum_{i=2}^{n} \gamma_i h_i \in \mathrm{rad}_\Omega^2(X, Y)$. But $\gamma_1 - \sum_{i=2}^{n} \gamma_i h_i$ is a morphism clearly admitting a decomposition containing γ_1, hence

by Theorem 2.4.4 every decomposition of $\gamma_1 - \sum_{i=2}^{n} \gamma_i h_i$ contains γ_1. By the first part of the theorem we conclude that $\gamma_1 - \sum_{i=2}^{n} \gamma_i h_i$ must be irreducible in Ω, which is a contradiction. Thus the cosets in $\mathrm{Irr}_\Omega(X, Y)$ of sectional paths in χ such that no module along the path is in Ω are linearly independent. $\qquad\square$

Later we see that the converse statement, i.e. every non-sectional path between X and Y gives rise to a reducible morphism in Ω, is not always true, but we provide conditions under which it holds. Note that if χ equals A-mod in the last and most the following theorems, the statements are still true even if Y is projective. This follows from the fact that these theorems are based on Theorem 2.4.1 and Theorem 2.4.3.

Lemma 2.5.3 *Given two short exact sequences*

$$0 \longrightarrow X \xrightarrow{(f_1, f_2)} U \oplus Y \xrightarrow{(g_1, g_2)} Z \longrightarrow 0$$

and

$$0 \longrightarrow U \xrightarrow{(g_1, g_3)} Z \oplus V \xrightarrow{(h_1, h_2)} W \longrightarrow 0,$$

then there is a short exact sequence

$$0 \longrightarrow X \xrightarrow{(g_3 f_1, f_2)} V \oplus Y \xrightarrow{(-h_2, h_1 g_2)} W \longrightarrow 0.$$

Proof:
Let $(g_3 f_1, f_2)(x) = 0$ for some $x \in X$, then, in particular, $f_2(x) = 0$. We assume that $f_1(x) \neq 0$ and deduce $g_1 f_1(x) \neq 0$ from $g_3 f_1(x) = 0$ using that (g_1, g_3) is injective. On the other hand, we know that $g_1 f_1(x) = -g_2 f_2(x)$, contradicting $f_2(x) = 0$. Thus $f_1(x) = 0$ and, consequently, $x = 0$ by injectivity of (f_1, f_2). Hence $(g_3 f_1, f_2)$ is a monomorphism.

For some $w \in W$ there is an element $z + v \in Z \oplus V$ such that $h_1(z) + h_2(v) = w$. Repeating the procedure for z we get $g_2(y) + g_1(u) = z$. Using $h_1 g_1(u) = -h_2 g_3(u)$ we know that $h_1 g_2(y) - h_2(g_3(u) - v) = c$. Therefore, $(-h_2, h_1 g_2)$ is surjective.

We now prove that $\mathrm{Im}(g_3 f_1, f_2) \subset \mathrm{Ker}(-h_2, h_1 g_2)$. For an $x \in X$ we get $h_1 g_2 f_2(x) - h_2 g_3 f_1(x) = h_1 g_2 f_2(x) + h_1 g_1 f_1(x) = h_1 g_2 f_2(x) - h_1 g_2 f_2(x) = 0$ using the exactness of the given sequences once each.

In the last step of the proof we verify that in fact $\mathrm{Im}(g_3 f_1, f_2) = \mathrm{Ker}(-h_2, h_1 g_2)$ using the Jordan-Hölder length of its modules. We have $l(V) = l(U) + l(W) - l(Z)$ and $l(Y) = l(X) + l(Z) - l(U)$. Thus
$$l(V \oplus Y) = l(X) + l(W)$$
and the sequence

$$0 \longrightarrow X \xrightarrow{(g_3 f_1, f_2)} V \oplus Y \xrightarrow{(-h_2, h_1 g_2)} W \longrightarrow 0$$

is exact. $\qquad\square$

The new sequence splits if and only if both of the given sequences split. This is easily seen because $(-h_2, h_1g_2)$ and (g_3f_1, f_2) are radical morphisms if (h_1, h_2) and (f_1, f_2) are radical morphisms respectively.

Lemma 2.5.4 *Let*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

be a sectional path in χ such that X_i is not projective for $i = 0, \ldots, n$ and let α_i name the almost split sequences

$$0 \longrightarrow \tau_\chi(X_i) \longrightarrow M_i \oplus \tau_\chi(X_{i+1}) \oplus X_{i-1} \longrightarrow X_i \longrightarrow 0,$$

where we require that $M = \bigoplus_{i=0}^n M_i \neq 0$ and define $\tau_\chi(X_{n+1})$ and X_{-1} to be the zero module. Then there is a non-split exact sequence

$$0 \longrightarrow \tau_\chi(X_0) \longrightarrow M \xrightarrow{f} X_n \longrightarrow 0$$

such that for all modules X in χ that do not contain a direct summand isomorphic to X_i for $i = 0, \ldots, n$, all morphisms $g : X \to X_n$ factor through f.

Proof:
We prove the lemma by induction on n. For $n = 0$ both existence and the factorization property follow immediately from the fact that α_0 is an almost split sequence in χ. Suppose the statement has been proved for $n - 1$, then by induction there is a non-split short exact sequence

$$0 \longrightarrow \tau_\chi(X_0) \longrightarrow \bigoplus_{i=0}^{n-1} M_i \oplus \tau_\chi(X_n) \xrightarrow{(h,h')} X_{n-1} \longrightarrow 0.$$

Applying Lemma 2.5.3 to this sequence and α_n we obtain another non-split short exact sequence

$$0 \longrightarrow \tau_\chi(X_0) \longrightarrow \bigoplus_{i=0}^{n} M_i \xrightarrow{f} X_n \longrightarrow 0.$$

Let X be an indecomposable module in χ not isomorphic to X_i for $i = 0, \ldots, n$ and let $g : X \to X_n$ be a morphism. Then there is a morphism $(g_n, g_n')^T : X \to X_{n-1} \oplus M_n$ such that $g = (f_n, f_n')(g_n, g_n')^T$, where $(f_n, f_n') : X_{n-1} \oplus M_n \to X_n$ is a minimal right almost split morphism in χ. By induction there is a factorization $g_n = (h, h')(s, s')^T$ with $(s, s') : X \to \bigoplus_{i=0}^{n-1} M_i \oplus \tau_\chi(X_n)$ and $(h, h') : \bigoplus_{i=0}^{n-1} M_i \oplus \tau_\chi(X_n) \to X_{n-1}$.

If $M_n = 0$, then the composition $f_n h' s'$ is zero and by construction f equals the composition $f_n(h, h')$, which gives us $g = f_n(h, h')(s, s')^T = f(s, s')^T$ as stated. On the other hand, if $M_n \neq 0$, then there is a morphism $f' : \tau_\chi(X_n) \to M_n$ such that $f_n h' = f_n' f'$ and we have $g = (f_n, f_n')(g_n, g_n')^T = f_n g_n + f_n' g_n' = f_n hs + f_n h's' + f_n' g_n' = f_n hs + f_n' f's' + f_n' g_n' = f_n hs + f_n'(f's' + g_n') = f(s, f's' + g_n')^T$, where $f = (f_n h, f_n') : M \to X_n$.
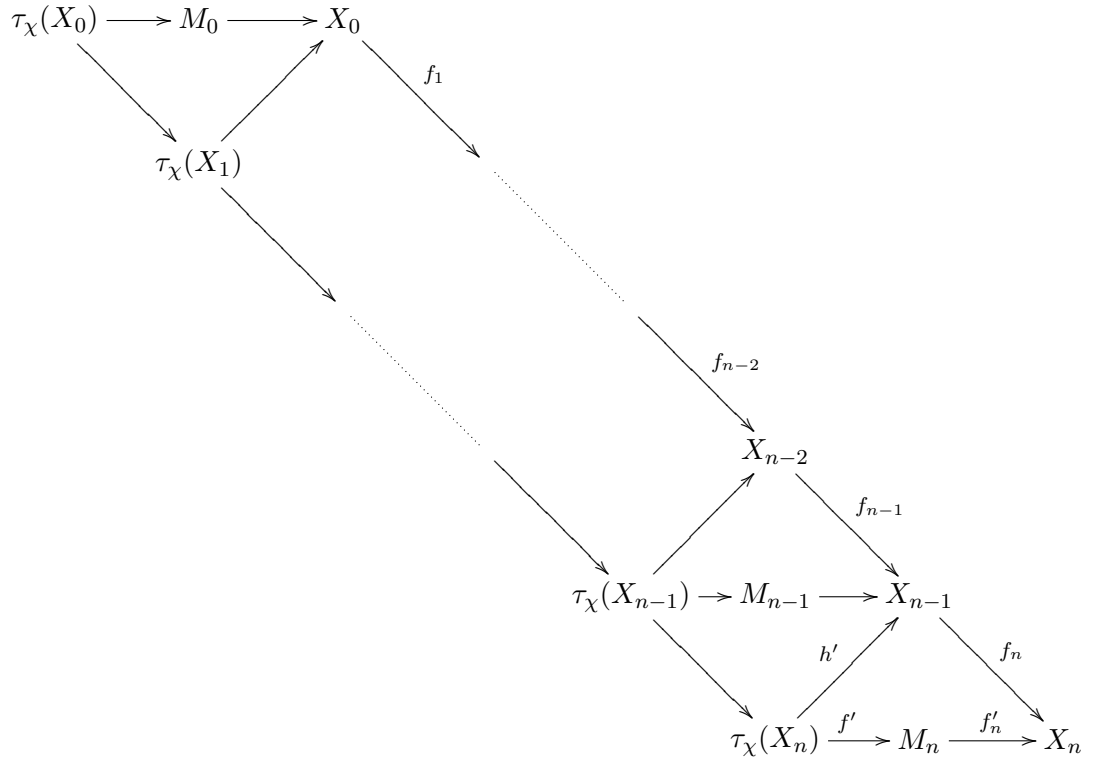
This proves the statement for an indecomposable module X not isomorphic to X_i for $i = 0, \ldots, n$. Let X now be any module in χ such that none of its direct summands is isomorphic to X_i for $i = 0, \ldots, n$ and let $g : Y \to X_n$ be a morphism. Clearly, all induced morphisms from a direct summand of Y to X_n factor through f, which gives rise to a factorization of g through f, which completes the proof. $\qquad\square$

Theorem 2.5.5 *Let X, Y be indecomposable modules in Ω such that X is not Ext-injective in Ω, Y is not projective and let $f : X \to Y$ be an Ω-irreducible morphism. Then f does not factor over $\tau_\chi(Y)$ non-trivially, i.e. if there are morphisms $h : X \to \tau_\chi(Y)$ and $g : \tau_\chi(Y) \to Y$ such that $f = gh$, then $g : \tau_\chi(Y) \to Y$ is an isomorphism.*

Proof:
Suppose there is a factorization $f = gh$ where $h : X \to \tau_\chi(Y)$ and $g : \tau_\chi(Y) \to Y$. By definition h factors through the minimal right Ω-approximation $h_{\tau_\chi(Y)} : X_{\tau_\chi(Y)} \to \tau_\chi(Y)$, so we have $f = g h_{\tau_\chi(Y)} h'$ for some $h' : X \to X_{\tau_\chi(Y)}$. Since f is irreducible, either $g h_{\tau_\chi(Y)}$ is a split epimorphism or h' is a split monomorphism. But if $g h_{\tau_\chi(Y)}$ is a split epimorphism, then, in particular, g is an isomorphism and the factorization $f = gh$ is trivial. Suppose now h' is a split monomorphism. It follows that $X \cong \tau_\Omega(Y)$ as it is the only direct summand of $X_{\tau_\chi(Y)}$ that is not Ext-injective by Theorem 2.1.15. This gives rise to a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \overset{\varphi}{\longrightarrow} & M' & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow & & \| & & \\
0 & \longrightarrow & \tau_\chi(Y) & \overset{g'}{\longrightarrow} & M & \longrightarrow & Y & \longrightarrow & 0,
\end{array}
$$

by Lemma 2.1.11. Clearly, $g'h$ is an Ω-section by Corollary 2.5.1, so φ must be a split monomorphism, which is a contradiction as by construction the upper row is the almost split sequence in Ω ending in Y. $\qquad\square$

Now we show that if a connected component Γ of Γ_χ is in some sense large, then it does not contain Ω-irreducible morphisms given by non-sectional paths in Γ.

Definition 2.5.6 *Let X, Y be indecomposable modules in Ω in a connected component Γ of Γ_χ. Suppose there is an immediate successor Z of X such that either there is a sectional path*

$$Z = Z_0 \longrightarrow Z_1 = \tau_\chi^{-1}(X) \longrightarrow \cdots \longrightarrow Z_{k-1} \longrightarrow Z_k = Y$$

from Z to Y such that each Z_i is not Ext-projective for $i = 1, \ldots, k$ and where we define $l = 0$ or there exist an $l > 0$ such that there are two paths

$$Z = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_{k+l-1} \longrightarrow Z_{k+l} = Y$$

and

$$Z = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{k+l-1} \longrightarrow Y_{k+l} = Y \ .$$

from Z to Y with the following properties.

1. *Both paths are not sectional in precisely one module, i.e. we have $\tau_\chi(Z_{l+1}) = Z_{l-1}$ and $\tau_\chi(Y_{k+1}) = Y_{k-1}$, while the induced paths from Z to Z_l, from Z_l to Y, from Z to Y_k and from Y_k to Y are all sectional.*

2. *$\tau_\chi(Y_1) = X$, $Z_1 \neq Y_1$ and $Z_{k+l-1} \neq Y_{k+l-1}$.*

3. *$Y_1, \ldots, Y_k$ are not projective.*

4. *If $l \leq k$ we have $Y_i = \tau_\chi^l(Z_{i+2l})$ for $i = 0, \ldots, k - l$ and $Y_{k-i} = \tau_\chi^i(Y_{k+i})$, $Z_{l-i} = \tau_\chi^i(Z_{l+i})$ for $i = 0, \ldots, l$.*

5. *If $k \leq l$ we have $Z_i = \tau_\chi^k(Y_{i+2k})$ for $i = 0, \ldots, l - k$ and $Y_{k-i} = \tau_\chi^i(Y_{k+i})$, $Z_{l-i} = \tau_\chi^i(Z_{l+i})$ for $i = 0, \ldots, k$.*

*We then say that Γ is **large between X and Y**.*
*Moreover, we call the following the **inner modules of X and Y**:*

(a) *If $l \leq k$, the inner modules are $\tau_\chi^j(Z_{i+2l})$ for $i = 0, \ldots, k - l$, $j = 0, \ldots, l$,*
   $\tau_\chi^j(Y_{k+i})$ for $i = 0, \ldots, l$, $j = 0, \ldots, i$ and
   $\tau_\chi^j(Z_{l+i})$ for $i = 1, \ldots, l$, $j = 0, \ldots, i - 1$.

(b) *If $k \leq l$, the inner modules are $\tau_\chi^j(Y_{i+2k})$ for $i = 0, \ldots, l - k$, $j = 0, \ldots, k$,*
   $\tau_\chi^j(Y_{k+i})$ for $i = 0, \ldots, k$, $j = 0, \ldots, i$ and
   $\tau_\chi^j(Z_{l+i})$ for $i = 1, \ldots, k$, $j = 0, \ldots, i - 1$.

The following is an example for a large component between X and Y for $k = 5$ and $l = 2$. Note that in general in a large component the number of middle terms for an almost split sequence does not have to be 2. Inner modules of the given paths are marked in red.
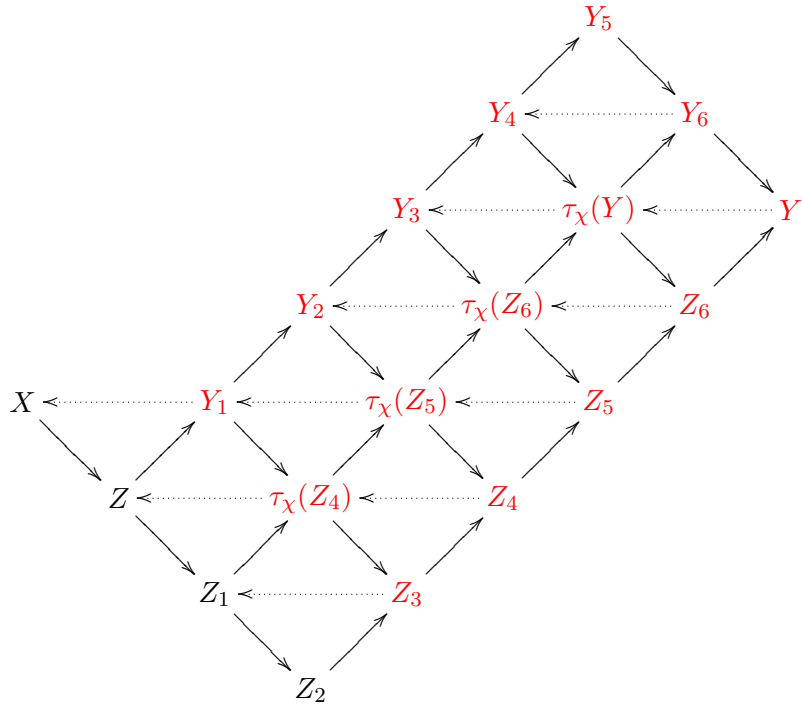
Note that a component Γ may be large between X and Y using different paths in the Auslander-Reiten quiver.

Example 2.5.7 *Let A be the path algebra of*

$$\overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longrightarrow \overset{3}{\cdot}$$

The preprojective component of its Auslander-Reiten quiver is illustrated on the next page. It is easy to see that there are $\frac{n}{2}$ or $\frac{n+1}{2}$ different combinations of paths that satisfy the definition of largeness between P_3 and Y_n for n even and odd respectively. Furthermore, the inner modules of a large component are not unique either, but they are uniquely determined by the paths from Z_l and Y_k to Y. Let us denote these paths by γ_l and γ_k respectively. We then say that Γ is large between X and Y **with respect to γ_l and γ_k** and refer to inner modules of γ_l and γ_k if necessary.

[P_2] ← Y_2

[P_3] ← Y_1 ← Y_4

[P_1] ← Y_3 ← Y_6

[P_2] ← Y_2 ← Y_5 ← Y_8

[P_3] ← Y_1 ← Y_4 ← Y_7

[P_1] ← Y_3 ← Y_6

[P_1] ← Y_2 ← Y_5

[P_2] ← Y_1 ← Y_4
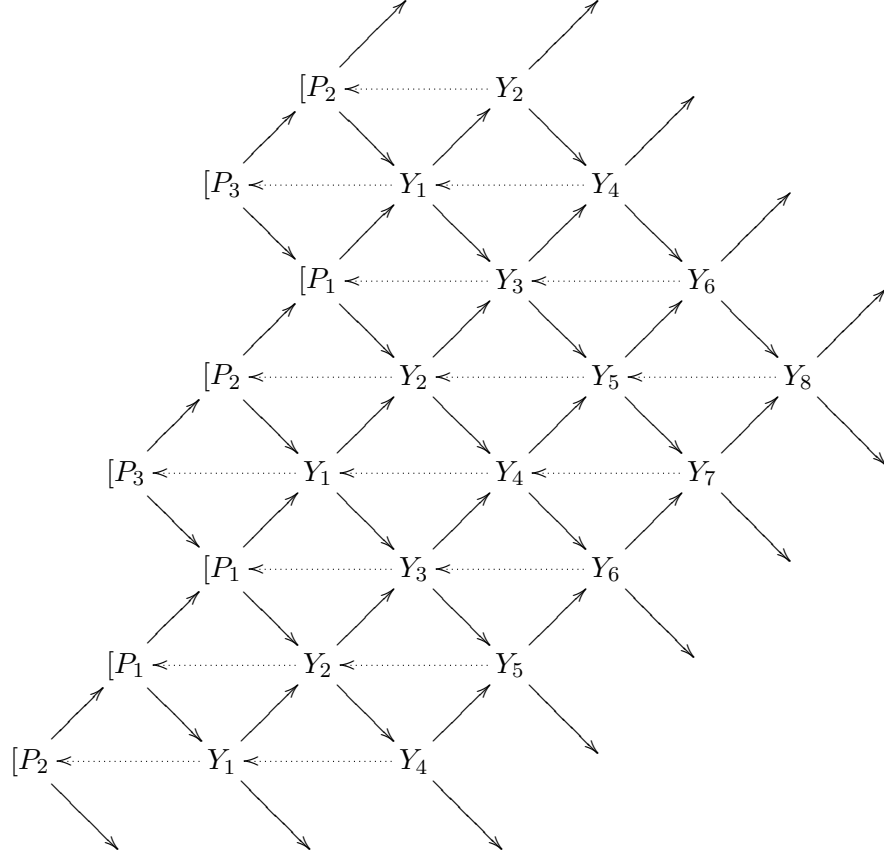
Lemma 2.5.8 *Let X and Y be indecomposable modules in χ such that their connected component Γ is large between them. Then there is a non-split short exact sequence*

$$0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$$

and every morphism $f : N \to Y$ such that N does not contain a direct summand isomorphic to an inner module of X and Y factors over M.

Proof:

We prove the lemma by induction on l as defined for large components. For $l = 0$ we are precisely in the situation of Lemma 2.5.4 and there is nothing left to prove. Suppose now that $l > 0$ and both statements are true for $l - 1$. Since $l > 0$, there are paths

$$\gamma_l : \qquad Z = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_{k+l-1} \longrightarrow Z_{k+l} = Y,$$

$$\gamma_k : \qquad Z = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{k+l-1} \longrightarrow Y_{k+l} = Y$$

such that Γ is large between X and Y with respect to γ_l and γ_k. By Theorem 2.5.4 there are l short exact sequences of the form

$$0 \longrightarrow Z_i \longrightarrow Z_{i+1} \oplus Y_{k+i} \oplus M_i \longrightarrow Y_{k+1+i} \longrightarrow 0$$

for $i = 0, \ldots, l - 1$ and one short exact sequence

$$0 \longrightarrow X \longrightarrow Z_0 \oplus M_{-1} \xrightarrow{(f_0, f_{-1})} Y_k \longrightarrow 0 \ .$$

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The connected component of X and Y is also large between Z and Y either since

$$Z_1 \longrightarrow \cdots \longrightarrow Z_{k+l-1} \longrightarrow Z_{k+l} = Y$$

is a sectional path for $l = 1$ or with respect to the paths

$$Z_l \longrightarrow \cdots \longrightarrow Z_{k+l-1} \longrightarrow Z_{k+l} = Y$$

and

$$Y_{k+1} \longrightarrow Y_{k+2} \longrightarrow \cdots \longrightarrow Y_{k+l-1} \longrightarrow Y_{k+l} = Y$$

for $l \geq 2$. So by inductive construction there is a short exact sequence

$$0 \longrightarrow Z \longrightarrow Y_k \oplus Z_l \oplus \bigoplus_{i=0}^{l-1} M_i \xrightarrow{(g_k, g)} Y \longrightarrow 0.$$

with $g_k : Y_k \to Y$ and $g : Z_l \oplus \bigoplus_{i=0}^{l-1} M_i \to Y$. We apply Lemma 2.5.3 and obtain a short exact sequence

$$0 \longrightarrow X \longrightarrow Z_l \oplus \bigoplus_{i=-1}^{l-1} M_i \longrightarrow Y \longrightarrow 0,$$

where we define $M = Z_l \oplus \bigoplus_{i=-1}^{l-1} M_i$. Let N be a module such that none of its direct summands is isomorphic to an inner module of X and Y, in particular, none of its direct summands is isomorphic to an inner module of Z and Y. Then, by induction, every morphism $f : N \to Y$ has a factorization $f = (g_k, g)(g_k', g')^T$ for some $g_k' : N \to Y_k$ and $g' : N \to Z_l \oplus \bigoplus_{i=0}^{l-1} M_i$. By Lemma 2.5.4 g_k' factors through $(f_0, f_{-1}) : Z_0 \oplus M_{-1} \to Y_k$ since $Y_1, \ldots, Y_k$ are inner modules of X and Y, i.e. there is an $(f_0', f_{-1}')^T : N \to Z_0 \oplus M_{-1}$ such that $g_k' = (f_0, f_{-1})(f_0', f_{-1}')^T$. Moreover, by construction $g_k f_0 : Z_0 \to Y$ factors through g, i.e. $g_k f_0 = gh$ for some $h : Z_0 \to Z_l \oplus \bigoplus_{i=0}^{l-1} M_i$. Composing these morphisms we have a factorization $f = g_k f_{-1} f_{-1}' + g(g' + h f_0')$, where $(g_k f_{-1}, g)$ is a morphism mapping from M to Y which completes the proof. $\square$

For the following theorem we define the distance in Γ_χ from an indecomposable module X in χ to another indecomposable module Y in χ. Suppose there are non-negative integers l and n with $n + l$ minimal such that there is a sectional path

$$\tau_\chi^{-l}(X) = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n = Y$$

Then $n + l$ is the **distance from X to Y** in the Auslander-Reiten quiver of χ.

Theorem 2.5.9 *Let X and Y be indecomposable modules in Ω that are in the same connected component of Γ_χ and let l be a positive and n a non-negative integer with $l + n$ minimal such that there is a sectional path*

$$\gamma : \qquad \tau_\chi^{-l}(X) = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n = Y.$$

Moreover, we assume that l is also minimal, i.e. for all non-negative integers l' and n'

with $l + n = l' + n'$ such that there is a sectional path

$$\tau_\chi^{-l'}(X) = Y_0' \longrightarrow Y_1' \longrightarrow \cdots \longrightarrow Y_{n'-1}' \longrightarrow Y_{n'}' = Y$$

we have $l' \geq l$. Suppose there are sectional paths

$$\gamma_1 : \qquad Y_{-t} \longrightarrow Y_{-t+1} \longrightarrow \cdots \longrightarrow Y_{-1} \longrightarrow Y_0 = \tau_\chi^{-l}(X),$$

$$\gamma_2 : \qquad X_{-s} \longrightarrow X_{-s+1} \longrightarrow \cdots \longrightarrow X_{-1} \longrightarrow X_0 = Y$$

such that

1. $\gamma\gamma_1$ is sectional,

2. X_{-1} is not isomorphic to Y_{n-1},

3. Either $t = l$ and $s = l - 1$ or $s = l$ and $t = l - 1$ and

4. X_j is not projective for $j = 1 - l, \ldots, 0$ and Y_j is not projective for $j = 1 - l, \ldots, n$.

Then there is no Ω-irreducible morphism from X to Y.

Proof:

We consider the following diagram for the case that $s = l$ to visualize the assumptions.
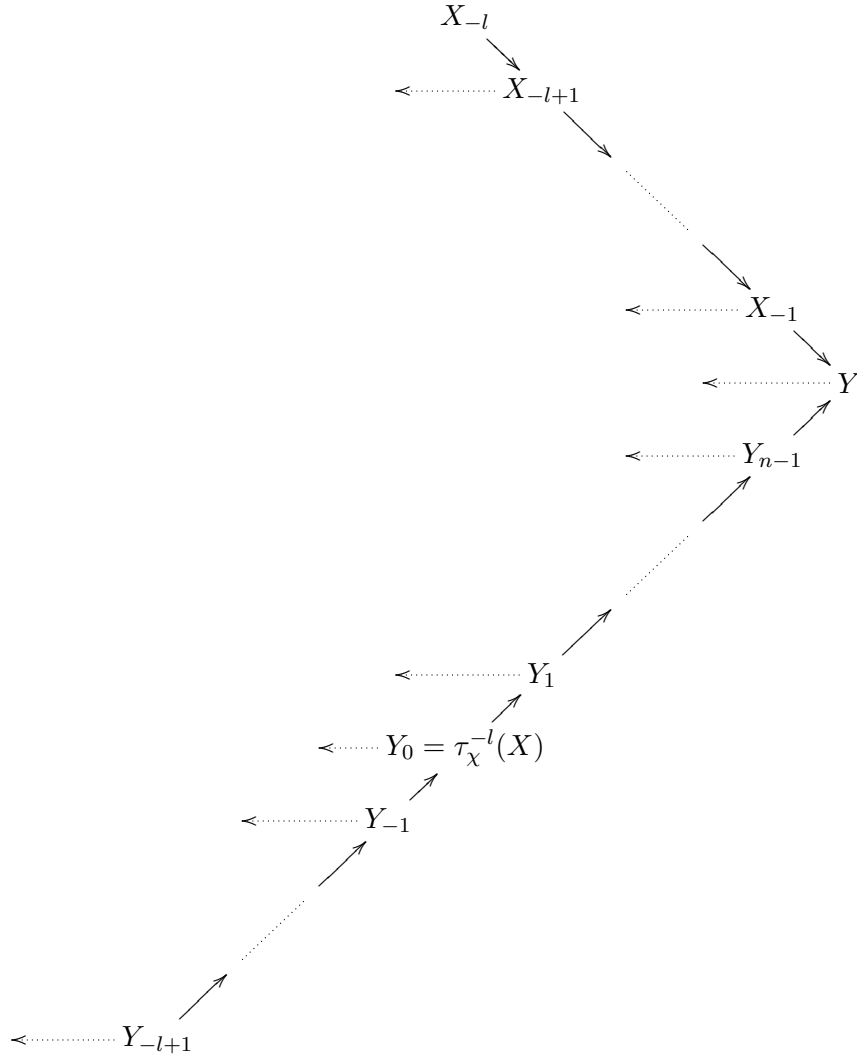
Clearly, X and Y are in the same connected component Γ of Γ_χ as there is a sectional path from $\tau_\chi^{-l}(X)$ to Y. If Γ is large between X and Y with respect to $\gamma\gamma_1$ and γ_2, then by Lemma 2.5.8 there is a module M in Ω such that all morphisms $Z \to Y$ factor over M if no inner module of X and Y is isomorphic to a direct summand of Z. By minimality of $n + l$ we know that X cannot be an inner module as the distance from an inner module of $\gamma\gamma_1$ and γ_2 to Y is always strictly smaller than $n + l$ by definition.

Suppose N is an indecomposable direct summand of M such that there is a sectional path whose distance to Y is given by $l' + n'$. By construction in Lemma 2.5.8 we have $n' + l' \leq n + l$ and $l' < l$. Since we have chosen $n + l$ and l to be minimal, X is not a direct summand of M.

Suppose now Γ is not large between X and Y with respect to $\gamma\gamma_1$ and γ_2, i.e. for some $l' \geq 1$ and some module $Y' = Y_i$ or $Y' = X_j$ we have $\tau_\chi^{l'}(Y') = P$ is projective. Let n' denote the length of the implicit path from Y' to Y given by either $\gamma\gamma_1$ or γ_2. Without loss of generality, we can assume that P is the projective module obtained in this way such that $n' + l'$ is minimal. Clearly, $n' + l' < l + n$, so by Lemma 2.5.8 there is a module M in Ω such that all morphisms $Z \to Y$ factor over M if no inner module of P and Y is isomorphic to a direct summand of M. Since we have chosen $n + l$ to be minimal, X can neither be an inner module of P and Y nor a direct summand of M.

In both cases M is in Ω as it is closed under extensions and all morphisms $f : X \to Y$ have a factorization $f = gh$ where $h : X \to M$ and $g : M \to Y$ are radical morphisms. Hence the existence of an irreducible morphism $f : X \to Y$ is impossible. $\qquad\square$

Theorem 2.5.10 *Let X, Y be indecomposable modules in Ω such that X is not Ext-injective in Ω, Y is not projective and $f : X \to Y$ an Ω-irreducible morphism given by a path*

$$\gamma : \qquad X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n = Y \ .$$

Then γ is sectional if one of the following conditions hold.

(a) Every X_i has at most 2 predecessors.

(b) Every domain of each right Ω-approximation of $\tau_\chi(X_i)$ has at most one direct summand that is not Ext-injective in Ω.

Proof:
In the whole proof we assume that γ is not sectional and find various contradictions. Therefore, let $j \leq n - 2$ be the largest integer such that $X_j = \tau_\chi(X_{j+2})$. Firstly, suppose that every X_i has at most 2 predecessors. Clearly, we have a sectional path from X_{j+2} to X_{n-1} such that all modules on this path have exactly 2 predecessors. By Theorem 2.5.4 there is a short exact sequence

$$0 \longrightarrow X_j \longrightarrow X_{j+1} \oplus \tau_\chi(Y) \longrightarrow X_{n-1} \longrightarrow 0$$

where the induced morphisms are given by $f_{j+1} : X_j \to X_{j+1}$ and $f_{n-1} \cdots f_{j+2} : X_{j+1} \to X_{n-1}$. Hence f factors over $\tau_\chi(Y)$ and, consequently, cannot be irreducible by Theorem 2.5.5.

Let us now suppose every domain of each right Ω-approximation of $\tau_\chi(X_i)$ has at most one direct summand that is not Ext-injective in Ω, then, in particular, the domain of the approximation $f_{X_j} : X_{X_j} \to X_j$ has only one direct summand that is not Ext-injective. Since $f_j \cdots f_1 : X \to X_j$ is an Ω-section, we know that X must be the aforementioned only direct summand of X_{X_j} that is not Ext-injective in Ω and, without loss of generality, we can assume that f_{X_j} is an extension of $f_j \cdots f_1$ to a minimal right Ω-approximation. By Theorem 2.5.4 there is a non-split short exact sequence

$$0 \longrightarrow X_j \longrightarrow X_{j+1} \oplus M \longrightarrow Y \longrightarrow 0$$

such that the induced morphism from X_j to X_{j+1} is given by f_{j+1}. By Lemma 2.1.11 there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X_{X_j} & \xrightarrow{\ h\ } & X_{X_{j+1}} \oplus X_M & \longrightarrow & Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_{X_j}} & & \downarrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & X_j & \longrightarrow & X_{j+1} \oplus M & \longrightarrow & Y & \longrightarrow & 0
\end{array}
$$

such that that the upper row does not split. Since X is the only direct summand of X_{X_j} that is not Ext-injective, it follows that $h|_X$ cannot be a split monomorphism. On the other hand, by construction we have that $gh|_X$ equals $f_{j+1} \cdots f_1$, which is an Ω-section. Since Ω is closed under extensions, M is in Ω. Therefore, $h|_X$ must be a split monomorphism, which is a contradiction. $\qquad\square$

Corollary 2.5.11 *Let X, Y be indecomposable modules in Ω such that X is not Ext-injective in Ω and Y is not projective. Moreover, we suppose that X and Y are in the same connected component Γ of Γ_Ω and $f : X \to Y$ is an Ω-irreducible morphism given by a path*

$$\gamma: \qquad X = X_0 \xrightarrow{\ f_1\ } X_1 \xrightarrow{\ f_2\ } \cdots \xrightarrow{\ f_{n-1}\ } X_{n-1} \xrightarrow{\ f_n\ } X_n = Y \ .$$

If γ is not sectional, then the following statements hold.

(a) There is an X_i with at least 3 predecessors.

(b) The domain of the Ω-approximation of the greatest j such that $\tau_\chi(X_{j+2}) = X_j$ has at least two direct summands that are not Ext-injective in Ω.

(c) Γ is not large between X and Y.

Proof:
The statements are just the contrapositives of Theorem 2.5.10 and Theorem 2.5.9. $\qquad\square$

Corollary 2.5.12 *Let $f : X \to Y$ be an Ω-irreducible morphism between indecomposable modules, where X is not Ext-injective whereas Y is not projective. If $f = v + \sum u_i$ denotes*

a decomposition such that $v \in \mathrm{rad}^\infty_\chi(X,Y)$ *and* $v \in \mathrm{rad}^2_\Omega(X,Y)$, *then* X *and* Y *are in the same connected component* Γ *of* Γ_χ *and the decomposition contains a path*

$$\gamma : \qquad X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n = Y$$

such that $f_n \cdots f_1$ *is irreducible. Moreover,* γ *is either sectional or it is non-sectional and the following holds*

(a) *There is an* X_i *with at least 3 predecessors.*

(b) *The domain of the* Ω-*approximation of the greatest* j *such that* $\tau_\chi(X_{j+2}) = X_j$ *has at least two direct summands that are not Ext-injective in* Ω.

(c) Γ *is not large between* X *and* Y.

Proof:
Since f is irreducible by assumption and $v \in \mathrm{rad}^2_\Omega(X,Y)$, we conclude that $\sum u_i \notin \mathrm{rad}^2_\Omega(X,Y)$. In particular, it must contain an individual path

$$\gamma : \qquad X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n = Y$$

such that $f_n \cdots f_1 \notin \mathrm{rad}^2_\Omega(X,Y)$, hence $f_n \cdots f_1$ is Ω-irreducible. All other statements follow immediately from Corollary 2.5.11. □

The previous corollary suggests a way of labeling arrows in a functorially finite resolving subcategory Ω. We first label the Auslander-Reiten quiver of χ as described in Section 2.3. Let X and Y be modules in Ω such that Y is not projective and there is a sectional path

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = Y$$

in χ such that no module along this path is in Ω. Then we label an arrow from X to Y with $f_n \cdots f_1$. Doing this for all modules and sectional paths we have labeled all arrows in Ω but those who either end in projective modules and those that have to be labeled either with a morphism in $\mathrm{rad}^\infty_\chi(X,Y)$ or a morphism whose decomposition does not contain a sectional path. We fix this induced labeling for the rest of the chapter.

Theorem 2.5.13 *Let* $f : X \to Y$ *be a morphism between indecomposable modules in* Ω, *that is given by a path*

$$\gamma_\chi : \qquad X = Y_0 \xrightarrow{g_1} Y_1 \xrightarrow{g_2} \cdots \longrightarrow Y_{m-1} \xrightarrow{g_m} Y_m = Y$$

in χ *and a path*

$$\gamma_\Omega : \qquad X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = Y$$

in Ω.

(a) *If* γ_χ *is sectional and all* Y_i *are non-projective for* $i \geq 1$, *then* γ_Ω *is also sectional.*

(b) *Suppose γ_Ω is sectional and all X_i are neither projective for $i \geq 1$ nor Ext-injective in Ω for $i < n$. Furthermore, we assume that every module in the connected component of Γ_χ containing X and Y has at most 2 immediate predecessors or the domain of every right Ω-approximation of Y_i has at most one direct summand that is not Ext-injective in Ω. Then γ_χ is sectional.*

Proof:

Let $\{i_0, i_1, \ldots, i_n\} \subset \{0, \ldots, m\}$ be the set indexing all modules on γ_χ in Ω. Note that in both in (a) and (b) these modules are exactly all modules on γ_Ω by Theorem 2.4.4.

In order to prove the first statement, let γ_χ be sectional and suppose

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = Y$$

is not sectional, then there is a j such that $X_j = \tau_\Omega(X_{j+2})$. Let $M \oplus X_{j+1}$ denote the whole middle term of the almost split sequence

$$0 \longrightarrow X_j \longrightarrow M \oplus X_{j+1} \longrightarrow X_{j+2} \longrightarrow 0.$$

The module M is in Ω as it is closed under extensions and, consequently, M is also a module in χ. Note that $M \neq 0$ since f is non-zero by Corollary 2.4.7 and, in particular, the morphism from X_j to X_{j+2} induced by f is non-zero. Therefore, f factors over M, so there is a decomposition of f not containing γ_χ, which contradicts Theorem 2.4.4. Hence

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = Y$$

is sectional, which proves (a).

By the assumptions that we have made in (b) it is clear that we can apply Theorem 2.5.10 to each path from $Y_{i_j} = X_j$ to $Y_{i_{j+1}} = X_{j+1}$. Consequently, these paths must be sectional and, if we suppose γ_χ to be non-sectional, there is a Y_{i_j} in Ω such that $\tau_\chi(Y_{i_j+1}) = Y_{i_j-1}$. Let us first suppose that the domain of the right Ω-approximation of each Y_i has only one direct summand that is not Ext-injective. Then X_{j-1} is the only direct summand of $X_{Y_{i_j-1}}$ that is not Ext-injective and we can rewrite $X_{Y_{i_j-1}} = X_{j-1} \oplus I$. By Theorem 2.5.4 there is a non-split short exact sequence

$$0 \longrightarrow Y_{i_j-1} \longrightarrow X_j \oplus M \longrightarrow X_{j+1} \longrightarrow 0$$

such that the induced morphism from X_j to X_{j+1} is given by f_{j+1}. By Lemma 2.1.11 there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X_{j-1} \oplus I & \xrightarrow{\left(f_j \quad \right)} & X_j \oplus X_M & \longrightarrow & X_{j+1} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_{Y_{i_j-1}}} & & \downarrow & & \| & & \\
0 & \longrightarrow & Y_{i_j-1} & \longrightarrow & X_j \oplus M & \xrightarrow{(f_{j+1}, \,)} & X_{j+1} & \longrightarrow & 0.
\end{array}
$$

Since X_j is in Ω, its minimal right Ω-approximation must be an isomorphism. Hence, without loss of generality, we can assume that the same irreducible morphism $f_{j+1} : X_j \to X_{j+1}$ also occurs in the top row after reducing it to

$$0 \longrightarrow X_{j-1} \xrightarrow{(f_j,\ )^T} X_j \oplus N \xrightarrow{(f_{j+1},\ )} X_{j+1} \longrightarrow 0.$$

The morphism $(f_{j+1},\ ) : X_j \oplus N \to X_{j+1}$ factors through a minimal right almost split morphism which we obtain by an extension of f_{j+1}, so we can extend the diagram by the almost split sequence ending in X_{j+1}.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_\Omega(X_{j+1}) & \xrightarrow{(g,\ )^T} & X_j \oplus M' & \xrightarrow{(f_{j+1},\ )} & X_{j+1} & \longrightarrow & 0 \\
& & {\scriptstyle f}\uparrow & & {\scriptstyle \left(\begin{smallmatrix} \mathrm{Id} & 0 \end{smallmatrix}\right)}\uparrow & & \| & & \\
0 & \longrightarrow & X_{j-1} & \xrightarrow{(f_j,\ )^T} & X_j \oplus N & \xrightarrow{(f_{j+1},\ )} & X_{j+1} & \longrightarrow & 0
\end{array}
$$

The irreducible morphism f_j factors over the indecomposable module $\tau_\Omega(X_{j+1})$, it follows that either f or g is an isomorphism. In the latter case the top row would split, which is a contradiction as it is an almost split sequence. In the former we have $X_{j-1} \cong \tau_\Omega(X_{j+1})$, which is a contradiction to the assumptions made in (b). Thus γ_χ must be a sectional path.

Let us now suppose that every module in the connected component of Γ_χ containing X and Y has at most 2 immediate predecessors. Then, as the composition $f_{j+1}f_j = g_{i_{j+1}} \cdots g_{i_{j-1}+1}$ is non-zero, the component in χ between $Y_{i_j-1} = X_{j-1}$ and $Y_{i_j+1} = X_{j+1}$ is large and there is a short exact sequence

$$0 \longrightarrow X_{j-1} \longrightarrow X_j \oplus M \longrightarrow X_{j+1} \longrightarrow 0$$

by Lemma 2.5.8 where M is an indecomposable module. By our assumptions this is not an almost split sequence because γ_Ω is sectional. By construction the morphism $f : M \to X_{j+1}$ is given by a sectional path which factors through an irreducible morphism $f' : M' \to X_{j+1}$, where M' must be a module on the sectional path from M to X_{j+1} by Theorem 2.4.4. But then $(f', f_{j+1}) : M' \oplus X_j \to X_{j+1}$ is a minimal right almost split morphism in Ω whose kernel is an indecomposable module isomorphic to some Y_k, $i_{j-1} < k < i_j$. This contradicts f_j to be irreducible since Ω is closed under kernels of surjective morphisms and f_j factors over Y_k as the following example diagram shows.
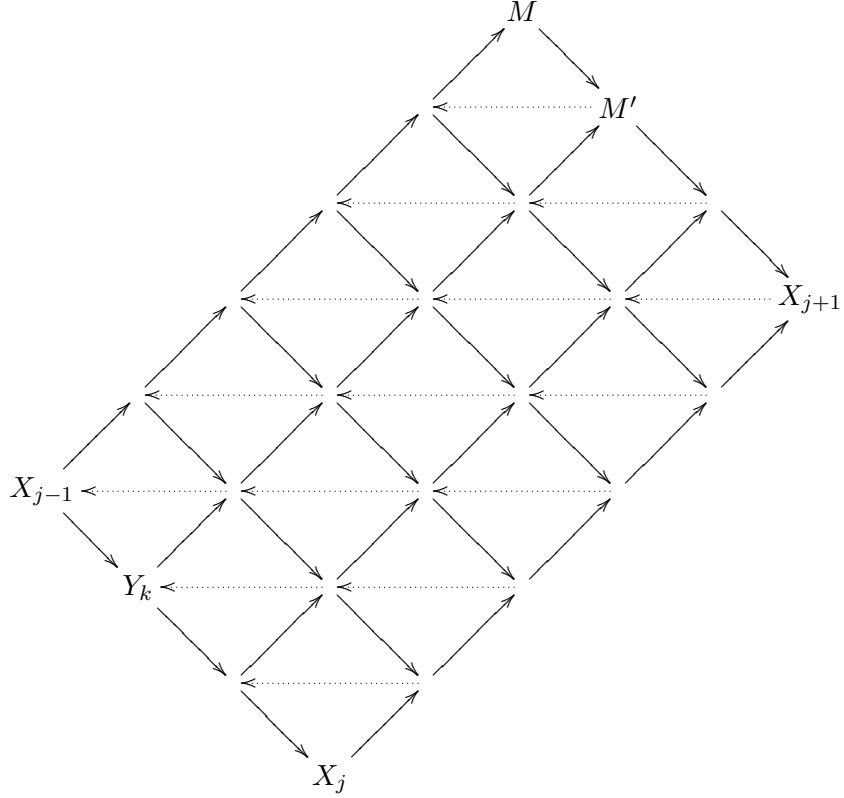
2.6 Applications and Examples

The main purpose of the theorems in Section 2.5 is to derive Γ_Ω from the Auslander-Reiten quiver of a larger category such as A-mod. If the larger quiver is finite, the procedure always works and one can get subquivers without computation of approximations. However, in infinite cases the theorems are not always applicable. Certain subcategories of the standard example show that the restriction to not Ext-injective and non-projective modules is necessary.

Example 2.6.1 *We derive the Auslander-Reiten quiver of $\mathfrak{F}(\Delta)$ from the Auslander-Reiten quiver of A-mod of the quasi-hereditary algebra A whose Jordan-Hölder composition series as a left A-module is given by*

$$
{}_A A = \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \\ S_2 \oplus S_1 & S_3 \oplus S_2 & S_4 \oplus S_3 \\ S_1 & S_2 & S_3 \end{array}.
$$

Recall the Auslander-Reiten quiver of A-mod that has already been calculated in Example 1.2.13 and modules in $\mathfrak{F}(\Delta)$ are again marked red.

$$[P_1]$$

$$[P_4 \leftarrow \Delta_2 \leftarrow \nabla_2 \leftarrow I_4]$$

$$S_3 \leftarrow X \leftarrow S_2 \leftarrow Y \leftarrow S_3$$

$$N_2 \leftarrow M_2 \leftarrow N_3 \leftarrow M_3 \leftarrow N_2$$

$$S_1 \leftarrow \nabla_3 \leftarrow S_4 \leftarrow \Delta_3 \leftarrow S_1$$

$$[P_2] \qquad [P_3]$$
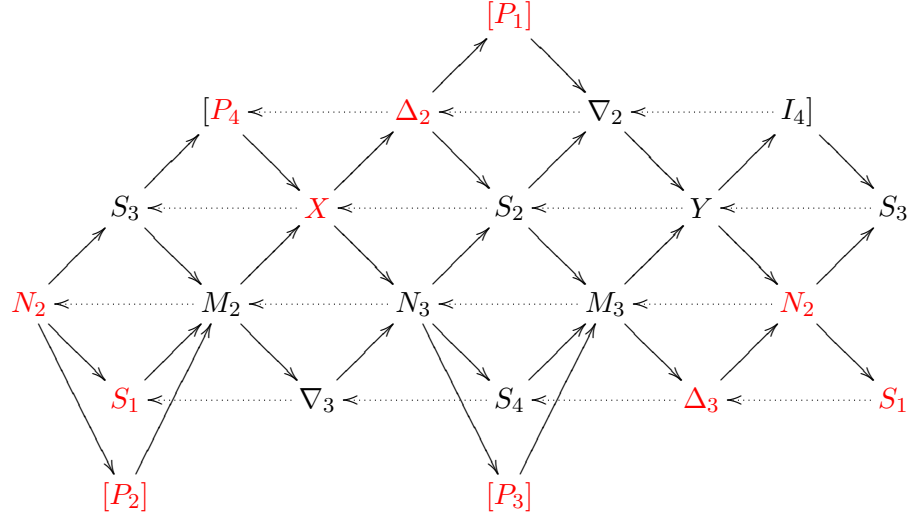
There are sectional paths from X to P_3, Δ_2 to Δ_3, P_1 to N_2, P_3 to Δ_3, N_2 to P_4, S_1 to X and P_2 to X. Since these are sectional paths in A-mod, we can apply Theorem 2.5.2 to all these paths including those that end in a projective module and obtain that they all give rise to irreducible morphisms in $\mathfrak{F}(\Delta)$. We can now easily derive the Auslander-Reiten quiver of $\mathfrak{F}(\Delta)$.

$$[P_1] \qquad [P_4$$

$$[P_4 \leftarrow \Delta_2 \leftarrow N_2 \leftarrow X$$

$$X \leftarrow \Delta_3 \leftarrow S_1]$$

$$[P_3] \qquad [P_2]$$
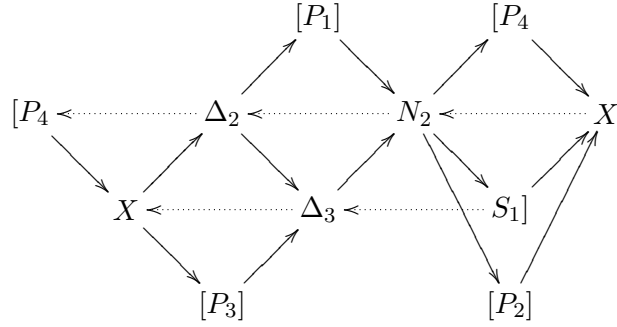
In order to prove that certain restrictions to non-projective modules in Section 2.4 are necessary, we continue by computing the Auslander-Reiten quiver of the functorially finite resolving subcategory $^{\perp}T = \{\mathrm{Ext}^i_A(-,T) = 0\}$ for $T = P_1 \oplus P_2 \oplus P_3 \oplus \Delta_2$, which is clearly a generalized cotilting module. The indecomposable modules in this category are $\Delta_2, \Delta_3, P_1, P_2, P_3, P_4$ and X. The Auslander-Reiten quiver is easily obtained in the same manner as before.

$$[P_1]$$

$$[P_4 \leftarrow \Delta_2]$$

$$\Delta_3 \leftarrow X \leftarrow \Delta_3$$

$$[P_2] \qquad [P_3]$$

$$[P_1]$$
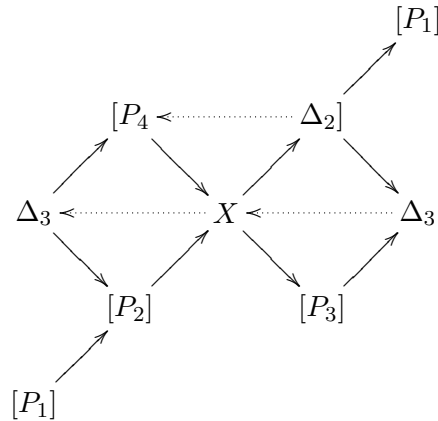
Considering the Jordan-Hölder decompositions of Δ_2 and P_2 it is obvious that there is only one morphism from Δ_2 to P_2 up to scalar multiplication. This morphism factors over

both P_1 and Δ_3 and can clearly be given by two different sectional paths. This contradicts neither Theorem 2.4.4 nor Corollary 2.4.6 as P_2 is projective.

In the Auslander-Reiten quiver of $^\perp T$ there is a sectional path γ from P_1 to itself. This shows that sectional cycles exist if they contain both an Ext-injective and Ext-projective module. As γ^2 is still a sectional path but $\gamma^2 = 0$, we can see that Corollary 2.4.7 does not hold for Ext-projective modules.

If we consider another generalized cotilting module $T' = P_1 \oplus P_2 \oplus P_3 \oplus \Delta_3$, we obtain another subcategory $^\perp T'$ containing the indecomposable modules P_1, P_2, P_3, P_4 and Δ_3. Hence $^\perp T'$ is a subcategory of $^\perp T$. Although it does not follow from previous corollaries, it is easy to see that there is an irreducible morphism $f : P_2 \to P_3$ in $^\perp T'$ since it only factors over X in $^\perp T$.

The morphism f considered in A-mod clearly corresponds to a non-sectional path factoring over N_3, X and $M_2 = \tau(N_3)$. All domains of the right $^\perp T'$-approximations of these modules have at most one direct summand which is not Ext-injective in $^\perp T'$. This shows that Theorem 2.5.10 is not true when not restricted to modules that are not projective and Ext-injective respectively.

One of the main questions arising is whether there are irreducible morphisms in Ω given by non-sectional paths. We can answer this question considering the following example.

Example 2.6.2 *Let A be the path algebra of the quiver of Dynkin type D_5 with the following orientation.*
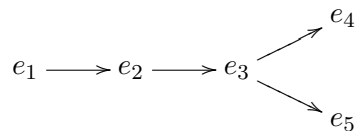
$$e_1 \longrightarrow e_2 \longrightarrow e_3 \overset{\displaystyle \nearrow e_4}{\underset{\displaystyle \searrow e_5}{}}$$

Then there is a functorially finite resolving subcategory Ω such that there is an Ω-irreducible morphism from P_2 to S_2.

As usual, the projective, injective and simple modules are named P_i, I_i and S_i respectively. Note that $S_4 = P_4$, $S_5 = P_5$ and $I_1 = S_1$. The Auslander-Reiten quiver of A is given by
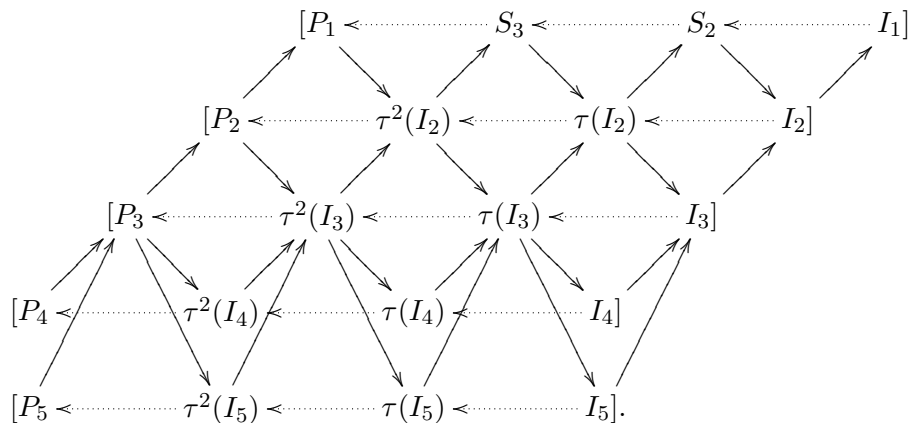
For a better understanding, we provide the Jordan-Hölder composition series of the non-simple, indecomposable modules.

$$P_1 = \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \quad S_5 \end{array} \qquad P_2 = \begin{array}{c} S_2 \\ S_3 \\ S_4 \quad S_5 \end{array} \qquad P_3 = \begin{array}{c} S_3 \\ S_4 \quad S_5 \end{array}$$

$$I_5 = \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_5 \end{array} \qquad I_4 = \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \qquad I_3 = \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \qquad I_2 = \begin{array}{c} S_1 \\ S_2 \end{array}$$

$$\tau^2(I_2) = \begin{array}{c} S_1 \\ S_2 \\ S_3^2 \\ S_4 \quad S_5 \end{array} \qquad \tau(I_3) = \begin{array}{c} S_1 \\ S_2^2 \\ S_3^2 \\ S_4 \quad S_5 \end{array}$$

$$\tau^2(I_3) = \begin{array}{c} S_2 \\ S_3^2 \\ S_4 \quad S_5 \end{array} \qquad \tau(I_4) = \begin{array}{c} S_2 \\ S_3 \\ S_5 \end{array} \qquad \tau(I_5) = \begin{array}{c} S_2 \\ S_3 \\ S_4 \end{array}$$

$$\tau(I_2) = \begin{array}{c} S_2 \\ S_3 \end{array} \qquad \tau^2(I_4) = \begin{array}{c} S_3 \\ S_4 \end{array} \qquad \tau^2(I_5) = \begin{array}{c} S_3 \\ S_5 \end{array}$$

It can easily be checked that $T = P_1 \oplus P_4 \oplus P_5 \oplus S_2 \oplus I_2$ is a generalized cotilting module and the functorially finite resolving subcategory $\Omega = {}^{\perp}T$ contains P_2, P_3 and I_1 in addition to the direct summands of T. The morphism f given by the path

$$P_2 \longrightarrow \tau^2(I_3) \longrightarrow \tau(I_5) \longrightarrow \tau(I_3) \longrightarrow \tau(I_2) \longrightarrow S_2$$

does not factor over $\tau^2(I_2)$. Therefore, the only other decomposition of f is given by the path

$$P_2 \longrightarrow \tau^2(I_3) \longrightarrow \tau(I_4) \longrightarrow \tau(I_3) \longrightarrow \tau(I_2) \longrightarrow S_2$$

and hence f is Ω-irreducible. The Auslander-Reiten quiver of Ω also contains an irreducible morphism from P_1 to I_2 and looks as follows.
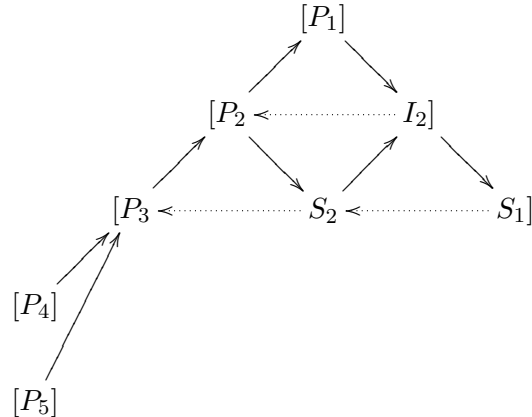
Moreover, let us observe what Corollary 2.5.11 means for this example. All decompositions of f factor over $\tau(I_3)$, which is a module with 3 predecessors and must occur by the aforementioned corollary. Furthermore, the approximation of $\tau^2(I_3)$ is a morphism $f_{\tau^2(I_3)} : P_3 \oplus P_2 \to \tau^2(I_3)$, so its domain contains 2 non-isomorphic indecomposable direct summands that are not Ext-injective.

Chapter 3

Degrees of irreducible morphisms in functorially finite subcategories

Degrees of irreducible morphisms have been established for irreducible morphism in A-mod by Liu in [L92]. In this chapter we generalize this notion for functorially finite subcategories and see that irreducible morphisms and paths behave similarly to A-mod when Ext-projective modules are not involved. We give a proof for a generalized version of the Happel-Preiser-Ringel theorem, which says that infinite connected components of Γ_A that contain a τ-periodic module are stable tubes [HPR80]. Moreover, we analyze the shape of left stable connected components that contain an oriented cycle but no periodic modules. The whole chapter closely follows [L92] and [L93]. In order to do this, we introduce some additional notation.

Let Z be an indecomposable module in χ that is not Ext-projective, we then denote by $\mathcal{E}_\chi(Z)$ the almost split sequence in χ ending in Z,

$$0 \longrightarrow \tau_\chi(Z) \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

Moreover, we denote the almost split sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow \tau_\chi^{-1}(X) \longrightarrow 0$$

by $\mathcal{E}_\chi'(X)$ if $\tau_\chi^{-1}(X)$ exists for a given indecomposable module X in χ. The module Y is called the **middle term** of $\mathcal{E}_\chi(Z)$ and $\mathcal{E}_\chi'(X)$.

Let $f : X \to Y$ and $g : Y \to Z$ be irreducible morphisms in χ. We say that a pair $\{f, g\}$ is a **component** of an almost split sequence if either there is an almost split sequence

$$0 \longrightarrow X \xrightarrow{\ g\ } Y \xrightarrow{\ f\ } Z \longrightarrow 0$$

in χ or if there are χ-irreducible morphisms $f' : X \to Y'$ and $g' : Y' \to Z$ such that

$$0 \longrightarrow X \xrightarrow{(g,g')^T} Y \oplus Y' \xrightarrow{(f,f')} Z \longrightarrow 0$$

is an almost split sequence in χ.

3.1 Definitions and basic properties

Definition 3.1.1 *Let $f : X \to Y$ be an irreducible morphism in χ.*

(a) *We define the **left degree** $d^l_\chi(f)$ of f in χ to be the least integer n such that there is a module Z and a morphism $g \in \mathrm{rad}^n_\chi(Z, X) \backslash \mathrm{rad}^{n+1}_\chi(Z, X)$ such that $fg \in \mathrm{rad}^{n+2}_\chi(Z, Y)$. If none such integer exists, we define $d^l_\chi(f)$ to be ∞.*

(b) *Dually, we define the **right degree** $d^r_\chi(f)$ of f in χ to be the least integer n such that there is a module Z and a morphism $g \in \mathrm{rad}^n_\chi(Y, Z) \backslash \mathrm{rad}^{n+1}_\chi(Y, Z)$ such that $gf \in \mathrm{rad}^{n+2}_\chi(X, Z)$. If none such integer exists, we define $d^r_\chi(f)$ to be ∞.*

Lemma 3.1.2

(a) *Let $n \geq 1$ be an integer and let $p : X \to Y$ and $f : Y \to Z$ be morphisms in χ, where f is χ-irreducible and Z is indecomposable and not Ext-projective. If $p \notin \mathrm{rad}^{n+1}_\chi(X, Y)$, $fp \in \mathrm{rad}^{n+2}_\chi(X, Z)$ and*

$$0 \longrightarrow \tau_\chi(Z) \xrightarrow{(g,g')^T} Y \oplus Y' \xrightarrow{(f,f')} Z \longrightarrow 0$$

is an almost split sequence in χ, then there exists a morphism $q : X \to \tau_\chi(Z)$ such that $q \notin \mathrm{rad}^n_\chi(X, \tau_\chi(Z))$, $p + gq \in \mathrm{rad}^{n+1}_\chi(X, Y)$ and $g'q \in \mathrm{rad}^{n+1}_\chi(X, Y')$.

(b) *Dually, let $n \geq 1$ be an integer and let $g : X \to Y$ and $p : Y \to Z$ morphisms in χ, where g is χ-irreducible and X is indecomposable and not Ext-injective. If $p \notin \mathrm{rad}^{n+1}_\chi(Y, Z)$, $pg \in \mathrm{rad}^{n+2}_\chi(X, Z)$ and*

$$0 \longrightarrow X \xrightarrow{(g,g')^T} Y \oplus Y' \xrightarrow{(f,f')} \tau^{-1}_\chi(X) \longrightarrow 0$$

is an almost split sequence in χ, then there exists a morphism $q : \tau^{-1}_\chi(X) \to Z$ such that $q \notin \mathrm{rad}^n_\chi(\tau^{-1}_\chi(X), Z)$, $p + qf \in \mathrm{rad}^{n+1}_\chi(Y, Z)$ and $qf' \in \mathrm{rad}^{n+1}_\chi(Y', Z)$.

Proof:
There is a factorization $fp = ts$ with $s \in \mathrm{rad}^{n+1}_\chi(X, W)$ and $t \in \mathrm{rad}_\chi(W, Z)$. The morphism t factors through (f, f') as the latter is a minimal right almost split morphism. If $t = (f, f')(u, u')^T$, then $(f, f')(us - p, u's)^T = 0$. Since $\mathrm{Im}(us - p, u's)^T \subset \mathrm{Im}(g, g')^T \cong \tau_\chi(Z)$, there exists a morphism $q : X \to \tau_\chi(Z)$ such that $(us - p, u's)^T = (g, g')^T q$, which is equivalent to $(p + gq, g'q)^T = (us, u's)^T \in \mathrm{rad}^{n+1}_\chi(X, Y \oplus Y')$ as $s \in \mathrm{rad}^{n+1}_\chi(X, W)$. Thus $p + gq \in \mathrm{rad}^{n+1}_\chi(X, Y)$ and $g'q \in \mathrm{rad}^{n+1}_\chi(X, Y')$. Moreover, as $p \notin \mathrm{rad}^{n+1}_\chi(X, Y)$ and $p + gq \in \mathrm{rad}^{n+1}_\chi(X, Y)$, we know that $gq \notin \mathrm{rad}^{n+1}_\chi(X, Y)$ and hence $q \notin \mathrm{rad}^n_\chi(X, \tau_\chi(Z))$. $\square$

Corollary 3.1.3

(a) *Let $f : Y \to Z$ be an irreducible morphism in χ with finite left degree in χ, where Z is an indecomposable module that is not Ext-projective. If $Y \oplus Y'$ is a direct summand of the whole middle term of $\mathcal{E}_\chi(Z)$ with $Y' \neq 0$, then there is an irreducible morphism $g' : \tau_\chi(Z) \to Y'$ with $d^l_\chi(g') < d^l_\chi(f)$. Consequently, if $d^l_\chi(f) = 1$, then f is a surjective minimal right almost split morphism.*

(b) *Dually, let $g : X \to Y$ be an irreducible morphism in χ with finite right degree in χ, where X is an indecomposable module that is not Ext-injective. If $Y \oplus Y'$ is a direct summand of the whole middle term of $\mathcal{E}'(X)$ with $Y' \neq 0$, then there is an irreducible morphism $f' : Y' \to \tau_\chi^{-1}(X)$ with $d_\chi^r(f') < d_\chi^r(g)$. Consequently, if $d_\chi^r(g) = 1$, then g is an injective minimal left almost split morphism.*

Proof:

Suppose that $d_\chi^l(f) = n$, i.e. there is a morphism $p \in \mathrm{rad}_\chi^n(X,Y) \backslash \mathrm{rad}_\chi^{n+1}(X,Y)$ such that $fp \in \mathrm{rad}_\chi^{n+2}(X,Z)$. Since $Y \oplus Y'$ is a direct summand of the middle term of $\mathcal{E}_\chi(Z)$, we know there are irreducible morphisms $(g,g')^T : \tau_\chi(Z) \to Y \oplus Y'$ and $(f,f') : Y \oplus Y' \to Z$ such that $\{(g,g')^T, (f,f')\}$ is a component of $\mathcal{E}_\chi(Z)$. If $(g,g',g'')^T : \tau_\chi(Z) \to Y \oplus Y' \oplus Y''$ denotes an extension of $(g,g')^T$ to a minimal left almost split morphism, then there is a morphism $q \notin \mathrm{rad}_\chi^n(X,\tau_\chi(Z))$ such that $(g',g'')^T q \in \mathrm{rad}_\chi^{n+1}(X,Y' \oplus Y'')$ by Lemma 3.1.2. Consequently, we also have $g'q \in \mathrm{rad}_\chi^{n+1}(X,Y')$, which implies that $d_\chi^l(g') \leq n-1 < d_\chi^l(f)$.
$\square$

Definition 3.1.4 *A path*

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$$

*in Γ_χ is said to be **pre-sectional in χ** if for all $i = 2,\ldots,n$ such that $\tau_\chi(X_i)$ exists, $X_{i-2} \oplus \tau_\chi(X_i)$ is a direct summand of the domain of a minimal right almost split morphism mapping to X_{i-1}.*

Equivalently, one can define that the path is pre-sectional in χ if for all $i = 2,\ldots,n$ such that $\tau_\chi^{-1}(X_{i-2})$ exists, $\tau_\chi^{-1}(X_{i-2}) \oplus X_i$ is a direct summand of the codomain of a minimal left almost split morphism mapping from X_{i-1}. Clearly, every sectional path is pre-sectional.

Lemma 3.1.5 *Let*

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$$

be a pre-sectional path in Γ_χ and let m be an integer. Then the following paths are also pre-sectional if they are defined.

(a) $\tau_\chi^n(X_n) \longrightarrow \tau_\chi^{n-1}(X_{n-1}) \longrightarrow \cdots \longrightarrow \tau_\chi(X_1) \longrightarrow X_0$

(b) $X_n \longrightarrow \tau_\chi^{-1}(X_{n-1}) \longrightarrow \cdots \longrightarrow \tau_\chi^{1-n}(X_1) \longrightarrow \tau_\chi^{-n}(X_0)$

(c) $\tau_\chi^m(X_0) \longrightarrow \tau_\chi^m(X_1) \longrightarrow \cdots \longrightarrow \tau_\chi^m(X_{n-1}) \longrightarrow \tau_\chi^m(X_n)$

Proof:

The lemma follows immediately from Corollary 2.2.7.
$\square$

Lemma 3.1.6

(a) *Let* $f : X \to Y$ *be an irreducible morphism in* χ *such that* Y *is indecomposable and* $d_\chi^l(f) < \infty$. *Let*

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = Y$$

be a pre-sectional path in Γ_χ *such that no* X_i *is Ext-projective and* $n \geq 1$. *If* $X \oplus X_1$ *is a direct summand of the whole middle term of* $\mathcal{E}_\chi(Y)$, *then for each* $1 \leq i \leq n$ *there is an irreducible morphism* $f_i : \tau_\chi(X_{i-1}) \to X_i$ *such that*

$$d_\chi^l(f_n) < d_\chi^l(f_{n-1}) < \cdots < d_\chi^l(f_1) < d_\chi^l(f).$$

Consequently, $n < d_\chi^l(f)$.

(b) *Dually, suppose* $g : X \to Y$ *be an irreducible morphism in* χ *such that* X *is indecomposable and* $d_\chi^r(g) < \infty$. *Let*

$$X = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n$$

be a pre-sectional path in Γ_χ *such that no* Y_i *is Ext-injective and* $n \geq 1$. *If* $Y \oplus Y_1$ *is a direct summand of the whole middle term of* $\mathcal{E}_\chi'(X)$, *then for each* $1 \leq i \leq n$ *there is an irreducible morphism* $g_i : Y_i \to \tau_\chi^{-1}(Y_{i-1})$ *such that*

$$d_\chi^r(g_n) < d_\chi^r(g_{n-1}) < \cdots < d_\chi^r(g_1) < d_\chi^r(g).$$

Consequently, $n < d_\chi^r(g)$.

Proof:

Let $X \oplus X_1$ be a direct summand of the whole middle term of $\mathcal{E}_\chi(Y)$. Then by Corollary 3.1.3 there is an irreducible morphism $f_1 : \tau_\chi(Y) \to X_1$ with $d_\chi^l(f_1) < d_\chi^l(f)$. Suppose that $1 \leq m < n$ and for each i such that $1 \leq i \leq m$ there is an irreducible morphism $f_i : \tau_\chi(X_{i-1}) \to X_i$ such that $d_\chi^l(f_m) < \cdots < d_\chi^l(f_1) < d_\chi^l(f)$. By definition of pre-sectional paths $\tau_\chi(X_{m-1}) \oplus X_{m+1}$ is a direct summand of the whole middle term of $\mathcal{E}_\chi(X_m)$. We can apply Corollary 3.1.3 again to find an irreducible morphism $f_{m+1} : \tau_\chi(X_m) \to X_{m+1}$ such that $d_\chi^l(f_{m+1}) < d_\chi^l(f_m)$. Hence the result follows inductively. $\square$

Corollary 3.1.7

(a) *Let* Y *be indecomposable and let* $f : X \to Y$ *be an irreducible morphism in* χ. *Let*

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = Y$$

be an infinite pre-sectional path in Γ_χ *such that no* X_i *is Ext-projective and* $X \oplus X_1$ *is a direct summand of* $\mathcal{E}_\chi(Y)$. *Then* f *has infinite left degree in* χ.

(b) Dually, let $g : X \to Y$ be an irreducible morphism in χ for some indecomposable module X. Let

$$X = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n \longrightarrow \cdots$$

be an infinite pre-sectional path in Γ_χ such that no Y_i is Ext-injective and $Y \oplus Y_1$ is a direct summand of $\mathcal{E}'_\chi(X)$. Then g has infinite right degree in χ.

Proof:

As the path is pre-sectional, we obtain an infinite collection of irreducible morphisms $f_i : \tau_\chi(X_{i-1}) \to X_i$ such that

$$\cdots < d^l_\chi(f_n) < d^l_\chi(f_{n-1}) < \cdots < d^l_\chi(f_1) < d^l_\chi(f).$$

by Lemma 3.1.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 3.1.8 *Let Y and Y' be indecomposable modules in χ that are not Ext-projective and let $(f, f') : \tau_\chi(Y) \oplus \tau_\chi(Y') \to X$ and $(g, g')^T : X \to Y \oplus Y'$ be irreducible morphisms. If $\{f, g\}$ and $\{f', g'\}$ are components of $\mathcal{E}_\chi(Y)$ and $\mathcal{E}_\chi(Y')$ respectively, we say that (f, f') is a **left neighbour** of $(g, g')^T$ and $(g, g')^T$ is a **right neighbour** of (f, f').*

Lemma 3.1.9 *Let $f : X \to Y$ be an irreducible morphism in χ.*

(a) If f has finite left degree and $Y = Y_1 \oplus Y_2$ such that both Y_i are indecomposable and not Ext-projective, then f has a left neighbour

$$g = (g_1, g_2) : \tau_\chi(Y_1) \oplus \tau_\chi(Y_2) \to X$$

with $d^l_\chi(g) < d^l_\chi(f)$

(b) If f has finite right degree and $X = X_1 \oplus X_2$ such that both X_i are indecomposable and not Ext-injective, then f has a right neighbour

$$g = (g_1, g_2)^T : Y \to \tau_\chi^{-1}(X_1) \oplus \tau_\chi^{-1}(X_2)$$

with $d^r_\chi(g) < d^r_\chi(f)$.

Proof:

Assume that $d^l_\chi(f) = n$. Then there is a $p \in \mathrm{rad}^n_\chi(Z, X) \backslash \mathrm{rad}^{n+1}_\chi(Z, X)$ such that $fp \in \mathrm{rad}^{n+2}_\chi(Z, Y)$. Let

$$g = (g_1, g_2) : \tau_\chi(Y_1) \oplus \tau_\chi(Y_2) \to X$$

be a left neighbour of f. By definition $\{g_i, f_i\}$ is a component of $\mathcal{E}_\chi(Y_i)$. So by Lemma 3.1.2 there are $q_i \notin \mathrm{rad}^n_\chi(Z, \tau_\chi(Y_i))$ such that $p + g_i q_i \in \mathrm{rad}^{n+1}_\chi(Z, X)$. Consequently, $g(q_1, -q_2)^T = g_1 q_1 - g_2 q_2 = p + g_1 q_1 - (p + g_2 q_2) \in \mathrm{rad}^{n+1}_\chi(Z, X)$. Since $(q_1, q_2)^T \notin \mathrm{rad}^n_\chi(Z, \tau_\chi(Y_1) \oplus \tau_\chi(Y_2))$, we obtain $d^l_\chi(g) \le n - 1 < d^l_\chi(f)$, which completes the proof. $\square$

Definition 3.1.10 *Let X be an indecomposable module in χ.*

(a) *We call X **left stable** in χ if $\tau_\chi^n(X)$ exists for all $n \geq 0$.*

(b) *Dually, X is called **right stable** in χ if $\tau_\chi^n(X)$ exists for all $n \leq 0$.*

(c) *X is called **stable** in χ if it is both left and right stable.*

(d) *If there is a positive integer $n \geq 1$ such that $\tau_\chi^n(X) = X$ we call X $\boldsymbol{\tau_\chi}$**-periodic** or, for convenience, periodic.*

Moreover, we call a subquiver Γ of the Auslander-Reiten quiver of χ left stable, right stable, stable or τ_χ-periodic if all their modules are left stable, right stable, stable or τ_χ-periodic respectively and, if X is in Γ, then the module $\tau_\chi^n(X)$ is also Γ for all $n \in \mathbb{Z}$ such that $\tau_\chi^n(X)$ exists.

Note that a module that is stable in a subcategory does not have to be stable in neither a larger nor a smaller subcategory. Consider the standard example 1.2.13 and let $\chi = \mathfrak{F}(\Delta)$ be the category of standard-filtered modules and $\Omega = {}^{\perp}T$ the Ext-orthogonal of the generalized cotilting module $T = P_1 \oplus P_2 \oplus P_3 \oplus \Delta_2$. In 2.6.1 we have shown that Ω is a subcategory of χ just as in our general setup. The indecomposable module X is stable and periodic in A-mod as $\tau^4(X) = X$, but X is neither left or right stable in χ as $\tau_\chi^3(X)$ is projective and $\tau_\chi^{-3}(X)$ is Ext-injective in χ. On the other hand, in Ω we have $\tau_\Omega^2(X) = X$, hence X is τ_Ω-periodic and, consequently, stable in Ω.

Lemma 3.1.11 *Let $X \to Y$ be an arrow in Γ_χ with valuation (a, a). If X and Y are left stable in χ and there is an irreducible morphism $f : X \to Y$ with finite left degree or if X and Y are right stable in χ and there is an irreducible morphism $f : X \to Y$ with finite right degree, then $a = 1$.*

Proof:
Suppose X and Y are left stable and f has finite left degree, hence we prove the statement by induction on $d_\chi^l(f)$. If $d_\chi^l(f) = 1$, then f must be a minimal right almost split morphism by Corollary 3.1.3, so $a = 1$. Assume the statement is true for $d_\chi^l(f) < m$. If $d_\chi^l(f) = m$ and $a > 1$, then $X \oplus X$ is a direct summand of the whole middle term of $\mathcal{E}_\chi(Y)$. We use Corollary 3.1.3 to obtain a morphism $g : \tau_\chi(Y) \to X$ with $d_\chi^l(g) < m$. By induction the arrow from $\tau_\chi(Y)$ to X must be valued $(1, 1)$. On the other hand, the same arrow must be valued (a, a) by Corollary 2.2.7, which is a contradiction. The proof for the other condition works dually. $\qquad\square$

Corollary 3.1.12 *Let $X \to Y$ be an arrow in Γ_χ and let $f : X \to Y$ and $g : X \to Y$ be irreducible morphisms in χ.*

(a) *If the valuation of the arrow is $(1, 1)$ or X and Y are left stable in χ, then $d_\chi^l(f) = d_\chi^l(g)$.*

(b) *Dually, if the valuation of the arrow is $(1, 1)$ or X and Y are right stable in χ, then $d_\chi^r(f) = d_\chi^r(g)$.*

Proof:

First we assume that X and Y are left stable in χ. Let (a, a) be the valuation of the arrow from X to Y, if $a > 1$, then $d_\chi^l(f) = d_\chi^l(g) = \infty$ by Lemma 3.1.11. So let $a = 1$ and, without loss of generality, suppose $d_\chi^l(f) < \infty$, i.e. there is a $p \in \mathrm{rad}_\chi^n(Z, X) \backslash \mathrm{rad}_\chi^{n+1}(Z, X)$ such that $fp \in \mathrm{rad}_\chi^{n+2}(Z, Y)$. As $a = 1$, there is an isomorphism $\varphi : X \to X$ such that $f - g\varphi \in \mathrm{rad}_\chi^2(X, Y)$. Therefore, $g\varphi p = fp - (f - g\varphi)p \in \mathrm{rad}_\chi^{n+2}(Z, Y)$ and hence $d_\chi^l(g) \leq d_\chi^l(f) < \infty$ as $\varphi p \notin \mathrm{rad}_\chi^{n+1}(Z, X)$. We similarly obtain $d_\chi^l(f) \leq d_\chi^l(g)$, which completes the proof in this case. $\qquad\square$

Now the following definition makes sense.

Definition 3.1.13 *Let $X \to Y$ be an arrow in Γ_χ and $f : X \to Y$ an irreducible morphism.*

(a) *If the valuation of the arrow is $(1, 1)$ or X and Y are left stable in χ, we then define the **left degree of the arrow $X \to Y$** to be $d_\chi^l(f)$.*

(b) *Dually, if the valuation of the arrow is $(1, 1)$ or X and Y are right stable in χ, we then define the **right degree of the arrow $X \to Y$** to be $d_\chi^r(f)$.*

Example 3.1.14 *We compute the left degrees of all arrows of the Auslander-Reiten quiver of A-mod and $\mathfrak{F}(\Delta)$ of the algebra discussed in the standard example.*

Note that for this algebra all arrows that are given by an irreducible monomorphisms have left degree ∞. The almost split sequences $\mathcal{E}(\Delta_2)$, $\mathcal{E}(I_4)$, $\mathcal{E}(\Delta_3)$, $\mathcal{E}(S_4)$, $\mathcal{E}(\nabla_3)$ and $\mathcal{E}(S_1)$ have an indecomposable middle term, hence the arrows from these middle terms to $\Delta_2, I_4, \Delta_3, S_4, \nabla_3$ and S_1 respectively must have left degree 1. Let $f : X \to Y$ be an irreducible morphism between indecomposable A-modules such that we have not obtained the left degree of the corresponding arrow yet. It is not hard to see that in this case if

$$ X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = X $$

is the shortest sectional path such that $X_1 = \tau(Y)$ and $ff_1 \cdots f_n = 0$, then $d_\chi^l(f) = n$ and corresponding arrow also has left degree n. This completes the calculation and gives us the following quiver with arrows labeled with their left degrees.
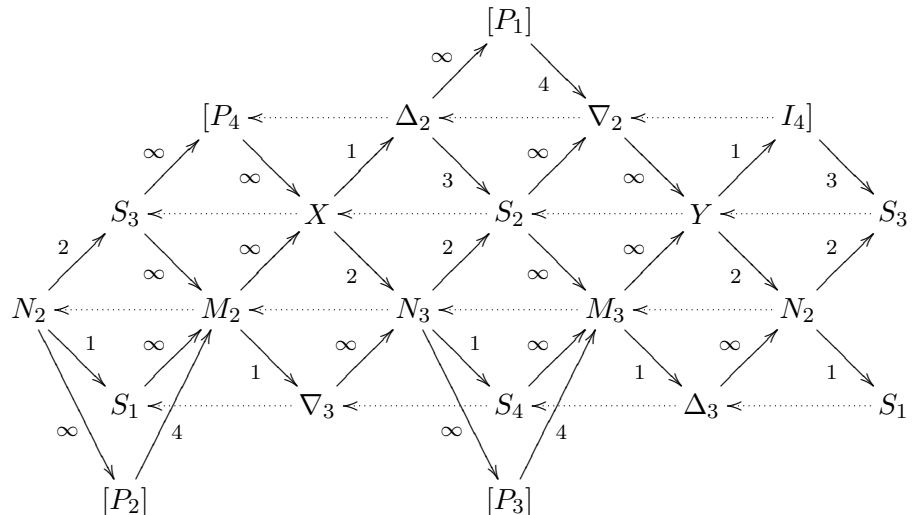
In $\mathfrak{F}(\Delta)$, just as in A-mod, all irreducible monomorphisms have infinite left degree. Moreover, all $\mathfrak{F}(\Delta)$-irreducible morphisms between irreducible modules in $\mathfrak{F}(\Delta)$ that are minimal right almost split morphisms have left degree 1. We obtain the left degrees of the remaining arrows similarly to A-mod, except for the arrow given by the irreducible morphism $f : P_2 \to X$. Composing this morphism with a sectional path in χ of length n starting in some module M always gives a morphism in $\mathrm{rad}_{\mathfrak{F}(\Delta)}^{n+1}(M,X)\backslash\mathrm{rad}_{\mathfrak{F}(\Delta)}^{n+2}(M,X)$. On the other hand, if we compose f with the up to a scalar unique non-zero morphism $g \in \mathrm{rad}_{\mathfrak{F}(\Delta)}^3(\Delta_2, P_2)\backslash\mathrm{rad}_{\mathfrak{F}(\Delta)}^4(\Delta_2, P_2)$, which can be considered as a non-sectional path, we obtain $fg = 0$. Hence the left degree of the arrow from P_2 to X is 3 and we have finished the computations.
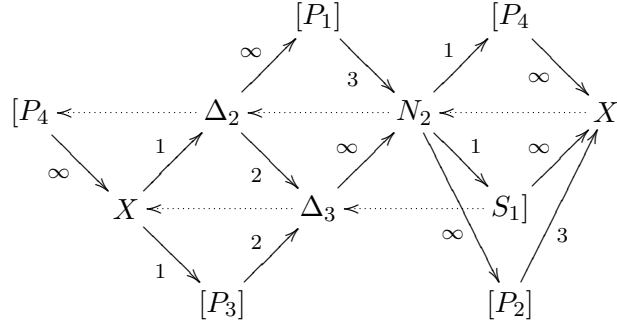
Contrary to A-mod, where arrows pointing at a projective module always have infinite left degree, the arrow in the Auslander-Reiten quiver of $\mathfrak{F}(\Delta)$ from N_2 to P_4 has left degree 1.

Lemma 3.1.15

(a) Let

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

be an infinite pre-sectional path in Γ_χ with all X_i left stable. If there is some integer $n \geq 0$ such that the almost split sequence $\mathcal{E}_\chi(X_n)$ has three left stable middle terms or the arrow $X_{n+1} \to X_n$ has non-trivial valuation, then all arrows $\tau_\chi^j(X_{i+1}) \to \tau_\chi^j(X_i)$ and $\tau_\chi^{j+1}(X_i) \to \tau_\chi^j(X_{i+1})$ with $j \geq 0$ and $i > n+1$ have infinite left degree.

(b) Let

$$Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_i \longrightarrow \cdots$$

be an infinite pre-sectional path in Γ_χ with all Y_i right stable. If there is some integer $n \geq 0$ such that the almost split sequence $\mathcal{E}_\chi'(Y_n)$ has three right stable middle terms or the arrow $Y_n \to Y_{n+1}$ has non-trivial valuation, then all arrows $\tau_\chi^j(Y_i) \to \tau_\chi^j(Y_{i+1})$ and $\tau_\chi^j(Y_{i+1}) \to \tau_\chi^{j-1}(Y_i)$ with $j \leq 0$ and $i > n+1$ have infinite right degree.

Proof:

As all X_i are left stable, it is sufficient to prove that all arrows $X_{i+1} \to X_i$ and $\tau(X_i) \to X_{i+1}$ with $i > n+1$ have infinite left degree. Since $\tau_\chi(X_i) \oplus X_{i+2}$ is a direct summand of the whole middle term of $\mathcal{E}_\chi(X_{i+1})$ by definition, all arrows $\tau_\chi(X_i) \to X_{i+1}$ with $i \geq 0$ have infinite left degree by Corollary 3.1.7.

Suppose that $\mathcal{E}_\chi(X_n)$ has three left stable middle terms X_{n+1}, Y_n and Z_n and there is an irreducible morphism $f : X_{i+1} \to X_i$ with finite left degree in χ. We can find a pre-sectional path

$$\cdots \longrightarrow \tau_\chi^{i-n}(X_n) \longrightarrow \tau_\chi^{i-n-1}(X_{n+1}) \longrightarrow \cdots \longrightarrow \tau_\chi(X_{i-1}) \longrightarrow X_i$$

in Γ_χ such that $X_{i+1} \oplus \tau_\chi(X_{i-1})$ is a direct summand of the whole middle term of $\mathcal{E}_\chi(X_i)$. By Lemma 3.1.6 there exists an irreducible morphism $f' : \tau_\chi^{i-n}(X_{n+1}) \to \tau_\chi^{i-n}(X_n)$ such that $d_\chi^l(f') < d_\chi^l(f)$. Since X_{n+1}, Y_n and Z_n are left stable, the whole middle term of $\mathcal{E}_\chi(\tau_\chi^{i-n}(X_n))$ has a direct summand $\tau_\chi^{i-n}(X_{n+1}) \oplus \tau_\chi^{i-n}(Y_n) \oplus \tau_\chi^{i-n}(Z_n)$. There is an irreducible morphism $h : \tau_\chi^{i-n+1}(X_n) \to \tau_\chi^{i-n}(Y_n) \oplus \tau_\chi^{i-n}(Z_n)$ with $d_\chi^l(h) < d_\chi^l(f')$ by Corollary 3.1.3 and hence another irreducible morphism $h' : \tau_\chi^{i-n+1}(Y_n) \oplus \tau_\chi^{i-n+1}(Z_n) \to \tau_\chi^{i-n+1}(X_n)$ such that $d_\chi^l(h') < d_\chi^l(h)$ by Lemma 3.1.9. On the other hand, there is a pre-sectional path

$$\cdots \longrightarrow \tau_\chi^{i-n+1}(X_{n+k}) \longrightarrow \tau_\chi^{i-n+1}(X_{n+k-1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{i-n+1}(X_{n+1}) \longrightarrow \tau_\chi^{i-n+1}(X_n)$$

such that $\tau_\chi^{i-n+1}(X_{n+1}) \oplus \tau_\chi^{i-n+1}(Y_n) \oplus \tau_\chi^{i-n+1}(Z_n)$ is a direct summand of the whole middle term of $\mathcal{E}_\chi(\tau_\chi^{i-n+1}(X_n))$. Therefore, by Corollary 3.1.7, h' has infinite left degree, which is a contradiction.

Suppose now there is an arrow $X_{n+1} \to X_n$ valued (a, a) with $a > 1$. Then for each integer $i > n$ there is a pre-sectional path

$$\cdots \longrightarrow \tau_\chi^{i-n}(X_{n+k}) \longrightarrow \cdots \longrightarrow \tau_\chi^{i-n}(X_{n+1}) \longrightarrow \tau_\chi^{i-n}(X_n) \longrightarrow$$

$$\longrightarrow \tau_\chi^{i-n-1}(X_{n+1}) \longrightarrow \cdots \longrightarrow \tau_\chi(X_{i-1}) \longrightarrow X_i$$

in Γ_χ such that $\tau_\chi(X_{i-1}) \oplus X_{i+1}$ is a summand of the whole middle term of $\mathcal{E}_\chi(X_i)$. Thus by Corollary 3.1.7 the arrow $X_{i+1} \to X_i$ has infinite left degree. $\qquad\square$

Lemma 3.1.16 *Let*

$$\cdots \longrightarrow X_{-n} \longrightarrow \cdots \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$$

be a be-infinite pre-sectional path in Γ_χ.

(a) *If all X_i are left stable, then all arrows $\tau_\chi^j(X_i) \to \tau_\chi^j(X_{i+1})$ and $\tau_\chi^{j+1}(X_{i+1}) \to \tau_\chi^j(X_i)$ with $j \geq 0$ and $i \in \mathbb{Z}$ have infinite left degree.*

(b) *Dually, if all X_i are right stable, then all arrows $\tau_\chi^j(X_i) \to \tau_\chi^j(X_{i+1})$ and $\tau_\chi^{j+1}(X_{i+1}) \to \tau_\chi^j(X_i)$ with $j \leq 0$ and $i \in \mathbb{Z}$ have infinite right degree.*

Proof:
Assume that all X_i are left stable. There are infinite pre-sectional paths

$$\cdots \longrightarrow \tau_\chi^j(X_{i-2}) \longrightarrow \tau_\chi^j(X_{i-1}) \longrightarrow \tau_\chi^j(X_i) \longrightarrow \tau_\chi^j(X_{i+1})$$

and

$$\cdots \longrightarrow \tau_\chi^{j+2}(X_{i+3}) \longrightarrow \tau_\chi^{j+1}(X_{i+2}) \longrightarrow \tau_\chi^{j}(X_{i+1})$$

in Γ_χ for each $j \geq 0, i \in \mathbb{Z}$ such that $\tau_\chi^{j+1}(X_{i+1}) \oplus \tau_\chi^{j}(X_{i-1})$ and $\tau_\chi^{j+1}(X_{i+2}) \oplus \tau_\chi^{j}(X_i)$ are direct summands of the whole middle terms of $\mathcal{E}_\chi(\tau_\chi^{j}(X_i))$ and $\mathcal{E}_\chi(\tau_\chi^{j}(X_{i+1}))$, respectively. Then the statement follows immediately from Corollary 3.1.7. $\qquad\square$

Lemma 3.1.17 *Let*

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$$

be a pre-sectional path in Γ_χ such that X_i is not Ext-projective for $i = 1, \ldots, n$ or not Ext-injective for $i = 0, \ldots, n-1$. Then there are χ-irreducible morphisms $f_i : X_{i-1} \to X_i$ such that $f_n \cdots f_1 \notin \operatorname{rad}_\chi^{n+1}(X_0, X_n)$. In particular, $f_n \cdots f_1$ is non-zero.

Proof:

Suppose none of the X_i is Ext-projective for $i = 1, \ldots, n$. For convenience, we define $\tau_\chi(X_{n+1}) = 0$, so that $X_{i-1} \oplus \tau_\chi(X_{i+1})$ is always a direct summand of the domain of a minimal right almost split morphism mapping to X_i. We use induction to prove the following statement from which we conclude the lemma.

For each $i \in \{1, \ldots, n\}$ there is an irreducible morphism $(f_i, g_i) : X_{i-1} \oplus \tau_\chi(X_{i+1}) \to X_i$ such that $f_i \cdots f_1 + g_i p_{i-1} \notin \operatorname{rad}_\chi^{i+1}(X_0, X_i)$ for every morphism $p_{i-1} : X_0 \to \tau_\chi(X_{i+1})$ in χ.

For $i = 1$ we choose $(f_1, g_1) : X_0 \oplus \tau_\chi(X_2) \to X_1$ to be the restriction of a minimal right almost split morphism mapping to X_1. Let $p_0 : X_0 \to \tau_\chi(X_2)$ be an arbitrary morphism. If $X_0 \not\cong \tau_\chi(X_2)$, it is clear that $f_1 + g_1 p_0 \notin \operatorname{rad}_\chi^2(X_0, X_1)$. If $X_0 \cong \tau_\chi(X_2)$, then f_1 and g_1 are linearly independent in $\operatorname{Irr}_\chi(X_0, X_1)$ as a $T_{X_0}^{op}$-module by Theorem 2.2.6, hence $f_1 + g_1 p_0 \notin \operatorname{rad}_\chi^2(X_0, X_1)$.

Suppose now $1 < i < n$ and we have found a χ-irreducible morphism (f_i, g_i) as required. We may choose $p_i = 0$ and obtain $f_i \cdots f_1 \notin \operatorname{rad}_\chi^{i+1}(X_0, X_i)$. Since $X_i \oplus \tau_\chi(X_{i+2})$ is a direct summand of the domain of a minimal right almost split morphism mapping to X_{i+1}, we know there are χ-irreducible morphisms $(g_i, h_i)^T : \tau_\chi(X_{i+1}) \to X_i \oplus \tau_\chi(X_{i+2})$ and $(f_{i+1}, g_{i+1}) : X_i \oplus \tau_\chi(X_{i+2}) \to X_{i+1}$ such that $\{(g_i, h_i)^T, (f_{i+1}, g_{i+1})\}$ is a component of the almost split sequence $\mathcal{E}_\chi(X_{i+1})$. Suppose there is a $p_i : X_0 \to \tau_\chi(X_{i+2})$ such that $f_{i+1} \cdots f_1 + g_{i+1} p_i \in \operatorname{rad}_\chi^{i+2}(X_0, X_{i+1})$. Then by Lemma 3.1.2 there is a $p_{i-1} : X_0 \to \tau_\chi(X_{i+1})$ such that $(f_i \ldots f_1, p_i) + (g_i, h_i) p_{i-1} \in \operatorname{rad}_\chi^{i+1}(X_0, X_i \oplus \tau_\chi(X_{i+2}))$. In particular, $f_i \ldots f_1 + g_i p_{i-1} \in \operatorname{rad}_\chi^{i+1}(X_0, X_i)$, which contradicts our inductive assumption. The case where none of the X_i is Ext-injective for $i = 0, \ldots, n-1$ can be proved dually. $\qquad\square$

Corollary 3.1.18 *Let*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n$$

be a pre-sectional path in Γ_χ such that X_i is not Ext-projective for $i = 1, \ldots, n$ or not Ext-injective for $i = 0, \ldots, n-1$. Then for each given constant b there are less than 2^b integers $i \in \{0, \ldots, n\}$ such that $l(X_i) \leq b$.

Proof:

Suppose that each X_i is not Ext-projective for $i = 1, \ldots, n$. Without loss of generality, we assume that $f_n \cdots f_1 \neq 0$ by Lemma 3.1.17. Suppose the statement is not true, i.e. there is a positive integer b such that there are at least 2^b modules $X_{i_1}, X_{i_2}, \ldots, X_{i_{2^b}}$ with $i_j < i_{j+1}$ for $j = 1, \ldots, 2^b - 1$ and $l(X_{i_j}) \leq b$ for $j = 1, \ldots, 2^b$. We set $g_j = f_{i_{j+1}} \cdots f_{i_j+1} : X_{i_j} \to X_{i_{j+1}}$ for $j = 1, \ldots, 2^b - 1$, so the composition $g_{2^b-1} \cdots g_1 : X_{i_1} \to X_{i_{2^b}}$ is zero by Lemma 2.2.8. By construction this is a contradiction to $f_n \cdots f_1 \neq 0$. Again, the other case is proved dually. $\qquad\square$

Corollary 3.1.19 *Every pre-sectional cycle in Γ_χ contains at least one Ext-projective and one Ext-injective module.*

Proof:

A pre-sectional cycle in an Auslander-Reiten quiver is a pre-sectional path

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = X_0$$

from some indecomposable X_0 in χ to itself such that the composition of the path with itself is again pre-sectional. Let $f = f_n \cdots f_1$, then by the lemma of Harada and Sai there is an integer k such that $f^k = 0$. Suppose none of the X_i is Ext-projective, then $f^k \neq 0$ by Lemma 3.1.17, which is a contradiction. If we assume that none of the X_i is Ext-injective, we can apply the same arguments to

$$\tau_\chi^{-1}(X_0) \xrightarrow{f_1} \tau_\chi^{-1}(X_1) \xrightarrow{f_2} \cdots \longrightarrow \tau_\chi^{-1}(X_{n-1}) \xrightarrow{f_n} \tau_\chi^{-1}(X_n) = \tau_\chi^{-1}(X_0).$$

$\qquad\square$

3.2 The Happel-Preiser-Ringel theorem for subcategories

Recall that an **oriented cycle** is a path in an Auslander-Reiten quiver that starts and ends in the same module.

Lemma 3.2.1 *Every oriented cycle in Γ_χ contains both an arrow of finite left degree and an arrow of finite right degree.*

Proof:
Let

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = X_0$$

be an oriented cycle in Γ_χ. By Lemma 2.2.8 there is a $k \geq 1$ such that $(f_n \cdots f_1)^k = 0$, which completes the proof. $\qquad\square$

Theorem 3.2.2 *Let Γ be a connected component of Γ_χ that is left stable or right stable and assume every almost split sequence in Γ has at least 2 middle terms. Then for every path*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

in Γ we have $f_n \cdots f_1 \notin \mathrm{rad}_\chi^{n+1}(X_0, X_n)$. In particular, $f_n \cdots f_1$ is non-zero.

Proof:

Suppose there are no Ext-projective modules in Γ. We prove by induction that for each $n \in \mathbb{N}$ the left degree of each irreducible morphism $f : X \to Y$ between indecomposable modules X and Y in Γ is greater then n. Clearly, f cannot be a surjective right almost split morphism because $\mathcal{E}_\chi(Y)$ has at least two middle terms. Hence $d_\chi^l(f) > 1$ by 3.1.3.

We assume that the statement is true for $n - 1$, but false for n, i.e. there exists an irreducible morphism $f : X \to Y$ such that $d_\chi^l(f) = n$. Since every almost split sequence has at least two middle terms, there is a module X' such that $X \oplus X'$ is a direct summand of the whole middle term of $\mathcal{E}_\chi(Y)$. So by Corollary 3.1.3 there is an irreducible morphism $f' : \tau_\chi(Y) \to X'$ with $d_\chi^l(f') < n$, which is a contradiction.

We have proved that every arrow in Γ has infinite left degree and hence every path

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

in Γ of length n is not in $\mathrm{rad}_\chi^{n+1}(X_0, X_n)$. If Γ does not contain any Ext-injective modules, the theorem can be proved dually. $\qquad\square$

Lemma 3.2.3 *Let Γ be a connected stable subquiver of Γ_χ and let X and Y be modules in Γ. Then the following holds.*

(a) There is a path in Γ from X to $\tau_\chi^n(Y)$ for some $n \in \mathbb{Z}$.

(b) If there is a path from X to Y in Γ, then either $X = \tau_\chi^n(Y)$ for some $n \geq 1$ or there is a sectional path in Γ from X to $\tau_\chi^n(Y)$ for some $n \geq 0$.

Proof:

Since Γ is connected, there is a walk

$$X = X_0 \text{——} X_1 \text{——} \cdots \text{——} X_{s-1} \text{——} X_s = Y$$

in Γ. We prove the first statement by induction on s. For $s = 1$ the statement is trivial. Suppose now $s > 0$ and there exists a path in Γ from X to $\tau_\chi^n(X_{s-1})$ for some integer n. Since there is an edge between X_{s-1} and Y, there is either an arrow $X_{s-1} \to Y$ or $Y \to X_{s-1}$. In the first case there is also an arrow from $\tau_\chi^n(X_{s-1}) \to \tau_\chi^n(Y)$ as Γ is stable and we obtain a path from X to $\tau_\chi^n(Y)$ in Γ. Otherwise there is an arrow $\tau_\chi^n(X_{s-1}) \to \tau_\chi^{n-1}(Y)$ and a path from X to $\tau_\chi^{n-1}(Y)$ again using that Γ is stable.

In order to prove the second statement, let

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{s-1} \longrightarrow X_s = Y$$

be a path in Γ. Again the statement is trivial for $s = 1$. Assume $s > 0$, then by the inductive hypothesis we either have $X = \tau_\chi^n(X_{s-1})$ for some $n \geq 1$ or there is a sectional path

$$X = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{k-1} \longrightarrow Y_k = \tau_\chi^m(X_{s-1})$$

for some $m \geq 0$. In the first case $X \to \tau_\chi^n(Y)$ is a sectional path. In the second case if $Y_{k-1} = \tau_\chi^{m+1}(Y)$, then either $X = \tau_\chi^{m+1}(Y)$ with $m + 1 > 0$ or

$$X = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{k-1} = \tau_\chi^{m+1}(Y)$$

is a sectional path. If $Y_{k-1} \neq \tau_\chi^{m+1}(Y)$, then

$$X = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{k-1} \longrightarrow Y_k \longrightarrow \tau_\chi^m(Y)$$

is a sectional path. $\qquad \square$

In order to proof the main theorem of this section, we need another definition and lemma.

Definition 3.2.4 *Let X and Y be indecomposable modules in χ. Then the set $\{\tau_\chi^n(X) | n \in \mathbb{Z}\}$ is called the $\boldsymbol{\tau_\chi}$-**orbit** of X. We call a τ_χ-orbit left stable, right stable, stable or periodic if all the modules it contains are left stable, right stable, stable or periodic respectively. A τ_χ-orbit that is neither left stable nor right stable is a finite set and hence called a finite τ_χ-orbit. Moreover, we say two orbits $\{\tau_\chi^n(X) | n \in \mathbb{Z}\}$ and $\{\tau_\chi^n(Y) | n \in \mathbb{Z}\}$ are **adjacent** if there are integers $i, j \in \mathbb{Z}$ such that there is an irreducible morphism $f : \tau_\chi^i(X) \to \tau_\chi^j(Y)$ or $g : \tau_\chi^j(Y) \to \tau_\chi^i(X)$.*

It follows directly from the definitions that if one module X is left stable, right stable, stable or periodic, then all modules in the τ_χ-orbit of X are left stable, right stable, stable or periodic respectively. Note that although periodic τ_χ-orbits only contain a finite number of non-isomorphic indecomposable modules, we do not call them finite τ_χ-orbits.

Lemma 3.2.5 *Let Γ_χ be the Auslander-Reiten quiver of χ with a module X in a periodic τ_χ-orbit, i.e. there is an $n \in \mathbb{N}$ such that $\tau_\chi^n(X) = X$. Then any τ_χ-orbit adjacent to the τ_χ-orbit of X is either periodic or finite.*

Proof:
Suppose there is an τ_χ-orbit $\{\tau_\chi^n(Y) | n \in \mathbb{Z}\}$ that is adjacent to the τ_χ-orbit of X which is neither periodic nor finite. Without loss of generality, we assume that this τ_χ-orbit contains a module Y such that there is an arrow from X to Y in the Auslander-Reiten quiver of χ and that $\tau_\chi^m(Y)$ exists for all $m \geq 0$. By our assumption we know that $\tau_\chi^m(Y) \neq Y$ for all $m \geq 1$. Since both orbits are left stable, we conclude that there is an arrow from $\tau_\chi^n(X) = X$ to $\tau_\chi^n(Y)$. Proceeding like this it is easy to see that there are arrows from X to $Y, \tau_\chi^n(Y), \tau_\chi^{2n}(Y), \ldots$, i.e. the almost split sequence $\mathcal{E}'_\chi(X)$ has infinitely many middle terms, which is impossible. $\qquad \square$

Definition 3.2.6 *A stable subquiver Γ of Γ_χ is called a **stable tube** if every module in Γ is τ_χ-periodic and it contains an infinite sectional path*

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = Y$$

such that each orbit in Γ is generated by a unique module X_i.

Theorem 3.2.7 (Happel-Preiser-Ringel Theorem) *Let Γ be an infinite connected stable subquiver of Γ_χ. If there is a τ_χ-periodic module in Γ, then Γ is a stable tube.*

Proof:

Since Γ is stable and connected, if one module in Γ is τ_χ-periodic, then every module in Γ is τ_χ-periodic by Lemma 3.2.5. Thus there must be infinitely many τ_χ-orbits in χ. Fix a module X_0 in Γ, then for every module X that is not in the same τ_χ-orbit as X_0 there is a sectional path in Γ from $\tau_\chi^n(X)$ to X_0 for some $n \in \mathbb{Z}$ by Lemma 3.2.3. Consequently, there are arbitrarily long sectional paths in Γ ending in X_0 and hence there exists an infinite sectional path

$$\gamma: \qquad \cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

in Γ. For each integer $i \geq 0$, there is an oriented cycle

$$X_{i+1} = \tau_\chi^{n_i}(X_{i+1}) \longrightarrow \tau_\chi^{n_i}(X_i) \longrightarrow \tau_\chi^{n_i-1}(X_{i+1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi(X_i) \longrightarrow X_{i+1}$$

in Γ. By Lemma 3.2.1 there is an integer $j_i \geq 0$ such that $\tau_\chi^{j_i}(X_{i+1}) \to \tau_\chi^{j_i}(X_i)$ or $\tau_\chi^{j_i+1}(X_i) \to \tau_\chi^{j_i}(X_{i+1})$ has finite left degree. As a consequence, every arrow in γ has trivial valuation and X_i has two immediate predecessors X_{i+1} and $\tau_\chi(X_{i-1})$ and two immediate successors X_{i-1} and $\tau_\chi^{-1}(X_{i+1})$ for all integers $i \geq 1$ by Lemma 3.1.15. Moreover, γ cannot be contained in a bi-infinite sectional path

$$\cdots \longrightarrow X_{-n} \longrightarrow \cdots \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$$

by Lemma 3.1.16. Without loss of generality, we can assume that γ is a maximal sectional path, i.e. X_0 has only one immediate predecessor X_1 and one immediate successor $\tau_\chi^{-1}(X_1)$. Therefore, γ contains at least one module of every τ_χ-orbit and all X_i have the same τ_χ-periodicity.

It remains to prove that the X_i belong to pairwise different τ_χ-orbits. So let X_m and X_j be in the same τ_χ-orbit with $m < j$ and $j - m$ minimal. Then X_{j-1} and X_{m-1} also belong to the same τ_χ-orbit as every module has at most two successors and we have chosen $j - m$ to be minimal. We continue inductively and obtain that X_0 and X_{j-m} are contained in the same τ_χ-orbit. Since these modules only have 1 predecessor and successor respectively, X_1 and X_{j-m-1} must belong to the same τ_χ-orbit as well, which is a contradiction to the minimality of $j - m$. $\qquad\square$

Lemma 3.2.8 *Let Γ be a connected stable subquiver of Γ_χ and let X be a module in Γ. If there is a sectional path in Γ from $X \to \tau_\chi^n(X)$ for some $n \geq 1$, then Γ is a finite subquiver of τ_χ-periodic modules.*

Proof:
Let

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{k-1} \longrightarrow X_k = \tau_\chi^n(X)$$

be a sectional path in Γ. Without loss of generality, we assume the path to be minimal, i.e. if $k \geq 3$ then $X_{k-1} \neq \tau_\chi^m(X_1)$ for all $m \geq 0$. We construct a bi-infinite path

$$\gamma: \qquad \cdots \longrightarrow \tau_\chi^{-2n}(X_{k-1}) \longrightarrow \tau_\chi^{-n}(X_0) \longrightarrow \tau_\chi^{-n}(X_1) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{-n}(X_{k-1}) \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots$$

$$\cdots \longrightarrow X_{k-1} \longrightarrow \tau_\chi^n(X_0) \longrightarrow \tau_\chi^n(X_1) \longrightarrow \cdots$$

in Γ. Note that there is also an oriented cycle in Γ as follows

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{k-1} \longrightarrow X_k \longrightarrow \tau_\chi^{-1}(X_{k-1}) \longrightarrow \tau_\chi^{-1}(X_k) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{-n}(X_{k-1}) \longrightarrow \tau_\chi^{-n}(X_k) = X_0$$

By Lemma 3.2.1 there are some $0 \leq i \leq k-1$ and $0 \leq j \leq n$ such that $\tau_\chi^{-j}(X_i) \rightarrow \tau_\chi^{-j}(X_{i+1})$ or $\tau_\chi^{-j}(X_{i+1}) \rightarrow \tau_\chi^{-j-1}(X_i)$ has finite left degree. Then by Lemma 3.1.7 γ cannot be sectional, which gives us $1 \leq k \leq 2$. If $k = 1$, then there is a non-sectional path

$$X_0 \longrightarrow \tau_\chi^n(X_0) \longrightarrow \tau_\chi^{2n}(X_0)$$

and, consequently, $X_0 = \tau_\chi^{2n+1}(X_0)$. Similarly, if $k = 2$, then the path

$$X_1 \longrightarrow \tau_\chi^n(X_0) \longrightarrow \tau_\chi^n(X_1)$$

is not sectional and we have $X_1 = \tau_\chi^{n+1}(X_1)$. In both cases there are periodic modules in Γ. If Γ is infinite, then it is a stable tube by Theorem 3.2.7 and there is no sectional path

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{k-1} \longrightarrow X_k = \tau_\chi^n(X).$$

Thus Γ must be a finite subquiver consisting of τ_χ-periodic modules. $\qquad \square$

Theorem 3.2.9 *Let Γ be a connected stable subquiver of Γ_χ. Then Γ contains an oriented cycle if and only if it consists of τ_χ-periodic modules.*

Proof:
Clearly, Γ contains an oriented cycle if it consists of τ_χ-periodic modules. On the other hand, suppose Γ contains a non-trivial oriented cycle from X to itself. Then, by Lemma 3.2.3, there is either an $n \geq 1$ such that $X = \tau_\chi^n X$ or there is a sectional path from X to $\tau_\chi^n(X)$ where $n \geq 0$. Hence Γ contains τ_χ-periodic modules by 3.2.8. $\qquad \square$

Corollary 3.2.10 *Let X be a module in Γ_χ such that there is an oriented cycle*

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X$$

such that all X_i are stable modules. Then all X_i are τ_χ-periodic modules.

Proof:
We obtain the result by applying Theorem 3.2.9 to the connected stable subquiver generated by the τ_χ-orbits of the X_i. $\qquad \square$

3.3   Global degrees of irreducible morphisms

Definition 3.3.1 *Let $X \to Y$ be an arrow in Γ_χ.*

(a) *Let X and Y be left stable modules. We then define the **global left degree** in χ of the arrow $X \to Y$ to be the minimum of left degrees in χ of all arrows $\tau_\chi^{n+1}(Y) \to \tau_\chi^n(X)$ and $\tau_\chi^n(X) \to \tau_\chi^n(Y)$ for all $n \geq 0$.*

(b) *Dually, if X and Y are right stable modules, we define the **global right degree** in χ of the arrow $X \to Y$ to be the minimum of right degrees in χ of all arrows $\tau_\chi^n(Y) \to \tau_\chi^{n-1}(X)$ and $\tau_\chi^n(X) \to \tau_\chi^n(Y)$ for all $n \leq 0$.*

These definitions allow us to rephrase Lemmas 3.1.15 and 3.1.16 as follows.

Lemma 3.3.2

(a) *Let*

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

be a sectional path in Γ_χ with all X_i left stable. If the path contains infinitely many arrows with finite global left degree in χ, then for each integer $i \geq 0$ the arrow $X_{i+1} \to X_i$ has trivial valuation and the module X_i has at most two left stable immediate predecessors in Γ_χ.

(b) *Let*

$$Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_i \longrightarrow Y_{i+1} \longrightarrow \cdots$$

be a sectional path in Γ_χ with all Y_i right stable. If the path contains infinitely many arrows with finite global right degree in χ, then for each integer $i \geq 0$ the arrow $Y_i \to Y_{i+1}$ has trivial valuation and the module Y_i has at most two right stable immediate successors in Γ_χ.

Lemma 3.3.3 *Let*

$$\cdots \longrightarrow X_{-n} \longrightarrow \cdots \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$$

be a bi-infinite sectional path in Γ_χ.

(a) *If all modules X_i are left stable in χ then all arrows $X_{i+1} \to X_i$ have infinite global left degree in χ.*

(b) *Dually, if all modules X_i are right stable in χ, then all arrows $X_{i+1} \to X_i$ have infinite global right degree in χ.*

The following lemma is an easy consequence of Corollary 3.1.7.

Lemma 3.3.4

(a) Let

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

and

$$\cdots \longrightarrow Y_{i+1} \longrightarrow Y_i \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0$$

be infinite sectional paths in Γ_χ containing only left stable modules. If $X_0 = Y_0$ and $X_1 \neq Y_1$, then both arrows $X_{i+1} \to X_i$ and $Y_{i+1} \to Y_i$ have infinite global left degree in χ for each $i \geq 0$ where $Z = X_0 = Y_0$.

(b) Let

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

and

$$Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_i \longrightarrow Y_{i+1} \longrightarrow \cdots$$

be infinite sectional paths in Γ_χ containing only right stable modules. If $X_0 = Y_0$ and $X_1 \neq Y_1$, then both arrows $X_i \to X_{i+1}$ and $Y_i \to Y_{i+1}$ have infinite global right degree in χ for each $i \geq 0$.

Proof:

Clearly, it is sufficient to prove the statement for an arrow $X_i \to X_{i-1}$. For any arrow $\tau_\chi^n(X_i) \to \tau_\chi^n(X_{i-1})$ we can construct an infinite sectional path

$$\cdots \longrightarrow \tau_\chi^{n+i}(Y_2) \longrightarrow \tau_\chi^{n+i}(Y_1) \longrightarrow \tau_\chi^{n+i}(Z) \longrightarrow \tau_\chi^{n+i-1}(X_1) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{n+2}(X_{i-2}) \longrightarrow \tau_\chi^{n+1}(X_{i-1}) \longrightarrow \tau_\chi^n(X_i).$$

It then follows from Corollary 3.1.7 that the arrow $\tau_\chi^n(X_i) \to \tau_\chi^n(X_{i-1})$ has infinite left degree in χ. Similarly, we use the sectional path

$$\cdots \longrightarrow \tau_\chi^n(X_{i+2}) \longrightarrow \tau_\chi^n(X_{i+1}) \longrightarrow \tau_\chi^n(X_i) \longrightarrow \tau_\chi^n(X_{i-1})$$

to verify that $\tau_\chi^n(X_{i-1}) \to \tau_\chi^{n-1}(X_i)$ also has infinite left degree in χ again by Corollary 3.1.7. Hence by definition the arrow $X_i \to X_{i-1}$ has infinite global left degree in χ. □

Lemma 3.3.5

(a) Let Γ be a connected left stable subquiver of Γ_χ and let X and Y be modules in Γ. If there is a non-trivial path from X to Y in Γ, then either $X = \tau_\chi^n(Y)$ for some $n \geq 1$ or there is a sectional path in Γ from X to $\tau_\chi^n(Y)$ for some $n \geq 0$.

(b) Dually, if Γ is a connected right stable subquiver of Γ_χ and there is a non-trivial path from X to Y in Γ, then either $X = \tau_\chi^n(Y)$ for some $n \geq 0$ or there is a sectional path in Γ from $\tau_\chi^n(X)$ to Y for some $n \leq 0$.

Proof:

We prove the Lemma by induction on s. Let

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{s-1} \longrightarrow X_s = Y$$

be a path in Γ. The statement is trivial for $s = 1$. Assume $s > 0$, then by the inductive hypothesis we either have $X = \tau_\chi^n(X_{s-1})$ for some $n \geq 1$ or there is a sectional path

$$X = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{k-1} \longrightarrow Y_k = \tau_\chi^m(X_{s-1})$$

for some $m \geq 0$. In the first case $X \to \tau_\chi^n(Y)$ is a sectional path. In the second case if $Y_{k-1} = \tau_\chi^{m+1}(Y)$, then either $X = \tau_\chi^{m+1}(Y)$ with $m + 1 > 0$ or

$$X = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{k-1} = \tau_\chi^{m+1}(Y)$$

is a sectional path. If $Y_{k-1} \neq \tau_\chi^{m+1}(Y)$, then

$$X = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{k-1} \longrightarrow Y_k \longrightarrow \tau_\chi^m(Y)$$

is a sectional path. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.3.6

(a) *Let Γ be a left stable component of Γ_χ containing no τ_χ-periodic module. If there is an oriented cycle in Γ, then there is an infinite sectional path*

$$\cdots \longrightarrow \tau_\chi^{2r}(X_2) \longrightarrow \tau_\chi^{2r}(X_1) \longrightarrow \tau_\chi^r(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^r(X_2) \longrightarrow \tau_\chi^r(X_1) \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1$$

in Γ containing infinitely many arrows of finite global left degree in χ, where $r > s$, the X_i belong to pairwise different τ_χ-orbits and at least one of the X_i is not stable.

(b) *Dually, let Γ be a right stable component of Γ_χ containing no τ_χ-periodic module. If there is an oriented cycle in Γ, then there is an infinite sectional path*

$$X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_s \longrightarrow \tau_\chi^{-r}(X_1) \longrightarrow \tau_\chi^{-r}(X_2) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{-r}(X_s) \longrightarrow \tau_\chi^{-2r}(X_1) \longrightarrow \tau_\chi^{-2r}(X_2) \longrightarrow \cdots$$

in Γ containing infinitely many arrows of finite global right degree in χ, where $r > s$, the X_i belong to pairwise different τ_χ-orbits and at least one of the X_i is not stable.

Proof:

Suppose that Γ contains an oriented cycle from X to itself. Then there is a sectional path from X to $\tau_\chi^r(X)$ for some $r \geq 0$ by Lemma 3.3.5. Let γ denote a sectional path

$$Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_{s-1} \longrightarrow Y_s \longrightarrow Y_{s+1}$$

in Γ of minimal positive length such that $Y_{s+1} = \tau_\chi^n(Y_1)$ for some $n \geq 0$. We prove that $n > 0$. Suppose that $n = 0$, i.e. we have $Y_1 = Y_{s+1}$. By Corollary 2.4.8 we know that if the composition

$$Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_{s-1} \longrightarrow Y_s \longrightarrow Y_1 \longrightarrow \cdots$$

$$\cdots \longrightarrow Y_{s-1} \longrightarrow Y_s \longrightarrow Y_1 \longrightarrow \cdots$$

was sectional, it would contain both an Ext-projective and an Ext-injective module. But by our assumptions Γ is left stable and hence does not contain Ext-projective modules. It follows that the composition of γ with itself is not sectional and, therefore, $\tau_\chi(Y_2) = Y_s$. For $s = 1$ this composition is

$$Y_1 \longrightarrow Y_1 \longrightarrow Y_1 \longrightarrow \cdots$$

while for $s = 2$ we trivially obtain $\tau_\chi(Y_2) = Y_2$. So in both cases we have $\tau_\chi(Y_s) = Y_s$, which contradicts our assumption that there is no τ_χ-periodic module in Γ. It necessarily follows that $s \geq 3$, but then the path

$$Y_2 \longrightarrow Y_3 \longrightarrow \cdots \longrightarrow Y_s \longrightarrow$$

is of positive length and $\tau_\chi(Y_2) = Y_s$, which contradicts γ to be of minimal length with this property. We conclude that n is strictly greater than 0.

The next step is to show that all modules $Y_1, \ldots, Y_s$ belong to pairwise different τ_χ-orbits. Since γ is of minimal length, if $1 \leq i < j \leq s$, then $Y_j \neq \tau_\chi^p(Y_i)$ for any $p \geq 0$. Hence if Y_i an Y_j belong to the same τ_χ-orbit, then $Y_i = \tau_\chi^q(Y_j)$ for some $q > 0$. By the minimality of γ we know that $Y_s \neq \tau_\chi^{n+1}(Y_2)$ and thus the path

$$Y_j \longrightarrow Y_{j+1} \longrightarrow \cdots \longrightarrow Y_s \longrightarrow \tau_\chi^n(Y_1) \longrightarrow \tau_\chi^n(Y_2) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^n(Y_{i-1}) \longrightarrow \tau_\chi^n(Y_i) = \tau_\chi^{n+q}(Y_j)$$

must be sectional. However, it is of length $s - j + i < s$, which contradicts γ to be of minimal length. Consequently, each Y_i belongs to a different τ_χ-orbit.

Now let $r = s + n$ and $X_i = \tau_\chi^{i-1}(Y_i)$ for $i = 1, \ldots, s$. We then have $r > s$, the X_i belong to pairwise different τ_χ-orbits and

$$\cdots \longrightarrow \tau_\chi^{2r}(X_2) \longrightarrow \tau_\chi^{2r}(X_1) \longrightarrow \tau_\chi^r(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^r(X_2) \longrightarrow \tau_\chi^r(X_1) \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1$$

is an infinite sectional path in Γ. For every $j \geq 0$ denote the sectional path

$$\tau_\chi^{(j+1)r}(X_1) \longrightarrow \tau_\chi^{jr}(X_s) \longrightarrow \cdots \longrightarrow \tau_\chi^{jr}(X_2) \longrightarrow \tau_\chi^{jr}(X_1)$$

by γ_j. Moreover, for each j there are paths

$$\tau_\chi^{jr+s}(X_1) \longrightarrow \tau_\chi^{jr+s-1}(X_2) \longrightarrow \cdots \longrightarrow \tau_\chi^{jr+1}(X_s) \longrightarrow \tau_\chi^{(j+1)r}(X_1)$$

and, since $r > s$,

$$\tau_\chi^{(j+1)r}(X_1) \longrightarrow \tau_\chi^{(j+1)r-1}(X_2) \longrightarrow \tau_\chi^{(j+1)r-1}(X_1) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{jr+s+1}(X_1) \longrightarrow \tau_\chi^{jr+s}(X_2) \longrightarrow \tau_\chi^{jr+s}(X_1).$$

The last two paths form an oriented cycle through $\tau_\chi^{(j+1)r}(X_1)$ that does not contain an Ext-projective module. By Lemma 3.2.1 an oriented cycle in χ always contains an arrow of finite left degree in χ, hence γ_j always contains an arrow of finite global left degree in χ, which can easily be seen in the picture on the next page.

Since the sectional path

$$\cdots \longrightarrow \tau_\chi^{2r}(X_2) \longrightarrow \tau_\chi^{2r}(X_1) \longrightarrow \tau_\chi^r(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^r(X_2) \longrightarrow \tau_\chi^r(X_1) \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1$$

is nothing but the composition $\gamma_0 \gamma_1 \cdots \gamma_{j-1} \gamma_j \gamma_{j+1} \cdots$, it contains infinitely many arrows of finite global left degree in χ.

In the last step we show that at least one of the X_i is not stable. Suppose all X_i are stable, then there is an infinite sectional path

$$\cdots \longrightarrow \tau_\chi^r(X_2) \longrightarrow \tau_\chi^r(X_1) \longrightarrow X_s \longrightarrow \cdots$$

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow \tau_\chi^{-r}(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{-r}(X_2) \longrightarrow \tau_\chi^{-r}(X_1) \longrightarrow \tau_\chi^{-2r}(X_s) \longrightarrow \cdots$$

which only contains arrows of infinite global left degree in χ by Lemma 3.3.3. This clearly contradicts that

$$\cdots \longrightarrow \tau_\chi^r(X_2) \longrightarrow \tau_\chi^r(X_1) \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1$$

contains arrows of finite global left degree in χ, which we have proved previously in this lemma. $\qquad\square$

$$\tau_\chi^{(j+1)r}(X_1) \longrightarrow \tau_\chi^{jr}(X_s) \cdots\cdots \tau_\chi^{jr}(X_2) \longrightarrow \tau_\chi^{jr}(X_1)$$

$$\tau_\chi^{jr+1}(X_s)$$

$$\tau_\chi^{jr+1}(X_2)$$

$$\tau_\chi^{jr+s-1}(X_2) \qquad \tau_\chi^{jr+s}(X_1)$$

$$\tau_\chi^{(j+1)r-1}(X_2) \qquad \tau_\chi^{(j+1)r}(X_1)$$

Theorem 3.3.7 *Let Γ be a left or right stable component of Γ_χ that contains an oriented cycle but does not contain a τ_χ-periodic module. Then*

(a) Γ is not a stable component of Γ_χ,

(b) Γ contains only finitely many τ_χ-orbits,

(c) every arrow in Γ has trivial valuation and

(d) each module in Γ has at most two immediate predecessors and at most two immediate successors in Γ.

Proof:
As usual, we prove the theorem only in the case where Γ is left stable. Since there is an oriented cycle in Γ and no τ_χ-periodic module, there is an infinite sectional path

$$\cdots \longrightarrow \tau_\chi^{2r}(X_2) \longrightarrow \tau_\chi^{2r}(X_1) \longrightarrow \tau_\chi^{r}(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{r}(X_2) \longrightarrow \tau_\chi^{r}(X_1) \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1$$

in Γ by Lemma 3.3.6, which we denote by γ. Furthermore, by the same lemma, γ contains infinitely many arrows of finite global left degree in χ, all X_i belong to pairwise different τ_χ-orbits and at least one of the X_i is not stable. Thus Γ is not a stable component. In addition, each arrow in γ has trivial valuation and each module in γ has at most two immediate predecessors in Γ by Lemma 3.3.2.

We prove that in fact each τ_χ-orbit in Γ is generated by some X_i in γ. Since Γ is left stable, it is sufficient to show that each immediate predecessor of a module in a τ_χ-orbit generated by some X_i is again in a τ_χ-orbit of some X_j. First note that each module in γ clearly belongs to the τ_χ-orbit of some module X_i and has at most two immediate predecessors in Γ. Hence every module in γ other than X_1 has precisely two immediate predecessors in Γ, each of which belongs to the τ_χ-orbit of one of the X_i. Let Y be an immediate predecessor of X_1, then either $\tau_\chi^r(Y) = \tau_\chi^r(X_2)$ or $\tau_\chi^r(Y) = \tau_\chi(X_s)$ as $\tau_\chi^r(X_1)$ has two immediate predecessors. In both cases $\tau_\chi^r(Y)$ and Y belong to the τ_χ-orbit of one of the X_i.

Now assume that Z is an immediate predecessor of $\tau_\chi^p(X_j)$ in Γ for some $p \geq 0$ and $1 \leq j \leq s$. Then $\tau_\chi^{pr-p}(Z)$ is an immediate predecessor of $\tau_\chi^{pr}(X_j)$ in Γ. Since $\tau_\chi^{pr}(X_j)$ is a module in γ, the modules $\tau_\chi^{pr-p}(Z)$ and Z belong either to the τ_χ-orbit of X_{j-1} or the orbit of X_{j+1}, where $X_0 = X_s$ and $X_1 = X_{s+1}$. Consequently, every orbit in Γ is generated by some X_i for $1 \leq i \leq s$.

So if Z now denotes an arbitrary module in Γ, then we have $Z = \tau_\chi^m(X_i)$ for some $m \in \mathbb{Z}$ and $1 \leq i \leq s$. It follows that there exists some $p > 0$ such that $\tau_\chi^p(Z)$ is a module in γ. Therefore, Z has at most two immediate predecessors in Γ since Γ is left stable. Furthermore, all arrows ending in Z have trivial valuation and so have all arrows in the whole component Γ, which completes the proof. $\qquad\square$

Corollary 3.3.8

(a) *For each module Z in a left stable component Γ of Γ_χ that contains an oriented cycle but no τ_χ-periodic module there is at most one infinite sectional path*

$$\cdots \longrightarrow Z_j \longrightarrow \cdots \longrightarrow Z_1 \longrightarrow Z_0 = Z$$

ending in Z.

(b) *Dually, if Γ is a right stable component of Γ_χ containing an oriented cycle but no τ_χ-periodic module, then for every module Z in Γ there is at most one infinite sectional path*

$$Z = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_j \longrightarrow \cdots$$

starting in Z.

Proof:

We know by Lemma 3.3.6 that Γ contains an infinite sectional path

$$\cdots \longrightarrow \tau_\chi^{2r}(X_2) \longrightarrow \tau_\chi^{2r}(X_1) \longrightarrow \tau_\chi^r(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^r(X_2) \longrightarrow \tau_\chi^r(X_1) \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1,$$

and every module Z in Γ equals $\tau_\chi^m(X_j)$ for some $m \in \mathbb{Z}$ and $1 \leq j \leq s$ by Theorem 3.3.7. Moreover, we can find a non-negative integers n and p such that $\tau_\chi^n(Z) = \tau_\chi^{pr}(X_j)$. So if we have an infinite sectional path

$$\cdots \longrightarrow Z_j \longrightarrow \cdots \longrightarrow Z_1 \longrightarrow Z_0 = Z$$

ending in Z, we also obtain an infinite sectional path

$$\cdots \longrightarrow \tau_\chi^n(Z_j) \longrightarrow \cdots \longrightarrow \tau_\chi^n(Z_1) \longrightarrow \tau_\chi^n(Z_0)$$

ending in $\tau_\chi^n(Z) = \tau_\chi^{pr}(X_j)$. We show that this path equals

$$\cdots \longrightarrow \tau_\chi^{p(r+1)}(X_2) \longrightarrow \tau_\chi^{p(r+1)}(X_1) \longrightarrow \tau_\chi^{pr}(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{pr}(X_{j+1}) \longrightarrow \tau_\chi^{pr}(X_j).$$

Suppose they are not equal and let k be the lowest positive integer such that $\tau_\chi^n(Z_k) = \tau_\chi^{qr}(X_i)$ but $\tau_\chi^n(Z_{k+1}) \neq \tau_\chi^{qr}(X_{i+1})$ for some $q \geq p$ and $1 \leq i \leq s$, where we define $X_{s+1} = \tau_\chi^r(X_1)$. Then there are two infinite sectional paths ending in $\tau_\chi^n(Z_k)$ and by Lemma 3.3.4 all arrows in

$$\cdots \longrightarrow \tau_\chi^{q(r+1)}(X_2) \longrightarrow \tau_\chi^{q(r+1)}(X_1) \longrightarrow \tau_\chi^{qr}(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{qr}(X_{i+1}) \longrightarrow \tau_\chi^{qr}(X_i).$$

have infinite global left degree in χ. This contradicts the previous result from Lemma 3.3.6 that

$$\cdots \longrightarrow \tau_\chi^{2r}(X_2) \longrightarrow \tau_\chi^{2r}(X_1) \longrightarrow \tau_\chi^{r}(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{r}(X_2) \longrightarrow \tau_\chi^{r}(X_1) \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1$$

contains infinitely many arrows of finite global left degree in χ. Hence

$$\cdots \longrightarrow \tau_\chi^{n}(Z_j) \longrightarrow \cdots \longrightarrow \tau_\chi^{n}(Z_1) \longrightarrow \tau_\chi^{n}(Z_0)$$

must equal

$$\cdots \longrightarrow \tau_\chi^{p(r+1)}(X_2) \longrightarrow \tau_\chi^{p(r+1)}(X_1) \longrightarrow \tau_\chi^{pr}(X_s) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\chi^{pr}(X_{j+1}) \longrightarrow \tau_\chi^{pr}(X_j)$$

and

$$\cdots \longrightarrow Z_j \longrightarrow \cdots \longrightarrow Z_1 \longrightarrow Z_0 = Z$$

is the only infinite sectional path ending in Z. □

Corollary 3.3.9 *Let Γ be a left or right stable component that contains an oriented cycle but no τ_χ-periodic module. Then Γ does not contain a bi-infinite sectional path*

$$\cdots \longrightarrow Z_j \longrightarrow \cdots \longrightarrow Z_1 \longrightarrow Z_0 \longrightarrow Z_{-1} \longrightarrow \cdots \longrightarrow Z_{-j} \longrightarrow \cdots$$

Proof:

We prove the statement in the case that Γ is left stable, the other follows by duality. Let us assume there is an infinite sectional path

$$\cdots \longrightarrow Z_j \longrightarrow \cdots \longrightarrow Z_1 \longrightarrow Z_0 \longrightarrow Z_{-1} \longrightarrow \cdots \longrightarrow Z_{-j} \longrightarrow \cdots$$

then we have two different infinite sectional paths

$$\cdots \longrightarrow Z_j \longrightarrow \cdots \longrightarrow Z_1 \longrightarrow Z_0$$

and

$$\cdots \longrightarrow \tau_\chi^{j}(Z_{-j}) \longrightarrow \cdots \longrightarrow \tau_\chi^{2}(Z_{-2}) \longrightarrow \tau_\chi(Z_{-1}) \longrightarrow Z_0$$

ending in Z_0, which clearly contradicts Corollary 3.3.8. □

Chapter 4

Shapes of Auslander-Reiten quivers of functorially finite resolving subcategories

In this chapter we find criteria for Auslander-Reiten quivers of resolving subcategories to be finite or infinite. Therefore, we introduce two different concepts of assigning a graph to a connected component of an Auslander-Reiten quiver, that coincide in some cases. However, some crucial examples show why both ideas are necessary.

4.1   Left stable components of Auslander-Reiten quivers

Let Γ_Ω be the Auslander-Reiten quiver of a functorially finite resolving subcategory Ω. The subquivers Γ_l, Γ_r and Γ_s of Γ_Ω containing all left stable, right stable and stable τ_Ω-orbits are called the stable, left stable and right stable Auslander-Reiten quivers of Ω respectively. Connected components of Γ_l, Γ_r and Γ_s are defined analogously to connected components of the whole Auslander-Reiten quiver. Note that if a connected component Γ of either Γ_l or Γ_r contains a τ_Ω-periodic module, then by Lemma 3.2.5 every module in Γ is τ_Ω-periodic and Γ is a connected component of Γ_s.

Definition 4.1.1 *Let Γ be a subquiver of Γ_Ω and let Σ be a connected subquiver of Γ. We call Σ a **sectional subgraph** if all paths of length two contained in Σ are sectional. Σ is called a **full sectional subgraph** of Γ if any connected subquiver Σ' of Γ such that $\Sigma \subsetneq \Sigma'$ is not a sectional subgraph. The undirected graph $\overline{\Sigma}$ associated to Σ is called the **type of** Σ and for a vertex X in Σ the corresponding vertex in $\overline{\Sigma}$ is denoted by $\overline{X}$.*

Naturally, we call a sectional subgraph and its type finite if it consists of only finitely many vertices. Note that the same module can occur several times in an Auslander-Reiten quiver and also in a sectional subgraph. In particular, there can be two arrows in a sectional subgraph whose composition is a non-sectional path if they are not adjacent in the subgraph, as the following example shows.

Example 4.1.2 *Let A be the path algebra of*

$$e_1 \xrightarrow{\ \alpha\ } e_2 \circlearrowright \beta$$

with the relation $\beta^2 = 0$. Then the Auslander-Reiten quiver of A-mod contains a sectional path which contains two arrows whose composition is not sectional.

Recall from Example 1.2.12 that the Auslander-Reiten quiver of A-mod looks as follows.
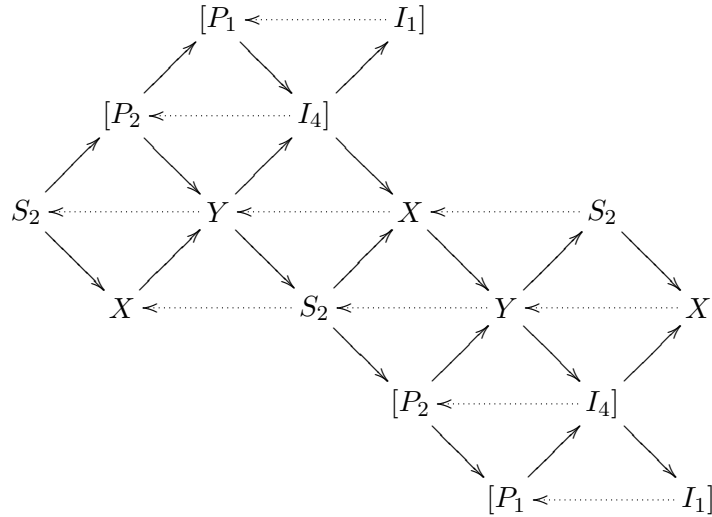
We denote the sectional path

$$P_2 \xrightarrow{\ f\ } Y \longrightarrow S_2 \xrightarrow{\ g\ } P_2$$

by γ. Although the composition fg is not sectional, γ is a sectional subgraph since the arrows of f and g are not adjacent and do not form a subpath of γ. Note that γ is not a full sectional subgraph as the arrow from P_2 to P_1 can be added to γ on both sides. In the following example we see that the differentiation between vertices that are given by the same module is very important in order to assign a graph to a connected component of Γ_l.

Example 4.1.3 *Let A be the hereditary path algebra given by*

$$e_1 \overset{\frown}{\longrightarrow} e_2 \longrightarrow e_3.$$

In its preinjective component Γ there are sectional subgraphs of type A_∞^∞ that contain non-consecutive arrows that form a non-sectional path.

The injective modules are given by the following Jordan-Hölder compositions.

$$
I_1 = \ S_1 \qquad
I_2 = \begin{matrix} S_1 \\ S_2 \end{matrix} \qquad
I_3 = \begin{matrix} & S_1 \\ S_1 & S_2 \\ S_3 \end{matrix}
$$

This information is already enough to determine the structure of the preinjective component Γ.

$$\cdots \longleftarrow \tau_\Omega^2(I_1) \longleftarrow \tau_\Omega(I_1) \longleftarrow I_1]$$

$$\cdots \longleftarrow \tau_\Omega^2(I_3) \longleftarrow \tau_\Omega(I_3) \longleftarrow I_3]$$

$$\cdots \longleftarrow \tau_\Omega^2(I_2) \longleftarrow \tau_\Omega(I_2) \longleftarrow I_2]$$

$$\cdots \longleftarrow \tau_\Omega^2(I_1) \longleftarrow \tau_\Omega(I_1) \longleftarrow I_1]$$

$$\cdots \longleftarrow \tau_\Omega^2(I_3) \longleftarrow \tau_\Omega(I_3) \longleftarrow I_3]$$

$$\cdots \longleftarrow \tau_\Omega^2(I_2) \longleftarrow \tau_\Omega(I_2) \longleftarrow I_2]$$
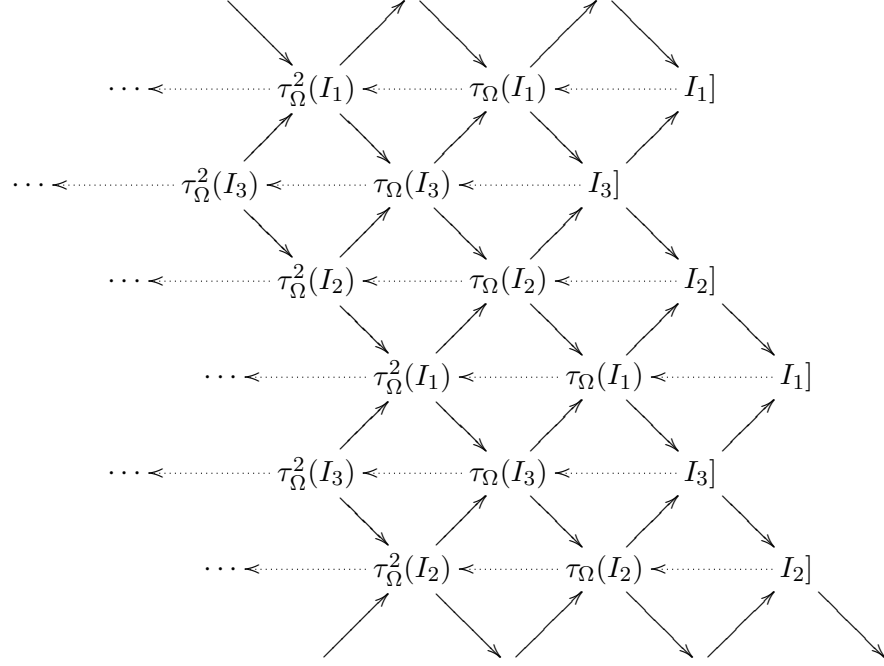
We denote the subgraph

$$\cdots \longrightarrow \tau_\Omega^n(I_3) \longrightarrow \tau_\Omega^n(I_2) \longrightarrow \tau_\Omega^n(I_1) \longrightarrow \tau_\Omega^{n-1}(I_3) \longrightarrow \cdots$$

$$\cdots \longrightarrow I_3 \longrightarrow I_2 \longrightarrow I_1 \longleftarrow I_3 \longleftarrow \tau_\Omega(I_2) \longleftarrow \tau_\Omega^2(I_1) \longleftarrow \cdots$$

$$\cdots \longleftarrow \tau_\Omega^n(I_3) \longleftarrow \tau_\Omega^{n+1}(I_2) \longleftarrow \tau_\Omega^{n+2}(I_1) \longleftarrow \tau_\Omega^{n+2}(I_3) \longleftarrow \cdots$$

by Σ and note that its type is A_∞^∞. It clearly contains arrows that, if we compose them, form a non-sectional path, but since these arrows are not adjacent in Σ the subgraph is a full sectional subgraph. However, Γ also contains a full sectional subgraph

$$\cdots \longrightarrow I_1 \longleftarrow I_3 \longrightarrow I_2 \longrightarrow I_1 \longleftarrow I_3 \longrightarrow I_2 \longrightarrow \cdots$$

of type A_∞^∞ which does not contain arrows whose composition is non-sectional.

Suppose now we would identify vertices of an Auslander-Reiten quiver with each other if they are given by the same module. Then we would have full sectional subgraphs

$$I_1 \longleftarrow I_3 \longrightarrow I_2 \longrightarrow I_1$$

and

$$\cdots \longrightarrow \tau_\Omega^n(I_2) \longrightarrow \tau_\Omega^n(I_1) \longrightarrow \tau_\Omega^{n-1}(I_3) \longrightarrow \cdots \longrightarrow I_3 \longrightarrow I_2 \longrightarrow I_1,$$

which would be of type $\widetilde{A}_2$ and A_∞ respectively. In particular, Γ would contain full sectional subgraphs of different types, which we avoid in our setup as the subsequent statements show.

Lemma 4.1.4 *Let X be a vertex in a full sectional subgraph Σ of a connected component Γ in the Auslander-Reiten quiver of Γ_Ω. Then the number of vertices adjacent to $\overline{X}$ in $\overline{\Sigma}$ equals the number of immediate predecessors of X in Γ plus the number of projective successors of X in Γ. Furthermore, let Y be an immediate predecessor or immediate successor of X in Σ, the number of undirected edges between $\overline{X}$ and $\overline{Y}$ equals the number of arrows from Y to X or X to Y in Γ respectively.*

Proof:

We prove the second statement first. Let X be a vertex in Σ and, without loss of generality, let Y be an immediate predecessor of X in Σ. Since Σ is full, it must necessarily contain all arrows from Y to X, and so the number of undirected edges between $\overline{X}$ and $\overline{Y}$ coincides with the number of arrows from Y to X.

It is easy to see that every projective successor of X in Γ must be contained in Σ as Σ is a full sectional subgraph. Assume there is a predecessor Y of X in Γ that is not contained in Σ. Then, as Σ is full, $\tau_\Omega^{-1}(Y)$ exists and all arrows from X to $\tau_\Omega^{-1}(Y)$ are contained in Σ. On the other hand, if both Y and $\tau_\Omega^{-1}(Y)$ are contained in Σ, then Σ also contains a path of length two from Y to $\tau_\Omega^{-1}(Y)$, which is a contradiction. $\qquad\square$

Theorem 4.1.5

(a) *Let Γ be a connected component of the left stable Auslander-Reiten quiver Γ_l and let Σ and Σ' be two full sectional subgraphs of Γ such that for each Ext-injective module I in Γ there is no path from I to any vertex in Σ or Σ'. Then Σ and Σ' are of the same type.*

(b) *If for every full sectional subgraph Σ of Γ there is an Ext-injective module I and a path from I to some vertex of Σ, then Γ contains an oriented cycle but no τ_Ω-periodic module and Σ is of type A_∞.*

(c) *Dually, let Γ denote a connected component of Γ_r, then two full sectional subgraphs Σ and Σ' have the same type if there is no path from Σ or Σ' to a projective module in Γ.*

(d) *If for every full sectional subgraph Σ of Γ there is a projective module P and a path from some vertex of Σ to P, then Γ contains an oriented cycle but no τ_Ω-periodic module and Σ is of type A_∞.*

Proof:

Suppose Σ and Σ' are full sectional subgraphs in Γ such that there is no path from an Ext-injective module I to Σ or Σ'. Due to this property and since Γ is a connected component of Γ_l, we can choose modules X in Σ, X' in Σ' and an integer $m \in \mathbb{Z}$ such that there is a sectional path

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = \tau_\Omega^m(X')$$

of minimal length n. If n does not equal zero, then X_1 is not a vertex of Σ by minimality of n. Consider the subgraph Σ_1 obtained by adding the module X_1 and the arrow from
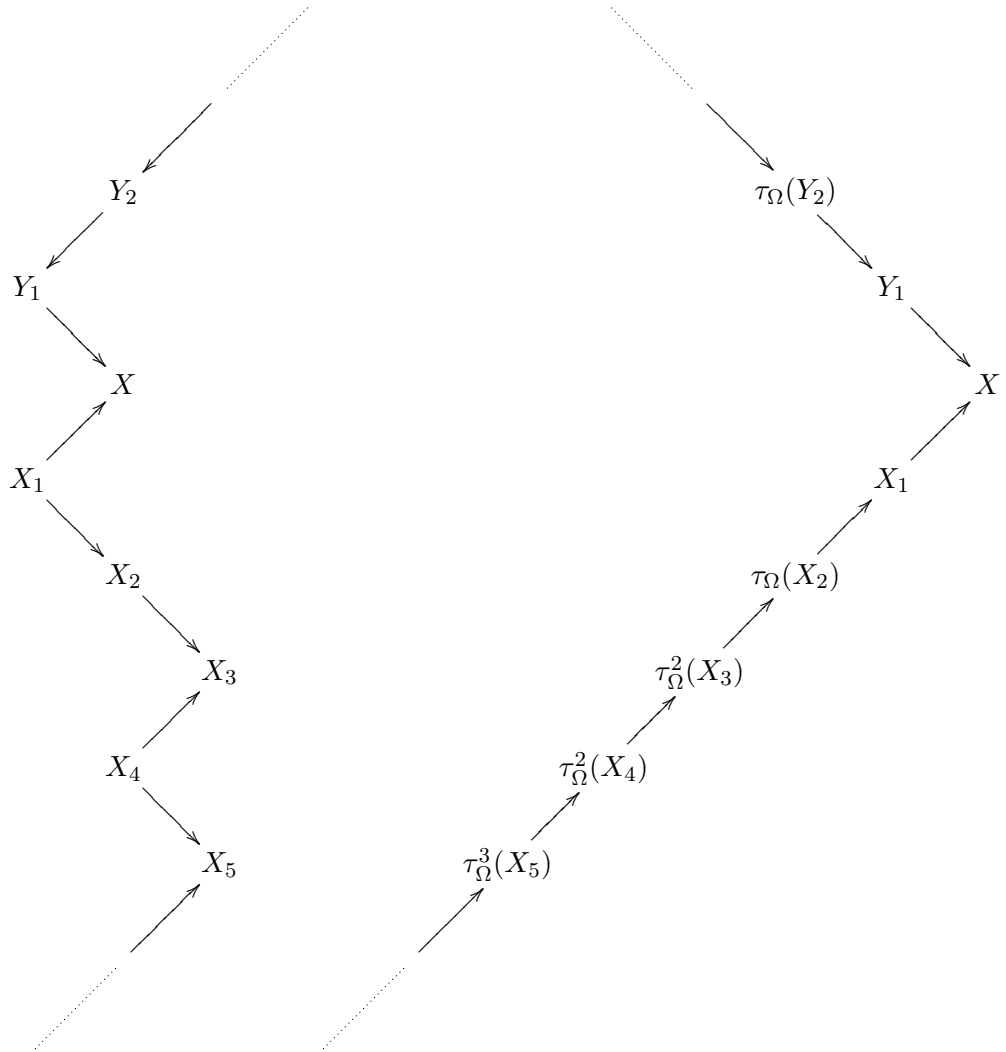
X to X_1 to the full sectional subgraph Σ. By definition of full sectional subgraphs there must be a path in Σ_1 that is not sectional and ends in X_1. Hence the path also contains $\tau_\Omega(X_1)$, which then must be a vertex of Σ and there is a sectional path from $\tau_\Omega(X_1)$ to $\tau_\Omega^{m+1}(X')$, which contradicts n to be minimal. Hence we can find modules X and X' such that $X = \tau_\Omega^m(X')$.

Without loss of generality, we can assume that $m \geq 0$. Since Γ is left stable, the numbers of vertices and undirected edges adjacent to $\overline{X'}$ and $\overline{X}$ equal the numbers of immediate predecessors of X' and X and arrows from immediate predecessors to X' and X respectively by Lemma 4.1.4. Therefore, as $m \geq 0$ and Γ is left stable, the number of vertices adjacent to $\overline{X}$ is greater than or equal to the number of vertices adjacent to $\overline{X'}$. Suppose this number is strictly higher, then there is an immediate predecessor Y of X such that $\tau_\Omega^{-m}(Y)$ is not an immediate predecessor of X'. Consequently, $\tau_\Omega^{-m}(Y)$ does not exist, the orbit of Y must be Ext-injective and there is a path from some Ext-injective module I to Σ', which is a contradiction. So let Y' be an immediate predecessor or immediate successor of X' in Σ' and without loss of generality, let $Y = \tau_\Omega^m(Y')$ be an adjacent vertex to X in Σ. Then the numbers of undirected edges between $\overline{X'}$ and $\overline{Y'}$ and between $\overline{X}$ and $\overline{Y}$ coincide by Corollary 2.2.7, which completes the proof of statement (a).

Assume now that Γ is a connected component of Γ_l with the property that for every full sectional subgraph Σ there is an Ext-injective module I such that there is a path from I to some vertex in Σ. Since every module X is contained in some full sectional subgraph, it follows that for every module X in Γ there is an Ext-injective module I such that there is a path from I to X as Γ is left stable. In particular, Γ contains an Ext-injective module I_0 and there is an Ext-injective module I_1 with a non-trivial path to I_0. Furthermore, there is also an Ext-injective module I_2 with a non-trivial path to I_1 which can be extended to a path from I_2 to I_0. We continue inductively and obtain an infinite collection of paths $I_j \to I_i$ for all $j > i \geq 0$. Since there are up to isomorphism only finitely many indecomposable Ext-injective modules in Ω, there is an oriented cycle in Γ.

Suppose Γ contains a τ_Ω-periodic module X. Then all adjacent orbits are either τ_Ω-periodic or finite by Lemma 3.2.5. Since Γ is left stable, it cannot contain finite orbits and hence every orbit must be τ_Ω-periodic. This is a contradiction as Γ contains at least one Ext-injective module.

Now by Lemma 3.3.6 Γ contains an infinite sectional path of type A_∞ and by Theorem 3.3.7 every module in Γ has at most two immediate predecessors. Hence the type of a full sectional subgraph is either A_∞ or A_∞^∞. Suppose Σ is a full sectional subgraph of type A_∞^∞ and let X be any module in Σ. Since Γ is left stable, we can then construct two different sectional paths ending in X in the way the following example suggests.

$$Y_2 \qquad\qquad\qquad\qquad\qquad\qquad \tau_\Omega(Y_2)$$

$$Y_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad Y_1$$

$$X \qquad\qquad\qquad\qquad\qquad\qquad\qquad X$$

$$X_1 \qquad\qquad\qquad\qquad\qquad\qquad X_1$$

$$X_2 \qquad\qquad\qquad\qquad\qquad \tau_\Omega(X_2)$$

$$X_3 \qquad\qquad\qquad\qquad \tau_\Omega^2(X_3)$$

$$X_4 \qquad\qquad\qquad \tau_\Omega^2(X_4)$$

$$X_5 \qquad\qquad \tau_\Omega^3(X_5)$$

This is impossible as it clearly contradicts the results from Corollary 3.3.8. It follows that every full sectional subgraph of Γ is of type A_∞ $\qquad\qquad\qquad\qquad\square$

The following example shows that there are left stable components such that for every module X there is an Ext-injective module I such that there is a path from I to X. Moreover, it can be seen why the proof for Theorem 4.1.5(a) cannot be applied to these components.

Example 4.1.6 *Let A be the path algebra of the quiver*

$$e_1 \quad \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \quad e_2 \quad \overset{\gamma}{\longrightarrow} \quad e_3 \,,$$

with the relation $\gamma\beta = 0$. Then the Auslander-Reiten quiver of A contains an infinite left stable component Γ such that for all modules X in Γ there is a path from I_3 to X.

First of all we write down the Jordan-Hölder composition series of the projective and injective modules.

$$P_1 = \begin{matrix} & S_1 & \\ S_2 & & S_2 \\ & S_3 & \end{matrix} \qquad P_2 = \begin{matrix} S_2 \\ S_3 \end{matrix} \qquad P_3 = S_3$$

$$I_1 = S_1 \qquad I_2 = \begin{matrix} S_1 & S_1 \\ & S_2 \end{matrix} \qquad I_3 = \begin{matrix} S_1 \\ S_2 \\ S_3 \end{matrix}$$

We easily verify $\tau(I_3) \cong I_3/S_3$. Then the component of I_3 looks the following way:
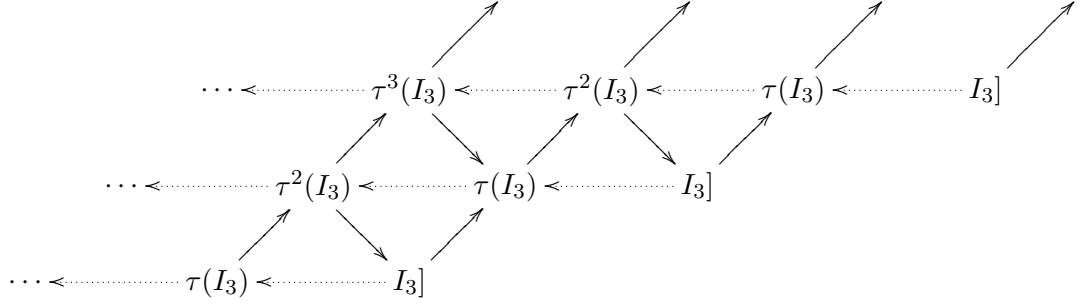
Clearly, the component must be infinite because all indecomposable projective modules are adjacent in the Auslander-Reiten quiver. Furthermore, we can see that any full sectional subgraph is of type A_∞ in this component. Its shape suggests the following name.

Definition 4.1.7

(a) *We call a connected component Γ of the left stable Auslander-Reiten quiver Γ_l **helical** if for every module X there is an Ext-injective module I in Γ such that there is a path from I to X.*

(b) *Dually, we call a connected component Γ of Γ_r **cohelical** if for every module X there is a projective module P in Γ and a path from X to P.*

Due to the last theorem the following definition makes sense.

Definition 4.1.8

(a) *Let Γ be a connected component of Γ_l. If Γ is helical, we define the **left subgraph type of Γ** to be A_∞ as any full sectional subgraph is of that type. On the other hand, if Γ is not helical, then the **left subgraph type of Γ** is defined as the type of a full sectional subgraph Σ such that there is no path from an Ext-injective module to any vertex in Σ.*

(b) *Dually, let Γ denote a connected component of Γ_r. If Γ is cohelical, we define the **right subgraph type of Γ** to be A_∞. If Γ is not cohelical, then the **right subgraph type of Γ** is defined as the type of a full sectional subgraph Σ such that there is no path from any vertex in Σ to a projective module.*

(c) *If Γ denotes a connected component of the stable Auslander-Reiten quiver Γ_s, then the **subgraph type of Γ** is defined as the type of any full sectional subgraph Σ in Γ.*

It is not hard to see that in a connected component of Γ_s subgraph type, left subgraph type and right subgraph type coincide. Let Γ denote a connected component of the left stable Auslander-Reiten quiver Γ_l of Ω. With some basic matrix calculations we show that Γ is finite if its left subgraph type is given by a Dynkin diagram. If Γ is finite, it clearly only contains τ_Ω-periodic orbits. An infinite component must have infinitely many orbits or some non-periodic, infinite, left stable orbits.

We assume Γ is infinite, so if Γ contains a τ_Ω-periodic orbit, we know by Theorem 3.2.7 that Γ is a stable tube. In particular, a full sectional subgraph is of type A_∞ and the left subgraph type is not Dynkin. Secondly, if Γ is helical, then every full sectional subgraph is of type A_∞ and the left subgraph type is not Dynkin.

Definition 4.1.9

(a) *Let Γ be a non-helical connected component of Γ_l that is not τ_Ω-periodic and let Σ be a full sectional subgraph of Γ. We say that Σ is **left of all projective and Ext-injective modules** if there is neither a projective module P nor an Ext-injective module I such that there is a path in the whole Auslander-Reiten quiver Γ_Ω from P or I to a module in Σ respectively.*

(b) *Dually, let Γ be a non-cohelical connected component of Γ_r that is not τ_Ω-periodic and let Σ be a full sectional subgraph of Γ. We say that Σ is **right of all projective and Ext-injective modules** if there is neither a projective module P nor an Ext-injective module I such that there is a path in the whole Auslander-Reiten quiver Γ_Ω from a module in Σ to P or I respectively.*

Note that a full sectional subgraph left of all projective and Ext-injective modules always exists since Γ is not τ_Ω-periodic but there are only finitely many projective and Ext-injective modules in Ω. Note that if Γ is of finite left subgraph type, then it cannot be helical, so the former is a stronger condition on Γ. Recall that the **dimension vector** of a module X is the column vector such that its i-th entry is given by the number of simple modules S_i in the Jordan-Hölder decomposition of X.

Definition 4.1.10

(a) *Let Γ be a connected component of Γ_l of finite left subgraph type that is not τ_Ω-periodic. If $X_1, \ldots, X_n$ denote all modules of a full sectional subgraph left of all projective and Ext-injective modules and $x_1, \ldots, x_n$ their dimension vectors, then τ_Ω generates a matrix with its ij-th entry being the number of x_i appearing in the dimension vector of $\tau_\Omega(X_j)$ given by $\mathcal{E}_\Omega(X_j)$. We call this matrix a **translation matrix** of Γ and denote it by M.*

(b) *Dually, let Γ be a connected component of Γ_r of finite right subgraph type that is not τ_Ω-periodic. If $X_1, \ldots, X_n$ denote all modules of a full sectional subgraph right of all projective and Ext-injective modules and $x_1, \ldots, x_n$ their dimension vectors, then τ_Ω^{-1} generates a matrix $M^- = (m_{ij}^-)$ such that m_{ij}^- is the number of x_i appearing in the dimension vector of $\tau_\Omega^{-1}(X_j)$ given by $\mathcal{E}'_\Omega(X_j)$. We call M^- a **cotranslation matrix** of Γ.*

Example 4.1.11 *Let A be the hereditary path algebra of the quiver*

$$
\begin{array}{ccc}
 & & e_6 \\
 & & \downarrow \\
e_1 & & e_3 \\
 & \searrow & \downarrow \swarrow \quad \searrow \\
 & e_2 & e_4 \qquad e_5.
\end{array}
$$
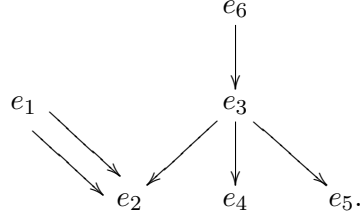
We calculate a translation matrix $M = (m_{ij})$ of the preinjective component of A.

Note that in this algebra we have $I_1 = S_1$ and $I_6 = S_6$. The other injective modules have the following Jordan-Hölder composition series.

$$
I_2 = \begin{array}{c} S_6 \\ S_1^2 \quad S_3 \\ S_2 \end{array} \qquad
I_3 = \begin{array}{c} S_6 \\ S_3 \end{array} \qquad
I_4 = \begin{array}{c} S_6 \\ S_3 \\ S_4 \end{array} \qquad
I_4 = \begin{array}{c} S_6 \\ S_3 \\ S_5 \end{array}
$$

For convenience, let us denote $\tau(I_j)$ by X_j for all $1 \le j \le 6$. The preinjective component Γ looks as follows.
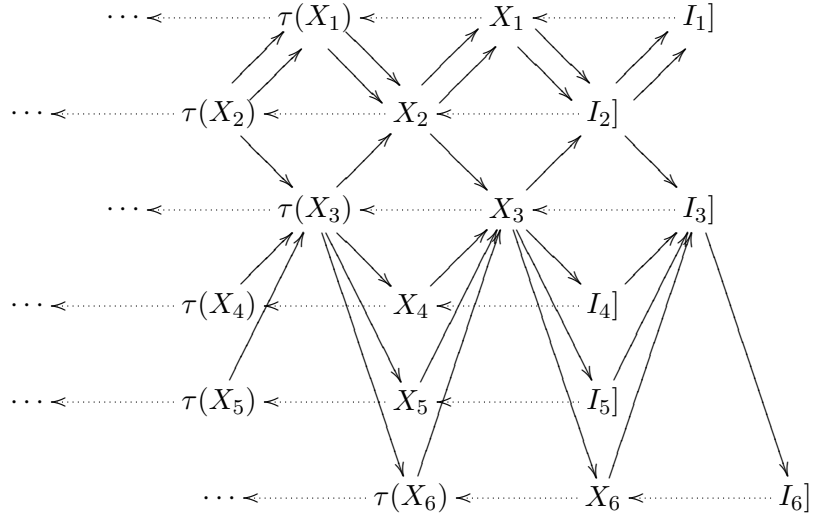
Clearly, the subquiver consisting of all modules X_j and all arrows between them is a full sectional subgraph of Γ that is left of all projective and Ext-injective modules. We denote the dimension vectors of X_j by x_j and the dimension vector of $\tau(X_j)$ by $\tau(x_j)$. It follows from the almost split sequence

$$
0 \longrightarrow \tau(X_6) \longrightarrow X_3 \longrightarrow X_6 \longrightarrow 0
$$

that $\tau(x_6) = x_3 - x_6$. By definition of the translation matrix we get $m_{36} = 1$, $m_{66} = -1$ while $m_{i6} = 0$ for $i = 1, 2, 4, 5$. We similarly obtain $\tau(x_1) = 2x_2 - x_1$ and the entries of the first column of M. We conclude from

$$
0 \longrightarrow \tau(X_3) \longrightarrow X_2 \oplus X_4 \oplus X_5 \oplus \tau(X_6) \longrightarrow X_3 \longrightarrow 0
$$

that $\tau(x_3) = x_2 + x_4 + x_5 + \tau(x_6) - x_3 = x_2 + x_4 + x_5 - x_6$ by substituting in $x_3 - x_6$ for $\tau(x_6)$. The other dimension vectors are $\tau(x_2) = 2\tau(x_1) + \tau(x_3) - x_2 = 4x_2 - 2x_1 + x_4 + x_5 - x_6$, $\tau(x_4) = \tau(x_3) - x_4 = x_2 + x_5 - x_6$ and $\tau(x_5) = \tau(x_3) - x_5 = x_2 + x_4 - x_6$, which gives rise to

$$
M = \begin{pmatrix}
-1 & -2 & 0 & 0 & 0 & 0 \\
2 & 4 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}.
$$

Lemma 4.1.12

(a) *Let Γ be a connected component of Γ_l that is not τ_Ω-periodic and is of finite left subgraph type. Let $X_1, \ldots, X_n$ denote the modules of a full sectional subgraph left of all projective and Ext-injective modules and $M = (m_{ij})$ its translation matrix. Then the ij-th entry of M^k equals the number of x_i in the dimension vector of $\tau_\Omega^k(X_j)$.*

(b) *Dually, let Γ be a connected component of Γ_r that is not τ_Ω-periodic and is of finite right subgraph type. Let $X_1, \ldots, X_n$ denote the modules of a full sectional subgraph right of all projective and Ext-injective modules and $M^- = (m_{ij}^-)$ its cotranslation matrix. Then the ij-th entry of $(M^-)^k$ equals the number of x_i in the dimension vector of $\tau_\Omega^{-k}(X_j)$.*

Proof:

For convenience, we speak about the number of x_i in $\tau_\Omega(X_j)$ instead of the number of x_i in the dimension vector of $\tau_\Omega(X_j)$. Furthermore, we denote the dimension vector of $\tau_\Omega^l(X_j)$ by $\tau_\Omega^l(x_j)$. We prove the statement by induction on k. The statement follows immediately from the definitions for $k = 0$ and $k = 1$.

Suppose now the statement is true for k. We can write the ij-th entry of M^{k+1} as $(y_{i1}, \ldots, y_{in})(m_{1j}, \ldots, m_{nj})^T = \sum_{t=1}^{n} y_{it} m_{tj}$, where $(y_{i1}, \ldots, y_{in})$ denotes the i-th row vector of M^k. By the induction hypothesis y_{ij} is the number of x_i in $\tau_\Omega^k(X_j)$, and by definition m_{ij} is the number of x_i in $\tau_\Omega(X_j)$, which coincides with the number of $\tau_\Omega^k(x_i)$ in $\tau_\Omega^{k+1}(X_j)$ as Γ is left stable. It follows that

$$
\sum_{t=1}^{n} y_{it} m_{tj} = \sum_{t=1}^{n} (\sharp \text{ of } x_i \text{ in } \tau_\Omega^k(X_t))(\sharp \text{ of } \tau_\Omega^k(x_t) \text{ in } \tau_\Omega^{k+1}(X_j)),
$$

so the ij-entry of M^{k+1} is the number of x_i in $\tau_\Omega^{k+1}(X_j)$. $\qquad \square$

Corollary 4.1.13

(a) *Let Γ be a connected component of Γ_l that is not τ_Ω-periodic and is of finite left subgraph type. Let $X_1, \ldots, X_n$ denote the modules of a full sectional subgraph left of all projective and Ext-injective modules and M its translation matrix. If v_i for $i = 1, \ldots, n$ denotes the row vector with its t-th entry v_{it} being the multiplicity of the*

simple module S_i in the Jordan-Hölder composition series of X_t, then the multiplicity of S_i in $\tau_\Omega^k(X_j)$ is given by $v_i \cdot M^k \cdot e_j$, where e_j is the standard basis column vector where the j-th entry is 1.

(b) Dually, let Γ be a connected component of Γ_r that is not τ_Ω-periodic and is of finite right subgraph type. Let $X_1, \ldots, X_n$ denote the modules of a full sectional subgraph right of all projective and Ext-injective modules and M^- its cotranslation matrix. If v_i for $i = 1, \ldots, n$ denotes the row vector with its t-th entry v_{it} being the multiplicity of the simple module S_i in the Jordan-Hölder composition series of X_t, then the multiplicity of S_i in $\tau_\Omega^{-k}(X_j)$ is given by $v_i \cdot (M^-)^k \cdot e_j$.

Proof:

For convenience, let us set $(y_{1j}, \ldots, y_{nj})^T = M^k \cdot e_j$, which is the j-th column vector of M^k. Therefore, y_{tj} is the number of x_t in $\tau_\Omega^k(X_j)$ by the previous lemma. We conclude that

$$\sum_{t=1}^n v_{it} y_{tj} = \sum_{t=1}^n (\sharp \text{ of } S_i \text{ in } X_t)(\sharp \text{ of } x_t \text{ in } \tau_\Omega^k(X_j))$$

and $v_i \cdot M^k \cdot e_j$ equals the number of S_i in $\tau_\Omega^k(X_j)$. $\qquad\square$

Obviously, M and M^- depend on the choice of the full sectional subgraph, but for our purposes it suffices to consider an arbitrary translation matrix, which we choose in a way that the computations are least complicated. If Γ is a finite connected component of Γ_l, then there cannot be a a full sectional subgraph Σ such that there is no path in Γ_Ω from a projective module to a module in Σ by Theorem 2.2.9. So if we try to calculate dimension vectors in finite components via M, then there must be an error after finitely many steps. This error can express itself in the following way.

Lemma 4.1.14

(a) Let M be a translation matrix of a connected component Γ of Γ_l that is not τ_Ω-periodic and of finite left subgraph type. Then there are no standard basis vectors e_j, e_l such that $M^k e_j = -e_l$ for some $k \in \mathbb{N}$.

(b) Dually, let M^- be a cotranslation matrix of a connected component Γ of Γ_r that is not τ_Ω-periodic and of finite right subgraph type. Then there are no standard basis vectors e_j, e_l such that $(M^-)^k e_j = -e_l$ for some $k \in \mathbb{N}$.

Proof:

The modules of a full sectional subgraph left of all projective and Ext-injective modules are again denoted by $X_1, \ldots, X_n$. By Corollary 4.1.13 the multiplicity of S_i in $\tau_\Omega^k(X_j)$ is given by $v_i \cdot M^k \cdot e_j$, where v_i denotes the row vector with its t-th entry being the multiplicity of the simple module S_i in the Jordan-Hölder composition series of X_t. Suppose there are standard basis vectors e_j, e_l and some $k \in \mathbb{N}$ such that $M^k e_j = -e_l$. Since X_l is non-zero, there exists a v_i with its l-th entry being greater than 0. We obtain that the multiplicity of S_i in $\tau_\Omega^k(X_j)$ is

$$v_i \cdot M^k \cdot e_j = v_i \cdot -e_l < 0,$$

which is impossible.

$\qquad\square$

Theorem 4.1.15

(a) *Let Γ be a connected component of Γ_l such that its left subgraph type is given by a Dynkin diagram. Then Γ is finite and, in particular, τ_Ω-periodic.*

(b) *Dually, if Γ is a connected component of Γ_r such that its right subgraph type is given by a Dynkin diagram, then Γ is finite and τ_Ω-periodic.*

Proof:

Firstly, assume Γ is of left subgraph type A_1, i.e. Γ consists of just one τ_Ω-orbit. Every module in this orbit has a minimal right almost split morphism whose domain consists completely of modules that are not left stable. Hence the only orbit must be τ_Ω-periodic and Γ is finite.

Let the left subgraph type of Γ now be given by a Dynkin diagram other than A_1. Suppose Γ is infinite, then it is clearly not τ_Ω-periodic as Γ contains only finitely many τ_Ω-orbits. So there is a translation matrix M that we can analyze.

1. A_n

Let the left subgraph type of Γ be given by the Dynkin diagram A_n for $n \geq 2$, i.e a full sectional subgraph left of all projective and Ext-injective modules and its relative Auslander-Reiten translates are given by
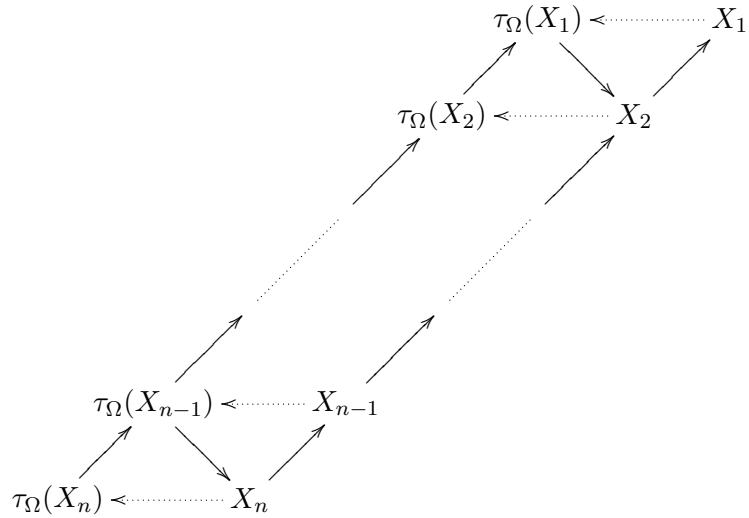
It is easy to see that the dimension vectors of $\tau_\Omega(X_1), \tau_\Omega(X_2), \ldots, \tau_\Omega(X_{n-1}), \tau_\Omega(X_n)$ are given by $x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1, -x_1$ respectively. This yields a translation matrix of Γ,

$$
M_n = \begin{pmatrix}
-1 & -1 & \cdots & -1 & -1 & -1 \\
1 & & & & & \\
& 1 & & & & \\
& & \ddots & & & \\
& & & 1 & & \\
& & & & 1 &
\end{pmatrix},
$$

where all blank entries are zero. It is easy to see that $M_n \cdot e_n = -e_1$, which is a contradiction by Lemma 4.1.14.

2. D_n

If Γ is of left subgraph type D_n for $n \geq 4$, a full sectional subgraph left of all projective and Ext-injective modules is obtained by attaching another module X_n to the path of type A_{n-1}. We reorder the module names to generate a nice translation matrix.

We obtain the multiplicities of x_i in $\tau_\Omega(X_j)$ in the same way as before by considering the almost split sequences ending in X_j. They generate translation matrices

$$
M_n = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
-1 & -1 & \cdots & -1 & -1 & -1 & -1 & 0 \\
-1 & -1 & \cdots & -1 & -1 & -1 & 0 & -1
\end{pmatrix}.
$$

We observe that $M_n^{n-1} = -\mathrm{Id}$ for n even and

$$
M_n^{n-1} = \begin{pmatrix}
-1 & & & & & & \\
 & -1 & & & & & \\
 & & \ddots & & & & \\
 & & & -1 & & & \\
 & & & & -1 & & \\
 & & & & & 0 & -1 \\
 & & & & & -1 & 0
\end{pmatrix}
$$

for n odd. Clearly, in both cases we can find standard basis vectors e_j and e_l such that $M^{n-1}e_j = -e_l$, which again contradicts Lemma 4.1.14.

3. E_n

Now we consider connected components whose left subgraph types are of Dynkin type E_n for $n = 6, 7, 8$. A full sectional subgraph left of all projective and Ext-injective modules of type E_n is given below.
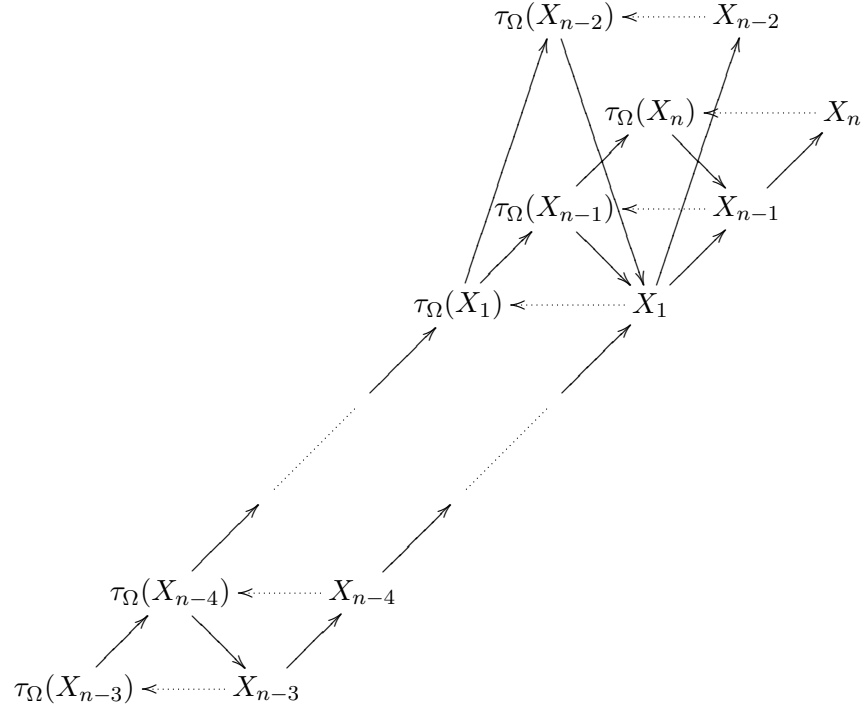
We again compute the dimension vectors of the relative Auslander-Reiten translates in terms of x_i and obtain the following translation matrices.

$$M_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & & & & & \\ & 1 & & & & \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & -1 \end{pmatrix}$$

$$M_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 \end{pmatrix}$$

$$M_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 \end{pmatrix}$$

Direct calculation shows that $M_8^{15} = -\mathrm{Id}$ and $M_7^9 = -\mathrm{Id}$. It also turns out that $M_6^6 \cdot e_1 = -e_1$; therefore, we can apply Lemma 4.1.14 to obtain a contradiction in all three cases. Hence Γ must be finite and τ_Ω-periodic if its left subgraph type is given by a Dynkin diagram. $\qquad \square$

4.2 Translation quivers and Riedtmann's structure theorem

Now we consider the concept of assigning a graph to a stable Auslander-Reiten quiver introduced by Riedtmann, which does not work for non-stable components. The whole section closely follows [R80]. Let Γ be a quiver with vertex set Γ_0 and arrow set Γ_1 and let τ be an injective map from a subset of Γ_0 into Γ_0. For any vertex x let x^- denote the set of immediate predecessors and x^+ the set of immediate successors, i.e.

$$x^- = \{y \in \Gamma_0 | \text{ there is an arrow } y \to x\},$$

$$x^+ = \{y \in \Gamma_0 | \text{ there is an arrow } x \to y\}.$$

Definition 4.2.1 *The pair (Γ, τ) is called a **translation quiver** if*

(a) Γ neither has loops nor multiple arrows between two vertices.

(b) Whenever $x \in \Gamma_0$ is such that $\tau(x)$ is defined then $x^- = \tau(x)^+$.

Moreover, we call τ the translation of the quiver and say that Γ is connected if for all vertices x and y in Γ there is a walk between x and y.

Definition 4.2.2 *We say a translation quiver (Γ, τ) is **locally finite** if x^+ and x^- are finite sets for every vertex x.*

Many stable Auslander-Reiten quivers of functorially finite resolving subcategories are locally finite translation quivers. However, there are two cases when stable Auslander-Reiten quivers are not translation quivers. The first case are components that contain arrows with non-trivial valuation, i.e. multiple arrows between modules. Secondly, although impossible in A-mod, we cannot eliminate the possibility that in a proper functorially finite resolving subcategory of A-mod there are irreducible morphisms from a stable module to itself, which would be considered a loop in the translation quiver by Riedtmann's construction.

Definition 4.2.3 *A morphism of quivers* $\varphi : \Gamma \to \Gamma'$ *assigns to each vertex x of Γ a vertex* $\varphi(x)$ *of Γ' and to each arrow α from x to y in Γ an arrow $\varphi(\alpha)$ from $\varphi(x)$ to $\varphi(y)$ in Γ'. We call φ a **morphism of translation quivers** if it commutes with the translation.*

Definition 4.2.4 *A **directed tree** is a directed graph without loops, multiple arrows or cycles.*

To a directed tree B we associate a translation quiver $\mathbb{Z}B$ as follows. The vertices of $\mathbb{Z}B$ are the pairs (n, x) with $n \in \mathbb{Z}$ and x a vertex of B. For each arrow $x \to y$ in B and each $n \in \mathbb{Z}$ we have two arrows $(n, x) \to (n, y)$ and $(n, y) \to (n - 1, x)$. The translation is defined via $\tau(n, x) = (n + 1, x)$. We consider B to be embedded in $\mathbb{Z}B$ as the subgraph consisting of vertices $(0, x)$ and the arrows connecting them.

Definition 4.2.5 *A group Π of automorphisms of a translation quiver Γ is said to be **admissible** if $\{\varphi(n, x) | \varphi \in \Pi\}$ does not contain two vertices with an arrow between them for all vertices (n, x) in Γ.*

We then construct the quotient quiver Γ/Π consisting of vertices $(\Gamma/\Pi)_0 = \Gamma_0/\Pi$ and arrows $(\Gamma/\Pi)_1 = \Gamma_1/\Pi$. Note that Γ/Π is a again a translation quiver which is stable if Γ is stable [R80].

Theorem 4.2.6 *[R80, Struktursatz] Let Γ be a connected translation quiver, then there is a directed tree B and an admissible group of automorphisms $\Pi \subset \mathrm{Aut}(\mathbb{Z}B)$ such that $\Gamma \cong \mathbb{Z}B/\Pi$. The undirected graph $\overline{B}$ associated to B is uniquely determined by Γ up to canonical isomorphism, and Π is uniquely defined up to conjugation in $\mathrm{Aut}(\mathbb{Z}B)$.*

Proof:
As it is necessary for further proofs, we give the construction of B. We fix a vertex x of Γ and consider the set of sectional paths starting in x. There is a vertex in B for each sectional path $(x = y_o \to y_1 \to \cdots \to y_n)$. The arrows of B are $(x = y_o \to y_1 \to \cdots \to y_n) \longrightarrow (x = y_o \to y_1 \to \cdots \to y_n \to y_{n+1})$. The rest of the proof can be found in [R80]. $\square$

Definition 4.2.7 *Under the assumptions of Theorem 4.2.6 we call the graph $\overline{B}$ associated to B the **tree type of Γ**.*

Corollary 4.2.8 *Let Γ be a connected component of the stable Auslander-Reiten quiver Γ_s that does not contain loops or multiple arrows, then there is a directed tree B and an admissible group of automorphisms $\Pi \subset \mathrm{Aut}(\mathbb{Z}B)$ such that $\Gamma \cong \mathbb{Z}B/\Pi$. The undirected graph $\overline{B}$ associated to B is uniquely determined by Γ up to canonical isomorphism, and Π is uniquely defined up to conjugation in $\mathrm{Aut}(\mathbb{Z}B)$.*

Proof:
Since Γ does not contain loops or multiple arrows, it must be a stable translation quiver, so the statement follows immediately from Theorem 4.2.6. $\square$

Corollary 4.2.9 *Let Γ be a connected component of the stable Auslander-Reiten quiver Γ_s that does not contain loops or multiple arrows. Then its tree type $\overline{B}$ coincides with its subgraph type $\overline{\Sigma}$.*

Proof:

We fix an indecomposable module X and let Σ be the subgraph of Γ consisting of all sectional paths starting in X. Then Σ is a full sectional subgraph and the undirected graph $\overline{\Sigma}$ associated to Σ coincides with $\overline{B}$ by the construction given in the proof of Theorem 4.2.6. □

We can now proceed preparing the proof of the main result using subadditive functions. Recall the definitions.

Definition 4.2.10 *Let T be a simple-laced graph consisting of edges T_1 and vertices T_0 such that each vertex has at most finitely many edges attached to it. The **Cartan matrix** $C(T) = (c_{ij})$ of T is defined as follows. If $i, j \in T_0$, then*

$$
c_{ij} = \begin{cases} 2 & \text{if i = j} \\ -1 & \text{if there is an edge between i and j} \\ 0 & \text{otherwise.} \end{cases}
$$

Definition 4.2.11 *A function $n : T_0 \to \mathbb{Q}$, $x \mapsto n_x$ is called **subadditive** if*

$$
\sum_{x \in T_0} c_{xy} n_x \geq 0 \text{ for all } y \in T_0.
$$

*It is called **additive** if equality holds for all $y \in T_0$.*

Let us now recall an important result on graphs and subadditive functions.

Theorem 4.2.12 *[B91, Theorem 4.5.8]*
Let T be a connected labeled graph and $x \mapsto n_x$ a subadditive function on T. If $x \mapsto n_x$ is not additive, then T is a finite Dynkin diagram or A_∞.

Theorem 4.2.13 *Suppose Γ is a finite, connected component of Γ_s without loops or multiple arrows. Then the tree type of Γ is a Dynkin diagram and coincides with its left subgraph type.*

Proof:

Since Γ is finite, we know that there are only finitely many orbits in Γ and there is an $n \in \mathbb{N}$ such that $\tau_\Omega^n(X) = X$ for all modules X in Γ. Let $X_1, \ldots, X_n$ denote the modules of an arbitrary orbit, while $Y_{11}, \ldots Y_{1n}, \ldots, Y_{k1}, \ldots Y_{kn}$ name the modules of the adjacent orbits. This means $Y_{1i} \oplus \cdots \oplus Y_{ki}$ is a direct summand of the middle term of $\mathcal{E}_\Omega(X_{i+1})$ and hence

$$
l(X_i) + l(X_{i+1}) \geq \sum_{j=1}^{k} l(Y_{ji}).
$$

Clearly, equality holds if and only if all modules in the corresponding almost split sequences are periodic. Adding these equations we obtain

$$
2 \sum_{i=1}^{n} l(X_i) \geq \sum_{j=1}^{k} \sum_{i=1}^{n} l(Y_{ji}).
$$

This gives rise to a subadditive function which assigns the sum of the length of n modules $X, \tau_\Omega(X), \ldots, \tau_\Omega^{n-1}(X)$ of an orbit to the corresponding vertex in Γ's tree type. Γ is finite but does not contain any projective modules, so it cannot be a connected component of Γ_Ω by Theorem 2.2.9. Consequently, there is a finite orbit in Γ_Ω adjacent to Γ by Lemma 3.2.5 and thus the aforementioned subadditive function is not additive. By Theorem 4.2.12 the tree type of Γ must be given by a Dynkin diagram or A_∞. But A_∞ can only be the tree type of an infinite component and hence the tree type of Γ is Dynkin.

By Lemma 3.2.5 every Ext-injective module adjacent to any orbit of Γ must be contained in a finite orbit. Consequently, Γ must also be a connected component of Γ_l, so left subgraph type, subgraph type and tree type of Γ coincide by definition and Corollary 4.2.9 respectively. $\qquad\square$

4.3 The main result

In this section we combine the results on the tree type and the left subgraph type and obtain that Ω is finite if and only if the left subgraph types of all connected components of the left stable Auslander-Reiten quiver are given by Dynkin diagrams. Firstly, we observe how the left subgraph type behaves if a connected component of Γ_l is not a translation quiver. Therefore, we consider the case of multiple arrows between a pair of indecomposable modules in the left stable part of the quiver. It is well known that all algebras of finite representation type do not have multiple arrows in their Auslander-Reiten quivers [ARS95, VII Proposition 2.2]. However, there are examples of finite Auslander-Reiten quivers of functorially finite resolving subcategories with multiple arrows between modules, where they occur between modules in finite orbits.

Example 4.3.1 *Let A be the path algebra of the Kronecker-quiver*

$$e_2 \rightrightarrows e_1 \;.$$

Then there is a functorially finite resolving subcategory of A-mod that is finite and contains multiple arrows in its finite component.

As A is a hereditary algebra, it has a unique preprojective component, which looks the following way. Note that the projective module P_i is given by Ae_i.

Since the projective modules are also the standard modules in this order of primitive idempotents, the whole category $\mathfrak{F}(\Delta)$ of standard-filtered modules consists only of P_1 and P_2. Consequently, its Auslander-Reiten quiver is given by

and contains a double arrow, although the subcategory is clearly finite.

Theorem 4.3.2

(a) *Let Γ be a connected component of Γ_l. If Γ contains multiple arrows from a module X to a module Y, then $l(\tau_\Omega^n(Y)) \to \infty$ as $n \to \infty$. In particular, Γ is infinite.*

(b) *Dually, if Γ is a connected component of Γ_r and contains multiple arrows from a module Y to a module X, then $l(\tau_\Omega^n(Y)) \to \infty$ as $n \to -\infty$. In particular, Γ is infinite.*

Proof:

Suppose first that $l(\tau_\Omega^n(X)) \leq l(\tau_\Omega^n(Y))$ and $l(\tau_\Omega^{n+1}(Y)) \leq l(\tau_\Omega^n(X))$ for every $n \geq 0$. It follows that there must be some $m \geq 0$ such that all modules $\tau_\Omega^n(X)$ and $\tau_\Omega^k(Y)$ have the same Jordan-Hölder length for all $n, k \geq m$. Without loss of generality, we can assume that $m = 0$. Since there are multiple arrows from X to Y, we have that $\mathcal{E}_\Omega(Y)$ must be of the form

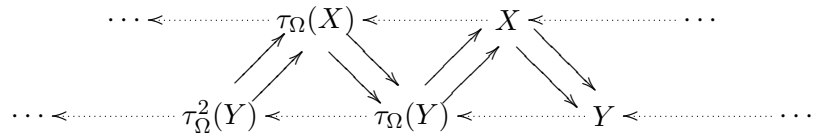$$0 \longrightarrow \tau_\Omega(Y) \longrightarrow X \oplus X \longrightarrow Y \longrightarrow 0$$

in order to satisfy $l(X \oplus X) = l(Y) + l(\tau_\Omega(Y))$. By Corollary 2.2.7 we inductively conclude that for every $n \geq 0$ the almost split sequence $\mathcal{E}_\Omega(\tau_\Omega^n(Y))$ is of the form

$$0 \longrightarrow \tau_\Omega^{n+1}(Y) \longrightarrow \tau_\Omega^n(X) \oplus \tau_\Omega^n(X) \longrightarrow \tau_\Omega^n(Y) \longrightarrow 0,$$

while $\mathcal{E}_\Omega(\tau_\Omega^n(X))$ looks as follows.

$$0 \longrightarrow \tau_\Omega^{n+1}(X) \longrightarrow \tau_\Omega^n(Y) \oplus \tau_\Omega^n(Y) \longrightarrow \tau_\Omega^n(X) \longrightarrow 0$$

It follows that the connected component of X and Y in Γ_l is of left subgraph type $\widetilde{A}_1$ and has the following shape.

On the other hand, we know that there must be a non-zero morphism $f \in \mathrm{rad}_\Omega^\infty(P, Y)$ for some projective module P. But then f factors over all modules $\tau_\Omega^n(X)$ and $\tau_\Omega^n(Y)$ with $n \geq 0$ and hence must be zero by Lemma 2.2.8, which is a contradiction. Consequently, there is an $n \geq 0$ such that $l(\tau_\Omega^n(X)) > l(\tau_\Omega^n(Y))$ or $l(\tau_\Omega^{n+1}(Y)) > l(\tau_\Omega^n(X))$. Without loss of generality, we can assume that $l(X) > l(Y)$ and there are k arrows from X to Y, where $k \geq 2$. There is an almost split sequence

$$0 \longrightarrow \tau_\Omega(Y) \longrightarrow Z \longrightarrow Y \longrightarrow 0 .$$

in which X occurs at least twice as a direct summand of Z, thus $l(\tau_\Omega(Y)) > l(X)$. Because there are k arrows from $\tau_\Omega(Y)$ to X by Corollary 2.2.7, we can continue inductively and obtain $l(\tau_\Omega^{n+1}(Y)) > l(\tau_\Omega^n(Y))$ for all $n \geq 0$. $\qquad\square$

Corollary 4.3.3 *If Γ is finite, then its left subgraph type does not contain multiple edges between vertices.*

Proof:

The statement is the contrapositive of Theorem 4.3.2. $\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The next step is to consider a left stable component that contains a loop, i.e. a module X such that there is an Ω-irreducible morphism to itself. Such morphisms do exist when the module X is projective and Ext-injective.

Example 4.3.4 *Let A be the path algebra*

$$e_1 \enspace \circlearrowright \beta$$

with the relation $\beta^2 = 0$. Then the subcategory $\Omega = \mathrm{add}(P_1)$ consisting only of copies of the one indecomposable projective module P_1 is functorially finite, resolving and there are Ω-irreducible morphisms from P_1 to itself.

The Auslander-Reiten quiver of A-mod is given by

$$
\begin{array}{ccccc}
 & [P_1] & & [P_1] & \\
 \nearrow & & \searrow \nearrow & & \searrow \\
S_1 \dashleftarrow & & S_1 \dashleftarrow & & S_1.
\end{array}
$$

Therefore, every module in A-mod has a right and a left Ω-approximation, so Ω is functorially finite. Moreover, it is closed under direct summands, extensions, kernels of epimorphisms and contains $_A A$. Hence Ω is a functorially finite resolving subcategory and, since P_1 is the only indecomposable module in Ω, the morphism from P_1 to itself factoring over S_1 is clearly irreducible in Ω.

Although there are to my best knowledge no examples of such morphisms in left stable components, we cannot rule out their existence. However, the existence of such a morphism provides enough information to determine the left subgraph type of the left stable connected component it is contained in.

Theorem 4.3.5 *Let X be an indecomposable module in Ω that is not projective or not Ext-injective. If there is an Ω-irreducible morphism $f : X \to X$, then $\tau_\Omega(X) = X$ and the connected components of Γ_l, Γ_r and Γ_s containing X coincide and are of subgraph type A_n for some $n < 2^{l(X)}$.*

Proof:

Let $f : X \to X$ be an irreducible morphism in Ω. Then there exists an $k \in \mathbb{N}$ such that the composition f^k is zero by Lemma 2.2.8. Hence if X is not projective then the path obtained by composing f with itself cannot be sectional by Corollary 2.4.7, so we have $\tau_\Omega(X) = X$. On the other hand, if X is projective, we then know by our assumptions that X is not Ext-injective. Therefore, we can apply the same arguments to $\tau_\Omega^{-1}(X)$ and analogously obtain $\tau_\Omega(X) = X$, which contradicts X to be projective. Consequently, $\tau_\Omega(X) = X$ holds under our assumptions and X is neither projective nor Ext-injective,

but stable and τ_Ω-periodic. Moreover, all orbits in the connected components of Γ_l and Γ_r containing X must in fact be periodic by Lemma 3.2.5. We denote the stable connected component of X by Γ.

Assume now the arrow from X to itself in the Auslander-Reiten quiver does not have valuation $(1,1)$. Then $X \oplus X$ is a direct summand of the middle term of $\mathcal{E}_\Omega(X)$, so by Jordan-Hölder length arguments the arrow's valuation is $(2,2)$ and $X \oplus X$ is in fact the whole middle term of $\mathcal{E}_\Omega(X)$. Consequently, we have a component of the Auslander-Reiten quiver Γ_Ω that contains only X, in particular, it does not contain a projective module, which contradicts Theorem 2.2.9. Thus the valuation of the arrow from X to itself must be $(1,1)$. Let

$$X = X_{-1} \xrightarrow{f} X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{m-1} \xrightarrow{f_m} X_m.$$

be a sectional path in the stable component of X, then the following holds.

1. $\tau_\Omega(X_j) = X_j$ for $j = 0, \ldots, m$.

2. $l(X_{j-1}) \geq l(X_j)$ for $j = 0, \ldots, m$.

3. The almost split sequence $\mathcal{E}_\Omega(X_j)$ contains at most two non-projective indecomposable middle terms for $j = 0, \ldots, m$.

We prove all three statements by induction on m. Let $m = 0$, then the first statement holds as we have seen earlier in the proof of this theorem. Moreover, $l(X_{-1}) \geq l(X_0)$ holds as $X_{-1} = X = X_0$. Lastly, we consider the almost split sequence

$$0 \longrightarrow X \longrightarrow X \oplus M \longrightarrow X \longrightarrow 0.$$

It is easy to see that $l(M) = l(X)$. Suppose now M has two non-projective indecomposable direct summands M_1 and M_2, then
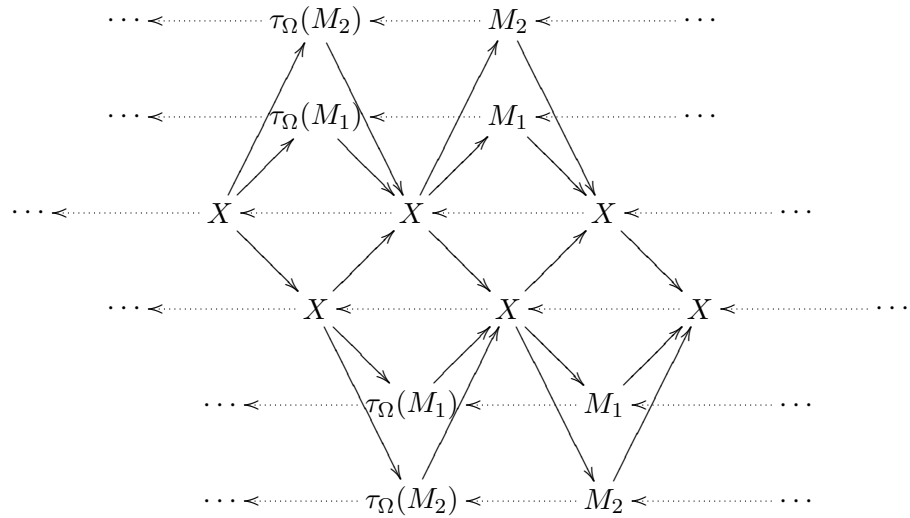
is an example of how the Auslander-Reiten quiver could be shaped around X. Furthermore, there are two almost split sequences

$$0 \longrightarrow \tau_\Omega(M_i) \longrightarrow X \oplus N_i \longrightarrow M_i \longrightarrow 0.$$

Since there is an irreducible morphism from $\tau_\Omega(M_i)$ to X, $\tau_\Omega(M_i)$ is a direct summand of M. Let us assume $\tau_\Omega(M_1)$ is not isomorphic to either M_1 or M_2. But then $l(M_1) + l(\tau_\Omega(M_1)) < l(X)$ by $\mathcal{E}_\Omega(X)$ and, on the other hand, $l(M_1) + l(\tau_\Omega(M_1)) \geq l(X)$ by $\mathcal{E}_\Omega(M_1)$. Thus $\tau_\Omega(M_1)$ and must be isomorphic to either M_1 or M_2. We repeat the argument for M_2, which leaves us with two cases.

The first one is $\tau_\Omega(M_1) = M_1$ and $\tau_\Omega(M_2) = M_2$, which gives us almost split sequences

$$0 \longrightarrow M_i \longrightarrow X \oplus N_i \longrightarrow M_i \longrightarrow 0$$

and, therefore, $l(X) \leq 2l(M_i)$ for $i = 1, 2$. Putting these inequations into $l(M) = l(X)$ we obtain $2l(M_1) + 2l(M_2) \leq 2l(X) \leq 2l(M_1) + 2l(M_2)$ and deduce $2l(M_i) = l(X)$. Thus N_i must be zero and the whole component of X only consists of X, M_1 and M_2 but does not contain any projective module, which contradicts Theorem 2.2.9. In the second case we have $\tau_\Omega(M_1) = M_2$ and $\tau_\Omega(M_2) = M_1$. We obtain an almost split sequence

$$0 \longrightarrow M_1 \longrightarrow X \oplus N_2 \longrightarrow M_2 \longrightarrow 0$$

from which we deduce $l(X) = l(M_1) + l(M_2)$ in the same way as in the first case. Hence the component of X is finite again but does not contain projective modules. Due to this contradiction we know that M has at most one indecomposable non-projective direct summand.

Suppose now the statements have been proved for $m - 1$. By the induction hypothesis there is an almost split sequence

$$0 \longrightarrow X_{m-1} \longrightarrow X_{m-2} \oplus X_m \oplus P \longrightarrow X_{m-1} \longrightarrow 0$$

that has two non-projective middle terms X_{m-2} and X_m, from which we deduce that $\tau_\Omega(X_m) = X_m$. This proves the first statement. Moreover, the induction hypothesis implies that $l(X_{m-2}) \geq l(X_{m-1})$. Consequently, $l(X_{m-1}) \geq l(P) \oplus l(X_m)$ and, in particular, $l(X_{m-1}) \geq l(X_m)$ hold, which proves the second statement.

Consider now the almost split sequence

$$0 \longrightarrow X_m \longrightarrow X_{m-1} \oplus M \longrightarrow X_m \longrightarrow 0.$$

Suppose now M has two indecomposable non-projective direct summands M_1 and M_2, then
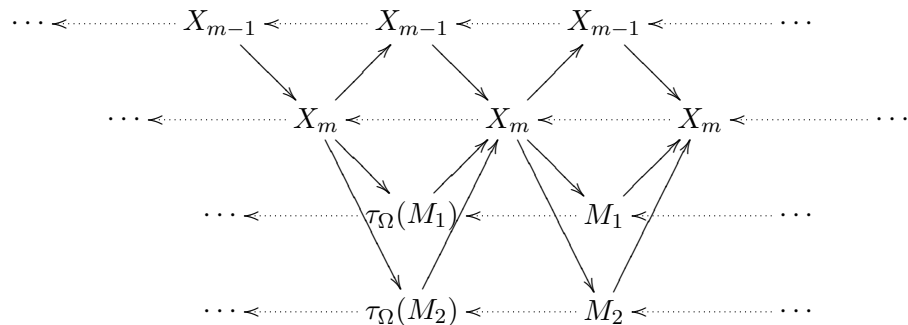
is an example of what the Auslander-Reiten quiver could look like around X_m. Since $\tau_\Omega(X_m) = X_m$, both $\tau_\Omega(M_1)$ and $\tau_\Omega(M_2)$ must be direct summands of M, which is a module whose length is shorter or equal than the length of X_m. On the other hand, by the almost split sequences

$$0 \longrightarrow \tau_\Omega(M_1) \longrightarrow X_m \oplus N_1 \longrightarrow M_1 \longrightarrow 0$$

and

$$0 \longrightarrow \tau_\Omega(M_2) \longrightarrow X_m \oplus N_2 \longrightarrow M_2 \longrightarrow 0$$

we know that $l(\tau_\Omega(M_1)) + l(M_1) \geq l(X_m)$ and $l(\tau_\Omega(M_2)) + l(M_2) \geq l(X_m)$. It follows that $2l(M) \geq l(\tau_\Omega(M_1)) + l(M_1) + l(\tau_\Omega(M_2)) + l(M_2) \geq 2l(X_m) \geq 2l(M)$, hence $M \cong M_1 \oplus M_2$ and $l(X_m) = l(M_1) + l(M_2)$. In particular, $\tau_\Omega(M_1) \oplus \tau_\Omega(M_2)$ is isomorphic to $M_1 \oplus M_2$, which leaves us with the same two cases as in the induction start. In the first case there are almost split sequences

$$0 \longrightarrow M_1 \longrightarrow X_m \longrightarrow M_1 \longrightarrow 0$$

and

$$0 \longrightarrow M_2 \longrightarrow X_m \longrightarrow M_2 \longrightarrow 0$$

with $2l(M_1) = 2l(M_2) = l(X_m)$. In the second case the almost split sequences are given by

$$0 \longrightarrow M_2 \longrightarrow X_m \longrightarrow M_1 \longrightarrow 0$$

and

$$0 \longrightarrow M_1 \longrightarrow X_m \longrightarrow M_2 \longrightarrow 0.$$

In both cases the component ends in these orbits, in particular, there are only finitely many modules in the stable component of X and the almost split sequence

$$0 \longrightarrow X_m \longrightarrow X_{m-1} \oplus M_1 \oplus M_2 \longrightarrow X_m \longrightarrow 0$$

gives us $l(X_m) = l(X_{m-1})$. Now we show that this cannot hold if there is a projective module in this component. Since $X, X_1, \ldots, X_m, M_1$ and M_2 are the only indecomposable non-projective modules in this component, there must be a projective module P such that there is an almost split sequence

$$0 \longrightarrow X_j \longrightarrow X_{j-1} \oplus X_{j+1} \oplus P \longrightarrow X_{j-1} \longrightarrow 0.$$

Without loss of generality, we can assume that j is maximal with that property. We obtain $l(X_j) > l(X_{j+1})$ by the almost split sequence and the induction hypothesis. As all the almost split sequences $\mathcal{E}_\Omega(X_{j+1}), \ldots, \mathcal{E}_\Omega(X_{m-1})$ have precisely two middle terms given by the original sectional path, it easily follows that $l(X_j) > l(X_{j+1}) > \cdots > l(X_{m-1}) > l(X_m)$, which contradict our previous result. Hence our assumption that $\mathcal{E}_\Omega(X_m)$ has three non-projective middle terms cannot be true, which completes the proof for the third statement.

We deduce that every module in Γ has length shorter or equal to $l(X)$ by statements 1 and 2 and is of τ_Ω-periodicity 1. In a stable component every full sectional subgraph is of the same type by Lemma 4.1.5. Therefore, let Σ be an arbitrary full sectional subgraph. By statement 3 we know that for every module Y in Σ the almost split sequence

$$0 \longrightarrow Y \longrightarrow M \longrightarrow Y \longrightarrow 0$$

contains at most two indecomposable non-projective middle terms, hence every orbit in Γ has at most two adjacent orbits in Γ. It follows that every vertex in the subgraph type of Γ has at most two edges connected to it, so the subgraph type of Γ is either A_n, A_∞ or A_∞^∞. Suppose it is not A_n for some $n < 2^{l(X)}$, then there is a sectional path

$$Y_0 \xrightarrow{g_1} Y_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{m-1}} Y_{m-1} \xrightarrow{g_m} Y_m$$

in Γ of length $m \geq 2^{l(X)} - 1$ as each module in Σ is stable. Since $l(Y_j) \leq l(X)$ for all $j = 0, \ldots, m$ as a consequence of statements 1 and 2, we conclude that $g_m \cdots g_1 = 0$ by 2.2.8. This is a contradiction as, on the other hand, we know $g_m \cdots g_1 \neq 0$ by 2.4.7. Hence the subgraph type of Γ is A_n for some $n < 2^{l(X)}$, which completes the proof. $\square$

We can now summarize the obtained results to the main theorem.

Theorem 4.3.6 *Let Ω be a functorially finite resolving subcategory. Then the following statements are equivalent.*

(a) Ω is finite.

(b) The left subgraph type of each connected component of Γ_l is given by a Dynkin diagram.

(c) The right subgraph type of each connected component of Γ_r is given by a Dynkin diagram.

Proof:
We only prove the equivalence of the first and second statement since the equivalence of the first and third statement can be shown dually.

Let Ω be finite and let Γ be a connected component of Γ_l. In particular, Γ is stable and does not contain multiple arrows between two fixed indecomposable modules. If there is a module in Γ with an irreducible morphism to itself, then the left subgraph type of Γ is A_n by Theorem 4.3.5. On the other hand, if none such morphism exists, Γ is a translation quiver. By Theorem 4.2.13 the tree type of Γ is given by a Dynkin diagram and coincides with the left subgraph type.

Suppose now that the left subgraph types of all connected components of Γ_l are given by Dynkin diagrams and let Γ denote such a component. We know that Γ is not helical as helical components have left subgraph type A_∞ by definition. Therefore, we can apply Theorem 4.1.15 and obtain that Γ is in fact stable and finite.

Suppose now that there are infinitely many stable, finite components Γ_i. In particular, none of these components contains an Ext-injective module, so every projective module is contained in a finite orbit since the number of non-isomorphic, indecomposable projective modules coincides with the number of non-isomorphic, indecomposable Ext-injective modules. Then, as for every module X there is a projective module P and a surjective morphism from P to X, each Γ_i must be adjacent to at least one finite orbit in the whole Auslander-Reiten quiver Γ_Ω. But there are only finitely many projective modules in Γ_Ω and hence only finitely many finite orbits. Each of these orbits can only be adjacent to finitely many Γ_i, which is a contradiction. Hence Ω is finite. $\qquad\square$

In order to show how this theorem generalizes Riedtmann's theorem, we deduce the following.

Theorem 4.3.7 *Let Ω be a functorially finite resolving subcategory such that every projective module in Γ_Ω is contained in a finite orbit and there is no Ω-irreducible morphism $f : X \to X$. Then Ω is finite if and only if Γ_s does not contain multiple arrows and the tree type of every connected component of Γ_s is given by a Dynkin diagram.*

Proof:
Let Γ be a connected component of Γ_s. If Ω is finite, then Γ does not contain multiple arrows by Lemma 4.3.2. Moreover, by our assumptions Γ does not contain an arrow $X \to X$, so Γ is a stable translation quiver. Then by Theorem 4.2.13 the tree type of Γ is a Dynkin diagram.

On the other hand, suppose that Γ_s does not contain multiple arrows and the tree type of each connected component Γ of Γ_s is a Dynkin diagram. Γ is a translation quiver as it neither contains multiple arrows nor an Ω-irreducible morphism $f : X \to X$. By Corollary 4.2.9 the subgraph type of Γ equals the tree type of Γ and, therefore, must be given by a Dynkin diagram. Since the numbers of projective modules and Ext-injective modules in Γ_Ω coincide and every projective module is contained in a finite orbit, every left stable orbit is in fact stable. It follows that $\Gamma_s = \Gamma_l$ and the left subgraph type of each connected component is also given by a Dynkin diagram. Thus Ω is finite by Theorem 4.3.6. $\qquad\square$

The following example shows why we need to assume that every projective module is contained in a finite orbit.

Example 4.3.8 *Let A be the path algebra of the quiver*

$$\alpha \,\circlearrowright\, e_1 \,\circlearrowleft\, \beta$$

where all paths of length greater than 1 are zero.

It is not hard to see that the only indecomposable injective and projective modules have the following Jordan-Hölder decompositions.

$$P_1 = \begin{matrix} & S_1 & \\ S_1 & & S_1 \end{matrix} \qquad I_1 = \begin{matrix} S_1 & & S_1 \\ & S_1 & \end{matrix}$$

Computing the Auslander-Reiten translate of S_1 we obtain

$$\tau(S_1) = \begin{array}{ccc} S_1 & S_1 & S_1 \\ & S_1 & S_1 \end{array}$$

Therefore, the connected component Γ of Γ_A containing I_1, S_1 and P_1 looks as follows.

Note that the orbits of Γ are not adjacent to any other orbits since P_1 and I_1 are the only indecomposable projective and injective modules respectively. Hence the middle term of all almost split sequences in Γ consists of two copies of the same module. It follows that the length of modules in the orbit of I is strictly increasing and given by the formula $l(\tau^n(I_1)) = 4(n+1) - 1$ for all $n \geq 0$. We deduce that I_1 and P_1 are in different orbits and, in particular, the orbit of I_1 is left stable. Consequently, Γ_l contains a connected component of the form

whose left subgraph type is $\widetilde{A}_1$. On the other hand, the stable part of this connected component consist only of the following orbit.

$$\cdots \longleftarrow \tau(S_1) \longleftarrow S_1 \longleftarrow \tau^{-1}(S_1) \longleftarrow \cdots$$

Hence its tree type is given by the Dynkin diagram A_1 although the component is infinite.

In Riedtmann's original paper [R80] she considered multiple arrows just as single arrows with non-trivial valuation. From that point of view every Auslander-Reiten quiver without loops is a translation quiver. The next example shows why we use a different approach to obtain equivalent statements in Theorem 4.3.7.

Example 4.3.9 *Let A be the path algebra of the quiver*

$$\alpha \, \substack{\curvearrowleft} \, e_1 \, \substack{\curvearrowright} \, \beta$$

with relations $\alpha^2 = \beta^2 = \alpha\beta - \beta\alpha = 0$.

Note that this algebra is isomorphic to the group algebra of the Klein four-group over a field of characteristic 2 [B91]. It is not hard to see that A is selfinjective, i.e. every projective module is also injective, and the only indecomposable projective and injective module has the following Jordan-Hölder decomposition.

$$P_1 = I_1 = \begin{matrix} & S_1 & \\ S_1 & & S_1 \\ & S_1 & \end{matrix}$$
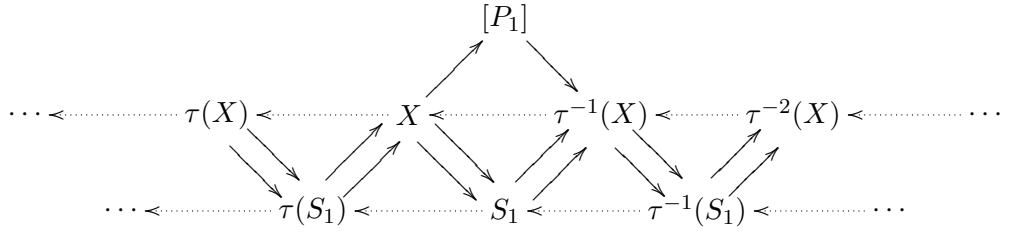
For convenience, let X denote the module $\mathrm{rad}(P_1)$. It follows from [ARS95, V. Proposition 5.5] that

$$0 \longrightarrow X \longrightarrow P_1 \oplus S_1 \oplus S_1 \longrightarrow \tau^{-1}(X) \longrightarrow 0$$

is an almost split sequence. Moreover, we compute

$$\tau(S_1) = \begin{matrix} S_1 & S_1 & S_1 \\ & S_1 & S_1 \end{matrix}$$

from which we deduce that the Auslander-Reiten quiver of the connected component containing P_1 looks as follows.

Let us denote the subgraph consisting of all stable modules in this component, i.e. every module except for P_1, by Γ. Then Γ is a connected component of both Γ_s and Γ_l and its left subgraph type is $\widetilde{A}_1$, which proves that the component is infinite by Theorem 4.3.6. If we consider the double arrows as single arrows with valuation, then the tree type of Γ is A_2. So Γ is infinite although its tree type is given by in Dynkin diagram in Riedtmann's point of view. This is why we consider connected components containing multiple arrows separately in Theorem 4.3.7.

Corollary 4.3.10 *[R80, Hauptsatz] Let A be an algebra of finite representation type. Then the tree type of each connected component of Γ_s is given by a Dynkin diagram.*

Proof:
The Corollary is an immediate consequence of Theorem 4.3.7 and Corollary 4.3.3. $\quad\square$

4.4 Coray tubes in functorially finite resolving subcategories

In this section we show that helical components in Γ_Ω are the same as coray tubes. Some of the statements are generalizations of results in [L93] and their proofs closely follow the original proofs in the aforementioned paper.

Definition 4.4.1 *Let Γ be a translation quiver. A vertex x of Γ is called a **coray vertex** if there is an infinite sectional path*

$$\cdots \longrightarrow x_n \longrightarrow x_{n-1} \longrightarrow \cdots \longrightarrow x_2 \longrightarrow x_1 = x$$

in Γ such that for each integer $n > 0$ the path

$$x_n \longrightarrow x_{n-1} \longrightarrow \cdots \longrightarrow x_2 \longrightarrow x_1 = x$$

*is the only sectional path of length n in Γ which ends at x. The aforementioned infinite sectional path is called a **coray** ending in x.*

Let x be a coray vertex in a translation quiver Γ with a coray as above. For a positive integer n we define the translation quiver $\Gamma[x, n]$ as follows. The vertices of $\Gamma[x, n]$ are those of Γ together with pairs (i, j) where $i \geq 1, 1 \leq j \leq n$. The arrows of $\Gamma[x, n]$ are those of Γ, excluding those ending at x_i other than $x_{i+1} \to x_i$ for $i \geq 1$, together with the following arrows:

1. $(i + 1, j) \to (i, j)$ for $i \geq 1, 1 \leq j \leq n$,

2. $(i, j + 1) \to (i + 1, j)$ for $i \geq 1, 1 \leq j \leq n$,

3. $(n + i - 1, 1) \to x_i$ for all $i \geq 1$,

4. $y \to (i, n)$ if $y \to x_i$ is an arrow in Γ other than $x_{i+1} \to x_i$.

Let τ be the translation of Γ. The translation τ' of $\Gamma[x, n]$ is defined as follows, if z is a vertex of Γ and $z \neq x_i$ for all $i \geq 1$ and $\tau(z)$ is defined, then $\tau'(z) = \tau(z)$. Moreover, we set $\tau'(x_i) = (n + i, 1)$ for all $i \geq 1$, $\tau'(i, j) = (i, j + 1)$ for $1 \leq j < n$, $i \geq 1$ and $\tau'(i, n) = \tau(x_i)$ if $\tau(x_i)$ is defined.

We inductively define a translation quiver $\Gamma[x_0, n_0][x_1, n_1] \ldots [x_r, n_r]$, where n_i are positive integers, x_0 is a coray vertex of Γ and x_i is a coray vertex of $\Gamma[x_0, n_0][x_1, n_1] \ldots [x_{i-1}, n_{i-1}]$. Where there is no need to emphasize the coray vertices we say that a translation quiver Γ is of the form $\Gamma[n_0, n_1, \ldots, n_r]$ if $\Gamma' \cong \Gamma[x_0, n_0][x_1, n_1] \ldots [x_r, n_r]$ for some coray vertices x_i. We say Γ' is a translation quiver obtained from Γ by **coray insertions**. Dually, one can define the concepts of ray vertices and ray insertions.

Definition 4.4.2

(a) *We call a translation quiver a **coray tube** if it is obtained from a stable tube by coray insertions.*

(b) *Dually, we call a translation quiver a **ray tube** if it is obtained from a stable tube by ray insertions.*

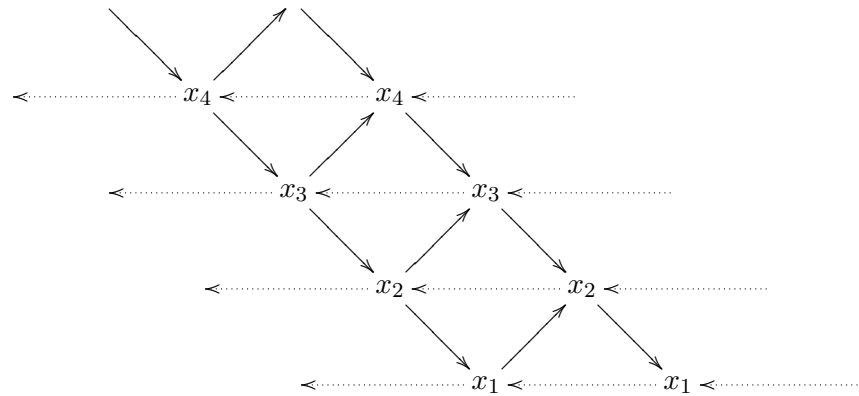**Example 4.4.3** *Let A be the path algebra of the quiver*

$$e_1 \quad \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \quad e_2 \quad \overset{\gamma}{\longrightarrow} \quad e_3 \ ,$$

with the relation $\gamma\beta = 0$. Then Γ_A contains a connected component which is a coray tube.
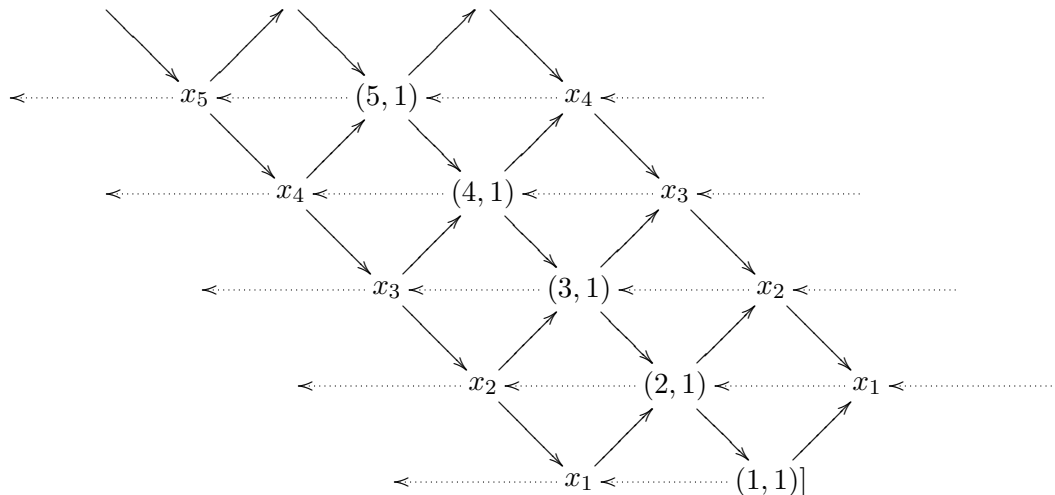
We have seen in Example 4.1.6 that the connected component Γ' of the injective module I_3 looks as follows.
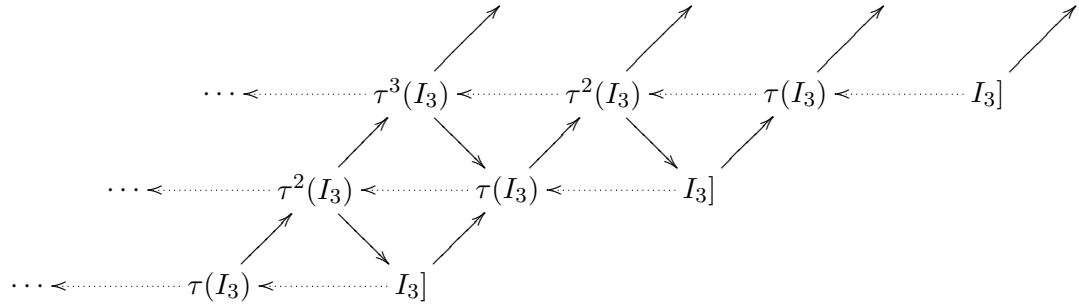
Γ' is obtained by coray insertions the following way. Given a stable tube Γ of type $\mathbb{Z}A_\infty/\tau$

such that x_1 is a coray vertex, then $\Gamma[x_1, 1]$

is a coray tube which is obviously isomorphic to Γ'.

Lemma 4.4.4 *Let Γ be a connected component of Γ_Ω with trivial valuation.*

(a) Assume that Γ does not contain projective modules and each module in Γ has at most two immediate predecessors. Suppose

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 = X$$

is an infinite sectional path in Γ containing an infinite number of arrows with finite global left degree. If n is a positive integer and $\tau_\Omega^{-n}(X_i)$ is defined for some $i \geq 1$, then $\tau_\Omega^{-n}(X_j)$ is also defined for all $j > i$. In particular, if some module X_i is stable, then all modules X_j with $j > i$ are also stable.

(b) Assume that Γ does not contain Ext-injective modules and each module in Γ has at most two immediate successors. Suppose

$$X = X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

is an infinite sectional path in Γ containing an infinite number of arrows with finite global right degree. If n is a positive integer and $\tau_\Omega^n(X_i)$ is defined for some $i \geq 1$, then $\tau_\Omega^n(X_j)$ is also defined for all $j > i$. In particular, if some module X_i is stable, then all modules X_j with $j > i$ are also stable.

Proof:

Suppose the statement is false, i.e. there are positive integers i and n for which $\tau_\Omega^{-n}(X_i)$ exists, but $\tau_\Omega^{-n}(X_{i+1})$ does not. Without loss of generality, we can assume that n is the least integer for which such an i exists. Hence Γ contains a subgraph which looks as follows.
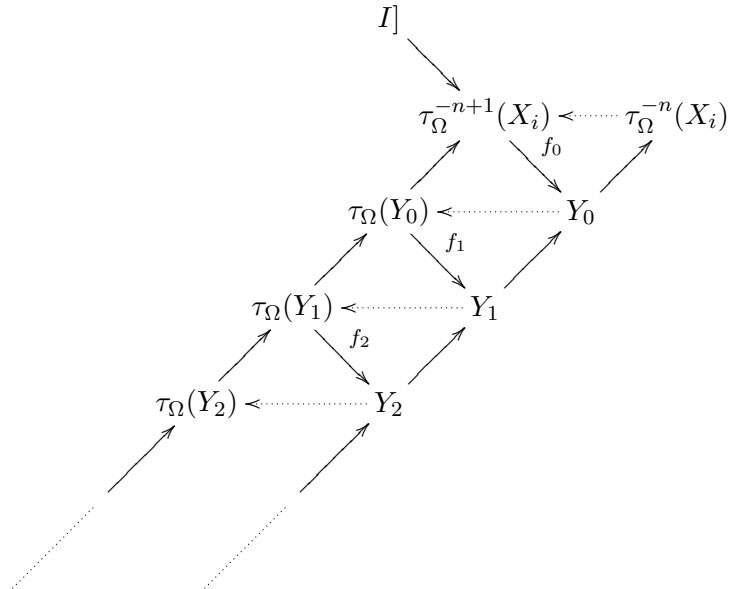
We prove by induction that there is a sectional path

$$Y_k \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow \tau_\Omega^{-n}(X_i)$$

for each length $k \geq 0$ and every irreducible morphism $f_i : \tau_\Omega(Y_{i-1}) \to Y_i$ is a monomorphism. Let

$$0 \longrightarrow \tau_\Omega^{-n+1}(X_i) \xrightarrow{f_0} Y_0 \longrightarrow \tau_\Omega^{-n}(X_i) \longrightarrow 0$$

be the almost split sequence ending in $\tau_\Omega^{-n}(X_i)$, then f_0 is a monomorphism. Since Γ does not contain projective modules and every module in Γ has at most 2 predecessors, Y_0 must be indecomposable, which proves the statement for $k = 0$. Suppose now the statement is true for an arbitrary k. Then there is a monomorphism $f_k : \tau_\Omega(Y_{k-1}) \to Y_k$, from which we conclude that the almost split sequence $\mathcal{E}_\Omega(Y_k)$ must have two indecomposable middle terms, $\tau_\Omega(Y_{k-1})$ and Y_{k+1}. Then by Lemma 2.5.3 we can construct a short exact sequence

$$0 \longrightarrow \tau_\Omega(Y_k) \xrightarrow{f_{k+1}} Y_{k+1} \longrightarrow \tau_\Omega^{-n}(X_i) \longrightarrow 0,$$

which gives us that the irreducible morphism $f_{k+1} : \tau_\Omega(Y_k) \to Y_{k+1}$ is again a monomorphism. As a consequence, there is an infinite sectional path

$$\cdots \longrightarrow Y_k \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0$$

ending in Y_0. However, as we have chosen n to be minimal and $Y_1 \not\cong \tau_\Omega^{-n+1}(X_i)$, there is another infinite sectional path

$$\cdots \longrightarrow \tau_\Omega^{-n+1}(X_{i+l}) \longrightarrow \cdots \longrightarrow \tau_\Omega^{-n+1}(X_{i+1}) \longrightarrow \tau_\Omega^{-n+1}(X_i) \longrightarrow Y_0.$$

Since Γ does not contain projective modules, we obtain two infinite sectional paths

$$\cdots \longrightarrow X_k \longrightarrow \cdots \longrightarrow X_{i-1} \longrightarrow X_i$$

and

$$\cdots \longrightarrow \tau_\Omega^n(Y_k) \longrightarrow \cdots \longrightarrow \tau_\Omega^n(Y_1) \longrightarrow \tau_\Omega^n(Y_0) \longrightarrow X_i$$

in which by Lemma 3.3.4 each arrow has infinite global left degree since $\tau_\Omega^n(Y_0) \not\cong X_{i-1}$. It follows that the infinite sectional path

$$\cdots \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 = X$$

contains only finitely many arrows of finite global left degree, which contradicts our assumptions. $\qquad\square$

Theorem 4.4.5 *Let Γ be a connected component of Γ_Ω containing an oriented cycle but no τ_Ω-periodic module.*

(a) If Γ does not contain a projective module, then Γ is a coray tube.

(b) Dually, if Γ does not contain an Ext-injective module, then Γ is a ray tube.

Proof:
By Theorem 3.3.7 we know that there is an infinite path of type A_∞ in Γ which is a full sectional subgraph. Moreover, the theorem says that Γ contains only finitely many different orbits and at least one of them is not right stable.

Suppose now that Γ contains a stable orbit, then its connected component Γ' in the right stable Auslander-Reiten quiver Γ_r is of right subgraph type A_n for some $n < s$, where s denotes the number of different τ_Ω-orbits in Γ. Then by Theorem 4.1.15 the component Γ' is finite and, in particular, τ_Ω-periodic, which contradicts our assumptions. Hence every orbit contains an Ext-injective module. Let

$$\cdots \longrightarrow \tau_\Omega^{2r}(X_1) \longrightarrow \tau_\Omega^r(X_s) \longrightarrow \cdots \longrightarrow \tau_\Omega^r(X_1) \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_1$$

be an infinite sectional path in Γ, which exists by Lemma 3.3.6. Then there exist positive integers $n_1, \ldots, n_s$ such that $X_j = \tau_\Omega^{n_j}(I_j)$ for some Ext-injective modules $I_1, \ldots, I_s$. It follows from Lemma 4.4.4 that $n_1 \leq n_2 \leq \cdots \leq n_s$. Let $m_1, m_2, \ldots, m_t$ be positive integers such that

$$n_1 = \cdots = n_{m_1} < n_{m_1+1} = \cdots = n_{m_2} < \cdots < n_{m_{t-1}+1} = \cdots = n_{m_t} = n_s.$$

It then follows that Γ as a translation quiver is of the form

$$(\mathbb{Z}A_\infty/\tau^n)[m_1, m_2 - m_1, \ldots, m_t - m_{t-1}],$$

where τ denotes the translation in $\mathbb{Z}A_\infty$ and $n = r - s$. $\qquad \square$

Lemma 4.4.6 *Let $n_1, n_2, \ldots, n_t, n_{t+1}$ be positive integers for some $t \geq 0$.*

(a) *If the coray tube $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t]$ contains a path from some vertex x to another vertex y, then $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t, n_{t+1}]$ also contains a path from x to y.*

(b) *Dually, if the ray tube $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t]$ contains a path from some vertex x to another vertex y, then $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t, n_{t+1}]$ also contains a path from x to y.*

Proof:
Let z_1 be the coray vertex where we insert n_{t+1} and let

$$\cdots \longrightarrow z_{i+1} \longrightarrow z_i \longrightarrow \cdots \longrightarrow z_2 \longrightarrow z_1$$

be the corresponding coray, while

$$\gamma: \qquad x = x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_{m-1} \longrightarrow x_m = y$$

denotes the path from x to y. If $x_{j-1} \to x_j$ is an arrow in γ such that either $x_j \neq z_i$ for all $i \geq 1$ or $x_{j-1} \to x_j = z_{i+1} \to z_i$ for some $i \geq 1$, then by definition the same arrows exist in $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t, n_{t+1}]$. On the other hand, if $x_{j-1} \to x_j$ is an arrow in γ such that $x_j = z_i$ but $x_{j-1} \neq z_{i+1}$, then there is path

$$x_{j-1} \longrightarrow (i, n_{t+1}) \longrightarrow (i+1, n_{t+1} - 1) \longrightarrow \cdots$$

$$\cdots \longrightarrow (n_{t+1} + i - 2, 2) \longrightarrow (n_{t+1} + i - 1, 1) \longrightarrow x_j$$

in $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t, n_{t+1}]$, which completes the proof. $\qquad \square$

Theorem 4.4.7 *Let Γ be a connected component of Γ_Ω.*

(a) Suppose Γ is left stable. Then Γ is helical if and only if it is a coray tube.

(b) Dually, if Γ is right stable, then Γ is cohelical if and only if it is a ray tube.

Proof:

Let Γ be helical, then by Theorem 4.1.6 it contains an oriented cycle. Moreover, since helical components always contain an Ext-injective module, there cannot be a τ_Ω-periodic module in Γ by Lemma 3.2.5. We deduce that Γ is a coray tube by Theorem 4.4.5.

Suppose now that Γ is a coray tube of the form

$$(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t].$$

We prove by induction on t that there is a vertex i in $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t]$ such that $\tau^{-1}(i)$ does not exist and there is a path from i to any vertex y in $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t]$. If $t = 1$, then we consider a coray tube of the form $(\mathbb{Z}A_\infty/\tau^n)[n_1]$ and let z_1 denote the coray vertex of the insertion n_1. Since $\mathbb{Z}A_\infty/\tau^n$ is a stable tube, there is a path from z_1 to y by Lemma 3.2.3 for every vertex y in $\mathbb{Z}A_\infty/\tau^n$. Additionally, there is also a path from z_1 to y in $(\mathbb{Z}A_\infty/\tau^n)[n_1]$ by Lemma 4.4.6. On the other hand, $\tau^{-1}(i)$ does not exist for the vertex $i = (n_1, 1)$ in $(\mathbb{Z}A_\infty/\tau^n)[n_1]$ by construction and there is an arrow $i \to z_1$ in $(\mathbb{Z}A_\infty/\tau^n)[n_1]$, which gives us a path from i to y.

Let y now be a vertex in $(\mathbb{Z}A_\infty/\tau^n)[n_1]$ that has been added in the coray insertion. Hence y is a vertex of the form (k, l) with $k \in \mathbb{N}$ and $1 \le l \le n_1$. If

$$\cdots \longrightarrow z_{i+1} \longrightarrow z_i \longrightarrow \cdots \longrightarrow z_2 \longrightarrow z_1$$

is the sectional path for the coray vertex z_1, then there is a path

$$\tau(z_k) \longrightarrow (k+1, n_1) \longrightarrow (k+2, n_1 - 1) \longrightarrow \cdots$$

$$\cdots \longrightarrow (k + n_1 - l, l + 1) \longrightarrow (k + n_1 - l + 1, l) \longrightarrow (k + n_1 - l, l) \longrightarrow \cdots$$

$$\cdots \longrightarrow (k+1, l) \longrightarrow (k, l) = y.$$

As we have seen earlier in this proof, there is a path from i to $\tau(z_k)$ that we now compose to a path from i to y. This completes the proof for $t = 1$.

Suppose the statement is true for some $t \ge 1$, then for every vertex y in the coray tube $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t]$ there is path from i to y, where i is a vertex such that $\tau^{-1}(i)$ does not exist. Consequently, there exists a path from i to y in $(\mathbb{Z}A_\infty/\tau^n)[n_1, n_2, \ldots, n_t, n_{t+1}]$ by Lemma 4.4.6. If $y = (k, l)$ is a vertex obtained by the coray insertion of n_{t+1} at the coray

$$\cdots \longrightarrow z_{i+1} \longrightarrow z_i \longrightarrow \cdots \longrightarrow z_2 \longrightarrow z_1,$$

then there is a path

$$\gamma: \quad \tau(z_k) \longrightarrow (k+1, n_{t+1}) \longrightarrow (k+1, n_{t+1} - 1) \longrightarrow \cdots$$

$$\cdots \longrightarrow (k + n_{t+1} - l, l-1) \longrightarrow (k + n_{t+1} - l + 1, l) \longrightarrow (k + n_{t+1} - l, l) \longrightarrow \cdots$$

$$\cdots \longrightarrow (k+1, l) \longrightarrow (k, l) = y.$$

By the induction hypothesis we conclude that there exists a path from i to $\tau(z_k)$ in $(\mathbb{Z}A_\infty / \tau^n)[n_1, n_2, \ldots, n_t]$ which by Lemma 4.4.6 gives rise to a path from i to $\tau(z_k)$ in $(\mathbb{Z}A_\infty / \tau^n)[n_1, n_2, \ldots, n_t, n_{t+1}]$. Hence we obtain a path from i to y by composing this path with γ, which proves that the statement is true for all $t \in \mathbb{N}$. Since Γ is a coray tube of the form $(\mathbb{Z}A_\infty / \tau^n)[n_1, n_2, \ldots, n_t]$, we deduce that for every module Y there is an Ext-injective module I such that there is a path from I to Y, which makes Γ a helical component. $\qquad\square$

4.5 Degrees of irreducible morphisms in functorially finite resolving subcategories

In this section we analyze Auslander-Reiten quivers of functorially finite resolving subcategories using degrees of irreducible morphisms. In particular, we observe under which assumptions we have $l(\tau_\Omega^n(X)) \to \infty$ as $n \to \infty$ for a module X in an infinite connected component. We give a complete description of all infinite connected components such that $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to \infty$. This section again closely follows [L92].

Lemma 4.5.1 *Let $f : X \to Y$ be an irreducible morphism in a functorially finite resolving subcategory Ω. Then f is a surjective minimal right almost split morphism if and only if $d_\Omega^l(f) = 1$ and all direct summands of Y are not projective.*

Proof:
We first assume that f is a surjective minimal right almost split morphism. Consequently, no direct summand of Y can be projective as f is surjective. Moreover, Y is indecomposable as f is an irreducible minimal right almost split morphism. So we compose f with the corresponding minimal left almost split morphism for $\tau_\Omega(Y)$ and obtain $d_\Omega^l(f) = 1$.

On the other hand, let $d_\Omega^l(f) = 1$ and suppose no direct summand of Y is projective. If Y is indecomposable, then f is a surjective minimal right almost split morphism by Corollary 3.1.3. In case that Y admits a direct summand $Y_1 \oplus Y_2$ we know that X is indecomposable as f is irreducible. Let $(f_1, f_2)^T : X \to Y_1 \oplus Y_2$ be the corestriction of f to $Y_1 \oplus Y_2$, then $d_\Omega^l((f_1, f_2)^T) = 1$. By Lemma 3.1.9 a left neighbour h of $(f_1, f_2)^T$ exists with $d_\Omega^l(h) < 1$, which is impossible. $\qquad\square$

Definition 4.5.2 *Let Q be a quiver without loops or cycles and let KQ be the hereditary path algebra of Q. Suppose Q has n vertices and let $p_1, \ldots, p_n$ and $i_1, \ldots, i_n$ denote the dimension vectors of the indecomposable projective and injective KQ-modules respectively. Then the **Coxeter-matrix of Q** is the matrix C defined via $C(i_j) = -p_j$.*

Note that C is invertible and that its inverse satisfies $C^{-1}p_j = -i_j$. Since the inverse of the Coxeter-matrix is more convenient for us, we set $C^{-1} = (c_{ij})$. Before we provide a combinatorial description of the entries in C^{-1} in terms of paths in Q, we need to establish some more notation. Let Q be a quiver with vertices $e_1, \ldots, e_n$. We say that e_j

is a predecessor of e_i if there is a non-trivial path from e_j to e_i in Q. If this path consists of only one arrow, then e_i is an immediate predecessor of e_j. Successors and immediate successors are defined dually. We denote an arrow from e_j to e_i in Q by $e_j \to e_i$, a path of arbitrary length is denoted by $\gamma : e_j \to e_i$. For example, for a fixed vertex e_i

$$\sum_{e_i \to e_j} \sum_{\gamma : e_k \to e_j}$$

is the sum over all paths in Q that end in an immediate successor of e_i. Moreover, for any column vector v we denote its i-th entry by $(v)_i$. This and other upcoming terminology is fixed for the rest of the section.

Lemma 4.5.3 *Let Q be a quiver without loops or cycles. If Q contains vertices $e_1, \ldots, e_n$ and $C^{-1} = (c_{ij})$ denotes the inverse of its Coxeter-matrix, then the entry c_{ij} is given by*

$$c_{ij} = -\sharp \text{ of paths from } e_i \text{ to } e_j + \Big(\sum_{e_j \to e_k} \sharp \text{ of paths from } e_i \text{ to } e_k \Big)$$

In particular, we have the following.

(a) $c_{ij} = 0$ if there is neither a path from e_i to e_j nor to an immediate predecessor of e_j.

(b) $c_{ij} = 0$ if e_j is a non-immediate predecessor of e_i.

(c) $c_{ij} = \sharp$ of arrows from e_j to e_i if e_j is an immediate predecessor of e_i.

If Q does not contain multiple arrows between two vertices then we also have the following.

(d) $c_{ij} = 1$ if e_j is an immediate predecessor of e_i.

(e) $c_{ij} = \sharp$ of arrows starting in $e_j - 1$ if $e_j = e_i$ or e_j is a successor of e_i.

(f) In particular, if e_j is a successor of e_i, we have $c_{ij} = c_{jj}$.

Dually, if we set $C = (d_{ij})$, then

$$d_{ij} = -\sharp \text{ of paths from } e_j \text{ to } e_i + \Big(\sum_{e_k \to e_j} \sharp \text{ of paths from } e_k \text{ to } e_i \Big).$$

Proof:

Let $P_1, \ldots, P_n$ and $I_1, \ldots, I_n$ denote the indecomposable projective and injective KQ-modules respectively. Alongside with p_j and i_j for dimension vectors of indecomposable projective and injective KQ-modules, we denote the dimension vector of the simple module $S_j = P_j/\text{rad}(P_j)$ by s_j. Clearly, c_{ij} is given by the i-th entry of the column vector of $C^{-1} s_j$. Note that since KQ is an hereditary algebra, P_j is given by all paths that start in e_j, while I_j is the module consisting of all paths ending in e_j.

Hence the dimension vector of a simple module can be expressed as $s_j = p_j - \text{rad}(p_j) = p_j - \sum_{e_j \to e_k} p_k$. By definition we have $C^{-1} p_k = -i_k$, so the i-th entry of this column vector is -1 times the number of paths from e_i to e_k. Hence

$$c_{ij} = (C^{-1} s_j)_i = (C^{-1} p_j)_i - \sum_{e_j \to e_k} (C^{-1} p_k)_i = \sum_{e_j \to e_k} (i_k)_i - (i_j)_i$$

$$= -\sharp \text{ of paths from } e_i \text{ to } e_j + (\sum_{e_j \to e_k} \sharp \text{ of paths from } e_i \text{ to } e_k).$$

The statements $(a), (b)$ and (c) are special cases of the formula above and follow immediately from the fact that Q contains neither loops nor cycles. Moreover, if Q in addition does not contain multiple arrows, then there is precisely one path from e_i to e_j if and only if $e_i = e_j$ or e_j is a successor of e_i. In all other cases there is no path from e_i to e_j, from which we deduce statements $(d), (e)$ and (f). $\qquad\square$

Let Γ be a connected component of Γ_Ω containing a sectional subgraph Σ that consists of modules $M_1, \ldots, M_n$ without multiple arrows between them. Since Σ neither contains loops nor cycles, there is a Coxeter-matrix $C = (d_{ij})$ of Σ and let $C^{-1} = (c_{ij})$ again denote its inverse. Furthermore, whenever Σ occurs in this section, let m denote the column vector $(l(M_1), \ldots, l(M_n))^T$. Then the i-th entry of $C^{-1}m$ is given by $(C^{-1}m)_i = \sum_{j=1}^n c_{ij}l(M_j)$. We split up the sum on the right hand side according to Lemma 4.5.3 $(d), (e)$ and obtain

$$(C^{-1}m)_i = \sum_{M_k \to M_i} l(M_k) + \sum_{\gamma: M_i \to M_k} c_{ik}l(M_k) = \sum_{M_k \to M_i} l(M_k) + \sum_{\gamma: M_i \to M_k} c_{kk}l(M_k),$$

where the last equality follows from Lemma 4.5.3 (f). This formula is used frequently in the following proofs.

Lemma 4.5.4 *Let M_i be a module in Σ, then we have*

(a)

$$\sum_{M_i \to M_j} (C^{-1}m)_j = l(M_i) + \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) + \sum_{\gamma: M_i \to M_k} c_{kk}l(M_k)$$

and

(b)

$$\sum_{M_j \to M_i} (Cm)_j = l(M_i) + \sum_{M_j \to M_i} \sum_{\substack{M_j \to M_k \\ k \neq i}} l(M_k) + \sum_{\gamma: M_k \to M_i} d_{kk}l(M_k).$$

Proof:

If M_i is a module without successors in Σ, then the left hand side of the equation is zero, while the right hand side is reduced to $l(M_i) + \sum_{\gamma: M_i \to M_k} c_{kk}l(M_k)$. Since only the trivial path starts in M_i, the last sum contains only one summand $c_{ii}l(M_i) = -l(M_i)$ by Lemma 4.5.4 and hence the right hand side is also zero. Let M_i now be any module in Σ, then we have

$$\sum_{M_i \to M_j} (C^{-1}m)_j = \sum_{M_i \to M_j} \left(\sum_{M_k \to M_j} l(M_k) + \sum_{\gamma: M_j \to M_k} c_{kk}l(M_k) \right)$$

$$= \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) + \sum_{M_i \to M_j} \left(l(M_i) + \sum_{\gamma: M_j \to M_k} c_{kk}l(M_k) \right)$$

$$= \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) + (c_{ii} + 1)l(M_i) + \sum_{M_i \to M_j} \sum_{\gamma: M_j \to M_k} c_{kk}l(M_k))$$

$$= l(M_i) + \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) + \sum_{\gamma : M_i \to M_k} c_{kk} l(M_k)$$

$\square$

Corollary 4.5.5 *For a module M_i in Σ we have*

(a)

$$(C^{-1}m)_i = -l(M_i) + \sum_{M_j \to M_i} l(M_j) + \sum_{M_i \to M_j} (C^{-1}m)_j - \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k)$$

and

(b)

$$(Cm)_i = -l(M_i) + \sum_{M_i \to M_j} l(M_j) + \sum_{M_j \to M_i} (Cm)_j - \sum_{M_j \to M_i} \sum_{\substack{M_j \to M_k \\ k \neq i}} l(M_k).$$

Proof:
Again we start by considering the case where M_i does not have successors in Σ. Then we have

$$-l(M_i) + \sum_{M_j \to M_i} l(M_j) + \sum_{M_i \to M_j} (C^{-1}m)_j - \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k)$$

$$= -l(M_i) + \sum_{M_j \to M_i} l(M_j) = (C^{-1}m)_i.$$

For an arbitrary M_i the statement follows from applying Lemma 4.5.3 to the right hand side of the equation, giving

$$-l(M_i) + \sum_{M_j \to M_i} l(M_j) + \sum_{M_i \to M_j} (C^{-1}m)_j - \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k)$$

$$= \sum_{M_j \to M_i} l(M_j) + \sum_{\gamma : M_i \to M_k} c_{kk} l(M_k) = (C^{-1}m)_i.$$

$\square$

Theorem 4.5.6 *Let Γ be a connected component of Γ_Ω containing a sectional subgraph Σ that consists of modules $M_1, \ldots, M_n$ without multiple arrows between them. Let $C^{-1} = (c_{ij})$ be the inverse of the Coxeter-matrix C of Σ. Furthermore, denote the column vectors $(l(M_1), \ldots, l(M_n))^T$, $(l(\tau_\Omega(M_1)), \ldots, l(\tau_\Omega(M_n)))^T$ and $(l(\tau_\Omega^{-1}(M_1)), \ldots, l(\tau_\Omega^{-1}(M_n)))^T$ by m, $\tau_\Omega(m)$ and $\tau_\Omega^{-1}(m)$ respectively.*

(a) If none of the M_i is projective, then

$$\tau_\Omega(m) = C^{-1}m + \dim(I),$$

where I is an injective $K\Sigma$-module.

(b) Dually, if none of the M_i is Ext-injective, then

$$\tau_\Omega^{-1}(m) = Cm + \dim(P),$$

where P is a projective $K\Sigma$-module.

Proof:

We use the same notation as before and prove

$$\tau_\Omega(m)_i = l(\tau_\Omega(M_i)) = (C^{-1}m)_i + \sum_{\gamma: M_i \to M_u} C_u,$$

where C_u are non-negative integers only depending on u. Let us define $M_i > M_j$ if there is a path from M_j to M_i and let M_i be a maximal module in this sense. We use induction on this ordering to prove the equation above. Consider the almost split sequence

$$0 \longrightarrow \tau_\Omega(M_i) \longrightarrow \bigoplus_{M_j \to M_i} M_j \oplus T \longrightarrow M_i \longrightarrow 0.$$

Therefore, we have

$$l(M_i) + l(\tau_\Omega(M_i)) = \sum_{M_j \to M_i} l(M_j) + l(T).$$

We reorder the summands in Corollary 4.5.5 and obtain

$$l(M_i) = -(C^{-1}m)_i + \sum_{M_j \to M_i} l(M_j)$$

since M_i does not have any successors. Substituting this into the previous equation we obtain

$$-(C^{-1}m)_i + \sum_{M_j \to M_i} l(M_j) + l(\tau_\Omega(M_i)) = \sum_{M_j \to M_i} l(M_j) + l(T).$$

We easily conclude $l(\tau_\Omega(M_i)) = (C^{-1}m)_i + l(T)$ and observe that we want $l(T) = \sum_{\gamma: M_i \to M_u} C_u$. Note that this sum contains only the summand obtained by the trivial path from M_i to M_i, hence we set $C_i = l(T)$, which is a non-negative integer. This shows the claim for a maximal M_i. Suppose now M_i is a module such that the statement has been proved for all its successors. There is an almost split sequence

$$0 \longrightarrow \tau_\Omega(M_i) \longrightarrow \bigoplus_{M_j \to M_i} M_j \bigoplus_{M_i \to M_j} \tau_\Omega(M_j) \oplus T_i \longrightarrow M_i \longrightarrow 0$$

from which we obtain

$$l(M_i) + l(\tau_\Omega(M_i)) = \sum_{M_j \to M_i} l(M_j) + \sum_{M_i \to M_j} l(\tau_\Omega(M_j)) + l(T_i).$$

We apply the induction hypothesis to all $l(\tau_\Omega(M_j))$ and obtain

$$l(M_i) + l(\tau_\Omega(M_i)) = \sum_{M_j \to M_i} l(M_j) + l(T_i) + \sum_{M_i \to M_j} \left((C^{-1}m)_j + \sum_{\gamma: M_j \to M_u} C_u \right).$$

In order to be able to use Corollary 4.5.5, we reorder the summands, add

$$\sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) - \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k)$$

to the equation and deduce

$$l(\tau_\Omega(M_i)) = -l(M_i) + \sum_{M_j \to M_i} l(M_j) + \sum_{M_i \to M_j} ((C^{-1}m)_j - \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k)$$

$$+ \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) + l(T_i) + \sum_{M_i \to M_j} \sum_{\gamma: M_j \to M_u} C_u$$

$$= (C^{-1}m)_i + \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) + l(T_i) + \sum_{\substack{\gamma: M_i \to M_u \\ i \neq u}} C_u.$$

We set

$$C_i = \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) + l(T_i),$$

which obviously is a non-negative integer, and obtain

$$\tau_\Omega(m)_i = l(\tau_\Omega(M_i)) = (C^{-1}m)_i + \sum_{\gamma: M_i \to M_u} C_u.$$

For convenience, let $v \in \mathbb{Z}^n$ denote the column vector whose i-th entry is given by $\sum_{\gamma: M_i \to M_u} C_u$. Consequently, we have

$$\tau_\Omega(m) = C^{-1}m + v,$$

so it remains to show that v is the dimension vector of an injective $K\Sigma$-module. Let p_{ij} denote the number of paths from M_i to M_j in Σ. Then we have

$$v = \sum_{u=1}^{n} (p_{1u}C_u, \ldots, p_{nu}C_u)^T = \sum_{u=1}^{n} C_u(p_{1u}, \ldots, p_{nu})^T,$$

but $(p_{1u}, \ldots, p_{nu})^T$ is precisely the dimension vector of the injective $K\Sigma$-module I_u corresponding to the vertex M_u since $K\Sigma$ is a hereditary algebra. This finishes the proof. □

For the rest of the section we choose a sectional subgraph Σ such that none of its vertices has more than one immediate predecessor, for example by considering a sectional subgraph that consists of all sectional paths starting in a fixed module. As a consequence, the formula

$$C_i = \sum_{M_i \to M_j} \sum_{\substack{M_k \to M_j \\ k \neq i}} l(M_k) + l(T_i)$$

reduces to $C_i = l(T_i)$, where T_i is again the module that completes the middle term of $\mathcal{E}_\Omega(M_i)$ in the following sense.

$$0 \longrightarrow \tau_\Omega(M_i) \longrightarrow \bigoplus_{M_j \to M_i} M_j \bigoplus_{M_i \to M_j} \tau_\Omega(M_j) \oplus T_i \longrightarrow M_i \longrightarrow 0$$

We deduce that in this case $\tau_\Omega(m) = C^{-1}m$ if and only if Σ is a full sectional subgraph or there are some indecomposable projective modules $P_1, \ldots, P_k$ and arrows $M_{i_j} \to P_j$ in Γ such that the union of Σ with these modules and arrows is a full sectional subgraph.

Definition 4.5.7 *Let Q be a quiver, then a map $\lambda : \mathbb{Z}Q \to \mathbb{Z}$ is called a subadditive function on $\mathbb{Z}Q$ if*

$$\lambda(i, k) + \lambda(i, k-1) \geq \sum_{(i,k) \to (j,k')} \lambda(j, k')$$

for all $(i, k) \in \mathbb{Z}Q$.

Note that this definition is different from the definition of subadditive functions on simple-laced graphs, which we introduced in Section 4.2. Let Γ be a connected component of Γ_l containing a sectional subgraph Σ consisting of modules $M_1, \ldots, M_n$. We can then define a subadditive function on $\mathbb{Z}\Sigma$ in the following way. First of all we set $\lambda(M_i, k) = l(\tau_\Omega^k(M_i))$ for all M_i and $k \in \mathbb{Z}$ such that $\tau_\Omega^k(M_i)$ exists. Note that since Γ is left stable, $\tau_\Omega^k(M_i)$ always exists for $k \geq 0$. If $\tau_\Omega^k(M_i)$ does not exist for some M_i and $k < 0$, we set $\lambda(i, k-1) = -\lambda(i, k) + \sum_{(i,k) \to (j,k')} \lambda(j, k')$. Hence we have inductively defined a subadditive function on $\mathbb{Z}\Sigma$.

Recall that the **defect** of a quiver Q is the lowest positive integer $d = d_Q$ such that $C^{-d}x - x$ is in rad_q for all $x \in \mathbb{Z}^n$, where q is the Euler quadratic form of the underlying graph of Q with undirected edges. The **defect of a vector** x is the integer $\partial(x)$ such that $C^{-d}x - x = \partial(x)h_q$, where h_q is the generator of rad_q. Note that $\partial : \mathbb{Z}^n \to \mathbb{Z}$ is clearly a morphism of abelian groups. For further detail we refer to [SS07, XI.1]. Analogously to sectional subgraphs, the type of a quiver Q is the undirected graph associated to Q.

Lemma 4.5.8 *[BB83, Proposition 4.4] Let Q be a quiver of Euclidean type with n vertices and let λ be a subadditive function on $\mathbb{Z}Q$. We denote the vector $(\lambda(1, k), \ldots, \lambda(n, k))$ by λ_k. For convenience, for two vectors $x, y \in \mathbb{Z}^n$ we say $x > y$ if and only if $x_i > y_i$ for all $i = 1, \ldots, n$.*

(a) If $\lambda_k > 0$ for $0 \leq k < n$ and $\partial(\lambda_0) \geq 0$, then $\lambda_k > 0$ for all $k \geq 0$. Moreover, if $\partial(\lambda_l) > 0$ for some $l \geq 0$, then $\lambda(i, k) \to \infty$ when $k \to \infty$ for all $i = 1, \ldots, n$.

(b) Dually, if $\lambda_k > 0$ for $0 \geq k > -n$ and $\partial(\lambda_0) \leq 0$, then $\lambda_k > 0$ for all $k \leq 0$. Moreover, if $\partial(\lambda_l) < 0$ for some $l \leq 0$, then $\lambda(i, k) \to \infty$ when $k \to -\infty$ for all $i = 1, \ldots, n$.

Lemma 4.5.9 *[SS07, XI. Proposition 1.2] Let M be an indecomposable $K\Sigma$-module.*

(a) If M is not contained in the preprojective component of $\Gamma_{K\Sigma}$, then we have

$$\dim(\tau^k(M)) = C^{-k}\dim(M)$$

for all $k \geq 0$.

(b) Dually, if M is not contained in the preinjective component of $\Gamma_{K\Sigma}$, then we have

$$\dim(\tau^{-k}(M)) = C^k \dim(M)$$

for all $k \geq 0$.

Lemma 4.5.10 *[SS07, XI. Lemma 1.5] Let Q be a Euclidean quiver with n vertices and let C^{-1} denote the inverse of its Coxeter matrix. Then*

$$\partial(x) = \partial(C^{-1}(x))$$

for all $x \in \mathbb{Z}^n$.

Note that the following theorem is a generalization of Theorem 4.2.13.

Theorem 4.5.11 *Let Σ be a sectional subgraph of Euclidean type in a connected component Γ of Γ_Ω.*

(a) *If all modules in Σ are left stable, then $l(\tau_\Omega^k(X)) \to \infty$ as $k \to \infty$ for each vertex X of Σ.*

(b) *If all modules in Σ are right stable, then $l(\tau_\Omega^k(X)) \to \infty$ as $k \to -\infty$ for each vertex X of Σ.*

Proof:

If Σ contains multiple arrows, then the statement follows immediately from Theorem 4.3.2. Hence we can assume that Σ only contains arrows with trivial valuation. Since $\overline{\Sigma}$ is Euclidean, the Auslander-Reiten quiver of $K\Sigma$ has an infinite preinjective component. By Lemma 4.5.9 we have $\dim(\tau^k(I')) = C^{-k} \dim(I')$ for an indecomposable injective $K\Sigma$-module I' and all $k \in \mathbb{N}$, so $C^{-k} \dim(I')$ is always the dimension vector of a $K\Sigma$-module. Therefore, $C^{-d} \dim(I') - \dim(I')$ cannot have only negative entries as all entries in $\dim(I')$ are at most one by choice of Q.

If $C^{-d} \dim(I') - \dim(I') = 0$, then by Lemma 4.5.9 we obtain $l(\tau_\Omega^k(I')) \leq p_{I'}$ for all $k \geq 0$ and a constant $p_{I'}$. There are precisely n orbits in the preinjective component of $K\Sigma$ since n denotes the number of vertices in Σ and, as we have seen in Lemma 2.2.13, there is a constant p such that $l(Z') \leq pl(Z)$ for any arrow $Z' \to Z$ in the Auslander-Reiten quiver of Ω. Hence we have $l(X) \leq p^n p_{I'}$ for every module X in the preinjective component of $K\Sigma$. As a result, every morphism from a projective module P to I' must be zero by Lemma 2.2.8, which is a contradiction. Thus $\partial(\dim(I'))$ is a positive integer and the defect of any non-zero injective module is also positive.

By Theorem 4.5.6 and Lemma 4.5.10 we have $\partial(\tau_\Omega(m)) = \partial(C^{-1}m) + \partial(\dim(I)) = \partial(m) + \partial(\dim(I))$ and, consequently, $\partial(\tau_\Omega^{k'}(m)) \geq \partial(\tau_\Omega^k(m))$ if $k' > k$. Suppose that $\partial(\tau_\Omega^k(m)) \leq 0$ for all $k \geq 0$. Hence there must be an integer $r \geq 0$ such that for all $k \geq r$ we have $\partial(\tau_\Omega^k(m)) = \partial(\tau_\Omega^r(m)) \leq 0$. Without loss of generality, we can assume $r = 0$ and that Σ is a full sectional subgraph left of all injective and projective modules, which gives us $\tau_\Omega^k(m) = C^{-k}m$ for all $k \geq 0$.

If $\partial(m) < 0$, then there is a $t \in \mathbb{N}$ such that $0 > C^{-t}m = \tau_\Omega^t(m)$, which is impossible as each entry in the latter is given by the length of an A-module. Hence $\partial(m) = 0$, then by definition the entries of $C^{-k}m = \tau_\Omega^k(m)$ are bounded for all $k \geq 0$ as $C^{-d}m - m = 0$. We know there is a non-zero morphism $f \in \mathrm{rad}_\Omega^\infty(P, M_1)$ for a projective module P. Since Σ is a full sectional subgraph left of all injective and projective modules, f must factor over infinitely many modules of the form $\tau_\Omega^k(M_i)$ with $k \geq 0$. So by Lemma 2.2.8 f must be zero, which is a contradiction. Hence there is a $k \geq 0$ such that $\partial(\tau_\Omega^k(m)) > 0$. Let λ be the map that sends a module to its Jordan-Hölder length. If necessary, we extend λ to a subadditive function $\lambda : \mathbb{Z}\Sigma \to \mathbb{Z}$ in the aforementioned way. Then by Lemma 4.5.8 $l(\tau_\Omega^k(M_i)) = \lambda(k, M_i) \to \infty$ as $k \to \infty$, which completes the proof. $\qquad\square$

Lemma 4.5.12 *Let Γ be a connected component of Γ_l. Assume there is a module X in Γ such that $l(\tau_\Omega^n(X))$ does not tend to infinity for $n \to \infty$. Then*

(a) $l(\tau_\Omega^n(Y))$ does not tend to infinity as $n \to \infty$ for every module Y in Γ,

(b) there is no sectional subgraph of Euclidean type in Γ and

(c) every arrow $Z \to Y$ in Γ has finite global left degree.

Dually, let Γ be a connected component of Γ_r. Assume there is a module X in Γ such that $l(\tau_\Omega^n(X))$ does not tend to infinity for $n \to -\infty$. Then

(d) $l(\tau_\Omega^n(Y))$ does not tend to infinity as $n \to -\infty$ for every module Y in Γ,

(e) there is no sectional subgraph of Euclidean type in Γ and

(f) every arrow $Y \to Z$ in Γ has finite global right degree.

Proof:
As we have seen in Lemma 2.2.13, there is a constant p such that $l(Z') \leq pl(Z)$ for any arrow $Z' \to Z$ in the Auslander-Reiten quiver of Ω. Moreover, by our assumptions there is a constant p_X such that $l(\tau_\Omega^n(X)) \leq p_X$ for infinitely many positive integers n. Let Y be a module in Γ that is not in the τ_Ω-orbit of X, then we know there is a sectional path of length k from $\tau_\Omega^m(Y)$ to $\tau_\Omega^n(X)$ for some $n, m \in \mathbb{N}$ by Lemma 3.3.5 as Γ is left stable. If we set $p_Y = p_X p^k$, we have $l(\tau_\Omega^n(Y)) \leq p_Y$ for infinitely many $n \geq m$, which proves the first statement. From this and Theorem 4.5.11 we immediately deduce that Γ cannot contain a sectional subgraph of Euclidean type.

Let Y and Z be modules in Γ such that there is an arrow from Z to Y. By our assumptions there exists a constant $p_Y \in \mathbb{N}$ such that there are infinitely many $\tau_\Omega^n(Y)$ such that $l(\tau_\Omega^n(Y)) \leq p_Y$. Let $m \in \mathbb{N}$ be an integer such that $l(\tau_\Omega^n(Y)) \leq p_Y$ for more than 2^{p_Y} integers $0 \leq n \leq m$. Consequently, the path

$$\tau_\Omega^m(Z) \xrightarrow{f_m} \tau_\Omega^m(Y) \xrightarrow{g_m} \cdots \xrightarrow{g_2} \tau_\Omega(Z) \xrightarrow{f_1} \tau_\Omega(Y) \xrightarrow{g_1} Z \xrightarrow{f_0} Y$$

is labeled with irreducible morphisms such that $f_0 g_1 f_1 \cdots g_m f_m = 0$ by Lemma 2.2.8. Hence every arrow in Γ has finite global left degree. $\qquad\square$

Corollary 4.5.13

(a) *Let Γ be a connected component of Γ_l such that its left subgraph type is A_∞^∞. Then $l(\tau_\Omega^n(X)) \to \infty$ as $n \to \infty$ for all modules X in Γ.*

(b) *Dually, if Γ is a connected component of Γ_r such that its right subgraph type is A_∞^∞, then $l(\tau_\Omega^n(X)) \to \infty$ as $n \to -\infty$ for all modules X in Γ.*

Proof:
Since Γ is left stable and of left subgraph type A_∞^∞, for every module X in Γ there is an $n \in \mathbb{N}$ such that there are two infinite sectional paths

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_0 = X$$

and

$$\cdots \longrightarrow Y_{i+1} \longrightarrow Y_i \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_0 = X$$

ending in X with $Y_1 \not\cong X_1$. Consequently, by Lemma 3.3.4 all arrows $X_{i+1} \to X_i$ have infinite global left degree. Hence the Corollary follows immediately from the contrapositive of the previous lemma. $\qquad\square$

Lemma 4.5.14

(a) *Suppose there is a module X in a connected component Γ of Γ_l such that $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to \infty$. If there is a sectional path in Γ which meets some τ_Ω-orbit more than once, then Γ consists of τ_Ω-periodic modules.*

(b) *Dually, let X be a module in a connected component Γ of Γ_r such that $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to -\infty$. If there is a sectional path in Γ which meets some τ_Ω-orbit more than once, then Γ consists of τ_Ω-periodic modules.*

Proof:
Let p be the constant from Lemma 2.2.13 such that for all modules Y and Z we have $l(Y) \leq pl(Z)$ if there is an Ω-irreducible morphism from Y to Z. By Lemma 4.5.12 and our assumptions there is a constant p_Y for every module Y in Γ such that $l(\tau_\Omega^n(Y)) \leq p_Y$ for infinitely many positive integers n. It follows that for every $r \in \mathbb{N}$ there are infinitely many positive integers m such that $l(\tau_\Omega^{mr}(Y)) \leq p^{2r} p_Y$. Suppose that

$$\tau_\Omega^r(Y_0) = Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0$$

is a sectional path in Γ with some $r \geq 0$. Without loss of generality, we may assume that Y_{n-1} and Y_1 are not contained in the same τ_Ω-orbit if $n > 2$. Then there is an infinite path

$$\cdots \longrightarrow \tau_\Omega^{2r}(Y_1) \longrightarrow \tau_\Omega^{2r}(Y_0) \longrightarrow \tau_\Omega^r(Y_{n-1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\Omega^r(Y_1) \longrightarrow \tau_\Omega^r(Y_0) \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0.$$

Since there are infinitely many modules $\tau_\Omega^{mr}(Y)$ on this path with $l(\tau_\Omega^{mr}(Y)) \leq p^{2r} p_Y$, the path cannot be sectional by Corollary 2.4.7 and Lemma 2.2.8. Hence Y_{n-1} and Y_1 belong

to the same τ_Ω-orbit and n is either 1 or 2. If $n = 1$, then $Y_0 = \tau_\Omega^{2r-1}(Y_0)$. On the other hand, if $n = 2$, then $r \neq 1$ as the path

$$\tau_\Omega^r(Y_0) = Y_2 \longrightarrow Y_1 \longrightarrow Y_0$$

is sectional and we obtain $Y_1 = \tau_\Omega^{r-1}(Y_1)$. In both cases Γ contains a τ_Ω-periodic module, and hence Γ entirely consists of τ_Ω-periodic modules by Lemma 3.2.5 as Γ is left stable. For $r < 0$ we prove the statement similarly using the infinite path

$$\tau_\Omega^r(Y_0) \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow \tau_\Omega^{-r}(Y_{n-1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \tau_\Omega^{-r}(Y_1) \longrightarrow \tau_\Omega^{-r}(Y_0) \longrightarrow \tau_\Omega^{-2r}(Y_{n-1}) \longrightarrow \cdots$$

$\square$

Corollary 4.5.15

(a) Let Γ be a coray tube, then $l(\tau_\Omega^n(X)) \to \infty$ as $n \to \infty$ for all modules X in Γ.

(b) Let Γ be a ray tube, then $l(\tau_\Omega^n(X)) \to \infty$ as $n \to -\infty$ for all modules X in Γ.

Proof:
Suppose $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to \infty$. Since a coray tube is the same as a helical component by Theorem 4.4.7, we know there is an oriented cycle from an injective module I to itself in Γ. Since $\tau_\Omega^n(I) = I$ is only possible for $n = 0$, there is a sectional path from I to $\tau_\Omega^n(I)$ in Γ for some $n \geq 0$ by Lemma 3.3.5. Consequently, Γ consists of τ_Ω-periodic modules by the previous lemma, which is a contradiction. $\square$

Lemma 4.5.16 *Let X be a non-periodic module in Γ_Ω.*

(a) *If X is left stable and the almost split sequence $\mathcal{E}_\Omega(X)$ admits three left stable middle terms, then $l(\tau_\Omega^n(X)) \to \infty$ as $n \to \infty$.*

(b) *If X is right stable and the almost split sequence $\mathcal{E}'_\Omega(X)$ admits three right stable middle terms, then $l(\tau_\Omega^n(X)) \to \infty$ as $n \to -\infty$.*

Proof:
Let Γ be the connected component of X in Γ_l, which does not contain τ_Ω-periodic modules by Lemma 3.2.5. Suppose $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to \infty$, then Γ does not contain a sectional subgraph of Euclidean type by Lemma 4.5.12. Moreover, every sectional path in Γ meets each τ_Ω-orbit at most once by Lemma 4.5.14. Consequently, the left subgraph type of Γ must be either A_∞, A_∞^∞ or D_∞. In all cases there is a infinite sectional path

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = \tau_\Omega^r(X)$$

for some $r \geq 0$. Since $\mathcal{E}_\Omega(X)$ has three left stable middle terms, so does $\mathcal{E}_\Omega(\tau_\Omega^r(X))$. Then by Lemma 3.1.15 the arrow $\tau_\Omega^r(X_2) \to \tau_\Omega^r(X_1)$ has infinite global left degree, which contradicts Lemma 4.5.12. $\square$

Theorem 4.5.17

(a) *Let Γ be an infinite connected component of Γ_l that contains a module X such that $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to \infty$. Then the left subgraph type of Γ is A_∞.*

(b) *Dually, if Γ is an infinite connected component of Γ_r containing a module X such that $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to -\infty$, then its right subgraph type is A_∞.*

Proof:

If Γ is τ_Ω-periodic, then it is a stable tube by Theorem 3.2.7 and its left subgraph type is A_∞. If Γ is not τ_Ω-periodic, then we know that its left subgraph type is not given by a Dynkin diagram by Theorem 4.3.6. Moreover, since $l(\tau_\Omega(X))$ does not tend to infinity as $n \to \infty$, Γ neither contains a sectional subgraph of Euclidean type nor of type A_∞^∞ by Lemma 4.5.12 and Corollary 4.5.13 respectively. The left subgraph type of Γ also cannot be D_∞ as Γ does not contain a module with three immediate predecessor in Γ by Lemma 4.5.16. We deduce that the left subgraph type of Γ is A_∞ by process of elimination. $\square$

Recall that we have already seen in Corollary 4.5.15 that the converse statement does not hold, i.e. there are connected components of Γ_l with left subgraph type A_∞ containing a module X such that $l(\tau_\Omega^n(X)) \to \infty$ as $n \to \infty$.

Corollary 4.5.18 *Let Γ be a connected component of Γ_Ω. Suppose there are only finitely many τ_Ω-orbits in Γ. Then for each positive integer k there are at most finitely many modules of length k in Γ.*

Proof:

We assume that there is a positive integer k such that there are infinitely many modules of length k in Γ. Since there are only finitely many τ_Ω-orbits in Γ, there exists a module X which is not τ_Ω-periodic such that $l(\tau_\Omega^n(X)) = k$ for infinitely many integers n. Without loss of generality, we assume that there are infinitely many positive integers n with this property. Then by Theorem 4.5.17 we know that X is contained in a connected component of Γ_l such that its left subgraph type is A_∞, which is a contradiction. $\square$

Corollary 4.5.19 *Suppose there are infinitely many indecomposable modules in Ω of length k for some $k \in \mathbb{N}$. Then Γ_Ω admits infinitely many τ_Ω-orbits.*

Proof:

Suppose the statement is false and Γ_Ω consists of finitely many τ_Ω-orbits. Then there are clearly only finitely many connected components in Γ_Ω, one of which must contain infinitely many indecomposable modules of length k. This contradicts Corollary 4.5.18. $\square$

We call the following statement the second Brauer-Thrall conjecture for subcategories. If Ω is infinite, then there are infinitely many positive integers $n_1, n_2, \ldots$ such that for each $i \in \mathbb{N}$ there are infinitely many non-isomorphic indecomposable modules of Jordan-Hölder length n_i in Ω. As a consequence of the last corollary and Theorem 2.2.14, one could disprove the second Brauer-Thrall conjecture for subcategories by finding an example of an infinite functorially finite resolving subcategory Ω such that its Auslander-Reiten quiver consists

of only finitely many τ_Ω-orbits. [EMM10] contains an example of an infinite functorially finite resolving subcategory, of which only one connected component with finitely many orbits is known, but it is not clear if this component contains all indecomposable modules in Ω.

Theorem 4.5.20 *Let Γ be a connected component of Γ_Ω.*

(a) If Γ is left stable and contains a module X such that $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to \infty$, then there is a full sectional subgraph

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

such that there exists a non-negative integer n with the property that X_i is stable for all $i \geq n$ while X_j is left stable but not right stable for all $0 \leq j < n$.

(b) Dually, if Γ is right stable and contains a module X such that $l(\tau_\Omega^n(X))$ does not tend to infinity as $n \to -\infty$, then there is a full sectional subgraph

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

such that there exists a non-negative integer n with the property that X_i is stable for all $i \geq n$ while X_j is right stable but not left stable for all $0 \leq j < n$.

(c) If Γ is stable and contains a module X such that there are infinitely many $\tau_\Omega^n(X)$ with Jordan-Hölder length at most k for some $k \in \mathbb{N}$, then Γ is either a stable tube or isomorphic to $\mathbb{Z}A_\infty$.

Proof:
By Theorem 4.5.17 the left subgraph type of Γ is A_∞ and Γ is either a stable tube or not τ_Ω-periodic. In the former case the statement is trivial, so we suppose that Γ is not τ_Ω-periodic. Let Σ be a full sectional subgraph of Γ, then Σ contains a module X_0 that has only one immediate predecessor in Γ. Since Γ is left stable, we obtain a sectional path

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

which is a full sectional subgraph by Theorem 4.1.5. Suppose now there is a $j \geq 1$ such that X_j is not right stable but X_{j-1} is right stable. Without loss of generality, we can assume that j is minimal with that property. Since Γ is a connected component of Γ_Ω, we know that

$$X_{j-1} \longrightarrow X_{j-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

is a full sectional subgraph of a connected component Γ' of Γ_r. Hence the right subgraph type of Γ' is A_j and it follows that the component is finite and τ_Ω-periodic by Theorem 4.3.6, which is a contradiction. Thus the proof of (a) is completed and (b) can be proved dually.

If Γ is stable and contains a module X such that there are infinitely many $\tau_\Omega^n(X)$ with Jordan-Hölder length at most k for some $k \in \mathbb{N}$, then $l(\tau_\Omega^n(X))$ or $l(\tau_\Omega^{-n}(X))$ does not tend to infinity as $n \to \infty$. So statement (c) follows immediately from (a) or (b). $\qquad \square$

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