Energy variation of free non-linear oscillations

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ENERGY VARIATION

OF

FREE NON-LINEAR OSCILLATIONS

by

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CONTENTS

Chapter I

INTRODUCTION

Consider the system of differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y) \qquad (1.1)$$

where x,y,t are real variables. This system is said to be "autonomous" because P(x,y), Q(x,y) do not contain the variable t explicitly. From this it follows that if x(t), y(t) is a solution and k is an arbitrary constant then x(t + k), y(t + k) is also a solution. If x_o, y_o is a solution of the equations P(x,y) = 0, Q(x,y) = 0 then $x = x_o$, $y = y_o$ is a constant solution of (1.1) and the point (x_o, y_o) in the (x,y) plane is called a "singular point" of (1.1). We shall consider only systems having a finite number of singular points (which are therefore isolated). If x(t), y(t) is a non-constant solution then as t varies, the point (x(t), y(t)) describes a curve C in the (x,y) plane called a "trajectory" of (1.1). The associated solution x(t + k), y(t + k) also travels along C as t varies. Hence, each trajectory corresponds to an infinite family of solutions. When P(x,y), Q(x,y) satisfy a local Lipschitz condition, there is one and only one trajectory through each point. Trajectories cannot pass through a singular point though they may approach one as $t \to +\infty$ or as $t \to -\infty$. If $x(t)$, $y(t)$ is a periodic solution then the corresponding trajectory C is a closed curve. Conversely, each closed trajectory arises from a periodic solution. The location of the closed trajectories of (1.1) is therefore a problem of some interest. Later on we shall use the following well-known result which is proved in [1], page 78.

<u>Theorem 1.1</u> Bendixon's Second Theorem: Let $x(t)$, $y(t)$ be the parametric equations of a half trajectory C which remains for $t \to +\infty$ inside a bounded domain D which has no singular points inside it or on its boundary. The theorem asserts that only two cases are then possible.

1. Either C is itself a closed trajectory, or
2. C approaches asymptotically a closed trajectory C_o.

In this thesis our interest is in the free oscillation equation

$$\frac{d^2x}{dt^2} + f(x, \frac{dx}{dt}) \frac{dx}{dt} + g(x) = 0 \qquad (1.2)$$

This is equivalent to the following autonomous system:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -yf(x,y) - g(x) \qquad (1.3)$$

The (x,y) plane is called the "phase plane" of (1.2).

The singular points of (1.3) are all of the form $(x^*, 0)$, where x^* is a root of $g(x) = 0$. If we assume that

$$g(x) \operatorname{sign} x > 0 \text{ for } x \neq 0 \tag{1.4}$$

then (0,0) is the only singular point of (1.3). The "phase energy" function of (1.3) is defined to be

$$E(x,y) = \tfrac{1}{2}y^2 + G(x) \tag{1.5}$$

where

$$G(x) = \int_0^x g(\xi)d\xi \tag{1.6}$$

Under suitable conditions on $f(x,y)$ and $g(x)$ we shall show how to obtain two constants $\bar{e}$ and $\underline{e}$ such that all closed trajectories of (1.3) lie in the region of the (x,y) plane defined by $\bar{e} \geq E(x,y) \geq \underline{e}$. If $g(x)$ satifies (1.4) and the condition

$$G(x) \to +\infty \text{ as } |x| \to \infty \tag{1.7}$$

then this region is the annulus between the two simple closed curves $E(x,y) = \bar{e}$, $E(x,y) = \underline{e}$. With the help of the tables produced in Chapter IV the numbers $\bar{e}$, $\underline{e}$ can be easily computed in practice.

The Liénard equation is

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0 \tag{1.8}$$

This is the special case of (1.2) in which the function $f(x)$ is independent of $\frac{dx}{dt}$. As well as being equivalent to its corresponding phase system (1.3) the Liénard equation is also equivalent to the autonomous system

$$\frac{dx}{dt} = z - F(x), \quad \frac{dz}{dt} = -g(x) \tag{1.9}$$

where

$$F(x) = \int_0^x f(\xi)d\xi \tag{1.10}$$

The (x,z) plane is called the "Liénard plane" of (1.8). The singular points of (1.9) are all of the form $(x^*, F(x^*))$ where x^* is a root of $g(x) = 0$. If g(x) satifies (1.4) then (0,0) is the only singular point in the Liénard plane. The "Liénard energy" function of (1.9) is defined to be

$$V(x,y) = \tfrac{1}{2}z^2 + G(x) \tag{1.11}$$

$$= \tfrac{1}{2}[y + F(x)]^2 + G(x) \tag{1.12}$$

Under suitable conditions on f(x) and g(x) we shall show how to obtain two constants $\bar{v}$, $\underline{v}$ such that all closed trajectories of (1.9) lie in the region of the (x,y) plane defined by $\bar{v} \geq V(x,y) \geq \underline{v}$. If g(x) satifies (1.4) and (1.7) then this region is the annulus between the two simple closed curves $V(x,y) = \bar{v}$, $V(x,y) = \underline{v}$. Later on we shall make use of the following well-known result which is proved in [1], page 108-110.

<u>Theorem 1.2</u> Levinson and Smith's Theorem:

1. f(x) and g(x) are continuous. f(x) is an even function of x, hence, F(x) is odd. g(x) is an odd

function of x, hence, $G(x)$ is even. $g(x)$ satifies (1.4).

2. $F(x)$ has a single positive zero X^*. It is negative for $0 < x < X^*$. For $x > X^*$ it increases monotonically and hence is positive.

3. $F(x) \to \infty$ with x.

Under these assumptions, the equation (1.8) possesses a unique periodic solution.

In Chapter IV it is required to find the maximum and minimum values of the phase energy $E(x,y)$ on a closed trajectory of a system of type (1.3). The derivative of $E(x,y)$ following any solution $x(t)$, $y(t)$ of (1.3) is

$$\begin{aligned} \frac{dE}{dt} &= y\dot{y} + g(x)\dot{x} \\ &= y\{-yf(x,y) - g(x)\} + g(x)y \\ &= -y^2 f(x,y) \qquad (1.13) \end{aligned}$$

This shows that $E(x,y)$ increases when $y^2 f(x,y) < 0$ and $E(x,y)$ decreases when $y^2 f(x,y) > 0$. The maxima and minima of $E(x,y)$ on a closed trajectory therefore occur at the points of it where $y^2 f(x,y)$ changes sign. Similarly the derivative of $V(x,z)$ following any solution $x(t)$, $z(t)$ of (1.9) is

$$\begin{aligned} \frac{dV}{dt} &= z\dot{z} + g(x)\dot{x} \\ &= z\{-g(x)\} + g(x)\{z - F(x)\} \end{aligned}$$

$$= - F(x)g(x) \tag{1.14}$$

The maxima and minima of $V(x,z)$ on a closed trajectory of (1.9) therefore occur at the points on it where $F(x)g(x)$ changes sign.

We now suppose the system (1.2) has a periodic solution whose associated closed trajectory is C_1. Let e^* be the maximum phase energy on C_1. If the equation

$$\frac{d^2x}{dt^2} + \chi(x,\frac{dx}{dt}) \frac{dx}{dt} + g(x) = 0 \tag{1.15}$$

where $g(x)$ satifies (1.4) and (1.7), has a closed trajectory C_2 lying wholly inside C_1, then C_2 must also lie wholly inside the curve $E(x,y) = e^*$. If $E(x,y) \geq \underline{e}$ at all points of C_2 then the curve with equation $E(x,y) = \underline{e}$ lies wholly inside C_2, and, therefore wholly inside C_1. This means that both the closed trajectories C_1 and C_2 lie in the region of the phase space defined by

$$\underline{e} \leq E(x,y) \leq e^* \tag{1.16}$$

The problem in this thesis is to find out if a given free oscillation equation

$$\frac{d^2x}{dt^2} + \phi(x, \frac{dx}{dt}) \frac{dx}{dt} + g(x) = 0 \tag{1.17}$$

has any closed trajectory and if so, where. Suppose we have found with the help of a comparison theorem (proved in Chapter II) that there is a closed trajectory Γ of (1.17) between C_1 and C_2. Then Γ must lie between the curves $E = e^*$, $E = \underline{e}$. These curves are therefore outer and inner boundary curves for Γ. So the problem is solved if we find e^* and $\underline{e}$. Computation of e^* and $\underline{e}$ for two standard equations is done in Chapter IV.

Later on in Chapter V we shall take a Liénard equation of the type (1.8). If it has any closed trajectory, we will transform the equation to the standard form by abscissial transformation (see Chapter III). With the help of a comparison theorem (see Chapter II) and the energy tables produced in Chapter IV, we will find the range of variation of energy on a closed trajectory of this new equation. This range of variation will also be the range of variation on the corresponding closed trajectory of the original Liénard equation because of the invariance of the energies under abscissial transformation (see Chapter III). If the so-found range of variation of phase energies be $\underline{e}$ and e^* and that of Liénard energies $\underline{v}$ and v^* then two phase energy curves $E = e^*$ and $E = \underline{e}$ or two Liénard energy curves $V = v^*$ and $V = \underline{v}$ give outer and inner boundary curves for the closed trajectory. e^* and $\underline{e}$, or v^* and $\underline{v}$ may not be the actual maximum and minimum phase or Liénard energies on the closed trajectory but we are sure that the closed trajectory is not certainly exterior to the curves $E = e^*$ and $E = \underline{e}$ or to the curves $V = v^*$ and $V = \underline{v}$. We have done it in details in Chapter V.

Chapter II

Comparison Theorems

(A) Two by two autonomous system

Suppose the 2 x 2 autonomous system

$$\dot{x} = P_1(x,y), \qquad \dot{y} = Q_1(x,y) \qquad (2.1)$$

has a closed curve C_1.

Consider another 2 x 2 autonomous system

$$\dot{x} = P_2(x,y), \qquad \dot{y} = Q_2(x,y) \qquad (2.2)$$

Theorem 2.1

Suppose C_1 is described in a clockwise sense as t increases. If $P_1Q_2 < P_2Q_1$ on C_1 then any trajectory of (2.2) which meets C_1 subsequently passes into the interior of C_1.

Suppose C_1 is described in an anticlockwise sense as t increases. If $P_1Q_2 > P_2Q_1$ on C_1 then any trajectory of (2.2) which meets C_1 subsequently passes into the interior of C_1.

Proof

It is sufficient to prove the first part of the theorem.. The vector (P_1,Q_1) points along tangent to C_1. The vector $(-Q_1,P_1)$ is perpendicular to (P_1,Q_1) and is therefore normal to C_1. Since (P_1,Q_1) points in clockwise sense, $(-Q_1,P_1)$ points along outward normal. The vector

(P_2,Q_2) points inwards if its scalar product with $(-Q_1,P_1)$ is negative, that is, if $P_1Q_2 - P_2Q_1 < 0$. This establishes the theorem.

Cor 1

Suppose C_1 is described in a clockwise sense as t increases. If $P_1Q_2 > P_2Q_1$ on C_1 then any trajectory of (2.2) which meets C_1 subsequently passes to the exterior of C_1.

Suppose C_1 is described in an anticlockwise sense as t increases. If $P_1Q_2 < P_2Q_1$ on C_1 then any trajectory of (2.2) which meets C_1 subsequently passes to the exterior of C_1.

Note In our discussion, hereafter, unless otherwise mentioned, C_1 will always be supposed as described in a clockwise sense.

Cor 2

If $P_1Q_2 < P_2Q_1$ on C_1 then the system (2.2) has at least one singular point inside C_1.

Proof

Suppose corollary false. That is there are no singular points of (2.2) inside C_1.

Any trajectory C_2 of (2.2) which meets C_1, thereafter passes into the interior of C_1 and remains inside C_1 for ever (Theorem 2.1). Since there are no singular points of (2.2) on or inside C_1 the system (2.2) has at least one closed trajectory inside C_1 by Theorem 1.1. Let Γ be a closed trajectory of (2.2) inside C_1. Hence there exists at least one singular point inside Γ by Index theorem.* This contradicts our assumption. Hence the corollary is not false.

Note I $P_1Q_2 < P_2Q_1$ on C_1 implies that any trajectory C_2 of (2.2) inside C_1 cannot meet C_1. Trajectories of (2.2) which meet C_1 come from outside C_1 and thereafter remain inside C_1 for ever.

Note II When the sign of inequality does not change on C_1 between P_1Q_2 and P_2Q_1, there is always at least one singular point of (2.2) inside C_1, no matter whether C_1 is described clockwise or anticlockwise.

Cor 3

If $P_1Q_2 < P_2Q_1$ on C_1, and if (2.2) has a single and unstable singular point (S, say) inside C_1 then there exists at least one closed trajectory of (2.2) lying wholly inside C_1.

Proof

Since the singular point S is unstable, trajectories of (2.2) emanate away from S. Since there are no other singular points, we can encircle S by a closed curve ℓ which is wholly inside C_1 such that all trajectories of

*Poincaré has established a series of Index theorems one of which is given below.

"A closed trajectory surrounds at least one singular point".

See [2], page 79, line 8.

(2.2) which start from S cross ℓ outwards.

By Theorem 2.1, all trajectories of (2.2) which meet C_1 cross C_1 inwards.

Now if the ring-shaped region between C_1 and ℓ be the interior of the domain D whose boundaries are C_1 and ℓ, then the domain D is bounded and contains no singular points of (2.2) inside it or on its boundary. Since any trajectory of (2.2) which meets C_1 or ℓ enters into D and remains inside D for ever, we conclude by Theorem 1.1 that there exists a closed trajectory of (2.2) in D.

<u>Cor 4</u>

If $P_1Q_2 \leq P_2Q_1$, the equality occuring only at a finite number of points on C_1 and if there are no singular points of (2.2) on C_1 then any trajectory of (2.2) which meets C_1 passes subsequently into C_1. Conversely if any trajectory of (2.2) which meets C_1, is not exterior to C_1 then $P_1Q_2 \nmid P_2Q_1$ on C_1.

<u>Proof</u>

The equality $P_1Q_2 = P_2Q_1$ occurs only at a finite number of points (which are therefore isolated) on C_1. At all other points on C_1, trajectories of (2.2) pass into C_1. Since solutions vary continuously with their initial conditions, the trajectories of (2.2) which meet C_1 at these points of equality are bound to pass into C_1.

Cor 5

If (i) $P_1Q_2 \leq P_2Q_1$ on C_1, the equality occuring only at a finite number of points on C_1,

(ii) there is a single and unstable singular point of (2.2) inside C_1

and (iii) there is no singular point of (2.2) on C_1, then there exists at least one closed trajectory of (2.2) which is not exterior to C_1.

Proof

Same as proof to Cor 3.

(B) Phase system of a free non-linear oscillation equation

Consider the general differential equation of the relaxation oscillations

$$\ddot{x} + f_1(x,\dot{x})\dot{x} + g(x) = 0 \qquad (2.3)$$

where f_1 and g are continuous in their arguments and g is lipschitzian, in every bounded domain, and such that $xg(x) > 0$ for $x \neq 0$, $\int_0^{\pm\infty} g(x)dx = +\infty$.

(2.3) can be written as the first order system

$$\dot{x} = v, \; \dot{v} = - f_1(x,v)v - g(x) \qquad (2.4)$$

Suppose that (2.4) has at least one periodic solution; that is, there exists at least one closed curve C_1 in the (x,v) plane.

Consider another differential equation

$$\ddot{x} + f_2(x,\dot{x})\dot{x} + g(x) = 0 \qquad (2.5)$$

with f_2 continuous in its arguments and lipschitzian in every bounded domain.

(2.5) can also be written as the phase system

$$\dot{x} = v, \; \dot{v} = - f_2(x,v)v - g(x) \qquad (2.6)$$

<u>Theorem 2.2</u> (De Castro's Comparison Theorem [3])

If (i) $f_2(x,v) \geq f_1(x,v)$ on C_1, the equality occuring at a finite number of points on C_1

and (ii) $f_2(0,0) < 0$,

then there exists at least one closed trajectory of (2.5) in the (x,v) plane which is not exterior to C_1.

<u>Proof</u>

The equation (2.3) is a particular case of the system (2.1) in the (x,v) plane where

$$P_1 = v, \; Q_1 = - f_1 v - g$$

So is equation (2.5) of (2.2) where

$$P_2 = v, \; Q_2 = - f_2 v - g$$

At a point of intersection of a trajectory of (2.5) with C_1 in the (x,v) plane, we have

$$P_1Q_2 - P_2Q_1 = - v^2(f_2 - f_1) \leq 0 \text{ by hypothesis (i)}$$

It follows from Cor 5 that (2.5) has a closed trajectory which is not exterior to C_1.

<u>(C) Liénard system</u>

Consider a Liénard equation

$$\ddot{x} + \dot{x}f_1(x) + g(x) = 0 \qquad (2.7)$$

Where $g(x)$ sign $x > 0$ so that any trajectory of (2.7) is described in a clockwise sense in the Liénard plane. Suppose the equation (2.7) has a closed trajectory C_1 in this plane.

Now consider another Liénard equation

$$\ddot{x} + \dot{x}f_2(x) + g(x) = 0 \qquad (2.8)$$

Let $F_i(x) = \int_0^x f_i(\xi)d\xi$, $i = 1,2$ and let f_1 and f_2 be even functions of x so that F_1 and F_2 are odd.

<u>Theorem 2.3</u>

If F_2 sign $x \geq F_1$ sign x on C_1, the equality occuring at a finite number of points on C_1, then any trajectory of (2.8) which meets C_1 subsequently passes into the interior of C_1. If, in addition, $f_2(0) < 0$, then there exists at least a closed trajectory of (2.8) in the Liénard plane which is not exterior to C_1.

<u>Proof</u>

The equations (2.7) and (2.8) in the Liénard plane (x,z) can be written as the Liénard systems

$$\left.\begin{aligned} \dot{x} &= z - F_1(x) = P_1, \text{ say} \\ \dot{z} &= -g(x) = Q_1, \text{ say} \end{aligned}\right\} \qquad (2.9)$$

$$\left.\begin{aligned} \dot{x} &= z - F_2(x) = P_2, \text{ say} \\ \dot{z} &= -g(x) = Q_2, \text{ say} \end{aligned}\right\} \qquad (2.10)$$

respectively.

At a point of intersection of a trajectory of (2.8) with C_1 in the (x,z) plane, we have

$$\begin{aligned} P_1Q_2 - P_2Q_1 &= - g(F_2 - F_1) \\ &\leq 0 \text{ on } C_1. \end{aligned}$$

The first part of the theorem follows by Cor 4. Since $f_2(0) < 0$, the origin which is the only singular point of (2.8), is unstable. Hence by Cor 5, there exists at least a closed trajectory of (2.8) in the Liénard plane which is not exterior to C_1. This establishes the theorem.

Chapter III

Abscissial Transformation

Let $W(x)$ be any function such that

(i) $W(x) \uparrow$ from $-\infty$ to $+\infty$ as $x \uparrow$ from $-\infty$ to $+\infty$

(ii) the derivative $\frac{dW}{dx} = w(x)$, say, is continuous in $-\infty < x < \infty$.

If $x_2 = W(x_1)$ and $y_2 = y_1$ then $(x_2, y_2) = T(x_1, y_1)$ defines a continuous one-to-one transformation of the (x_1, y_1) plane onto the (x_2, y_2) plane. Then we call T an abscissial transformation of the plane. Obviously T has an inverse transformation T' which is also abscissial and corresponds to the inverse function of $W(x)$. Moreover if (x_1, y_1) travels along a curve C_1 then $(x_2, y_2) = T(x_1, y_1)$ travels along some curve C_2. We write it as $C_2 = TC_1$.

Theorem 3.1

If C_1 is a trajectory of the system

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = -y_1 f(W(x_1), y_1)\, w(x_1) - g(W(x_1))\, w(x_1) \tag{3.1}$$

then $C_2 = TC_1$ is a trajectory of the system

$$\dot{x}_2 = y_2, \quad \dot{y}_2 = -y_2 f(x_2, y_2) - g(x_2) \tag{3.2}$$

Proof

Let $x_1(t)$, $y_1(t)$ be a solution of (3.1) whose locus is C_1. If $x_2(t) = W(x_1(t))$, $y_2(t) = y_1(t)$ then C_2 is the locus of $x_2(t)$, $y_2(t)$.

Now $\dot{x}_2(t) = w(x_1)\,\dot{x}_1 = w(x_1)\,y_1 = w(\Omega(x_2))\,y_2$ where $x_1 = \Omega(x_2)$ is the inverse of $x_2 = W(x_1)$.

Again $\dot{y}_2(t) = \dot{y}_1(t) = -y_1 f(W(x_1), y_1)\,w(x_1) - g(W(x_1))w(x_1)$

$$= -y_2 f(x_2, y_2)\,w(\Omega(x_2)) - g(x_2)\,w(\Omega(x_2))$$

That is, $x_2(t)$, $y_2(t)$ is a solution of the system

$$\dot{x}_2 = \frac{y_2}{\sigma(x_2,y_2)}, \qquad \dot{y} = \frac{-y_2 f(x_2,y_2) - g(x_2)}{\sigma(x_2,y_2)} \tag{3.3}$$

where $\sigma(x_2, y_2) = \dfrac{1}{w(\Omega(x_2))}$. So C_2 is a trajectory of (3.3).

But (3.2) and (3.3) have the same trajectories. Therefore, C_2 is a trajectory of (3.2).

Now we can prove that under abscissial transformation the energy at a point on C_1 is equal to the energy at a corresponding point on C_2. For this purpose we will establish the following two invariance theorems on energy which will be found useful in Chapter V. As we are interested in a periodic solution of an equation, the theorems will be confined to energies on closed trajectories only.

Theorem 3.2

If C_1 is a closed trajectory of (3.1) then C_2 is a closed trajectory of (3.2) and conversely.

Furthermore the greatest and least values of the phase energy of (3.1) on C_1 are equal to the greatest and least values of the phase energy of (3.2) on C_2.

Proof

$$E(x_2, y_2) = \tfrac{1}{2}y_2^2 + G(x_2) \tag{3.4}$$

is, by definition, the phase energy of (3.2) where

$$G(x_2) = \int_0^{x_2} g(\xi)d\xi$$

But

$$\int_0^{x_1} g(W(\xi))\, w(\xi)d\xi = \int_0^{x_1} \frac{dG(W(\xi))}{d\xi}\, d\xi = G(W(x_1))$$

Hence the phase energy of (3.1) is

$$\tfrac{1}{2}y_1^2 + G(W(x_1)) = E(W(x_1), y_1) \tag{3.5}$$

Therefore, if $x_2 = W(x_1)$, $y_2 = y_1$ then (3.4) and (3.5) are equal. That is,

phase energy of (3.1) at (x_1, y_1)

= phase energy of (3.2) at $T(x_1, y_1)$.

Therefore, maximum and minimum phase energies on C_1 and C_2 are the same.

Theorem 3.3

If f is function of x_2 only then (3.1) and (3.2) are Liénard equations and a Liénard energy is defined. In this case the greatest and least values of the Liénard energy of (3.1) on C_1 are equal to the greatest and least values of the Liénard energy of (3.2) on C_2.

Proof

$$V(x_2, y_2) = \tfrac{1}{2}(y_2 + F(x_2))^2 + G(x_2) \tag{3.6}$$

is the Liénard energy of (3.2) where $F(x_2) = \int_0^{x_2} f(\xi)d\xi$.

But $\int_0^{x_1} f(W(\xi))\, w(\xi)d\xi = F(W(x_1))$. So the Liénard energy of (3.1) is

$$\tfrac{1}{2}(y_1 + F(W(x_1)))^2 + G(W(x_1)) = V(W(x_1),y_1) \quad (3.7)$$

Therefore, if $x_2 = W(x_1)$, $y_2 = y_1$ then (3.6) and (3.7) are equal. That is, under abscissial transformation, Liénard energy is same at corresponding points. Therefore the maximum and minimum Liénard energies on C_1 and C_2 are the same.

Definition

We say that the oscillation equations

$$\ddot{x} + \dot{x}f(W(x),\dot{x})\, w(x) + g(W(x))\, w(x) = 0 \quad (3.8)$$

$$\ddot{x} + \dot{x}f(x,\dot{x}) + g(x) = 0 \quad (3.9)$$

are abscissially equivalent. We have just proved that the trajectories in the phase plane of (3.9) can be got from those of (3.8) by making the abscissial transformation corresponding to function $W(x)$. That is, if we know the trajectories of one, we can immediately find the trajectories of the other.

Theorem 3.4

Suppose that (i) $g(x)\,\mathrm{sign}\, x > 0$ for $x \neq 0$ (3.10)

(ii) $g'(0) > 0$ (3.11)

(iii) $G(x) = \int_0^x g(\xi)d\xi \to +\infty \begin{cases} \text{as } x \to +\infty \\ \text{as } x \to -\infty \end{cases}$ (3.12)

Then the function $W(x)$ can be chosen so as to make

$$g(W(x))\, w(x) = x.$$

[That is, (3.9) is abscissially equivalent to an equation of form $\ddot{x} + \dot{x}\phi(x,\dot{x}) + x = 0$].

Proof

As $x \uparrow$ from $-\infty$ to $+\infty$ the function $\sqrt{2G(x)}$ sign $x \uparrow$ from $-\infty$ to $+\infty$. Let $x = W(\xi)$ be the inverse of $\xi = \sqrt{2G(x)}$ sign x. Then $W(\xi) \uparrow$ from $-\infty$ to $+\infty$ as $\xi \uparrow$ from $-\infty$ to $+\infty$.

Now $\xi^2 = 2G(x) = 2G(W(\xi))$

$$\therefore \xi = \frac{d}{d\xi}\left(\tfrac{1}{2}\xi^2\right) = \frac{d}{d\xi}\left(G(W(\xi))\right)$$

$$= g(W(\xi))\,\frac{d(W(\xi))}{d\xi} \qquad (3.13)$$

$\frac{dW}{d\xi}$ is obviously continuous and positive for $\xi \neq 0$.

We have $g(x) = g(0) + xg'(0) + \frac{x^2}{2}g''(0) + \ldots.$

$$= 0 + xg'(0) + o(x^2) \text{ when } x \text{ is small}$$

But $G(x) = \int_0^x g(\xi)d\xi = \tfrac{1}{2}x^2 g'(0) + o(x^3)$

$$= \tfrac{1}{2}x^2\left[g'(0) + o(x)\right]$$

$$\therefore \sqrt{2G(x)} = |x|\sqrt{g'(0) + o(x)}$$

Again $\xi = \sqrt{2G(x)}$ sign $x = x\sqrt{g'(0) + o(x)}$

$$\frac{x}{\xi} = \frac{1}{\sqrt{g'(0) + o(x)}} \to \frac{1}{\sqrt{g'(0)}} \text{ as } \xi \to 0$$

$\therefore$ there exists $\left(\frac{dx}{d\xi}\right)_{\xi=0} = \frac{1}{\sqrt{g'(0)}} > 0$, by hypothesis

But $\frac{dW(\xi)}{d\xi} = \frac{dx}{d\xi} > 0$ at $\xi = 0$

Hence the derivative $w(\xi)$ of $W(\xi)$ is continuous and positive in $-\infty < \xi < +\infty$.

$\therefore$ We have now from (3.13)

$$\xi = g(W(\xi))\ w(\xi)$$

This establishes the theorem.

Chapter IV

Standard equations

In this Chapter the closed trajectories of certain standard equations are discussed. These standard equations have unique closed trajectories. At present, our object is to compute the greatest and least values of the phase and Liénard energies on the closed trajectory of a standard equation. They will be used in Chapter V in conjunction with the Comparison Theorem and Abscissial Transformation.

The first of the standard equations is the van der Pol equation

$$\frac{d^2x}{dt^2} + \frac{\lambda}{\mu^2}(x^2 - \mu^2)\frac{dx}{dt} + x = 0 \qquad (4.1)$$

where λ and μ are positive constants. It follows at once from Theorem 1.2 that this equation has one and only one closed trajectory in the Liénard plane. All other trajectories approach this closed trajectory asymptotically as $t \to +\infty$. It follows that the same is true of the trajectories in the phase plane of (4.1). Let $\bar{e}(\frac{\lambda}{\mu^2}, \mu^2)$, $\underline{e}(\frac{\lambda}{\mu^2}, \mu^2)$ denote the greatest and least values, respectively of the phase energy $\frac{1}{2}(x^2 + \dot{x}^2)$ as the point $(x, \dot{x})$ varies round the closed trajectory of (4.1) in the phase plane. The change of variable $x = \mu\xi$

reduces (4.1) to the normalised form

$$\frac{d^2\xi}{dt^2} + \lambda(\xi^2 - 1)\frac{d\xi}{dt} + \xi = 0 \tag{4.2}$$

Since $x^2 + \dot{x}^2 = \mu^2(\xi^2 + \dot{\xi}^2)$ it follows that

$$\begin{aligned} \bar{e}\left(\frac{\lambda}{\mu^2}, \mu^2\right) &= \mu^2\bar{e}(\lambda,1) \\ \underline{e}\left(\frac{\lambda}{\mu^2}, \mu^2\right) &= \mu^2\underline{e}(\lambda,1) \end{aligned} \tag{4.3}$$

It is therefore sufficient to tabulate the functions $\bar{e}(\lambda,1)$ and $\underline{e}(\lambda,1)$. For $\lambda = 0.1, 0.2, \ldots, 1.0$ and for $\lambda = 2, 3, \ldots, 10$ the periodic solution of (4.2) has been accurately computed by Urabe [4] and Urabe, Yanagiwara and Shinohara [5] respectively. It follows from (1.13) that the maximum and minimum values of the phase energy of the closed trajectory of (4.2) occur at points $(\xi,\dot{\xi})$ at which $\lambda(\xi^2 - 1)\dot{\xi}^2$ changes sign, that is, at points where the trajectory crosses the lines $\xi = \pm 1$. The exact coordinates of these intersection points were found from Urabe's tables by interpolation and the functions $\bar{e}(\lambda,1)$, $\underline{e}(\lambda,1)$ shown in table A were then calculated. Similarly let $\bar{v}(\frac{\lambda}{\mu^2},\mu^2)$, $\underline{v}(\frac{\lambda}{\mu^2},\mu^2)$ denote the greatest and least values of the Liénard energy function

$$V(x,z) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\left[\dot{x} - \lambda x + \frac{1}{3}\frac{\lambda}{\mu^2}x^3\right]^2$$

along the closed trajectory of (4.1). Since

$$V(x,z) = \mu^2\left[\tfrac{1}{2}\xi^2 + \tfrac{1}{2}(\dot{\xi} - \lambda\xi + \tfrac{1}{3}\lambda\xi^3)^2\right],$$

it follows that

$$\left.\begin{aligned} \bar{v}(\tfrac{\lambda}{\mu^2}, \mu^2) &= \mu^2\bar{v}(\lambda,1) \\ \underline{v}(\tfrac{\lambda}{\mu^2}, \mu^2) &= \mu^2\underline{v}(\lambda,1) \end{aligned}\right\} \qquad (4.4)$$

where $\bar{v}(\lambda,1)$, $\underline{v}(\lambda,1)$ are the numbers corresponding to (4.2). It follows from (1.14) that the maximum and minimum values of $v(x,z)$ for the periodic solution of (4.2) occur at the points where $\lambda(\frac{1}{3}\xi^3 - \xi)\xi$ changes sign, that is, at points where the trajectory crosses the lines $\xi = \pm\sqrt{3}$. The precise coordinates of these points of intersection were found from Urabe's tables by interpolation and the functions $\bar{v}(\lambda,1)$, $\underline{v}(\lambda,1)$ shown in Table A were then computed. From these tables and (4.3) and (4.4) we can readily calculate $\bar{e}(\frac{\lambda}{\mu^2},\mu^2)$, $\underline{e}(\frac{\lambda}{\mu^2},\mu^2)$, $\bar{v}(\frac{\lambda}{\mu^2},\mu^2)$, $\underline{v}(\frac{\lambda}{\mu^2},\mu^2)$ for a wide range of values of $\frac{\lambda}{\mu^2}$ and μ^2.

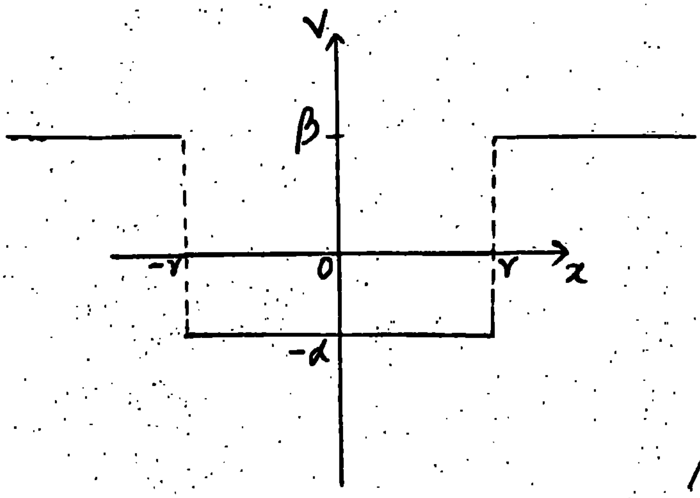
Our second standard equation is

$$\frac{d^2x}{dt^2} + \psi(x,\alpha,\beta,\gamma)\frac{dx}{dt} + x = 0 \qquad (4.5)$$

where α,β,γ are positive constants and $\psi(x,\alpha,\beta,\gamma)$ is the step function which is equal to β in the range $|x| > \gamma$ and is equal to $-\alpha$ in the range $|x| \le \gamma$. The Liénard system (1.9) associated with (4.5) is

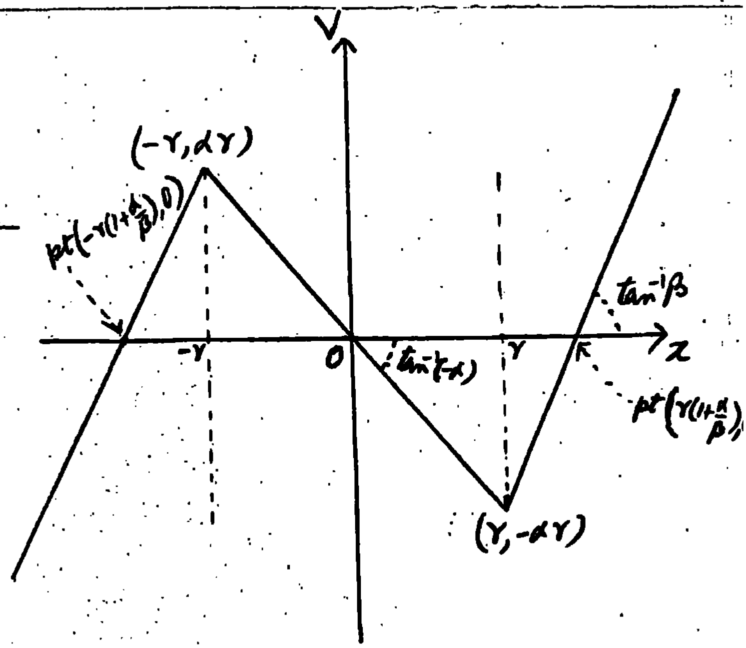
$$\frac{dx}{dt} = z - \Psi(x,\alpha,\beta,\gamma), \ \frac{dz}{dt} = -x, \qquad (4.6)$$

where $\Psi(x,\alpha,\beta,\gamma) = \int_0^x \psi(x,\alpha,\beta,\gamma)d\xi$ is equal to $-\alpha x$ in the range $|x| \leq \gamma$ and equal to $\beta x - (\alpha + \beta)\gamma$ sign x for $|x| \geq \gamma$. The graphs of the function $\psi(x,\alpha,\beta,\gamma)$, $\Psi(x,\alpha,\beta,\gamma)$



$V = \psi(x, \alpha, \beta, \gamma)$

Figure 1



$V = \Psi(x, \alpha, \beta, \gamma)$

Figure 2

are shown in Figures 1 and 2 respectively. Even though $\psi(x,\alpha,\beta,\gamma)$ is discontinuous at $x = \pm\gamma$ its integral $\Psi(x,\alpha,\beta,\gamma)$ is continuous and satifies a Lipschitz condition in $-\infty < x < \infty$. The solutions of (4.6) are therefore well behaved as regards the existence and uniqueness problems. It follows that the same is true of the solutions of (4.5). From Theorem 1.2 we deduce that the system (4.6) has a unique closed trajectory in the (x,z) plane which is approached asymptotically by all other trajectories as $t \to +\infty$. It follows that the same is true of the trajectories in the phase plane of the equation (4.5). Let $e^*(\alpha,\beta,\gamma)$, $e_*(\alpha,\beta,\gamma)$ be the greatest and least values, respectively, of the phase energy $\frac{1}{2}(x^2 + \dot{x}^2)$ as the point $(x, \dot{x})$ varies along the closed trajectory of (4.5). Similarly let $v^*(\alpha,\beta,\gamma)$, $v_*(\alpha,\beta,\gamma)$ denote the greatest and least values of the Liénard energy $\frac{1}{2}x^2 + \frac{1}{2}[\dot{x} + \Psi(x,\alpha,\beta,\gamma)]^2$ along the closed trajectory. The change of variable $x = \gamma\xi$ reduces (4.5) to the form

$$\frac{d^2\xi}{dt^2} + \psi(\xi,\alpha,\beta,1)\frac{d\xi}{dt} + \xi = 0. \qquad (4.7)$$

As before, we also deduce that

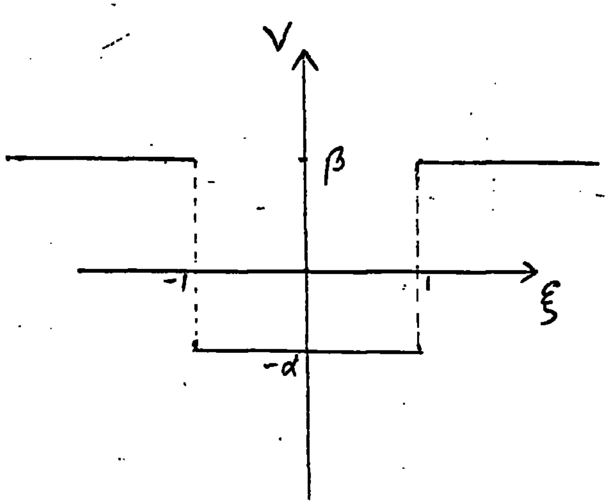
$$\begin{aligned} e^*(\alpha,\beta,\gamma) &= \gamma^2 e^*(\alpha,\beta,1), \quad e_*(\alpha,\beta,\gamma) = \gamma^2 e_*(\alpha,\beta,1) \\ v^*(\alpha,\beta,\gamma) &= \gamma^2 v^*(\alpha,\beta,1), \quad v_*(\alpha,\beta,\gamma) = \gamma^2 v_*(\alpha,\beta,1) \end{aligned} \qquad (4.8)$$

From (1.13) the maximum and minimum values of the phase energy are attained at points where $\psi(\xi,\alpha,\beta,1)\dot{\xi}^2$ changes sign, that is, at the points where the closed trajectory crosses the lines $\xi = \pm 1$. It was by finding these points of intersection that the functions $e^*(\alpha,\beta,1)$,

$e_*(\alpha,\beta,1)$ displayed in Table B were calculated. Similarly the maximum and minimum of the Liénard energy is attained at the points where $\Psi(\xi,\alpha,\beta,1)\xi$ changes sign, that is, at the points where the closed trajectory crosses the lines $\xi = \pm(\alpha + \beta)/\beta$. By finding these points of intersection the functions $v^*(\alpha,\beta,1)$, $v_*(\alpha,\beta,1)$ displayed in Figure B were computed.

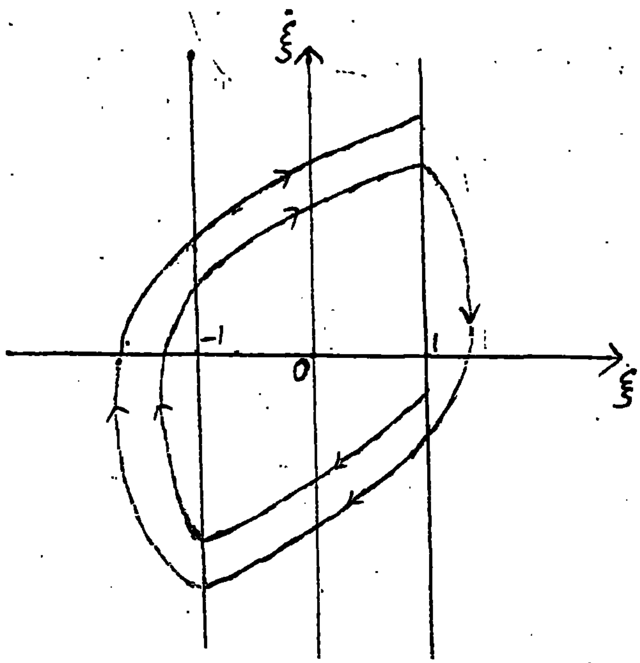
We now explain how to find the points of intersection of the lines $\xi = \pm 1$, $\xi = \pm(\alpha + \beta)/\beta$ with the closed trajectory in the phase plane of (4.7).

Trajectories of (4.7) in phase plane are made up (Figure 4)



$V = \Psi(\xi,\alpha,\beta,1)$

Figure 3



A trajectory of equation (4.7)
Red curves are trajectories of $\ddot{\xi} - \alpha\dot{\xi} + \xi = 0$
Green curves are trajectories of $\ddot{\xi} + \beta\dot{\xi} + \xi = 0$

Figure 4

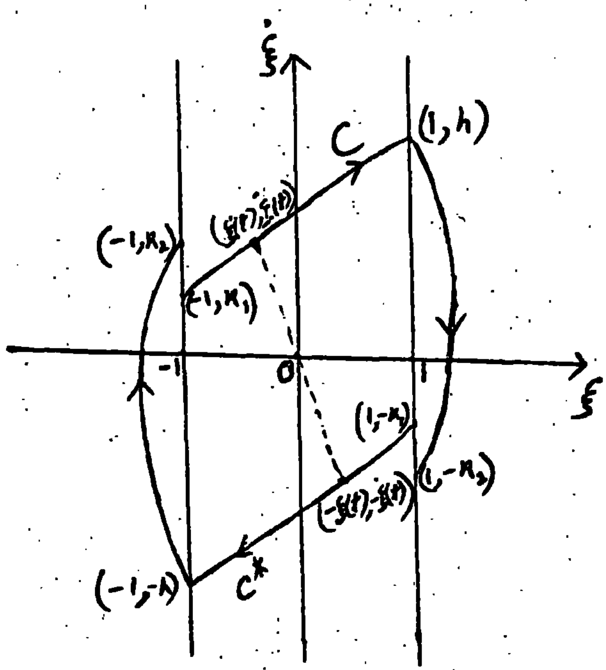
of pieces of trajectories of

$$\ddot{\xi} - \alpha\dot{\xi} + \xi = 0 \qquad (4.9)$$

in range $-1 \leq \xi \leq 1$ and pieces of trajectories of

$$\ddot{\xi} + \beta\dot{\xi} + \xi = 0 \qquad (4.10)$$

in range $|\xi| \geq 1$. Since $\psi(\xi,\alpha,\beta,1) = \psi(-\xi,\alpha,\beta,1)$, if $\xi(t)$ is a solution of (4.7), then so is $-\xi(t)$. Thus if the locus of $(\xi(t), \dot{\xi}(t))$ is C, then the locus C^* of



Trajectories of equation (4·7)

Figure 5

$(-\xi(t), -\dot{\xi}(t))$ is also a trajectory and is got by rotating C through π radians about origin. If C is trajectory passing through $(-1,k_1)$, $(1,h)$, $(1,-k_2)$ then C^* is the trajectory passing through $(1,-k_1)$, $(-1,-h)$, $(-1,k_2)$. In order that C be a piece of closed trajectory it is necessary and sufficient that $k_1 = k_2$.

For any point $(1, h)$ we can find k_1 and k_2 as functions of h. If we plot the graphs $V = k_1(h)$, $V = k_2(h)$ then

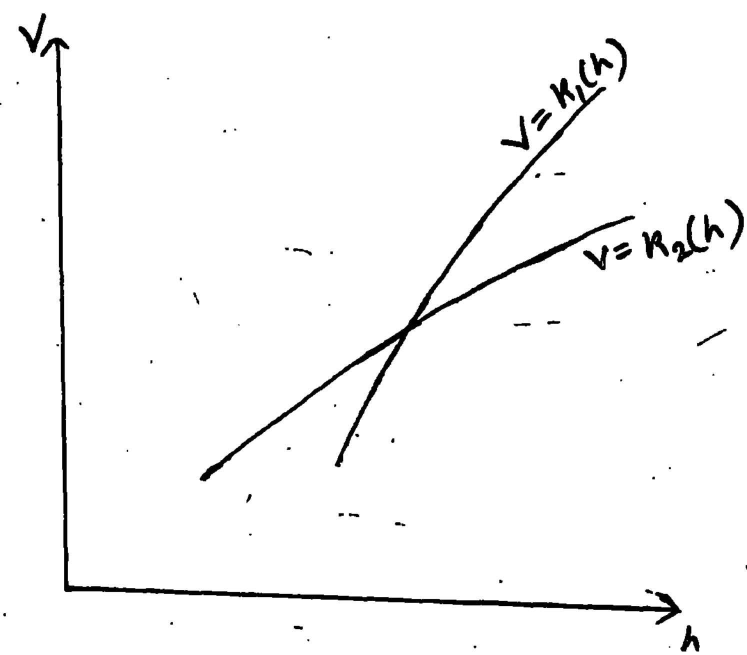


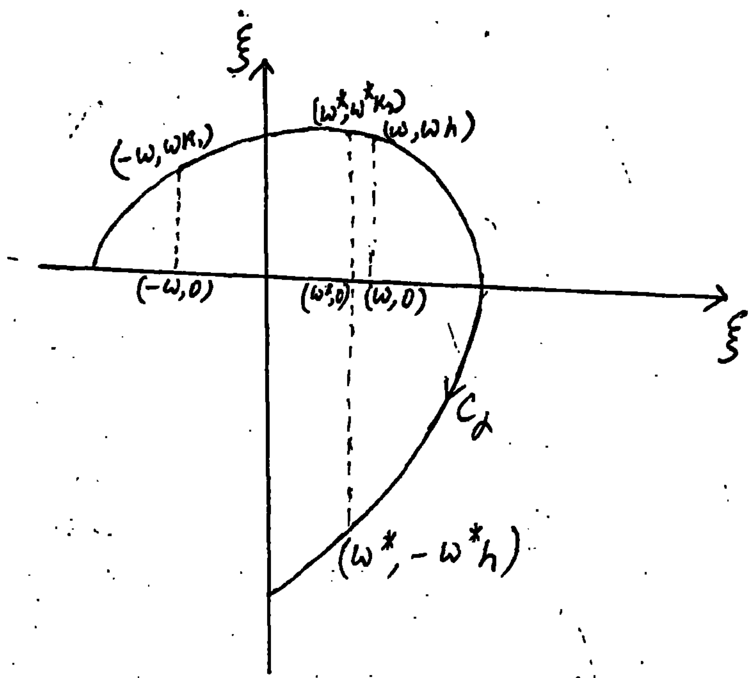
Figure 6

the point where these graphs cross has $k_1(h) = k_2(h) = k$, say. The point of intersection of graphs determines h, k for the closed trajectory. So problem is solved by computing functions $k_1(h)$, $k_2(h)$ and finding their point of intersection.

Now let us consider the equation

$$\ddot{\xi} + \psi(\xi,\alpha,\alpha,1)\dot{\xi} + \xi = 0. \qquad (4.11)$$

Suppose we have computed one trajectory C_α of (4.9). If C_α passes through points $(-w, wk_1)$, (w, wh) then by shrinking C_α by a factor $\frac{1}{w}$ we would get a trajectory through points $(-1, k_1)$, $(1, h)$. For each value of w, we



A trajectory of equation (4.9)

Figure 7

can determine from C_α a pair of values h, k_1. We can then plot a continuous graph of the function $k_1(h)$. The function $k_1(h)$ will be denoted by $k_1(h,\alpha)$ to indicate its dependence upon the parameter α in (4.9). We then plot the graph

$$V = k_1(h, \alpha) \tag{4.12}$$

in the (h, V) plane.

A trajectory C_α^* of

$$\ddot{\xi} + \alpha\dot{\xi} + \xi = 0 \tag{4.13}$$

would be obtained by reflecting C_α in the ξ-axis. If C_α^* is trajectory of (4.13) through points (w^*, w^*h), $(w^*, -w^*k_2)$ then by scaling by a factor $\frac{1}{w^*}$ we would get a trajectory of (4.13) through points (1, h), (1, $-k_2$). So for each value of w^* we can determine from C_α a pair of values h, k_2 and plot the graph

$$V = k_2(h, \alpha) \tag{4.14}$$

in the (h, V) plane.

By considering the equation

$$\ddot{\xi} + \psi(\xi,\beta,\beta,1)\dot{\xi} + \xi = 0, \tag{4.15}$$

we can similarly plot the graphs

$$V = k_1(h, \beta) \tag{4.16}$$

and

$$V = k_2(h, \beta) \tag{4.17}$$

in the (h, V) plane.

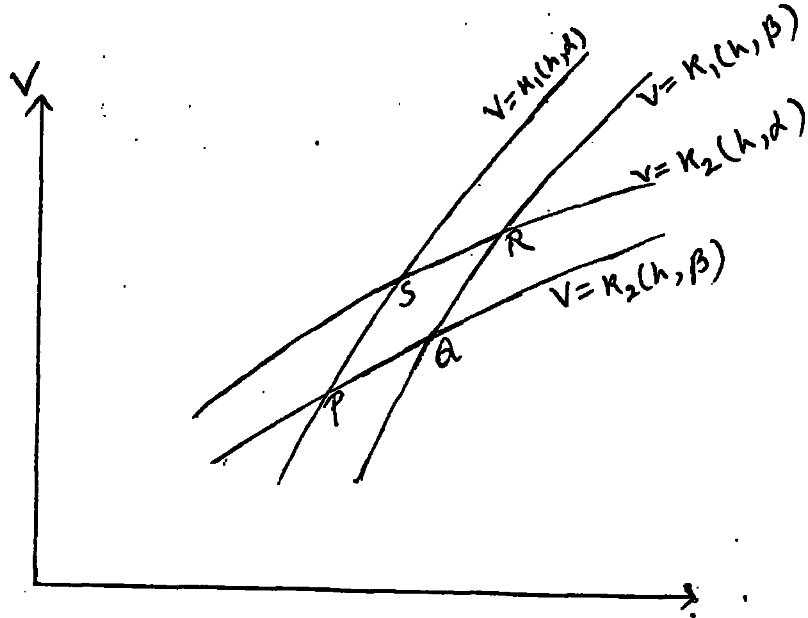


Figure 8

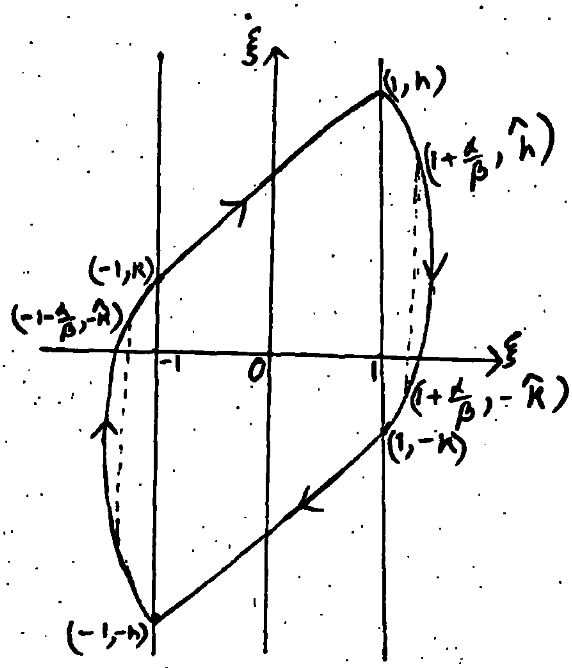
Then the point P where the graphs (4.12) and (4.17) intersect has coordinates (h, k) from which we deduce the points of intersection (1, h), (1, -k) of the line $\xi = +1$ with the closed trajectory of (4.7). The point Q gives the corresponding point for (4.15), the point R for the equation

$$\ddot{\xi} + \psi(\xi,\beta,\alpha,1)\dot{\xi} + \xi = 0 \qquad (4.18)$$

and the point S for (4.11).

From (1.13) the phase energy $\frac{1}{2}(\xi^2 + \dot{\xi}^2)$ of (4.7) takes its maximum and minimum values on the closed trajectory at the points where $\psi(\xi,\alpha,\beta,1)$ changes sign - that is, at the points $\xi = \pm 1$. Hence the point P whose coordinates are (h, k) determines the maximum energy $e^*(\alpha,\beta,1) = \frac{1}{2}(1 + h^2)$ and minimum energy $e_*(\alpha,\beta,1) = \frac{1}{2}(1 + k^2)$. By this method the functions $e^*(\alpha,\beta,1)$ and $e_*(\alpha,\beta,1)$ displayed in Figure B were computed.

Our next step will be to find out $v^*(\alpha,\beta,1)$ and $v_*(\alpha,\beta,1)$. For this purpose, we find out the points of intersection of the lines $\xi = \pm (1 + \frac{\alpha}{\beta})$ with the closed



closed trajectory of equation (4.7)

Figure 9

trajectory of (4.7). Suppose the line $\xi = +(1 + \frac{\alpha}{\beta})$ meets the closed trajectory at points $(1 + \frac{\alpha}{\beta}, \hat{h})$ and $(1 + \frac{\alpha}{\beta}, -\hat{k})$. Then

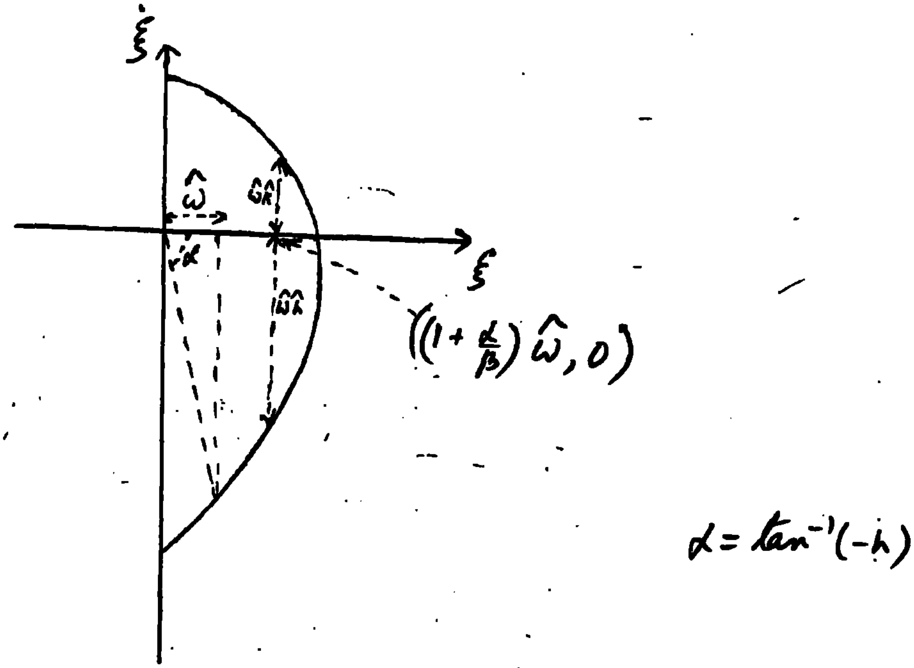
$$v^*(\alpha,\beta,1) = \tfrac{1}{2}\left[\left(1 + \frac{\alpha}{\beta}\right)^2 + \hat{h}^2\right]$$

and (4.19)

$$v_*(\alpha,\beta,1) = \tfrac{1}{2}\left[\left(1 + \frac{\alpha}{\beta}\right)^2 + \hat{k}^2\right]$$

we explain below the method of finding $\hat{h}$ and $\hat{k}$.

If a trajectory of (4.10) goes through points $(1, h)$, $(1 + \frac{\alpha}{\beta}, \hat{h})$, $(1 + \frac{\alpha}{\beta}, -\hat{k})$, $(1, -k)$ then the trajectory with scale factor $\hat{w}$ goes through points



A trajectory of $\ddot{\xi} - \beta\dot{\xi} + \xi = 0$

Figure 10.

$(\hat{w}, \hat{w}h)$, $((1 + \frac{\alpha}{\beta})\hat{w}, \hat{h}\hat{w})$, $((1 + \frac{\alpha}{\beta})\hat{w}, -\hat{k}\hat{w})$, $(\hat{w}, -k\hat{w})$.

On a loop of a trajectory of (4.9), draw the line with slope $-h$ where h is the value for periodic solution of equation (4.7). Suppose $(\hat{w}, -\hat{w}h)$ is the point where this line meets this loop. Draw a perpendicular to the ξ-axis through $((1 + \frac{\alpha}{\beta})\hat{w}, 0)$. The distances of this perpendicular to the ξ-axis from the curve are $\hat{w}\hat{h}$ and $\hat{w}\hat{k}$. So dividing these distances by $\hat{w}$, we get $\hat{h}$ and $\hat{k}$.

We have thus found out e^*, e_*, v^*, v_* for the closed trajectory of the equation (4.7) when α and β are both positive. The actual computed results for different α's and β's are shown in Table B.

Table A

$\bar{e}$, $\underline{e}$, $\bar{v}$, $\underline{v}$ for the closed trajectory of the van der pol equation

$$\ddot{x} + \lambda(x^2 - 1)\dot{x} + x = 0 \qquad (\lambda > 0).$$

λ	$\bar{e}$	$\bar{v}$	$\underline{v}$	$\underline{e}$
0.2	2.28	2.06	1.92	1.76
0.4	2.61	2.19	1.85	1.56
0.6	2.99	2.32	1.79	1.41
0.8	3.47	2.46	1.74	1.28
1.0	3.95	2.63	1.70	1.17
2.0	7.74	3.81	1.58	0.90
3.0	13.23	5.47	1.54	0.79
4.0	20.62	7.56	1.52	0.74
5.0	29.64	10.06	1.51	0.70
6.0	40.33	12.99	1.51	0.67
8.0	66.79	20.09	1.51	0.64
10.0	100.79	28.99	1.50	0.62

Table B

e^*, e_*, v^*, v_* for the closed trajectory of the equation

$$\ddot{x} + \dot{x}f(x,\alpha,\beta,1) + x = 0, \text{ where}$$

$$f(x,\alpha,\beta,1) \doteq -\alpha \text{ for } |x| < 1$$

$$\doteq \beta \text{ for } |x| > 1 .$$

$B = \frac{1}{4}$

α	e^*	v^*	v_*	e_*
0.25	3.80	3.14	2.91	2.34
0.50	10.80	7.23	6.69	6.38
1	30.53	23.63	21.11	15.62
1.5	60.01	48.76	40.43	30.92
2	95.72	74.12	61.50	47.06
2.5	136.62	99.87	87.19	66.39
3.5	245.81	176.26	161.31	117.08
4	333.32	228.17	212.47	158.03
5	545.00	386.41	352.15	257.918

$B = \frac{1}{2}$

α	e^*	v^*	v_*	e_*
0.25	2.21	1.72	1.52	1.28
0.50	4.58	3.42	2.89	2.08
1	12.02	8.16	6.76	4.16
1.50	22.94	15.02	12.43	6.90
2	35.36	21.28	18.04	9.83
2.50	52.62	27.87	24.01	14.17
3.50	92.98	46.47	41.90	22.41
4	120.94	55.63	51.31	28.85
5	188.10	87.88	79.26	44.40

$B = 1$

α	e^*	v^*	v_*	e_*
0.25	1.48	1.20	0.967	0.796
0.50	2.83	2.33	1.58	1.00
1	6.85	3.81	2.72	1.33
1.50	12.21	6.77	4.41	1.86
2	18.50	10.28	6.50	2.31
2.50	26.73	14.95	9.01	2.71
3.50	47.06	22.31	14.02	3.63
4	58.60	26.02	17.46	4.09
5	85.91	37.03	25.07	5.15

B = 1.5

α	e^*	v^*	v_*	e_*
0.25	1.26	0.96	0.82	0.64
0.50	2.30	1.59	1.08	0.74
1.00	5.21	2.88	1.76	0.86
1.50	9.36	4.78	2.58	0.94
2.00	14.18	6.74	3.42	1.04
2.50	20.88	9.02	4.65	1.14
3.50	35.36	14.11	7.12	1.23
4.00	45.06	17.71	8.91	1.32
5.00	66.63	24.22	12.37	1.44

B = 2

α	e^*	v^*	v_*	e_*
0.25	1.20	0.81	0.698	0.597
0.50	2.07	1.33	0.901	0.63
1.00	4.76	2.67	1.36	0.68
1.50	8.50	3.95	1.87	0.71
2.00	12.60	5.58	2.44	0.717
2.50	18.21	7.05	3.16	0.724
3.50	31.47	11.13	4.67	0.724
4.00	39.66	13.32	5.56	0.724
5.00	60.67	19.41	7.66	0.724

B = 2.5

α	e^*	v^*	v_*	e_*
0.25	1.13	0.74	0.659	0.55
0.50	1.97	1.06	0.812	0.58
1.00	4.50	2.14	1.13	0.60
1.50	8.02	3.11	1.48	0.61
2.00	12.12	4.38	1.89	0.615
2.50	17.86	5.78	2.32	0.62
3.50	30.53	8.95	3.34	0.625
4.00	38.43	11.02	3.93	0.625
5.00	59.36	15.97	5.23	0.625

B = 3.5

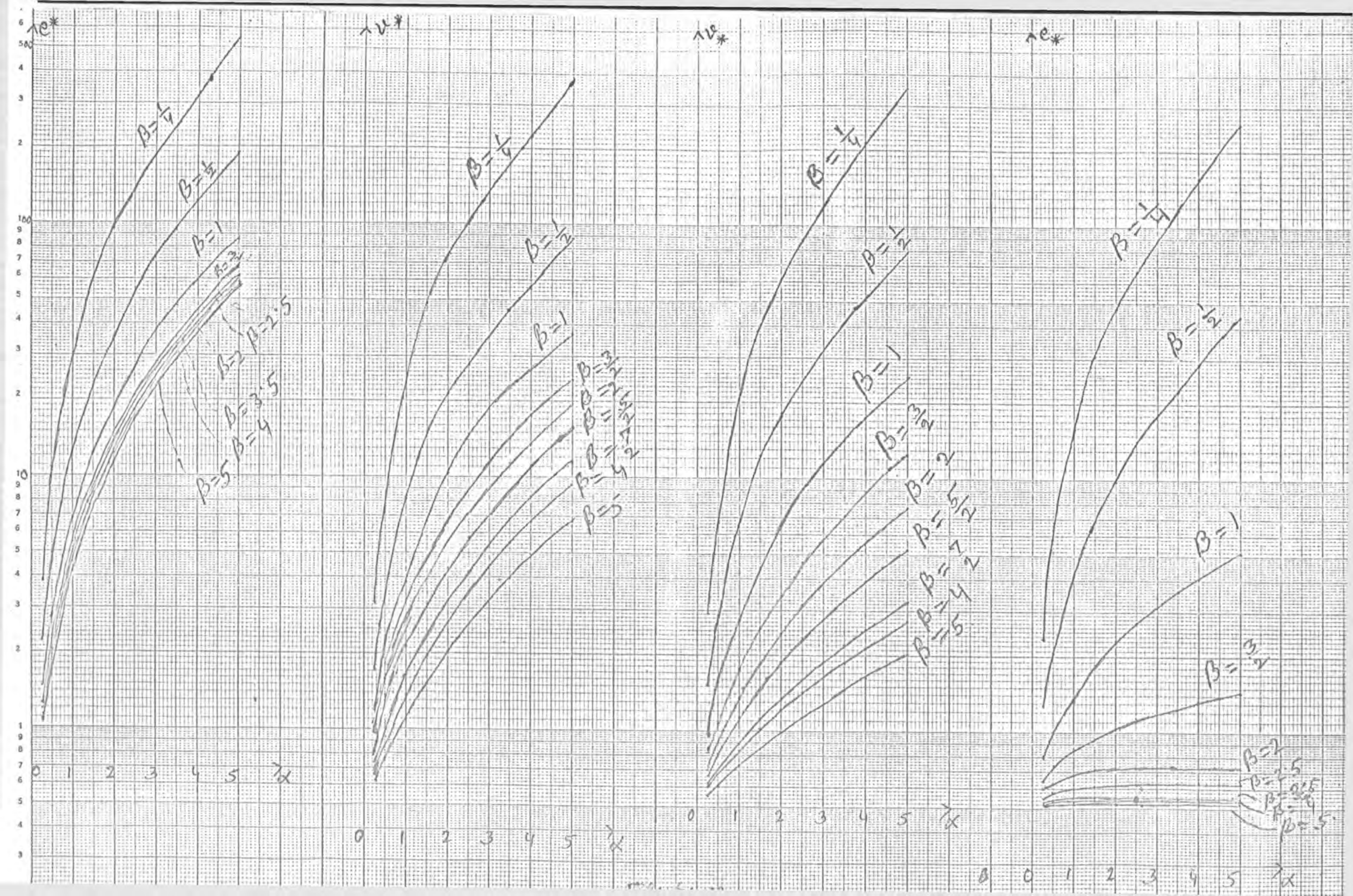
α	e^*	v^*	v_*	e_*
0.25	1.09	0.69	0.62	0.53
0.50	1.89	0.93	0.70	0.54
1.00	4.31	1.65	0.89	0.55
1.50	7.72	2.38	1.11	0.55
2.00	11.64	3.23	1.34	0.55
2.50	17.44	4.22	1.60	0.55
3.50	29.76	6.82	2.20	0.55
4.00	37.57	8.37	2.52	0.55
5.00	58.28	11.37	3.23	0.55

B = 4.00

α	e^*	v^*	v_*	e_*
0.25	1.08	0.65	0.61	0.53
0.50	1.86	0.82	0.678	0.53
1.00	4.23	1.37	0.836	0.53
1.50	7.61	1.95	1.01	0.54
2.00	11.55	2.65	1.20	0.54
2.50	17.20	3.52	1.41	0.54
3.50	29.38	5.53	1.88	0.54
4.00	37.31	6.61	2.14	0.54
5.00	57.85	9.32	2.72	0.54

B = 5.00

α	e^*	v^*	v_*	e_*
0.25	1.06	0.62	0.555	0.52
0.50	1.83	0.76	0.61	0.52
1.00	4.15	1.09	0.725	0.52
1.50	7.53	1.50	0.85	0.52
2.00	11.36	2.01	0.987	0.52
2.50	16.97	2.59	1.13	0.52
3.50	29.23	3.98	1.46	0.52
4.00	36.97	4.95	1.63	0.52
5.00	57.42	6.72	2.01	0.52



e^*
v^*
v_*
e_*
$\beta=\frac{1}{4}$
$\beta=\frac{1}{2}$
$\beta=1$
$\beta=\frac{3}{2}$
$\beta=2$
$\beta=2{\cdot}5$
$\beta=\frac{5}{2}$
$\beta=3{\cdot}5$
$\beta=\frac{7}{2}$
$\beta=4$
$\beta=5$
x

Chapter V

Outer and inner boundary curves of a closed trajectory.

Suppose we are given an oscillation equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \qquad (5.1)$$

Our idea in this Chapter is to find whether this equation has any closed trajectories and if so, where. We shall briefly outline the general procedure which will be followed by the illustration of an example.

Assuming that $g(x)$ satifies the conditions (3.10), (3.11) and (3.12), we can find from Theorem 3.4 an abscissially equivalent equation

$$\ddot{x} + \phi(x, \dot{x})\dot{x} + x = 0 \qquad (5.2)$$

In the special case when $f(x, \dot{x})$ is a function of x only the same will be true of $\phi(x, \dot{x})$ and the graph $v = \phi(x)$ can be easily drawn as a locus in the (x, v) plane of the point

$$x = \sqrt{2G(\theta)}\ \text{sign}\ \theta,\quad v = \frac{f(\theta)}{g(\theta)}\sqrt{2G(\theta)}\ \text{sign}\ \theta \qquad (5.3)$$

as θ varies from $-\infty$ to $+\infty$.

Now consider the equations

$$\ddot{x} + A(x)\dot{x} + x = 0 \qquad (5.4)$$

and

$$\ddot{x} + B(x)\dot{x} + x = 0 \qquad (5.5)$$

where $A(x)$, $B(x)$ satisfy the requirements of Levinson and Smith's Theorem of Chapter I so that the equations from outside. At all points on Γ_1^*, $\phi(x,y) > A(x)$. So by comparison theorem, trajectories of (5.2) cross Γ_1^* inwards. This is true for any loop of a trajectory of (5.4) outside Γ_1. So there cannot be any closed trajectory of (5.2) outside Γ_1. Similarly there cannot be any closed trajectory of (5.2) inside Γ_2. Hence, when (5.6) holds, any closed trajectory of (5.2) must lie in the ring between Γ_1 and Γ_2.

Suppose A(x) is a step function $\psi(x,\alpha,\beta,\gamma)$ and B(x) a parabola $\lambda(x^2 - \mu^2)$ so that (5.6) can be written as

$$\psi(x,\alpha,\beta,\gamma) < \phi(x,y) < \lambda(x^2 - \mu^2) \text{ for all } (x,y) \quad (5.7)$$

Using tables we now find maximum value $e^*(\alpha,\beta,\gamma)$ of phase energy $\frac{1}{2}(x^2 + y^2)$ on Γ_1 and minimum value $\underline{e}(\lambda,\mu^2)$ of phase energy on Γ_2. Then Γ (and all other closed trajectories of (5.2) if more than one) must lie in region

$$e^*(\alpha,\beta,\gamma) \leq \tfrac{1}{2}(x^2 + y^2) \leq \underline{e}(\lambda,\mu^2) \quad (5.8)$$

Let Λ be the closed trajectory of (5.1) corresponding to the closed trajectory Γ of (5.2). Since Theorem 3.2 shows that the energies on Γ and Λ are invariant under abscissial transformation Λ lies in the region

$$e^*(\alpha,\beta,\gamma) \leq \tfrac{1}{2}y^2 + G(x) \leq \underline{e}(\lambda,\mu^2) \quad (5.9)$$

In summary when g(x) satifies (3.10), (3.11) and (3.12) and the condition (5.7) holds then

(i) there exists a closed trajectory Λ of (5.1) in its phase plane,

(ii) Λ lies in the region defined by (5.9), and

(iii) all closed trajectories of (5.1) lie in this region.

An example

Locate the closed trajectory in the phase plane of

$$\ddot{x} + 4\dot{x}(x^2 - 1) + 2x^3 + x = 0 \qquad (5.10)$$

using (i) phase energy,

and (ii) Liénard energy.

Solution There exists abscissially equivalent equation of form

$$\ddot{x} + \phi(x)\dot{x} + x = 0 \qquad (5.11)$$

where graph of ϕ is the locus of the point (x, v) where

$$x = \theta\sqrt{\theta^2 + 1}\ \mathrm{sign}\ \theta, \quad v = \frac{4(\theta^2 - 1)}{2\theta^2 + 1}\sqrt{\theta^2 + 1}\ \mathrm{sign}\ \theta$$

as θ varies from $-\infty$ to $+\infty$. The curve $v = \phi(x)$ is plotted in Figure 2.

We see from the graph that

$$\psi(x,4,1,2.2) < \phi(x) < 1.2(x^2 - 1.3^2)$$

e^* for the closed trajectory of

$$\ddot{x} + \psi(x,\alpha,\beta,\gamma)\dot{x} + x = 0 \qquad (5.12)$$

when $\alpha = 4$, $\beta = 1$, $\gamma = 2.2$ is 283.6 and $\underline{e}$ for the closed trajectory of

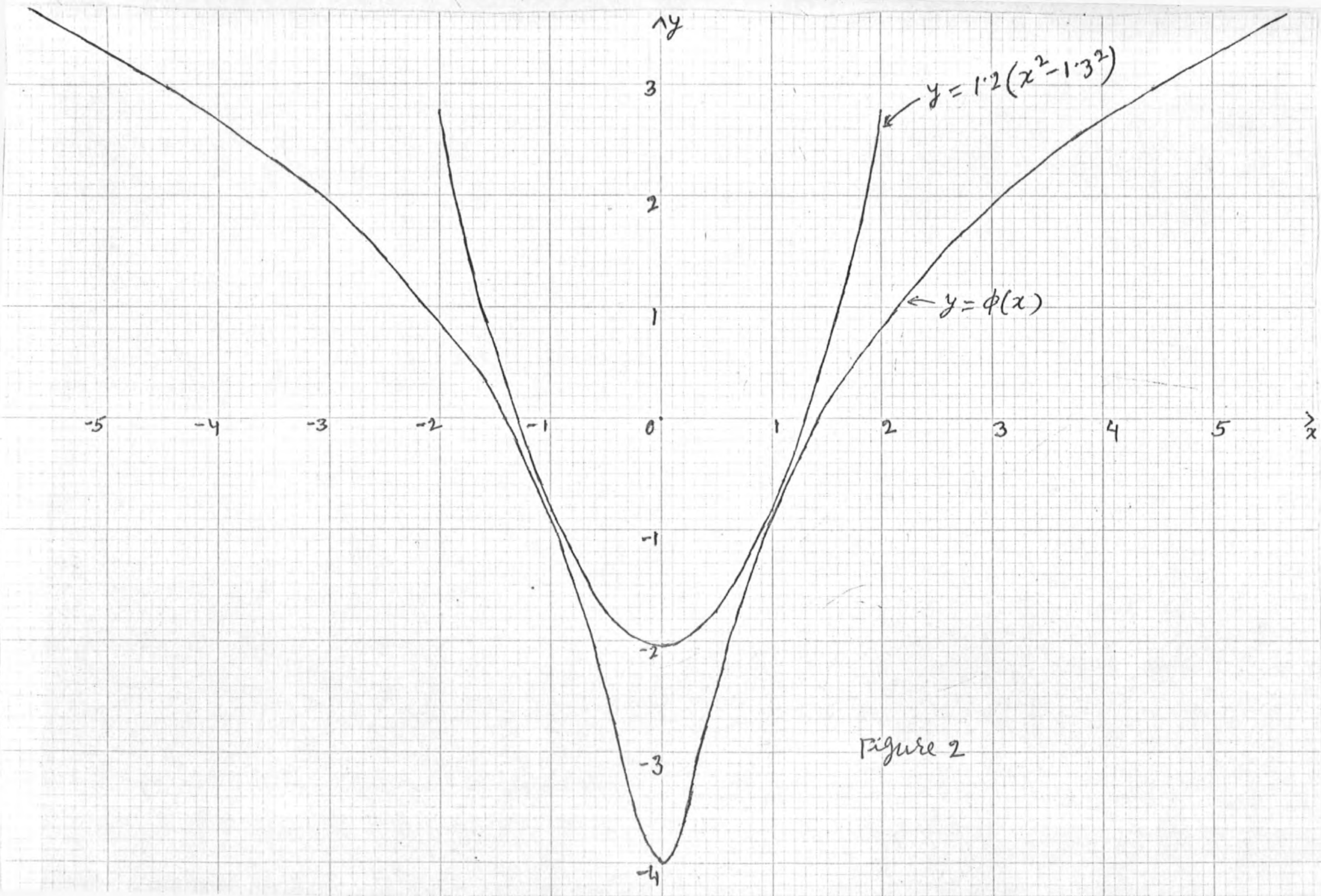


Figure 2

$$\ddot{x} + \lambda(x^2 - \mu^2)\dot{x} + x = 0 \qquad (5.13)$$

when $\lambda = 1.2$ and $\mu = 1.3$ is 1.50.

Therefore the closed trajectory of (5.11) lies between the circles

$\frac{1}{2}(x^2 + y^2) = 1.50$ and $\frac{1}{2}(x^2 + y^2) = 283.6$.

Hence because of the invariance of energies on the corresponding closed trajectories under abscissial transformation we conclude that the closed trajectory of (5.10) lies between the closed curves

$$\frac{1}{2}(y^2 + \frac{x^4}{2} + \frac{x^2}{2}) = 1.50 \text{ and } \frac{1}{2}(y^2 + \frac{x^4}{2} + \frac{x^2}{2}) = 283.6.$$

Also we see from the graph that

$$\Psi(x,4,1.5,2.2) < \phi(x) < 1.2(x^2 - 1.3^2)$$

The maximum Liénard energy $v^*(\alpha,\beta,\gamma)$ for the closed trajectory of (5.12) when $\alpha = 4$, $\beta = 1.5$, $\gamma = 2.2$ is 85.7 and the minimum Liénard energy $\underline{v}$ for the closed trajectory of (5.13) when $\lambda = 1.2$, $\mu = 1.3$ is 2.65.

Therefore the closed trajectory of (5.11) lies between the closed curves

$$\frac{1}{2}(y + \Phi(x))^2 + \frac{1}{2}x^2 = 2.65 \text{ and}$$

$$\frac{1}{2}(y + \Phi(x))^2 + \frac{1}{2}x^2 = 85.7 \text{ where}$$

$$\Phi(x) = \int_0^x \phi(\xi)d\xi$$

Hence for the same reason as above we conclude that the closed trajectory of (5.10) lies between the closed curves

$$\tfrac{1}{2}\left[y + 4\left(\frac{x^3}{3} - x\right)\right]^2 + \frac{x^4}{2} + \frac{x^2}{2} = 2.65 \text{ and}$$

$$\tfrac{1}{2}\left[y + 4\left(\frac{x^3}{3} - x\right)\right]^2 + \frac{x^4}{2} + \frac{x^2}{2} = 85.7$$

Remark I

The functions A(x) and B(x) need not necessarily be a parabola or a step function. They may both be parabolas or both step functions or some other functions so long as they satisfy (5.6) and we know the maximum and minimum energies on the closed trajectories of (5.4) and (5.5).

Remark II

It is obvious that to get a better approximation of the outer and inner boundary curves we must always choose A(x) and B(x) as close as possible to $\phi(x)$. If B(x) be a parabola the following method would give a very good approximation of the corresponding boundary curve.

Take a transparent sheet with parabolas $V = \lambda x^2$, $\lambda > 0$, on it. By conciding the V-axis of the parabola with the V-axis of the curve $V = \phi(x)$, shift the sheet up or down as necessary and find out which one of the parabolas $V = \lambda(x^2 - \mu^2)$ is closest to and above the graph $V = \phi(x)$.

References

1 N. Minorsky, "Introduction to Non-linear Mechanics", 78, 108-110 (1947).

2 N. Minorsky, "Non-linear Oscillations", 79, line8, (1962)

3 De Castro, "Un teorema di confronto per l'equazione differenziale delle oscillazioni di rillassamento", Bollettino della Unione Matematica Italiana, Serie III, 280-282, **9** (1954).

4 M. Urabe, "Periodic Solution of van der Pol's equation with damping coefficients $\lambda = 0(0.2)\ 1.0$", J. Sci. Hiroshima Univ., Ser. A, 193-207, **21** (1958).

5 M. Urabe, H. Yanagiwara, and Y. Shinohara, "Periodic solutions of van der Pol's equation with damping coefficient $\lambda = 2 \sim 10$", J. Sci. Hiroshima Univ., Ser. A, 325-366, **23** (1960).