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DIFFERENTIAL GEOMETRIC PROLONGATIONS OF SOLITON EQUATIONS

BY

MOSTAFA F. EL-SABBAGH

A thesis presented for the degree of Doctor of Philosophy at the University of Durham

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Mathematics Department
University of Durham

December

1980

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To

my parents,

my wife Eman and my son Ahmed.

ABSTRACT

This thesis is a study in the field of partial differential equations on differentiable manifolds. In particular non-linear evolution equations with soliton solutions are studied by means of differential geometric tools and methods. Differential geometric prolongation technique is applied to the A.K.N.S. system as a unifying system for known 2-dimension solitons. Soliton properties are studied in this differential geometric set up. The results are used to obtain a possible model for n-dimensional solitons.

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INTRODUCTION

In the notational guise of modern differential geometry and mapping theory, Cartan's calculus of exterior differential forms is a convenient mathematical tool for systematically applying many techniques of differential analysis. This calculus has been used for deriving important global results for systems of ordinary differential equations and especially for treating systems of partial differential equations.

Recently, Wahlquist and Estabrook (1975, 1976) introduced one of these techniques to handle in particular non-linear evolution equations with soliton solutions like the sine-Gordon equation, the k.DV. equation, ... etc. This technique is called differential geometric prolongation which, when used, pays off many algebraic and geometric structures closely related to the system of partial differential equations being treated.

In this thesis, we have attempted to unify the study of the subject in Chapters I to V. However, some results given in these chapters are already known, though they are to be found scattered throughout the literature and have often been obtained by ad hoc methods.

Chapter VI contains what we hope is original work - an attempt to find a model for n-dimensional solitons analogous to the part played by the group sL(2,R) and the pseudo-sphere for 2-dimensional solitons. We believe that the problem is difficult and claim only to have made some progress in this direction. The problem is presently receiving the attention of several mathematicians throughout the world, but no complete solution is yet in sight.

In Chapter I, necessary topics from exterior calculus, connection theory, fibre spaces, jet theory, differential equation theory and

Bäcklund maps are covered as preliminaries to lay the ground for the studies in the later chapters. In Chapter II, the relations between differential equations systems and exterior differential systems are formulated in two main theorems with proofs and examples. Also, soliton equations are introduced. In Chapter III, differential geometric prolongations are reformulated to yield interesting algebraic and geometric structures. Some interrelations between these structures are formulated in two theorems.

In Chapter IV, as the representative for the 2-dimensional soliton equations, the A.K.N.S. system is studied using the prolongation technique. The sL(2,R)-structure, the soliton connection and the pseudospherical surface property are shown as results of this geometric study of the A.K.N.S. system.

In Chapter V, the Bäcklund problem for the A.K.N.S. system is solved by means of prolongation. Also interesting interpretations of the A.K.N.S. system Bäcklund transformations are made.

Finally, in Chapter VI, we make use of the results of the study of 2-dimensional solitons and introduce a model for n-dimensional solitons.

CONTENTS

Page

CHAPTER I

PRELIMINARIES

I Differential forms and their operations

Let M be a differentiable manifold and V(M) denote the $C^\infty(M)$-module of smooth vector fields on M, where $C^\infty(M)$ denotes the ideal of functions of class C^∞, i.e. $\frac{\partial^\alpha f}{\partial x^\alpha}$ exist and are continuous for $[\alpha] \leq \infty$, for $f \in C^\infty$.

Definition: A k-form ω on M is an alternating multilinear map of V(M) into $C^\infty(M)$, i.e.

$$\omega : \underbrace{V(M) \times V(M) \times \cdots \times V(M)}_{\text{k-copies}} \longrightarrow C^\infty(M)$$

As we shall be concerned with differential geometric concepts applied to partial differential equations of physical interest, it is preferable to use another notation to fit the mathematical physics situations. We also mention that all results here are essentially local, and although we use the word manifold we really mean the Euclidean space, or, at least a space where it can be parametrized by a simple coordinate system. So, let

$\{x^i\}$, $i = 1, \ldots, n$ denote a coordinate system for some neighbourhood U around a point p of the manifold M, with n = dim M. Then a basis for V(U) is given by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$.

Now, if $v_1, \ldots, v_k \in V(M)$, we can choose

$$v_1(p) = \frac{\partial}{\partial x^{i_1}}, \quad v_2(p) = \frac{\partial}{\partial x^{i_2}}, \quad \ldots \quad , v_k(p) = \frac{\partial}{\partial x^{i_k}},$$

so $$\omega(v_1, \ldots, v_k) = \omega_p \left(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{i_2}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right),$$

and with $k = 1$ and $\omega_p \left(\frac{\partial}{\partial x^{i_1}}\right) = \delta^j_{i_1}$, then ω_p can be written as

$$\omega_p = dx^j$$

and a basis for 1-forms on M is $\{dx^1, \ldots, dx^n\}$.
So, a 1-form θ on M is an expression

$$\theta = \sum_{i=1}^{n} a_i dx^i, \quad a^i \text{ are functions on M.}$$

The multiplication of two 1-forms yields a 2-form. This sort of multiplication is called exterior multiplication and is denoted by $\wedge$, namely $dx^i \wedge dx^j$ is a 2-form on M. This exterior multiplication (wedge product), satisfies that

$$dx^i \wedge dx^j = - dx^j \wedge dx^i,$$

and hence $$dx^i \wedge dx^i = 0.$$

So, a basis for 2-forms on M can be written as the forms

$$\{dx^i \wedge dx^j, \ i < j, \ i,j = 1, \ldots, n\}$$

and a 2-form ω on M is an expression

$$\omega = a_{ij}\, dx^i \wedge dx^j ,$$

where a_{ij} are functions on M which are skew-symmetric in i,j and the summation convention is used "hereafter".

Similarly, a 3-form Ω on M is in the form

$$\Omega = a_{ijk} dx^i \wedge dx^j \wedge dx^k,$$

for skew-symmetric functions a_{ijk} on M, and so on. This wedge product is also used to describe multiplication of forms of different type. Simply it is expressed by

$$(\theta_1, \theta_2) \longrightarrow \theta_1 \wedge \theta_2,$$

and is characterized by the following:

(a) If $\theta_1 = f$, $\theta_2 = g$, zero forms, then $\theta_1 \wedge \theta_2 = fg$, the usual product of functions.

(b) $\theta_1 = f$, a zero form, $\theta_2 = a_i dx^i$, a 1-form, then $\theta_1 \wedge \theta_2 = f a_i dx^i$.

(c) $\theta_1 = a_i dx^i$, $\theta_2 = b_i dx^i$, then $\theta_1 \wedge \theta_2 = \frac{1}{2}(a_i b_j - a_j b_i)\, dx^i \wedge dx^j$

(d) $\theta_1 = a_i dx^i$, $\theta_2 = b_{ij} dx^i \wedge dx^j$, then

$\theta_1 \wedge \theta_2 = \frac{1}{3}(a_i b_{jk} + a_k b_{ij} + a_j b_{ki})\, dx^i \wedge dx^j \wedge dx^k$

and so on.

The exterior multiplication operation obeys the following rules:

$\theta_1 \wedge (\theta_2 + \theta_3) = \theta_1 \wedge \theta_2 + \theta_1 \wedge \theta_3$, for any forms θ_1, θ_2 and θ_3.

$\theta_1 \wedge \theta_2 = (-1)^{pq}\, \theta_2 \wedge \theta_1$ for θ_1 a p-form and θ_2 a q-form.

In fact exterior multiplication is one of several algebraic and differential operations which are admitted by differential forms and are natural, in the sense that they are invariant under arbitrary change of coordinates. For example, if $\theta = a_j dx^j$ is a 1-form on M, then the operation

$$\theta \longrightarrow \frac{\partial \theta}{\partial x^i} \equiv \left(\frac{\partial a_j}{\partial x^i}\right) dx^j$$

is not natural.

Here are the other natural operations:

Contraction:

If $A = A^i \frac{\partial}{\partial x^i}$ is a vector field on M, then denote by $A \rfloor \theta$ the contraction of θ by A, θ is a differential form on M. In the case of $\theta = a_i\, dx^i$, $A \rfloor \theta$ is given by

$$A \rfloor \theta = A^i a_i = \text{scalar field.}$$

For $\omega = a_{ij}\, dx^i \wedge dx^j$, one has

$$A \rfloor \omega = 2A^i a_{ij}\, dx^i \quad \text{which is a 1-form.}$$

For $\Omega = a_{ijk} dx^i \wedge dx^j \wedge dx^k$

$$A \rfloor \Omega = 3A^i a_{ijk}\, dx^j \wedge dx^k, \quad \text{a 2-form, and so forth.}$$

This contraction operation is related to the exterior multiplication by

$$A \rfloor (\theta_1 \wedge \theta_2) = (A \rfloor \theta_1) \wedge \theta_2 + (-1)^p\, \theta_1 \wedge (A \rfloor \theta_2),$$

where p is the degree of θ_1.

Exterior derivative:

The exterior derivative of a zero form f is denoted by

$$df = \frac{\partial f}{\partial x^i}\, dx^i .$$

For a 1-form $\theta = a_i\, dx^i$, the exterior derivative is

$$d\theta = da_i \wedge dx^i$$

$$= \tfrac{1}{2}\left(\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i}\right) dx^i \wedge dx^j .$$

For a 2-form $\omega = a_{ij}\, dx^i \wedge dx^j$,

$d\omega = da_{ij} \wedge dx^i \wedge dx^j$, and so on.

The exterior derivative operation obeys the following rules:

$$d(\theta_1 + \theta_2) = d\theta_1 + d\theta_2$$

$$d(\theta_1 \wedge \theta_2) = d\theta_1 \wedge \theta_2 + (-1)^p\, \theta_1 \wedge d\theta_2$$

for θ_1 a p-form.

Lie Derivative:

Let $A = A^i \frac{\partial}{\partial x^i}$ be a vector field on M and θ is a p-form on M. The Lie derivative of θ by A, denoted by $L_A\theta$ is a p-form defined as follows:

$$L_A(f) = A^i \frac{\partial f}{\partial x^i} \equiv A(f), \text{ a zero form.}$$

For θ, a 1-form, $\theta = a_i\, dx^i$, $L_A\theta$ is

$$L_A\theta = A(a_i)dx^i + a_i dA(x^i)$$

$$= \frac{\partial a^i}{\partial x^j} A^j\, dx^i + a_i \frac{\partial A^i}{\partial x^i} dx^i .$$

For $\omega = a_{ij}dx^i \wedge dx^j$, $L_A\omega$ is given by

$$L_A\omega = A(a_{ij})dx^i \wedge dx^j + a_{ij}d(A^i) \wedge dx^j + a_{ij}dx^i \wedge d(A^j)$$

and so on.

Some useful identities relating these operations are:

$$L_A(\theta_1 + \theta_2) = L_A\theta_1 + L_A\theta_2,$$

$$L_A(\theta_1 \wedge \theta_2) = (L_A\theta_1) \wedge \theta_2 + \theta_1 \wedge L_A\theta_2,$$

$$L_A(d\theta) = d(L_A\theta),$$

$L_A (\underline{B}\rfloor \theta) = [A,\underline{B}]\rfloor \theta + \underline{B}\rfloor (L_A\theta)$, where B is a vector field on M and [,] denotes the lie bracket of vector fields.

$$L_A(L_B\theta) = L_{[A,B]}\theta + L_B(L_A\theta),$$

$$L_A\theta = \underline{A}\rfloor d\theta + d(\underline{A}\rfloor \theta).$$

To see another important property of differential forms, let M and N be two differentiable manifolds with $\{x^i\}$ and $\{y^i\}$ coordinate neighbourhoods of M and N respectively. Let

$\phi : N \longrightarrow M$, be a map given by

$$\phi(y) = x(y).$$

Now, to a differential form θ on M, we assign a differential form, denoted by $\phi^*(\theta)$, on N, i.e. differential forms map backwards. For a zero form f on M, $\phi^*(f)$ is a zero form $y \longrightarrow f(x(y))$ on N.

If $\theta = a_i dx^i$ is a 1-form on M, then

$$\begin{aligned} \phi^*(\theta) &= \phi^*(a_i)\, d\phi*(x^i) \\ &= a_i(x(y))\, d(x^i(y)) \\ &= a_i(x(y))\, \frac{\partial x^i(y)}{\partial y^j}\, dy^j \end{aligned}$$

For a 2-form $\omega = a_{ij} dx^i \wedge dx^j$ on M,

$$\phi^*(\omega) = \phi^*(a_{ij}) d(x^i(y)) \wedge d(x^j(y))$$

$$= a_{ij}(x(y)) \, J \, dy^i \wedge dy^j ,$$

where J is the Jacobian,

$$J = \det\left(\frac{\partial x^i(y)}{\partial y^j}\right) .$$

The pull back map, ϕ^*, has the following general rules:

$$\phi^*(\theta_1 + \theta_2) = \phi^*(\theta_1) + \phi^*(\theta_2),$$

$$\phi^*(\theta_1 \wedge \theta_2) = \phi^*(\theta_1) \wedge \phi^*(\theta_2),$$

$$\phi^*(d\theta) = d(\phi^*(\theta)).$$

In fact exterior calculus, i.e. the calculus of differential forms, is increasingly being used to replace certain calculations which previously involved tensor calculus, and it is the more natural that it will find more and more applications because of its inner simplicity.

2 Fibre spaces

Let E and M be differentiable manifolds.
Let $\pi : E \longrightarrow M$ be a differentiable map.

Definition: (E, M, π) defines a fibre space with M as base space, E as total space and π as projection map if the following conditions are satisfied:

$\pi(E) = M$, and

π is a maximal rank map.

i.e. $\pi_*(E_p) = M_{\pi(p)}$ for all $p \in E$, where E_p denotes the tangent space of E at p and

$$\pi_* : T_pE \longrightarrow T_{\pi(p)}M.$$

This condition together with the implicit function theorem imply that each point $p \in E$ has an open neighbourhood U such that:

$\pi(U)$ is an open subset U' of M,

U is diffeomorphic to a product $U' \times F$ for some F, such that π restricted to U is the Cartesian projection $U' \times F \longrightarrow U'$.

If these local product neighbourhoods U can be taken as inverse image open sets $\pi^{-1}(U') = U$, then one speaks of (E,M,π) as a local product fibre space.

If these local products are tied together by an action of a Lie group on the fibres, one deals with a fibre bundle.

The fibre space (E,M,π) is usually deonoted by its total space E. For $p \in M$, $E(p) \equiv \pi^{-1}(p)$ denotes the fibre of E over p.

A cross section is a map $\gamma: M \longrightarrow E$ such that $\pi_0\gamma$ = identity map, id_M, of M. So $\gamma(p) \in E(p)$ for all $p \in M$, where $E(p) = \pi^{-1}(p)$ is the fibre of E over p.

As an example of the fibre space is the jet bundle which we shall need to work with here. So it is worthwhile to summarize the notation and facts about its theory.

3 Jet bundles

Let M and N be two differentiable manifolds, and let $C^\infty(M,N)$ denote the collection of C^∞ maps $f : M \longrightarrow N$.

Definition: Two maps $f, g \in C^\infty(M.N)$ are said to agree to order k at $x \in M$ if there are coordinate charts around $x \in M$ and $f(x) = g(x) \in N$ in which they have the same Taylor expansion up to and including order k.

It can be shown that this agreement is independent of the coordinates chosen and it is an equivalence relation.

Definition: The equivalence class of maps which agree with f to order k at $x \in M$ is called the k-jet of f at x and is denoted by $J^k_x f$.

If $\{x^a\}$ are local coordinates around $x \in M$ and $\{z^\mu\}$ around $f(x) \in N$, then $J^k_x f$ is determined by

$$x^a,\ z^\mu(x),\ z^\mu_a = \frac{\partial f^\mu}{\partial x^a}(x),$$

$$z^\mu_{ab} = \frac{\partial^2 f}{\partial x^a \partial x^b}(x),\ \ldots,\ z^\mu_{a_1\ldots a_k} = \frac{\partial^k f^\mu}{\partial x^{a_1}\cdots\partial x^{a_k}}(x)$$

where $z^\mu = f^\mu(x)$ is the presentation of f.

Definition: The k-jet bundle of M and N, denoted by $J^k(M,N)$, is the set of all k-jets, $J^k_x f$ with k fixed, $x \in M$ and $f \in C^\infty(M,N)$.

$J^k(M,N)$ will have a natural differentiable structure. The map

$$\alpha : J^k(M,N) \longrightarrow M,$$

by $\quad J^k_x f \longrightarrow x$, is called the source map.

The map

$$\beta : J^k(M,N) \longrightarrow N,$$

by $\quad J^k_x f \longrightarrow f(x)$, is called the target map.

Simply $f(x)$ is the target of $J^k_x f$ and x is its source.

A point $\xi \in J^k(M,N)$ is an equivalence class of maps with the same source, same target and the same derivatives up to k^{th} order in any coordinate presentation.

If $k > \ell$, there is a natural projection of the k-jet bundle on the ℓ-jet bundle, namely,

$$\pi^k_\ell : J^k(M,N) \longrightarrow J^\ell(M,N)$$

by $\quad J^k_x f \longrightarrow J^\ell_x f$

i.e. by ignoring the derivatives above the ℓ^{th} order.

A standard coordinate system on $J^k(M,N)$ may be taken as $(x^a, z^\mu, z^\mu_a, \ldots, z_{a_1 \ldots a_k})$.

A map from a jet bundle to another manifold induces maps of higher jet bundles called prolongations. Its construction is as follows:
Let M,N and P be manifolds and let

$$\phi : J^k(M,N) \longrightarrow P, \text{ be a smooth map.}$$

Definition: The s^{th} prolongation of ϕ is the unique map

$$\rho^s\phi : J^{k+s}(M,N) \longrightarrow J^s(M,P),$$

with the property that for every $f \in C^\infty(M,N)$, the diagram

$$\begin{array}{ccc} J^{k+s}(M,N) & \xrightarrow{\rho^s\phi} & J^s(M,P) \\ {\scriptstyle J^{k+s}f} \nwarrow & & \nearrow {\scriptstyle J^s(\phi \circ J^k f)} \\ & M & \end{array}$$

commutes.

This is the definition used in [22]. So, if $\{x^a\}$, $\{z^\mu\}$ and $\{y^A\}$ are local coordinates in M,N and P respectively, with

$$a,b,c,\ldots = 1,2,\ldots, \quad \dim M,$$
$$\mu,\nu,\ldots = 1,2,\ldots, \quad \dim N,$$
$$A,B,C,\ldots = 1,2,\ldots, \quad \dim P,$$

and if ϕ has the presentation $z^\mu = f^\mu(x^a)$,

then $\phi \circ J^k f$ has the presentation

$$y^A = \phi^A(x^a, f^\mu(x), \frac{\partial f^\mu(x)}{\partial x^a}, \ldots, \frac{\partial^k f^\mu}{\partial x^{a_1} \ldots \partial x^{a_k}})$$

so that $J^s(\phi \circ J^k f)$ has the presentation:

$$x^a = x^a$$

$$y^A = \phi^A(x^a, f^\mu(x), \ldots, \frac{\partial^k f^\mu}{\partial x^{a_1}, \ldots, \partial x^{a_k}}),$$

$$y^A_b = (\partial^{k+1\#}_b \pi^{k+1}_k * \phi^A)(x^a, f^\mu(x), \ldots, \frac{\partial^{k+1} f^\mu}{\partial x^{a_1} \ldots \partial x^{a_{k+1}}}),$$

$$\vdots$$

$$y^A_{b_1 \ldots b_s} = (\partial^{k+s\#}_{b_s} \pi^{k+1}_{k+s-1} * \partial^{k+s-1\#}_{b_{s-1}} \pi^{k+s-1}_{k+s-2} * \ldots$$

$$\ldots \partial^{k+1\#}_{b_1} \pi^{k+1}_k * \phi^A)(x^a, f^\mu(x), \ldots, \frac{\partial^{k+s} f^\mu}{\partial x^{a_1} \ldots \partial x^{a_{k+s}}}),$$

where $\partial^{h\#}_b = \frac{\partial}{\partial x^b} + z^\mu_b \frac{\partial}{\partial z^\mu} + z^\mu_{bb_1} \frac{\partial}{\partial z^\mu_{b_1}} + \ldots + z^\mu_{bb_1 \ldots b_{h-1}} \frac{\partial}{\partial z^\mu_{b_1 \ldots b_{h-1}}}$

is the total derivative operator, or the "hash operator".

Since this is the case for every f, $\rho^s\phi$ must have the presentation:

$$x^a = x^a,$$

$$y^A = \phi^A(x^a, z^\mu, z^\mu_a, \ldots z^\mu_{a_1\ldots a_k}),$$

$$y^A_b = (\partial^{k+1\,\#}_b \pi^{k+1*}_k \phi^A)(x^a, z^\mu, \ldots, z^\mu_{a_1\ldots a_{k+1}}),$$

$$\vdots$$

$$y^A_{b_1\ldots b_s} = (\partial^{k+s\,\#}_{b_s} \pi^{k+s*}_{k+s-1} \partial^{k+s-1\,\#}_{b_{s-1}} \pi^{k+s-1*}_{k+s-2} \ldots \partial^{k+1\,\#}_{b_1} \pi^{k+1*}_k \phi^A)(x^a, z^\mu, z^\mu_a, \ldots$$

$$\ldots, z^\mu_{a_1\ldots a_{k+s}}).$$

Definition: A 1-form θ on $J^k(M,N)$ is called a contact form if $(J^k f)^* \theta = 0$ for every $f \in C^\infty(M,N)$.

Contact forms on $J^k(M,N)$ comprise a module over $C^\infty(J^k(M,N),\mathbb{R})$, denoted by $\Omega^k(M,N)$ or Ω^k.

Clearly, if $k > \ell$, then $\pi^{k*}_\ell \Omega^\ell \subset \Omega^k$, so that Ω^ℓ is a submodule of Ω^k.

Each contact module is finitely generated, and Ω^k has a basis given by:

$$\theta^\mu = dz^\mu - z^\mu_a dx^a,$$

$$\theta^\mu_b = dz^\mu_b - z^\mu_{ba}\, dx^a,$$

$$\theta^\mu_{b_1\cdots b_{k-1}} = dz^\mu{}_{b_1\ldots b_{k-1}} - z^\mu_{b_1\ldots b_{k-1}\,a} dx^a.$$

The ideal generated by contact forms is not closed, that is $I(d\Omega^k) \not\subset I(\Omega^k)$.

The theory of jet bundles is very rich, but the properties described above will be enough for our requirements.

4 Cartan-Ehresmann connections

This is an important preliminary for later use. For the material of this section, it is enough to consider E and M differentiable manifolds with $\pi : E \longrightarrow M$ such that π_* has maximal rank, i.e.

$\pi_*(E_p) = M_{\pi(p)}$ for all $p \in E$, and not necessary for π to be a local product.

But for the requirements of our purposes, we let $\pi : E \longrightarrow M$ be a local product fibre space.

Definition: Let $p \in E$. A tangent vector $v \in E_p$ is said to be vertical if $\pi_*(v) = 0$.

Denote by $V(p)$ the space of such vertical vectors. Alternately $V(p)$ may be defined as the tangent space to the fibre $\pi^{-1}(\pi(p))$ passing through p.

Set

$$V = \{X \in \mathcal{H}(E) \mid X(p) \in V(p) \text{ for all } p \in E\}.$$

Definition: An element X of V is called a vertical vector field and V is a vector field system.

Note that V is an F(E) - submodule of $\mathcal{H}(E)$, where F(E) is the space of functions on E and $\mathcal{H}(E)$ is the space of vector fields on E.

Since π_* has maximal rank,

$$\dim V(p) = \dim E - \dim M.$$

Moreover,

$$V(p) = \{X(p) \mid X \in V\}.$$

Definition: An integral curve of V is a curve which lies entirely in one fibre of π.

Definition: A vector field system $H \subset \mathcal{H}(E)$ is said to be a horizontal system for the fibre space $\pi : E \longrightarrow M$, if

$$E_p = H(p) \oplus V(p) \text{ for all } p \in E.$$

Inwards, $p \longrightarrow H(p)$ assigns to each $p \in E$ a linear subspace $H(p) \subset E_p$ with the following properties:

$$\dim H(p) = \dim M \quad ,$$

$$H(p) \wedge V(p) = 0 \quad ,$$

π maps $H(p)$ isomorphically onto $M_{\pi(p)}$. i.e. H defines a horizontal distribution on E.

Definition: A curve $t \to \sigma(t)$ in E is horizontal with respect to H if

$$\sigma'(t) \equiv \frac{d\sigma}{dt}(t) \in H(\sigma(t)) \text{ for all } t, \text{ i.e. if } \sigma \text{ is an integral}$$

curve of H.

To keep as close as possible to the interesting physical applications, we use local coordinate systems, as the local coordinate formulae will be useful later on.

Suppose $\dim E = n + m$, and $\dim M = n$. We choose indices and the summation convention on these indices as follows:

$$1 \leqslant i,j,k, \ldots \leqslant n,$$

$$1 \leqslant a,b, \ldots \leqslant m.$$

If $\{z_i\}$ is a coordinate system of functions on M, set $\pi^*(z_i) = x_i$, then $\{x_1, \ldots, x_n\}$ are functionally independent functions on E.

Let $(y_1, \ldots, y_m)$ be additional functions on E such that $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ form a coordinate system for the fibre space E.
Let H* and V* be the pfaffian systems dual to H and V. The following results can be shown:

(i) The 1-forms $dx_1, \ldots dx_n$ form a basis for V* as an F(E)-module.

(ii) Each point of E is contained in an open subset E' such that H* restricted to E' has an F(E')-module basis $(\theta'_1, \ldots \theta'_m)$.

Moreover, we have the following:

Theorem 1:

Suppose that H* has at least one F(E)-module basis $(\theta_1, \ldots, \theta_m)$. Then one can choose this basis to have the form

$$\theta_a = dy_a - f_{ai} dx^i$$

with f_{ai} functions on E.
In this form it is unique.
For the proof one can see [12].

Now, if $t \to \sigma(t)$ is a curve in E which is horizontal relative to the system H, then

$$\theta(\sigma'(t)) = 0 \text{ for all } \theta \in H^*.$$

So the equation determining $\sigma(t)$ is

$$\frac{d}{dt} y_a(\sigma(t)) = f_{a_i} (x(\sigma(t)), y(\sigma(t))) \frac{d}{dt} x_i(\sigma(t))$$

and the following theorem is true for such curves:

Theorem 2:

Let $t \longrightarrow \sigma_1(t)$, $0 \leqslant t \leqslant \infty$, be a curve in M and let p be a point in E such that $\pi(p) = \sigma_1(0)$. Then there is a positive real number t_0 and a curve $t \longrightarrow \sigma(t)$, $0 \leqslant t \leqslant t_0$ in E such that

(i) σ is horizontal with respect to H,

(ii) $\pi(\sigma(t)) = \sigma_1(t)$, $0 \leqslant t \leqslant t_0$,

(iii) $\sigma(0) = p$.

Conditions (i), (ii) and (iii) determine σ uniquely.

Definition: Let $\pi : E \longrightarrow M$ be a fibre space and $H \subset \mathcal{H}(E)$ be a horizontal vector field system. H is said to determine a horizontally complete Ehresmann connection for the fibre space if for every curve $t \rightarrow \sigma_1(t)$, $0 \leqslant t \leqslant \infty$, in M and each point $p \in \pi^{-1}(\sigma_1(0))$, there is a horizontal curve $t \longrightarrow \sigma(t)$, $0 \leqslant t \leqslant \infty$ in E such that $\pi(\sigma(t)) = \sigma_1(t)$ and $\sigma(0) = p$.

Now, for $q \in M$, let $F(q) = \pi^{-1}(q)$ denote the fibre of π over q. Let $t \longrightarrow \sigma_1(t)$, $0 \leqslant t \leqslant 1$ be a curve in M such that $\sigma_1(0) = q_0$, $\sigma_1(1) = q_1$ for q_0, q_1 points in M. Given $p \in F(q_0)$, there is a unique horizontal curve $t \longrightarrow \sigma(t)$, $0 \leqslant t \leqslant 1$ in E such that:

$$\pi(\sigma(t)) = \sigma_1(t), \quad 0 \leqslant t \leqslant 1$$
$$\sigma(0) = p,$$

and so $\sigma(1) \in F(q_1)$ and we have a map:

$$\phi : F(q_0) \longrightarrow F(q_1)$$

called the parallel translation map.

If the curve $t \longrightarrow \sigma_1(t)$ in M is a loop, i.e. $\sigma_1(0) = \sigma_1(1) = q$, then the set of parallel translation maps $\{\phi\}$ defines a group of transformations on the fibre F(q) called the holonomy group of the connection H at q.

Definition: Let a geometric structure S be imposed on each fibre of the fibre space. We say that the connection H is an S-connection if the parallel translation maps between fibres all preserve this geometric structure S.

Now, as the lifting of curves on M, vector fields may also be lifted by means of the connection H, to horizontal vector fields on E. If X is a vector field on M, there is a vector field X_H on E which is uniquely determined by the two properties:

$$X_H \in H, \text{ and } \pi_*(X_H) = X.$$

X_H is called the horizontal lift of X.

The parallel transport maps fix the type of the connection, i.e. if the parallel transport along curves in the base always defines an affine automorphism between fibres, then H is said to be an affine connection. In a similar way we obtain other special kinds of Ehresmann connections, namely projective connections and conformal connections.

Definition: The vector field system H is said to be completely integrable if, and only if, $[H,H] \subset H$. In other words, integrability for H may be written as $dH^* \subset F^1(E) \wedge H^*$, where $F^1(E)$ is the space of 1-forms on E.

Definition: The curvature of the Ehresmann connection, determined by a horizontal vector field system H on a fibre space $\pi : E \longrightarrow M$, is the geometric object which decides complete integrability of H.

As the vector field systems H and V are complementary, we define the curvature tensor of the connection H to be the map

$$\Omega : H \times H \longrightarrow V$$

by $\Omega(X,Y)$ = projection of $[X,Y]$ in V with respect to the direct sum decomposition of H and V, where $X,Y \in H$.

And we have the following theorem:

Theorem 3:

Ω is an F(E)-bilinear skew-symmetric map. Ω is zero if, and only if, H is integrable.

In local coordinates (x_i, y_a) for E, and with $\theta_a = dy_a - f_{a_i} dx^i$ as basis for H*, we can write

$$\theta_a = dy_a - \omega_a - \omega_{ab}y_b - \omega_{abc}y_b y_c - \dots$$

by expanding the functions f_{a_i} as Taylor series in y with x as parameters. ω_a, ω_{ab}, ω_{abc}, ..., are 1-forms in the variables x_i alone, so they are 1-forms on M. They are called the connection 1-forms. Calculating $d\theta_a$, we get

$$d\theta_a = \Omega_a - \Omega_{ac}y_c - \Omega_{abc}y_b y_c - \dots$$

where $\Omega_a = d\omega_a - \omega_{ab} \wedge \omega_b$,

$\Omega_{ac} = d\omega_{ac} - \omega_{ab} \wedge \omega_{bc} - 2\omega_{ac} \wedge \omega_b$,

$\Omega_{abc} = d\omega_{abc} - \omega_{ad} \wedge \omega_{dcb} + \omega_{acd} \wedge \omega_{db} + \omega_{abd} \wedge \omega_{dc} + \dots$.

The 2-forms Ω_a, Ω_{ab}, Ω_{abc} , ... are the curvature 2-forms of the connection. Note that these forms depend on the choice of the coordinates $\{y_a\}$ which is called the moving frame for the fibre space. For example, if $y_a \to y_a'$, or in matrix notation, $Y \longrightarrow AY = Y'$, where A is an $m \times m$ matrix of

functions, $\theta = (\theta_a)$, $\zeta = (\omega_a)$, $\omega = (\omega_{ab})$ and $\omega^o = (\omega_{abc})$, ...
so that

$$\theta = dY - \eta - \omega Y - (\omega^o Y)Y - \ldots$$

Now, assuming that the new θ' is given by

$$\theta' = dY' - \eta' - \omega' Y' - (\omega^{o\prime} Y')Y' - \ldots$$

then substituting from $Y' = AY$, we get,

$$\eta' = A\eta ,$$

$$\omega' = dA\, A^{-1} + A\omega A^{-1},$$

$$\omega^{o\prime} = (A^t)^{-1}\omega^o ,$$

$$\ldots\ldots$$

where A^t is the transpose of A.

Also if $\underset{\sim}{\Omega} = (\Omega_a)$, then

$$\underset{\sim}{\Omega}' = A\, \underset{\sim}{\Omega}\, A^{-1},$$

and the transformation laws for (Ω_{ab}), (Ω_{abc}), ... may be calculated, but it looks complicated and we would not need it here.

5 Differential equations on fibre spaces

Let E be a differentiable manifold, which is the jet bundle of some manifolds M and N.

Definition: A system of differential equations with domain M and range N is a differentiable subset DE of $J^k(M,N)$, that is the zero set of finite number of functions on $J^k(M,N)$.

In the classical terminology, the variables of M are the independent variables of the system DE and the variables of N are the dependent variables of the system. The integer k determines the order of the system.

In terms of local coordinates for M,N and $J^k(M,N)$, say, $\{x^a\}$, $\{z^\mu\}$ and $(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1\ldots a_k})$ respectively, one can think of the system DE as being generated by a finite number of functions $F_1, \ldots F_q$ with

$$F_1(x^a, z^\mu, \ldots, z^\mu_{a_1\ldots a_k}) = 0,$$

$$F_2(x^a, z^\mu, \ldots, z^\mu_{a_1\ldots a_k}) = 0,$$

...

$$F_q(x^a, z^\mu, \ldots, z^\mu_{a_1\ldots a_k}) = 0,$$

and the system of partial differential equations DE is the zero set of these functions $F_1, \ldots, F_q$. Note that if $F_1, F_2, \ldots F_q$ are independent of the partial derivative variables $z^\mu_a, \ldots, z^\mu_{a_1\ldots a_k}$, then the system DE reduces to a set of ordinary equations.

If we are not concerned with the singularities of the functions $F_1, F_2, \ldots, F_q$, or of their solutions, then we may think of the system DE as a submanifold of $J^k(M,N)$ and not just a subset.

Definition: A solution of the system DE, $DE \subset J^k(M,N)$, is a map $f : M \dashrightarrow N$ such that $J^k f \subset DE$.

This means that

$$F_1\left(x^a, f^\mu(x), \frac{\partial f^\mu}{\partial x^a}, \ldots, \frac{\partial^k f^\mu}{\partial x^{a_1} \cdots \partial x^{a_k}}\right) = 0,$$

.....

$$F_q\left(x^a, f^\mu(x), \frac{\partial f^\mu}{\partial x^a}, \ldots, \frac{\partial^k f^\mu}{\partial x^{a_1} \cdots \partial x^{a_k}}\right) = 0,$$

where $f \equiv f^{\mu}(x) = z^{\mu}$ is the presentation of f.

Definition: The set $S = \{f : M \longrightarrow N \mid J^k f \subset DE\}$ is called the solutions space of DE.

Definition: A differential equation system $DE' \subset J^{k+1}(M,N)$ with S' as its space of solutions, is a prolongation of $DE \subset J^k(M,N)$ if

$$S' = S.$$

In this case DE' is called the 1st prolongation of DE in the classical sense. By this definition of prolongation, a clear fact is that all higher order prolongations of DE have the same space of solutions.

This idea of prolongation has been known for a long time (since S. Lie) and has had a modern treatment using the theory of jets and Lie pseudogroups. This has been done by a list of people, starting with Vessiot, Janet, Kuranishi, Ehresmann and recently by Pommaret in 1978.

But as we are concerned with the study of partial differential equations by means of differential geometric tools applied to particular systems of physical interest, we shall need a new definition of prolongation as will be shown in the following chapters.

We close this chapter with the following section about Bäcklund maps, which will be useful later on.

6 Bäcklund maps

Simply, a Bäcklund map is a transformation of the dependent variables in a system of differential equations.

Let us recall the notation of section 3, where we let M, N_1 and N_2 be C^{∞}-manifolds. We fix the following coordinate systems:

for M, $\{x^a\}$, $a = 1,2, \ldots,$ $\dim M = n$,

for N_1, $\{z^\mu\}$, $\mu = 1,2, \ldots,$ $\dim N_1 = m$,

and for

N_2, $\{y^A\}$, $A = 1,2, \ldots,$ $\dim N_2$

Definition: A Bäcklund map ψ is a C^∞-map

$$\psi: J^h(M,N_1) \times N_2 \longrightarrow J^1(M,N_2)$$

with the following requirements:

(i) ψ acts trivially on M, i.e. the diagram

$$\begin{array}{ccc} J^h(M,N_1) \times N_2 & \xrightarrow{\ \psi\ } & J^1(M,N_2) \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \alpha} \\ J^h(M,N_1) & \xrightarrow[\ \alpha\ ]{} & M \end{array}$$

commutes, where α is the source map and pr_1 is the projection onto the 1st component.

(ii) ψ acts trivially on N_2, i.e.

$$\begin{array}{ccc} J^h(M,N_1) \times N_2 & \xrightarrow{\ \psi\ } & J^1(M,N_2) \\ \downarrow{\scriptstyle pr_2} & \swarrow{\scriptstyle \beta} & \\ N_2 & & \end{array}$$

commutes where β is the target map.

These conditions on ψ imply that ψ is fixed completely if the coordinates y^A_a in (x^a, y^A, y^A_a) for $J^1(M,N_2)$ are given as functions, say ψ^A_a on $J^h(M,N_1) \times N_2$, i.e.

$$y^A_a = \psi^A_a\,(x^a,\ z^\mu,\ z^\mu_a,\ \ldots,\ z^\mu_{a_1\ldots a_h},\ y^B) \qquad (1)$$

and the $\{x^a\}$ and $\{y^A\}$ are unaltered by conditions (i) and (ii).

Now suppose that maps $f : M \longrightarrow N_1$ and $g : M \longrightarrow N_2$ are given, with $z^\mu = f^\mu(x)$ and $y^A = g^A(x)$ There are two ways to construct $\dfrac{\partial g^A}{\partial x^b}$, namely by differentiating $g^A(x)$ or by substituting in equation (1). The requirement that both ways give the same result, say,

$$\frac{\partial g^A}{\partial x^b}(x) = \psi^A_a\,\left(x^a,\ f^\mu,\ \frac{\partial f^\mu}{\partial x^a},\ \ldots\ ,\frac{\partial^h f^\mu}{\partial x^{a_1}\cdots\partial x^{a_h}},\ g^B(x)\right)$$

is that the map ψ satisfies its integrability conditions which follow from

$$\frac{\partial}{\partial x^c}\frac{\partial g^A}{\partial x^b}(x) - \frac{\partial}{\partial x^b}\frac{\partial g^A}{\partial x^c}(x) = 0$$

These integrability conditions may be written in the form:

$$\tilde{\partial}^{h+1}_a\ \tilde{\pi}^{h+1}_h{}^*\ \psi^A_b - \tilde{\partial}^{h+1}_b\ \tilde{\pi}^{h+1}_h{}^*\ \psi^A_a = 0, \qquad (2)$$

where $\tilde{\pi}^{h+1}_h = \pi^{h+1}_h \times id_{N_2} : J^{h+1}(M,N_1)\times N_2 \longrightarrow J^h(M,N_1)\times N_2$,

and $\tilde{\partial}^h_a$ is the extended hash operator to $J^h(M,N_1)\times N_2$ given by

$$\tilde{\partial}^{h+1}_a = \partial^{h+1\#}_a + \psi^A_a\frac{\partial}{\partial y^A},$$

i.e.
$$\tilde{\partial}^{h+1}_a = \frac{\partial}{\partial x^a} + z^\mu_a\frac{\partial}{\partial z^\mu} + z^\mu_{ac}\frac{\partial}{\partial z^\mu_c} + \ldots + z^\mu_{aa_1\ldots a_{h-1}}\frac{\partial}{\partial z^\mu_{a_1\ldots a_{h-1}}}$$
$$+\ z^\mu_{aa_1\ldots a_h}\frac{\partial}{\partial z^\mu_{a_1\ldots a_h}} + \psi^A_a\frac{\partial}{\partial y^A}.$$

The integrability conditions may also be written in a coordinate-free formula in terms of ideals of differential forms as in [22]. This is done as follows:

We have the diagram:

$$
\begin{array}{ccccc}
J^{h+1}(M,N_1) & \xleftarrow{\;pr_1\;} & J^{h+1}(M,N_1) \times N_2 & & \\
 & & \Big\downarrow \tilde{\pi}^{h+1}_{h} & & \\
 & & J^{h}(M,N_1) \times N_2 & \xrightarrow{\;\psi\;} & J^{1}(M,N_2)
\end{array}
\qquad (3)
$$

so that on $J^{h+1}(M,N_1) \times N_2$, there are two induced modules of forms, namely

$$I_1 = pr_1^{\ *}\ \Omega^{h+1}(M,N_1), \quad \text{and}$$

$$I_2 = \tilde{\pi}^{h+1\ *}_{h} \circ \psi^{*}\ \Omega^{1}(M,N_2).$$

Set $I_3 = \tilde{\Omega}^{h+1,\psi} = I_1 + I_2$,

I_3 lives also on $J^{h+1}(M,N_1) \times N_2$.

Now, because $\Omega^1(M,N_2)$ has a basis in the form $\theta^A = dy^A - y^A_a\, dx^a$, the integrability conditions have the equivalent form:

$$\tilde{\pi}^{h+1*}_{h}\ \psi^{*}\ d\Omega^{1}(M,N_2) \subset I(\tilde{\Omega}^{h+1,\psi}) \qquad (4)$$

where $I(\tilde{\Omega}^{h+1,\psi})$ is the ideal generated by the forms of $\tilde{\Omega}^{h+1,\psi}$.

It is clear that these integrability conditions also comprise a system of differential equations $\tilde{Z}$ on $J^{h+1}(M,N_1) \times N_2$.

If $\tilde{Z} = Z \times N_2$, where Z is a system of differential equations on $J^{h+1}(M,N_1)$ then ψ is called an ordinary Bäcklund map for the system Z.

Now, suppose that

$$\psi : J^{h}(M,N_1) \times N_2 \longrightarrow J^{1}(M,N_2)$$

is a Bäcklund map.

Definition: a map ψ^1,

$$\psi^1 : J^{h+1}(M,N_1) \times N_2 \longrightarrow J^2(M,N_2)$$

is said to be compatible with ψ if the following diagram

$$\begin{array}{ccc} J^{h+1}(M,N_1) \times N_2 & \xrightarrow{\psi^1} & J^2(M,N_2) \\ \downarrow \pi^{h+1}_h & & \downarrow \pi^2_1 \\ J^h(M,N_1) \times N_2 & \xrightarrow{\psi} & J^1(M,N_2) \end{array} \qquad (5)$$

commutes.

The map ψ^1 is determined completely by the specification of functions ψ^A_{bc} on $J^{h+1}(M,N_1) \times N_2$ such that under ψ^1

$$y^A_{bc} = \psi^A_{bc}\,(x^a,\ z^\mu,\ z^\mu_a,\ \ldots,\ z^\mu_{a_1 \ldots a_{h+1}},\ y^B) \qquad (6)$$

Because of (3), the appropriate choice of ψ^A_{bc} is

$$\psi^A_{bc} = \tilde{\partial}^{h+1}{}_{(b}\tilde{\pi}^{h+1*}_h\ \psi^A_{c)} \qquad (7)$$

where ( ) denotes symmetrization between the indices b and c.

Such a map ψ^1, satisfying equations (5), (6) and (7) is called the 1st prolongation of ψ.

In general, the map ψ^s,

$$\psi^s : J^{h+s}(M,N_1) \times N_2 \longrightarrow J^{s+1}(M,N_2)$$

which is compatible with ψ is called the s^{th} prolongation of ψ, and is fixed by the functions $\psi^A_{b_1 \ldots b_{s+1}}$ on $J^{h+s}(M,N_1) \times N_2$. The appropriate choice is

$$\psi^A_{b_1 \ldots b_{s+1}} = \tilde{\partial}^{h+s}{}_{(b_1}\tilde{\partial}^{h+s}_{b_2} \ldots \tilde{\partial}^{h+s}_{bs}{}_{)}\tilde{\pi}^{h+s*}_h\ \psi^A_{b_{s+1}} .$$

The integrability conditions of ψ^s, denoted by $\tilde{Z}^s$ are in the form of the system

$$\tilde{\partial}^{h+s}_{a_1} \cdots \tilde{\partial}^{h+s}_{a_{r-2}} \tilde{\partial}^{h+s}(_{a_{r-1}} \tilde{\pi}^{h+s}_h \psi^A_{a_r}) = 0$$

with r taking all values 2,3, ..., s+1 successively. (Note, r = 2 yields the integrability conditions for ψ itself.)

If there is a least integer s such that the image $\psi^s/\tilde{Z}^s$ is a system of differential equations Z' on $J^{s+1}(M,N_2)$ then the correspondence between Z and Z' is called the Bäcklund transformation determined by the Bäcklund map ψ.

Definition: If $J^o(M,N_1)$ and $J^o(M,N_2)$ are related by the identity diffeomorphism

$$(id)^o : J^o(M,N_1) \longrightarrow J^o(M,N_2)$$

and the ordinary Bäcklund map ψ determines a Bäcklund transformation between systems of equations Z on $J^{h+1}(M,N_1)$ and Z' on $J^{h+1}(M,N_2)$ in such a way that

$$(id)^{h+1}(Z) = Z'$$

then ψ is called a Bäcklund automorphism and the transformation is called a Bäcklund self-transformation. For construction of such Bäcklund automorphisms, see the Appendix.

Now, if there is given a system of partial differential equations Z on $J^{h+1}(M,N_1)$, then the Bäcklund problem for Z means the determination of manifolds N_2 and maps

$$\psi : J^h(M,N_1) \times N_2 \longrightarrow J^1(M,N_2)$$

which are ordinary Bäcklund maps for Z. If such ψ exists for a system Z, then an Ehresmann connection is associated to the system Z on the fibre space

$$(J^h(M,N_1) \times N_2,\ J^h(M,N_1),\ pr_1)$$

via the Bäcklund map ψ, by taking the module $\psi^*\Omega^1(M,N_2)$ as vertical forms on the product space.

This connection is called the Bäcklund connection. It depends on the map ψ and the solutions of the system Z. For more details one can consult [22]. The module $\psi^*\Omega^1(M, N_2)$ has a basis of the form

$$\theta^A = dy^A - \psi^A_a\, dx^a,$$

where ψ^A_a are the functions on $J^h(M,N_1) \times N_2$ determining the Bäcklund map .

If there is a Lie group G, say, acting effectively on the fibres of the fibre space and which is compatible with the local product structure of the fibre space, then it is argued that the functions ψ^A_a may be written in the form

$$\psi^A_a = X^A_\alpha(y)\ \omega^\alpha_a(x)$$

where $X^A_\alpha(y)$ are functions of y's alone and

$$X_\alpha = X^A_\alpha \frac{\partial}{\partial y^A}$$

are among a basis for the Lie algebra of vector fields on N_2. Moreover $\omega^\alpha_a(x)$ are functions of x's alone and

$$\omega^\alpha = \omega^\alpha_a(x)\ dx^a$$

is a set of 1-forms satisfying the same equations satisfied by the left invariant 1-forms of the Lie group G.

It is also worthwhile to mention here that a Bäcklund map ψ may be deformed to yield a 1-parameter family of Bäcklund maps [22], all having the same integrability conditions.

These facts will be useful for the theory of prolongations and the related Bäcklund transformations as it will be shown, where such a Bäcklund map ψ may lead to Bäcklund transformations between solutions of the differential equation under consideration for which ψ is an ordinary Bäcklund map.

CHAPTER II

PARTIAL DIFFERENTIAL EQUATIONS, EXTERIOR DIFFERENTIAL SYSTEMS AND SOLITONS

1 Exterior differential systems

The theory of exterior differential systems, which was developed by E. Cartan, is an effective way to study partial differential equation systems arising from a differential geometric background. This is now used particularly to study non-linear evolution equations which have physical applications.

Definition: Let M be a manifold. A collection of differential forms on M is said to be a differential (closed) ideal, denoted by I, if:

(i) For $\theta_1, \theta_2 \in I$, $f_1\theta_1 + f_2\theta_2 \in I$, where f_1, f_2 are functions on M.

(ii) For $\theta \in I$ and η any arbitrary form,

$$\theta \wedge \eta \in I.$$

(iii) For any $\theta \in I$, $d\theta \in I$.

An alternative name for the differential ideal is exterior differential system.

Definition: A subset of I is said to be generate I if the smallest exterior differential system containing that subset is I itself.

Definition: A differential form ω on M is said to be a conservation law for the exterior differential system I if $d\omega$ belongs to I.

Definition: An integral manifold N of I is a submanifold of M, $\phi : N \longrightarrow M$, such that

$$\phi^*(\theta) = 0 \text{ for all } \theta \in I.$$

The conditions $\phi^*(\theta) = 0$ for all $\theta \in I$, give rise to a differential equation system which should be satisfied on the submanifold N . This system is called the integral submanifold equations. For example, if $\theta = a_i dx^i$ is a 1-form on M, y is the coordinate of N and $y \longrightarrow x(y)$, the function that defines the submanifold, then the condition that $\phi^*(\theta) = 0$ gives:

$$a_i(x(y)) \frac{dx^i}{dy} = 0.$$

These are differential equations for x(y), so that the notion of an integral submanifold of an exterior differential system is nothing but a notion of a differential equation system in another guise.

To make this point clear, here are some examples:

Example 1:

Let $M = R^4$, with coordinates (x,y,u,v).

Consider the system of differential 2-forms

$$\theta^1 = du \wedge dx - dv \wedge dy,$$

$$\theta^2 = dv \wedge dx + du \wedge dy.$$

Since $d\theta^1 = d\theta^2 = 0$, the two forms θ^1 and θ^2 generate a differential ideal I. Assume that N is the submanifold of M, consisting of points (x,y,f(x,y),g(x,y)) where N is a 2-dimensional submanifold with coordinates (x,y). To determine the conditions satisfied by the functions u = f(x,y) and v = g(x,y), so that N is an integral submanifold of I, we have

ϕ: N $\longrightarrow$ M by

$$\phi : (x,y) \longrightarrow (x,y,f(x,y),g(x,y))$$

and hence

$$\phi^*(\theta^1) = (\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy) \wedge dx - (\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy) \wedge dy$$

$$= -(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\, dx \wedge dy$$

$$\phi^*(\theta^2) = (\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy) \wedge dx + (\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy) \wedge dy$$

$$= -(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x})\, dx \wedge dy.$$

So that $\phi^*(\theta^1) = 0 = \phi^*(\theta^2)$ give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

i.e. the functions f(x,y) and g(x,y) satisfy the above Cauchy Riemann conditions.

Thus the exterior differential system I generated by θ^1 and θ^2, when pulled back to one of its solution submanifolds, gives rise to a system of differential equations, which is the Cauchy Riemann system for the functions f(x,y) and g(x,y) on N.

Example 2:

Here we consider the manifold M to be given in the form of a product, M = N x Z.

Let I be an exterior differential system generated by the

1-form θ,

$$\theta = a(y,z)\ dy + b(y,z)\ dz$$

where z is the coordinate of Z.

Let the submanifold be given in the form

$$\phi : N \longrightarrow M = N \times Z$$

by $\qquad y \longrightarrow (y,z(y))$

i.e. graphs of mapping : $N \longrightarrow Z$.

Then θ will vanish on the integral submanifold N if $\phi^*(\theta) = 0$, which amounts to be

$$a(y,z(y))\ dy + b(y,z(y))\frac{\partial z}{\partial y}\ dy = 0$$

i.e. $\quad a(y,z(y)) + b(y,z(y))\dfrac{\partial z}{\partial y} = 0$

which is the integral submanifold equation. So we may state the following:

Theorem 1:

Let I be a given exterior differential system on a manifold M. Let $\phi : N \longrightarrow M$ be an integral submanifold of I. Then $\phi^*(I) = 0$ is a system of differential equations for the functions defining the map ϕ. This system is unique up to a prolongation.

This theorem has an alternative one for pfaffian systems, which was proved by Cartan using his notations of characters of exterior differential systems, genus, ... etc. We prove the theorem for the special case when M is a jet bundle, which will be useful for our requirements.

Proof: Assume M is the jet bundle $J^r(X,Z)$ where X is the space of independent variables, Z the space of dependent variables and r determines the highest degree of the forms generating I. We choose the coordinate systems $\{x^a\}$, $\{z^\mu\}$ and $\{x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_r}\}$ for X, Z and $J^r(X,Z)$ respectively.

Let the integral submanifold map ϕ,

$\phi : X \longrightarrow J^r(X,Z)$, be defined by

$$\phi : (x^a) \longrightarrow (x^a, \phi^\mu(x), \ldots, \phi^\mu_{a_1 \ldots a_r}(x)),$$

i.e. with $\phi^\mu(x) = z^\mu(x)$

and so
$$\phi^\mu_a = z^\mu_a ,$$
$$\phi^\mu_{ab} = z^\mu_{ab}$$
$$\phi^\mu_{a_1 \ldots a_r} = z^\mu_{a_1 \ldots a_r}.$$

The fact that $\phi^*(I) = 0$ gives rise to a differential equation system DE is obvious, by examples 1 and 2. What remains to prove is the uniqueness of this system DE.

So, suppose that $\phi^*(I) = 0$ gives rise to two systems DE and DE' with orders p and q, $p \neq q$, where p,q are positive integers. Assume that DE and DE', as differential equation systems on $J^r(X,Z)$, are generated by sets of functions:

$$F_1(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_p}) = 0,$$
$$F_2(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_p}) = 0,$$
....
$$F_\alpha(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_p}) = 0, \qquad \alpha < \infty$$

for DE, and

$$F_1'(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_q}) = 0,$$
$$F_2'(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_q}) = 0,$$
....

$$F_\beta'(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_q}) = 0 \qquad \beta < \infty$$

for DE'.

Now, since ϕ is an integral submanifold of I, then we must have:

$$\left.\begin{aligned} F_1(x^a, \phi^\mu(x), \ldots, \phi^\mu{}_{a_1 \ldots a_p}(x)) &= 0 \\ F_2(x^a, \phi^\mu(x), \ldots, \phi^\mu{}_{a_1 \ldots a_p}(x)) &= 0 \\ \ldots\ldots & \\ F_\alpha(x^a, \phi^\mu(x), \ldots, \phi^\mu{}_{a_1 \ldots a_p}(x)) &= 0 \end{aligned}\right\} \qquad (1)$$

and also

$$\left.\begin{aligned} F_1'(x^a, \phi^\mu(x), \ldots, \phi^\mu{}_{a_1 \ldots a_q}(x)) &= 0 \\ F_2'(x^a, \phi^\mu(x), \ldots, \phi^\mu{}_{a_1 \ldots a_q}(x)) &= 0 \\ \ldots\ldots & \\ F_\beta'(x^a, \phi^\mu(x), \ldots, \phi^\mu{}_{a_1 \ldots a_q}(x)) &= 0 \end{aligned}\right\} \qquad (2)$$

From equations (1) and (2), ϕ is a solution of both DE and DE'.

In fact, because ϕ was arbitrary integral submanifold of I, any solution of DE will be a solution for DE' and conversely any solution of DE' will be a solution of DE.

Hence DE and DE' have the same space of solutions. Thus DE = DE', or, one is a prolongation of the other, which proves the theorem.

Definition: A differential form ω, which is a conservation law for the exterior differential system I, is a conservation law for the underlying differential equation system, i.e. the integral submanifolds equation system.

Now, let DE be a given system of differential equations, the question is can one find an exterior differential system equivalent to DE in the above sense? The answer to this question is yes, and not only one exterior differential system but many.

Before we discuss this point, here are some examples:

Example 3:

Consider the hyperbolic wave equation

$$\frac{\partial^2 z}{\partial x^1 \partial x^2} = f(z, \frac{\partial z}{\partial x^1}, \frac{\partial z}{\partial x^2}) .$$

Construct a manifold with coordinates $(x^1, x^2, z, z_{x^1}, z_{x^2})$ and consider

$$\theta_1 = dz - z_{x^1}dx^1 - z_{x^2}dx^2,$$

$$\theta_2 = dz_{x^1} \wedge dx^1 - f(z, z_{x^1}, z_{x^2})dx^2 \wedge dx^1.$$

Then the differential ideal I generated by θ_1, θ_2, $d\theta_1$ and $d\theta_2$ has the hyperbolic wave equation as its integral submanifold equation.

Example 4:

The K.dV equation

$$z_{x^2} + z_{x^1x^1x^1} + 12zz_{x^1} = 0.$$

Consider the manifold of coordinates (x^1, x^2, z, p, q) with $p = z_{x^1}$, $q = z_{x^1x^1}$, and the set of differential forms:

$$\alpha^1 = dz \wedge dx^2 - pdx^1 \wedge dx^2,$$

$$\alpha^2 = dp \wedge dx^2 - qdx^1 \wedge dx^2,$$

$$\alpha^3 = -dz \wedge dx^1 + dq \wedge dx^2 - 12zpdx^1 \wedge dx^2.$$

Since

$$d\alpha^1 = dx^1 \wedge \alpha^2,$$

$$d\alpha^2 = dx^1 \wedge \alpha^3,$$

$$d\alpha^3 = -12(p\alpha^1 + z\alpha^2) \wedge dx^2$$

the set of forms $(\alpha^1,\alpha^2,\alpha^3)$ generate a differential ideal I.

If $\phi : (x^1,x^2) \longrightarrow (x^1,x^2,z(x^1,x^2),p(x^1,x^2),q(x^1,x^2))$ is an integral submanifold of I, i.e.

$$\phi^*(\alpha^i) = 0, \quad i = 1,2 \text{ and } 3, \text{ then}$$

$$\phi^*(\alpha^1) = \left(\frac{\partial z}{\partial x^1} - p\right) dx^1 \wedge dx^2 = 0,$$

$$\phi^*(\alpha^2) = \left(\frac{\partial p}{\partial x^1} - q\right) dx^1 \wedge dx^2 = 0,$$

$$\phi^*(\alpha^3) = \left(\frac{\partial z}{\partial x^2} + \frac{\partial q}{\partial x^1} + 12zp\right) dx^1 \wedge dx^2 = 0$$

and because $dx^1 \wedge dx^2 \neq 0$ on the integral submanifold, thus

$$p = \frac{\partial z}{\partial x^1}, \quad q = \frac{\partial p}{\partial x^1} = \frac{\partial^2 z}{\partial x^1 \partial x^1}$$

and
$$\frac{\partial z}{\partial x^2} + \frac{\partial q}{\partial x^1} + 12zp = 0$$

which is again the K.dv equation.

Example 5:

Consider the sine-Gordon equation:

$$\frac{\partial^2 z}{\partial x^1 \partial x^2} = \sin z$$

The 2-forms

$$\theta^1 = dz \wedge dx^2 - p\, dx^1 \wedge dx^2$$

$$\theta^2 = dp \wedge dx^1 + \sin z\, dx^1 \wedge dx^2$$

on the manifold M with coordinates (x^1,x^2,z,p), generate an exterior differential system I_1 since $d\theta^1 = -\theta^2 \wedge dx^2$ and $d\theta^2 = -\cos z\, dx^1 \wedge \theta^1$.

If $\phi : (x^1,x^2) \longrightarrow (x^1,x^2,z,p)$, is an integral submanifold of I_1,

then $\phi^*(\theta^i) = 0$, $i = 1,2$.

But $\phi^{\star}(\theta^1) = (\frac{\partial z}{\partial x^1} - p)\, dx^1 \wedge dx^2,$

$$\phi^{\star}(\theta^2) = -(\frac{\partial p}{\partial x^2} - \sin z)\, dx^1 \wedge dx^2$$

and as $dx^1 \wedge dx^2 \neq 0$ on the integral submanifold, we have

$$p = \frac{\partial z}{\partial x^1},$$

$$\frac{\partial^2 z}{\partial x^1 \partial x^2} = \sin z$$

which is the sine-Gordon equation.

Example 6:

Again, the sine-Gordon equation

$$\frac{\partial^2 z}{\partial x^1 \partial x^2} = \sin z.$$

Consider the manifold of dimension 5 with coordinates $(x^1, x^2, z, z_{x^1}=p, z_{x^2}=q)$ and the set of forms

$$\theta^1 = dz \wedge dx^2 - p\, dx^1 \wedge dx^2,$$

$$\theta^2 = dz \wedge dx^1 - q\, dx^1 \wedge dx^2,$$

$$\theta^3 = -dp \wedge dx^2 + dq \wedge dx^1 + 2 \sin z\, dx^1 \wedge dx^2$$

on this manifold.

We note that

$$d\theta^1 = -\theta^3 \wedge dx^2,$$

$$d\theta^3 = -2 \cos z\, \theta^1 \wedge dx^1,$$

$$d\theta^2 = \theta^3 \wedge dx^1.$$

So θ^1, θ^2 and θ^3 generate a differential I_2.

If $\phi : (x^1, x^2) \longrightarrow (x^1, x^2, z, \frac{\partial z}{\partial x^1}, \frac{\partial z}{\partial x^2})$

is an integral submanifold of I_2, then

$$\phi^{\star}(\theta^i) = 0, \quad i = 1, 2 \text{ and } 3.$$

This amounts about the equations

$$\left(\frac{\partial z}{\partial x^1} - p\right)dx^1 \wedge dx^2 = 0$$

$$\left(\frac{\partial z}{\partial x^2} - q\right)dx^1 \wedge dx^2 = 0,$$

$$\left(\frac{\partial p}{\partial x^2} + \frac{\partial q}{\partial x^1} - 2 \sin z\right)dx^1 \wedge dx^2 = 0,$$

so that on the integral submanifold where $dx^1 \wedge dx^2 \neq 0$, we have

$$p = \frac{\partial z}{\partial x^1}, \quad q = \frac{\partial z}{\partial x^2}$$

and $\dfrac{\partial^2 z}{\partial x^1 \partial x^2} = \sin z$

which is the sine-Gordon equation.

Now, from examples 5 and 6, it is seen that with the same differential equation, which is the sine-Gordon equation, two different ideals I_1 and I_2 are constructed. Clearly I_1 and I_2 have the same integral submanifolds which are specified by solutions of the sine-Gordon equation.

Thus, in general, we have the following:

Theorem 2:

Given a differential equation system DE, generated by a set of generators

$$F_1(x^a, z^\mu, z^\mu_a, \ldots) = 0,$$

$$F_2(x^a, z^\mu, z^\mu_a, \ldots) = 0,$$

$$\ldots\ldots$$

$$F_\alpha(x^a, z^\mu, z^\mu_a, \ldots) = 0, \quad \alpha<\infty \quad ,$$

one can construct exterior differential systems, not all necessarily equivalent, in terms of the variables x^a, z^μ, z^μ_a, ... such that the submanifolds

$$x^a \longrightarrow (x^a, z^\mu(x^a), \frac{\partial z^\mu}{\partial x^a}(x), \ldots)$$

are integral submanifolds of the exterior systems, if and only if,

$$x^a \longrightarrow z^\mu(x^a) = z^\mu(x)$$

is a solution of the differential equation system DE.

For more examples one may see the notes by T. Willmore [32], and Hermann [14].

The differential forms in the exterior differential systems, that one may construct from a given differential equation system, are classified into two basic types:

(a) Linearizing forms, which are introduced in order to reduce the differential equation system to a set of first order equations.

(b) Dynamic forms: these are the forms which represent the differential equation system itself. For example, the form θ^1 and θ^2 in example 6 and α^1, α^2 in example 4 are of the first type, while α^3 in example 4 and θ^3 in example 6 are dynamic forms.

The advantage of the use of exterior differential systems instead of differential equation systems is that they suit nicely the differential geometric study of non-linear partial differential equations.

As many exterior systems are constructed from a given differential equation system, there arises the question of which one to choose. A natural answer is that one may choose the smallest exterior differential system.

As soliton equations are the main objects of interest here, the following section is devoted to introducing solitons before we start their geometric study.

2 Soliton equations

In the study of non-linear wave equations, the term soliton is used to denote a single wave pulse (solitary wave), which emerges from a collision with a similar solitary wave having unchanged shape and speed.

A class of non-linear evolution equations with soliton solutions has been found. This class contains the sine-Gordon equation;

$$\frac{\partial^2 z}{\partial x \partial t} = \sin z,$$

the k.dV. equation;

$$z_t + z_{xxx} + 12zz_x = 0,$$

the modified k.dV. equation;

$$z_t + z_{xxx} + 6z^2z_x = 0,$$

in addition to some other equations which may be covered by the A.K.N.S. system

$$A_x = qC - rB,$$
$$q_t = B_x + 2Aq + 2\lambda iB,$$
$$r_t = C_x - 2Ar - 2\lambda iC,$$

where q, r are functions of x and t, A,B,C are functions of x, t, and λ, and λ is a parameter.

This system represents a class of non-linear evolution equations solvable by the scattering transformation [1].

Simply, the equations with soliton solutions are called "solitons". They have been shown to share several remarkable properties:

(i) They are solvable via the inverse scattering method.

(ii) They have an infinite number of conservation laws.

(iii) They have Bäcklund transformations.

(iv) They describe pseudospherical surfaces [23].

For the first property (i), all it amounts to is that the initial value problem for any soliton equation may be solved using linear methods only. In other words, it means that the given non-linear equation (soliton) can be obtained as the integrability condition of linear differential equations.

The second property (ii) tells us that solitons as waves do not die out over time and so they conserve energy. These conservation laws may be generated from the third property, that is the Bäcklund transformations which transform solutions of an equation to another, and that brought the fourth property (iv), that solitons describe pseudospherical surfaces, since Bäcklund transformations were originally defined as transformations between surfaces of constant negative curvature.

In this section "solitons" will mean 2-dimensional solitons, i.e. equations with two independent variables.

Geometrically, the first property of solitons, i.e. the inverse scattering method, is achieved by associating a pair of completely integrable pfaffian equations with the non-linear equation to be solved, namely

$$d\nu = \Omega\nu, \qquad \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{1}$$

where Ω is a 2×2 matrix, which is traceless

i.e. $\mathrm{tr}\Omega = 0$ (2)

and consists of 1-forms in the independent variables (x,t), the dependent variables and its derivatives.

Integrability of equation (1), i.e.

$$\Theta = d\Omega - \Omega \wedge \Omega = 0 \tag{3}$$

is by construction the original non-linear equation to be solved. Solutions of equation (1) will give rise to solutions of equation (3).

The matrix of 1-forms Ω is not unique for a given non-linear equation, because the scattering equations (1) and (2) and equation (3) are form invariant under the transformation

$$\left.\begin{aligned} v &\longrightarrow v' = Av \\ \Omega &\longrightarrow \Omega' = dAA^{-1} + A\Omega A^{-1}, \\ \Theta &\longrightarrow \Theta' = A\Theta A^{-1}, \end{aligned}\right\} \tag{4}$$

where A is an arbitrary 2×2 matrix with det A = 1.
Such a transformation (4) is called a gauge transformation.

It has been pointed out by Crampin [7], Hermann [16] and Dodd [9] that Ω can be interpreted as a connection 1-form for the principal sL(2,R)-bundle on R^2, and Θ its curvature 2-form. Also the matrix Ω is the key to associate the pseudospherical surface property to solitons as was shown by Sasaki [23] and by what will be shown for the A.K.N.S. system in following chapters.

A unifying class for soliton equations solvable by the 2-component inverse scattering method (as equations (1) and (2)) is given by the general A.K.N.S. system mentioned early in this section. In fact, with different identifications of the functions r, q, A, B and C, one obtains the different equations in this class.
For example

(a) $A = \dfrac{i}{4\lambda} \cos u, \ B = C = \dfrac{i}{4\lambda} \sin u$

$$r = -q = \frac{1}{2} u_x$$

for the equation $u_{xt} = \sin u$ (sine-Gordon).

(b) $q = u, \quad r = -1,$

$$A = -4\lambda^3 - 2\lambda u - u_x,$$

$$B = -u_{xx} - 2\lambda u_x - 4\lambda^2 u - 2u^2,$$

$$C = 4\lambda^2 + 2u$$

for the equation

$$u_t + 6uu_x + u_{xxx} = 0 \text{ (k,dV. equation).}$$

(c) $q = -r = u,$

$$A = -4\lambda^3 - 2\lambda u^2,$$

$$B = -u_{xx} - 2\lambda u_x - 4\lambda^2 u - 2u^2,$$

$$C = u_{xx} - 2\lambda u_x + 4\lambda^2 u + 2u^2$$

for the equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \text{ (modified k.dV. equation)}$$

and so on.

The analytic study of this system is given in detail in reference [1]. As a matter of fact, this system can be written in another form called the Lax form [18], using differential operators.

That is

$$\hat{L}\nu = \lambda\nu, \qquad \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

$$\nu_t = \hat{A}\nu,$$

$$\hat{L}_t = [\hat{A}, \hat{L}] \qquad \text{(the non-linear A.K.N.S. system)}$$

where $\hat{L} = i \begin{pmatrix} \frac{\partial}{\partial x} & -q \\ r & -\frac{\partial}{\partial x} \end{pmatrix},$

and

$$\hat{A} = \begin{pmatrix} A(x,t,\lambda) & B(x,t,\lambda) \\ C(x,t,\lambda) & -A(x,t,\lambda) \end{pmatrix} .$$

The system will be considered here to study differential geometric prolongations for solitons, as it covers all soliton equations and gives the unified results for it.

CHAPTER III

DIFFERENTIAL GEOMETRIC PROLONGATIONS

1 Prolongations

Similar to prolongations of differential equation systems, prolongations of exterior differential systems are defined. This prolongation is called differential geometric prolongation.

Definition: Let I be an exterior differential system on a manifold M, and I' an exterior differential system on a manifold M', with a map

$$\pi : M' \longrightarrow M, \text{ submersion,}$$

i.e. $\pi_* : TM' \longrightarrow TM$ is onto.

We say that I' is a prolongation of I via the map π if $\pi^*(I) \subset I'$.
We note that I' in this definition is not unique and there are many I's on M' that could satisfy the condition $\pi^*(\theta) \in I'$ for all $\theta \in I$.
But one may require I' to be the smallest exterior differential system between such I's on M'. Moreover the genus of I' in Cartan's sense (i.e. the maximum dimension of regular integral submanifolds) is equal to that of I.

Definition: In other words, a set of forms η^a on M', with $\pi : M' \longrightarrow M$, are said to define a prolongation of I on M via π if

$$d\eta^a \subset \text{(the ideal generated by } \pi^*(I) \text{ and } \eta^a)$$

i.e. $d\eta^a \subset \{\pi^*(I), \eta^a\}$.

η^a are called prolongation forms.

The forms η^a appear now as conservation laws for the ideal I' generated by $\pi^*(I)$ and η^a. Thus, when I is constructed from a differential equation with two independent variables, the set of forms η^a is in this case a set of 1-forms. This is the technique invented by Wahlquist and Estabrook [30, 31] to study conservation laws for non-linear evolution equations with soliton solutions.

Moreover, for geometric purposes and the sake of simplicity, the manifold M' will be taken as a product fibre space over M, say M' = MxN, with N another manifold with arbitrary dimension, say m. Let $\{x^i\}$, $\{y^a\}$ be coordinates for M and N respectively, with range of indices:

$$1 \leq i,j,k, \ldots \leq \dim M = n ,$$

$$1 \leq a,b,c, \ldots \leq \dim N = m.$$

The map $\pi : MxN \longrightarrow M$ is now given by

$$\pi(x,y) \longrightarrow x \text{ for } x \in M, \quad y \in N.$$

With this consideration, each $\theta \in I$ on M also belongs to I', so that one may say that I' is generated by I and the additional 1-forms η^a on MxN given by

$$\eta^a = dy^a - \omega^a, \quad 1 \leq a \leq m$$

where ω^a are 1-forms on M with coefficients functions of x^i and y^a, i.e. ω^a involve dx^i but not dy's.

In such a case, $I' = \{I, \eta^a\}$ is called a multiple prolongation of I and the y^a's appearing in η^a are called pseudopotentials.

Because the forms $\eta^a = dy^a - \omega^a$ are conservation laws for I', they may produce conservation laws for I and hence for the differential equation underlying it. This was the reason behind Wahlquist and Estabrook technique which is now called differential geometric prolongation.

Now, because I' is a differential ideal, we must have $d\eta^a \in I'$ for

$1 \leqslant a \leqslant m$. But $d\eta^a = d\omega^a$, and since ω^a contain only dx's,

i.e. $\omega^a = \omega^a_i (x,y)\, dx^i$, say

then $d\omega^a = \partial_{y^b} \omega^a \wedge dy^b + d_x \omega^a$,

where

$$d_x\omega^a = \left(\frac{\partial\omega^a_i}{\partial x^j} dx^j\right) \wedge dx^i,$$

and

$$\partial_{y^b}\omega^a = \frac{\partial\omega^a_i}{\partial y^b} dx^i.$$

Hence $d\eta^a = -\partial_{y^b} \omega^a \wedge (\eta^a + \omega^a) + d_x \omega^a$,

so that $d\eta^a \in I'$ requires that:

$$d_x \omega^a - \omega^b \wedge \partial_{y^b} \omega^a \in I'.$$

One may refer this condition to I by holding y's constant and say that

$$d_x \omega^a - \omega^b \wedge \partial_{y^b} \omega^a \in I \text{ for fixed y's.}$$

The part played by the 1-forms ω^a appearing in η^a is very important. It is the key for introducing many algebraic and geometric structures to a differential equation system. It also defines the type of prolongation one may be considering. Furthermore, knowing the form of ω^a gives one all that is needed to know about the algebraic and geometric structures that could be introduced to this differential equation system.

In the following sections, we show these points in detail.

2 Prolongation Lie algebras

Let us continue with same notation as in the last section. To see how algebraic structures may be associated to a differential equation system via prolongations, we take one of the possible choices for ω^a, which is the most effective for applications. That is the one in which ω^a take the form

$$\omega^a = A_1^a \beta^1 + A_2^a \beta^2 + \ldots + A_r^a \beta^r,$$

where $A_1^a, \ldots, A_r^a$ are functions of y's alone and $\beta^1, \ldots, \beta^r$ are 1-forms in x alone.

i.e. $\omega^a = A_u^a \beta^u, \qquad 1 \leqslant u, v, \ldots \leqslant r,$

where r is an integer which will have its significance afterwards.

Now, with this choice, we define

$$d_x \omega^a = A_u^a d\beta^u,$$

and $\partial_{y^b} \omega^a = \partial_{y^b}(A_u^a) \beta^u,$

so that the condition of prolongation becomes

$$d_x \omega^a - \omega^b \wedge \partial_{y^b} \omega^a = A_u^a \beta^u - \tfrac{1}{2}(A_v^b \partial_{y^b} A_u^a - A_u^a \partial_{y^b} A_v^a) \in I \text{ for fixed y's,}$$

i.e. $A_u^a \beta^u - \frac{1}{2}(A_v^b \partial_{y^b} A_u^a - A_u^a \partial_{y^b} A_v^a) \in I$ for fixed y's (1)

Let us introduce the notation

$$A_u = A_u^a \frac{\partial}{\partial y^a}, \qquad 1 \leqslant u, v, \ldots \leqslant r$$

Since A_u^a are functions of y's alone, this formula defines r vector fields on N.

Set $A_{\nu u} = [A_\nu, A_u]$, the Lie bracket of A_ν and A_u. Then

$$A_{\nu u} = [A^b_\nu \frac{\partial}{\partial y^b}, A^a_u \frac{\partial}{\partial y^a}]$$

$$= (A^b_\nu \frac{\partial A^a_u}{\partial y^b} - A^b_u \frac{\partial A^a_\nu}{\partial y^b}) \frac{\partial}{\partial y^a}$$

i.e.

$$A_{\nu u} = A^a_{\nu u} \frac{\partial}{\partial y^a},$$

with

$$A^a_{\nu u} = (A^b_\nu \frac{\partial A^a_u}{\partial y^b} - A^b_u \frac{\partial A^a_\nu}{\partial y^b}), \text{ functions of } y\text{'s.}$$

Thus it is seen that equations (1) take the form

$$A^a_u \, d\beta^u - \frac{1}{2} A^a_{\nu u} \, \beta^\nu \wedge \beta^u \in I \text{ for fixed } y \qquad (2)$$

If one knew the β^u's, equations (2) would give relations between A_ν and A_u, that is $[A_u, A_\nu] = A_{u\nu}$.
Moreover, from the set of vector fields A_u and A_ν, using the Lie bracket and the Jacobi identity, one may have a Lie algebra structure. In general this Lie algebra or "the like Lie algebra" seems to be infinite dimensional, as the processes may continue and it is not closed. But one may introduce conditions at some stage to force closure and has a finite dimensional Lie algebra as is the case in applications for some equations of physical interest, like the K,dV. equation, the sine-Gordon equation, ... etc., that have been obtained by many people starting with Wahlquist and Estabrook [30, 31], Dodd and Gibbon [9] and Morris [19, 20, 21]

Definition: The equations defining these algebraic structures are called the prolongation equations.

Since the vector fields $A_u = A_u^a \frac{\partial}{\partial y^a}$ are vector fields on N, the fiber of the fibre space π : M x N ⟶ M, the associated Lie algebras are Lie subalgebras of the Lie algebra of vector fields on N.

Moreover, the prolongation Lie algebra is unique in its dependence on y's up to maps of the vector fields A_u, which preserve the prolongation equations representing it.

In application to equations of interest, finite dimensional Lie algebras are shown to be homomorphic images of the infinite dimensional one, [26], [8]. A natural consequence is that one may look for representations of these Lie algebras and in this case the best choice is to take m = dim N to be the dimension of the representation space.

Having at hand such a prolongation Lie algebra, its automorphisms may be obtained from the symmetries of the exterior differential system I, i.e. from maps $\phi : M \longrightarrow M$ such that

$$\phi^*(I) = I$$

Also, as examples for geometric structures, associated to a differential equation system via prolongations, are Cartan-Ehresmann connections, to which the next section is devoted.

3 Prolongations and Cartan-Ehresmann connections

Assume that we have a finite dimensional prolongation Lie algebra, say $\underset{\sim}{G}$, of vector fields acting on the fiber of the fibre space

$$\pi: M \times N \longrightarrow M$$

with dim G = r.

Let A_u, $u = 1,2, \ldots, r$ be a basis for $\underset{\sim}{G}$, where each of A_u is of the form

$$A_u = A_u^a (y) \frac{\partial}{\partial y^a}$$

There are then real numbers $C_{u\nu}^w$, the structure constants of $\underset{\sim}{G}$ with respect to the basis A_u, such that

$$[A_u, A_\nu] = C_{u\nu}^w A_w$$

Denote by G the simply connected Lie group corresponding to $\underset{\sim}{G}$. It is the structure group of the fibre space.

A geometric interpretation of prolongation is that it defines a Cartan-Ehresmann connection on the fibre space $\pi: M \times N \longrightarrow M$, where η^a determine the connection 1-forms.

Thus, if $P \longrightarrow M$ is a submanifold of M which is an integral submanifold of the exterior differential system I, then on this submanifold the parallel transport defined by the connection is independent of the path. To work this out in more detail, let

$$\eta^a = dy^a - A_u^a \beta^u$$

be the prolongation 1-forms, with $\omega^a = A_u^a \beta^u$.

The prolongation conditions (2) lead to

$$(A_u^a d\beta^u - \frac{1}{2} C_{\nu u}^w \beta^\nu \wedge \beta^u A_w^a) \in I \text{ for fixed } y.$$

Let us assume that $\underset{\sim}{G}$ acts effectively on N, i.e. no element of $\underset{\sim}{G}$ is the zero vector field, so that A_u^a are not identically zero, and we may have

$$d\beta^u - \frac{1}{2} C_{\nu w}^u \beta^\nu \wedge \beta^w \in I.$$

Now, by the definition of Cartan-Ehresmann connection in Chapter I, the 2-forms

$$\Omega^u = d\beta^u + \frac{1}{2} C_{\nu w}^u \beta^\nu \wedge \beta^w$$

are, by definition, the curvature 2-forms of the connection defined by

η^a as its dual horizontal vector field system on the fibre space

$\pi : M \times N \longrightarrow M$, or by the Lie algebra-valued 1-form

$\underset{\sim}{\eta} = \beta^u A_u$, A_u are basis for $\underset{\sim}{G}$.

Because G is the structure group of the fibre space, the connection is a G-connection. Clearly, this connection is flat, i.e. the pfaffian system $\eta^a = dy^a - A^a_u \beta^u = 0$ is completely integrable if, and only if, the curvature forms Ω^u vanish.

Moreover, the connection satisfies a Bianchi identity of the form

$$2d\underset{\sim}{\Omega} = [\underset{\sim}{\Omega}, \underset{\sim}{\eta}] - [\underset{\sim}{\eta}, \underset{\sim}{\Omega}] ,$$

where $\underset{\sim}{\Omega} = \Omega^u A_u$ is the Lie algebra-valued 2-form of the curvature.

An important property of these G-connections is that their holonomy groups are subgroups of G. This will be a key to tie up different connections. For that, let us assume that there is a symmetry of I, $\phi : M \longrightarrow M$ with $\phi^*(I) = I$. This map ϕ may be lifted (prolonged) to a map $\tilde{\phi}$ on $M \times N$ such that the diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad \tilde{\phi} \quad} & M \times N \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[\quad \phi \quad]{} & M \end{array}$$

commutes and $\tilde{\phi}$ is a symmetry of I' on $M \times N$, i.e. $\phi^*(I') = I'$.

Then the map ϕ may be used to give rise to an automorphism of the associated Lie algebra $\underset{\sim}{G}$ on $M \times N$, say ℓ,

$\ell : \underset{\sim}{G} \longrightarrow \underset{\sim}{G}$ automorphism.

Suppose we have another fibre space over M with fiber Z, where dim Z $\geqslant$ dim N, i.e. $Z = N \times R^t$ for some integer $t \geqslant 0$, with $\{z^q\}$ coordinates of Z.

Let I'_1 be the exterior differential system on $M \times Z$, prolongation of I via the map:

$$M \times Z \longrightarrow M$$

together with a set of 1-forms ζ^{α}, $\alpha = 1, \ldots,$ dim Z. Let $\underset{\sim}{G}_1$ be the associated Lie algebra of prolongation, with B_q, $q = 1, \ldots,$ dim $\underset{\sim}{G}_1$ as its basis and Γ^q_{sp} its structure constants.

So $\qquad \zeta^{\alpha} = dz^{\alpha} - B^{\alpha}_q (z)\, \gamma^q$

where γ^q are 1-forms on M depending on x's alone and $1 \leqslant q,s,p,\ldots, \leqslant$ dim $\underset{\sim}{G}_1$

Now the symmetry ϕ of I may give rise to a symmetry $\tilde{\phi}_1$ of I_1' on MxZ.

i.e. $\qquad \tilde{\phi}_1 : MxZ \longrightarrow MxZ,$

with $\qquad \tilde{\phi}_1{}^*(I_1') = I_1'$.

So we have the following commutative diagram:

$$\begin{array}{ccc} MxZ & \xrightarrow{\tilde{\phi}_1} & MxZ = MxNxR^t \\ \downarrow & & \downarrow id_M x h \\ MxN & \xrightarrow{\tilde{\phi}} & MxN \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\phi} & M \end{array}$$

, with h,

$h : Z \longrightarrow N$

a projection map.

The G-connection on M x N, given by

$$\eta^a = dy^a - A^a_u \beta^u$$

has curvature 2-forms,

$$\Omega^u = d\beta^u + \frac{1}{2} C^u_{vw} \beta^v \wedge \beta^w .$$

A G_1-connection on M x Z, with G_1 the corresponding Lie group of $\underset{\sim}{G}_1$, is given via the set of 1-forms

$$\zeta^{\alpha} = dz^{\alpha} - B^{\alpha}_q \gamma^q ,$$

and has curvature 2-forms, say Λ^q,

$$\Lambda^q = d\gamma^q + \frac{1}{2}\Gamma^q_{sp}\ \gamma^s \wedge \gamma^p \quad .$$

Assume that $h : G_1 \longrightarrow G$ is a Lie algebra homomorphism which may be taken in a simple case as

$$hB_q = h^u_q\ A_u, \qquad \text{with } h^u_q \text{ constants}$$

As mentioned earlier in section 2, such homomorphism exists, and so we may write

$$h\zeta^\alpha = h\, dz^\alpha - h(B^\alpha_q)\ \gamma^q$$

in terms of A_u, the basis of G, as

$$h\zeta^\alpha = dy^a - A^a_u\ \beta^u,$$

with

$$hB_q = h^u_q\ A_u \qquad \text{and} \qquad \beta^u = h^u_q\ \gamma^q.$$

Hence, using the homomorphism property,

i.e. $h^u_q\Gamma^q_{sp} = h^\nu_s\ h^w_p\ C^u_{\nu w}$,

one finds that

$$\Omega^u = h^u_q\ \Lambda^q.$$

Thus the G-connection curvature 2-forms are linear combinations of the curvature 2-forms of the G_1-connection.

Hence, we proved the following result:

Theorem 1:

All Cartan-Ehresmann connections associated to a differential equation system via its differential geometric prolongations are homomorphic, in the sense that if we have two such connections, then the curvature 2-forms of one connection are linear combinations of the curvature 2-forms of the other.

We remark that the theorem also applies to the case of Cartan-Ehresmann connections on the same fibre space but obtained from different exterior differential systems which are prolongations of the original system considered on this fibre space, since I' is not unique as prolongation for I.

Now, this relation between the curvatures of Cartan-Ehresmann connections may lead to a relation between the holonomy groups of these connections. To do that, we first state the main theorem we know here, that is the Ambrose-Singer Theorem. For its details one may consult [2].

Theorem: (Ambrose-Singer Theorem)

Let (P, M, π, G, H) be a principal bundle with connection H and group G. Let $b \in P$ and consider $G(b)$ and $G_o(b)$ the holonomy group and the null holonomy group of H at b respectively. Let

$$P(b) = \{c \in P \mid c \text{ can be joined to b by a horizontal curve in } P\}.$$

Let $\underset{\sim}{\omega}$ be the Lie algebra-valued 1-form of H and $\underset{\sim}{\Omega}$ its Lie algebra-valued 2-form of the curvature.
Then the set of linear maps

$$\{\underset{\sim}{\Omega}\ (v_1, v_2) \mid \ v_1, v_2 \in M_a, \quad a \in \pi(c), \quad c \in P(b)\}$$

is a subalgebra of the Lie algebra of G. Moreover $G(b)$ and $G_o(b)$ are Lie subgroups of G and the subgroup generated by this subalgebra is $G_o(b)$, the null holonomy group of H.

As consequences of this theorem we state the following results:

1 - The holonomy groups of a connection at points in the same fiber form a conjugacy class of subgroups of G.

2 - Different connections give rise to different classes of conjugacy in G.

3 - If two points in the bundle space can be joined by a horizontal curve, then the holonomy groups of a connection at the two points are the same.

This enables one to state the following:

Theorem 2:

The holonomy groups of Cartan-Ehresmann connections associated to a differential equation system via prolongations are homomorphic.

Proof:

It is a direct result of Ambrose-Singer theorem and the relation between the curvatures of these connections as was shown in Theorem 1.

Now we turn to another important property of the equations of interest here, that is being solvable with linear methods, or in other words to a non-linear differential equation (soliton) one associates to it integrable linear systems, with solutions corresponding to those of the non-linear equation.

In fact, differential geometric prolongation helps one to do that. We consider this point in the following section.

4 Linear systems associated to a non-linear one via prolongations

It is now clear that different types of Lie algebras should be reflected in different sorts of pseudopotentials and prolongations and hence the associated connections. This is the part played by ω^a in the prolongation 1-forms

$$\eta^a = dy^a - \omega^a.$$

To clear this point, let us continue with same notation and assume that $\underset{\sim}{G}$ is a linear Lie algebra of linear vector fields on N, where by a linear vector field we mean

$$A = \alpha_a^b\, y^a \frac{\partial}{\partial y^b} ,$$

with (α_a^b) constants, a,b = 1,2, ..., m. This linear vector field A generates a 1-parameter group of linear transformations on N and the orbit equation is

$$\frac{dy^b}{dt} = \alpha_a^b\, y^a , \qquad t \text{ is the parameter.}$$

The map $A \longrightarrow (\alpha_a^b)$ constitutes a matrix representation of the Lie algebra $\underset{\sim}{G}$ of such linear vector fields.

Now, let I be an exterior differential system on M and $\underset{\sim}{G}$ is acting linearly on N. Let I' on M x N be a prolongation of I via π,

$$\pi : M \times N \longrightarrow M,$$

together with the set of 1-forms

$$\eta^a = dy^a - A_u^a\, \beta^u,$$

where $A_u = A_u^a \dfrac{\partial}{\partial y^a}$ are basis for $\underset{\sim}{G}$, i.e. $A^a_u = {}_u\alpha^a_b\, y^b$, $1 \leqslant a,b \leqslant \dim N = m$

$1 \leqslant u \leqslant \dim \underset{\sim}{G}$.

The prolongation conditions are that $d\eta^a \in I'$. If $Z \longrightarrow M$ by $z \longrightarrow x(z)$ is an integral submanifold of I, then the pfaffian equations

$$\eta^a = 0$$

are satisfied on Z and they are completely integrable.

In fact

$\eta^a/Z = 0$ is just

$$dy^a - {}_u\alpha^a_b\, y^b\, \overset{u}{\beta}(x(z)) = 0 \tag{3}$$

Hence the system (3) is completely integrable and it is a linear system which we have associated to the original non-linear system represented by I on M.

The equations in the linear system (3) are to be identified with the linear inverse scattering equations of the non-linear wave theory when it is to be considered. In this case equations (3) may take the form, in matrix notation,

$$dY = \Omega Y,$$

where $\Omega = ({}_u\alpha^a_b\, \beta^u)$ is a matrix of 1-forms and

$$Y^t = (y^1, y^2, \ldots, y^m).$$

This will be clear when we study solitons in the following chapter.

Another important property of the equations of interest here, is the Bäcklund transformations of these equations. Thus differential geometric prolongation may suggest a formulation of these transformations. We do that in the next section.

5 Prolongations and Bäcklund transformations

In this section, we will consider the most general definition for Bäcklund transformations that is transformations between solutions of differential equation systems.

So, we start by giving an exterior differential system I on a manifold M, representing a system of differential equations on M. Assume a prolongation is given by a set of prolongation forms η^a on some fibre space $M \times N$, with $\pi : M \times N \longrightarrow M$, a fibre space map.

Let $\alpha : Z \longrightarrow M$ be a solution of the system, i.e. an integral submanifold of I.

Hence

$$\alpha^*(I) = 0$$

Now extend α to a map $\bar{\alpha}$,

$$\bar{\alpha} : Z \longrightarrow M \times N,$$

such that

$$(\bar{\alpha})^* \eta^a = 0 \quad .$$

Definition: A diffeomorphism β,

$$\beta : M \times N \longrightarrow M \times N,$$

which is a symmetry (Bäcklund symmetry),

i.e. $\beta^*(I) \subset \{I, \eta^a\}$, is said to be a Bäcklund transformation of the differential equation system, in the sense that given a solution α, another solution may be generated via β.

In fact the map $\pi\beta\bar{\alpha} : Z \longrightarrow M$, is a new solution defined by β from α, since

$$
\begin{aligned}
(\pi\beta\bar{\alpha})^{*}(I) &= \bar{\alpha}^{*}\beta^{*}\pi^{*}(I) \\
&= \bar{\alpha}^{*}\beta^{*}\,(I) \\
&= \bar{\alpha}^{*}(\{I,\ \eta^{a}\}) \\
&= 0.
\end{aligned}
$$

Because it is not necessary that

$$(\beta\bar{\alpha})^{*}(\eta^{a}) = 0,$$

such Bäcklund transformations β's do not form a group which is actually the symmetry group of I.

Another way of describing Bäcklund transformations in terms of prolongation, as correspondence between solutions of exterior differential systems, could be as follows:

Let I and I' be two exterior differential systems on manifolds M and M' respectively. Consider another system I" on a manifold M" together with a pair of maps

$$
\begin{aligned}
\phi &: M'' \longrightarrow M, \\
\phi' &: M'' \longrightarrow M'.
\end{aligned}
$$

If ϕ and ϕ' define I" as prolongation of I and I' via ϕ and ϕ', i.e.

$$\phi^{*}(I) \subset I'' \text{ and } \phi'^{*}(I') \subset I'',$$

then these data are said to define a Bäcklund transformation between I and I'.

The correspondence between integral submanifolds of I and I' with these data is defined as follows:
Let $Z \longrightarrow M$ be an integral submanifold of I and $Z' \longrightarrow M'$ an integral submanifold of I'. They are said to correspond under a Bäcklund transformatior if there is an integral submanifold $Z'' \longrightarrow M''$ of I" such that:

$$\phi(Z'') = Z, \text{ and } \phi'(Z'') = Z'.$$

Thus Bäcklund transformations in general are really pairs of prolongations and the practical problem is to find (M", I") given (M,I) and (M', I').

As a remark, if the prolongations I' and I", ... are interpreted in terms of connections, then changing these connections with some gauge transformations, the Bäcklund transformation will vary in general as they are not gauge invariant.

To complete the theory of differential geometric prolongations, we close this chapter by a remark about another type of prolongation in the next section.

6 Quadratic prolongation

Quadratic prolongation is another type of prolongation in which the differential forms ω^a in η^a take the form

$$\omega^a = \omega^a_o + \omega^a_b\, y^b + \omega^o_b\, y^a\, y^b,$$

so that

$$\eta^a = dy^a - \omega^a_o - \omega^a_b\, y^b - \omega^o_b\, y^a\, y^b,$$

where ω^a_o, ω^a_b, ω^o_b are 1-forms on M with $\pi: M \times N \longrightarrow M$, the fibre space. In matrix notation

$$\eta = dY - W_o - W_1 Y - Y^t W_2 Y,$$

where

$$\eta^t = (\eta^1, \eta^2, \ldots, \eta^m),\quad Y^t = (y^1, \ldots, y^m),$$

$$W_o = \begin{pmatrix} \omega^1_o \\ \vdots \\ \omega^m_o \end{pmatrix}, \qquad W_1 = \begin{pmatrix} \omega^1_1 & \cdots & \omega^m_1 \\ \vdots & & \vdots \\ \omega^1_m & \cdots & \omega^m_m \end{pmatrix}$$

and $$W_2 = \begin{pmatrix} \omega_1^o \\ \vdots \\ \omega_m^o \end{pmatrix}$$, with m = dim N.

In this case we say that each η^a is a quadratic prolongation. The associated connections via this type of prolongation are of the projective type and such a projective connection will be integrable if, and only if, the pfaffian system $\eta^a = 0$ is completely integrable, i.e. if, and only if, $d\eta^a \subset \{I, \eta^a\}$, which will give us again the curvature of this projective connection as a type of Cartan-Ehresmann connection.

A known fact is that each projective connection has an underlying linear connection. This gives rise to linear systems associated to the non-linear one with this type of prolongation. This is done by raising the dimension of the fiber by 1 and then considering the underlying linear connection. The parallel transport equations of this new linear connection are just the equations of the linear system [13].

Also, if an algebraic structure is associated via quadratic prolongation, then it will be a Lie subalgebra of the Lie algebra of the projective group acting on the fiber by fractional linear transformations. But this type of prolongation is not used very much in applications because of its difficulty.

As the prolongation technique was invented to handle, in particular, non-linear partial differential equations with physical interest, it is more efficient when applied to soliton equations.

This is what will be shown in the next chapter.

CHAPTER IV

PROLONGATIONS OF SOLITON EQUATIONS

1 SL(2,R)-structure for the A.K.N.S. system

The set of equations

$$\left.\begin{aligned} A_x &= qC - rB \\ q_t &= B_x + 2Aq + 2\lambda iB \\ r_t &= C_x - 2Ar - 2\lambda iC \end{aligned}\right\} \quad (1)$$

with q,r functions of (x,t), A,B,C are functions of (x,t,λ) and λ is a parameter, as mentioned earlier is known as the A.K.N.S. system which represents a class of non-linear evolution equations solvable via the 2-component inverse scattering method.

We have seen that with different identifications of the functions A,B,C,q and r, this system covers the 2-dimension soliton equations known so far. Hence the study of this system is preferable to the study of each individual equation covered by the system as it raises the common results of such study for the 2-dimension solitons in general.

An attempt to obtain 2-dimension soliton equations solvable by the multilinear component inverse scattering method has been made [21]. As it is not complete, the equations solvable via the 2-component inverse scattering, i.e. the A.K.N.S. system, are considered here.

Now, for the A.K.N.S. system (1), an equivalent ideal of forms I is given by the following set of forms:

$$\alpha_1 = dA \wedge dt + (rB - qC)dx \wedge dt,$$

$$\alpha_2 = dq \wedge dx + dB \wedge dt + (2\lambda iB + 2Aq)dx \wedge dt,$$

$$\alpha_3 = dr \wedge dx + dC \wedge dt - (2\lambda iC + 2Ar)dx \wedge dt,$$

on a manifold M with coordinate presentation (x, t, A, B, C, q, r).

Since

$$d\alpha_1 = B\alpha_3 \wedge dt - r\alpha_2 \wedge dx - q\alpha_3 \wedge dx - C\alpha_2 \wedge dx,$$

then $d\alpha_1 \in I$.

Also

$$d\alpha_2 = -2\lambda i\alpha_2 \wedge dx - 2q\alpha_1 \wedge dx + 2A\alpha_2 \wedge dt,$$

and

$$d\alpha_3 = 2\lambda i\alpha_3 \wedge dx + 2r\alpha_1 \wedge dx - 2A\alpha_3 \wedge dt$$

therefore

$d\alpha_2 \in I$ and $d\alpha_3 \in I$.

Hence I is a differential ideal.

The A.K.N.S. system (1) now is the integral submanifold equations for the exterior differential system I.

Following the prolongation technique, let M' be a fibre space over M with $\pi: M' \longrightarrow M$ of maximal rank.

We can take $M' = M \times N$, N is the fiber of π with dim N = m and a coordinate presentation $\{y^a\}$, a = 1, ..., m. Let π define a prolongation of I via a set of 1-forms η^a, a = 1, ..., m. Because of the product $M' = M \times N$, the set of 1-forms η^a could be written in the form

$$\eta^a = dy^a - \omega^a$$

where ω^a are 1-forms on $M \times N$ containing only dx and dt with coefficient functions on $M \times N$, i.e.

$$\eta^a = dy^a + F^a(x,t,A,B,C,q,r,y)dx + G^a(x,t,A,B,C,q,r,y)dt.$$

The prolongation conditions $d\eta^a \subset \{I, \eta^a\}$ imply that $d\eta^a$ must be in the form

$$d\eta^a = \sum_{i=1}^{3} f_i^a \alpha_i + \sum_{b=1}^{m} \theta_b^a \wedge \eta^b \tag{2}$$

where f_i^a are functions on $M \times N$ and θ_b^a are 1-forms on M, i.e.

$$\theta_b^a = \sum_{j=1}^{7} g_{bj}^a \, du^j,$$

and u^j stands for x, t, A, B, q, and r, with j = 1,2, ..., 7.

For simplicity, we take F^a and G^a to be independent of x and t. Moreover we take u^j to run only for x and t, i.e.

$$g_{b3}^a = g_{b4}^a = \ldots = g_{b7}^a = 0, \text{ and}$$

$$\theta_b^a = g_{b1}^a \, dx + g_{b2}^a \, dt,$$

since there is no loss with these assumptions. This is a multilinear prolongation and equations (2) give:

$$dF^a \wedge dx + dG^a \wedge dt - f_i^a \alpha_i - g_{bj}^a \, du^j \wedge \eta^b = 0,$$

i.e.

$$\frac{\partial F^a}{\partial A} dA \wedge dx + \frac{\partial F^a}{\partial B} dB \wedge dx + \frac{\partial F^a}{\partial C} dC \wedge dx + \frac{\partial F^a}{\partial q} dq \wedge dx + \frac{\partial F^a}{\partial r} dr \wedge dx$$

$$+ \frac{\partial F^a}{\partial y^b} dy^b \wedge dx$$

$$+ \frac{\partial G^a}{\partial A} dA \wedge dt + \frac{\partial G^a}{\partial B} dB \wedge dt + \frac{\partial G^a}{\partial C} dC \wedge dt + \frac{\partial G^a}{\partial q} dq \wedge dt + \frac{\partial G^a}{\partial r} dr \wedge dt$$

$$+ \frac{\partial G^a}{\partial y^b} dy^b \wedge dt$$

$$- f_i^a \alpha_i - g_{bj}^a \, du^j \wedge \eta^b = 0$$

where the summation convention is used with the range of indices

$$j = 1,2$$

$$a,\ b = 1,\ \ldots,\ m.$$

This gives rise to

$$\left.\begin{aligned}
&\frac{\partial F^a}{\partial A} = 0\,, && \frac{\partial G^a}{\partial A} = f_1^a \\
&\frac{\partial F^a}{\partial B} = 0\,, && \frac{\partial G^a}{\partial B} = f_2^a \\
&\frac{\partial F^a}{\partial C} = 0\,, && \frac{\partial G^a}{\partial C} = f_3^a \\
&\frac{\partial F^a}{\partial q} = f_2^a, && \frac{\partial G^a}{\partial q} = 0 \\
&\frac{\partial F^a}{\partial r} = f_3^a && \frac{\partial G^a}{\partial r} = 0 \\
&\frac{\partial F^a}{\partial y^b} = -g_{b1}^a && \frac{\partial G^a}{\partial y^b} = g_{b2}^a
\end{aligned}\right\} \qquad (3)$$

and

$$f_1^a(rB - qC) + f_2^a(2\lambda iB + 2Aq) - f_3^a(2\lambda iC + 2Ar) - g_{b1}^a F^b + g_{b2}^a G^b = 0 \qquad (4)$$

Equations (3) and (4) give the prolongation equations. They are:

$$\left.\begin{aligned}
&\frac{\partial G^a}{\partial B} - \frac{\partial F^a}{\partial q} = 0 \\
&\frac{\partial G^a}{\partial C} - \frac{\partial F^a}{\partial r} = 0
\end{aligned}\right\} \qquad (5)$$

and

$$\frac{\partial G^a}{\partial A}(rB - qC) + \frac{\partial G^a}{\partial B}(2\lambda iB + 2Aq) - \frac{\partial G^a}{\partial C}(2\lambda iC + 2Ar) + \frac{\partial G^a}{\partial y^b}F^b - \frac{\partial F^a}{\partial y^b}G^b = 0 \qquad (6)$$

Solving equations (3) and (5) for the functions F^a and G^a, we find that

$$F^a = X_1^a + X_2^a q + X_3^a r ,$$
$$G^a = X_2^a B + X_3^a C + X_4^a A + X_5^a ,$$

where X_i^a, i = 1, 2, ..., 5 are functions of y's alone, the partial integration functions. Now, substituting back in equation (6), we have the following equation:

$$rBX_4^a - qCX_4^a + 2\lambda iBX_2^a + 2AqX_2^a - 2\lambda iCX_3^a - 2ArX_3^a + \left(X_1^b \frac{\partial X_5^a}{\partial y^b} - X_5^b \frac{\partial X_1^a}{\partial y^b}\right)$$

$$+ B\left(X_1^b \frac{\partial X_2^a}{\partial y^b} - X_2^b \frac{\partial X_1^a}{\partial y^b}\right) + A\left(X_1^b \frac{\partial X_4^a}{\partial y^b} - X_4^b \frac{\partial X_1^a}{\partial y^b}\right) + C\left(X_1^b \frac{\partial X_3^a}{\partial y^b} - X_3^b \frac{\partial X_1^a}{\partial y^b}\right)$$

$$+ q\left(X_2^b \frac{\partial X_5^a}{\partial y^b} - X_5^b \frac{\partial X_2^a}{\partial y^b}\right) + Aq\left(X_2^b \frac{\partial X_4^a}{\partial y^b} - X_4^b \frac{\partial X_2^a}{\partial y^b}\right) + qC\left(X_2^b \frac{\partial X_3^a}{\partial y^b} - X_3^b \frac{\partial X_2^a}{\partial y^b}\right)$$

$$+ r\left(X_3^b \frac{\partial X_5^a}{\partial y^b} - X_5^b \frac{\partial X_3^a}{\partial y^b}\right) + rB\left(X_3^b \frac{\partial X_5^a}{\partial y^b} - X_5^b \frac{\partial X_3^a}{\partial y^b}\right) + Ar\left(X_2^b \frac{\partial X_4^a}{\partial y^b} - X_4^b \frac{\partial X_2^a}{\partial y^b}\right) = 0 \quad (7)$$

Assume by definition that

$$\left(X_i^b \frac{\partial X_j^a}{\partial y^b} - X_j^b \frac{\partial X_i^a}{\partial y^b}\right) = [X_i, X_j]^a, \quad i,j = 1, \ldots, 5$$

then equation (7) gives, by equating coefficients of powers A, B, C, q, r and their multiplications,

$$\left.\begin{array}{ll}
[X_1, X_4]^a = 0 , & [X_1, X_5]^a = 0 \\
[X_1, X_2]^a = -2\lambda i X_2^a , & [X_1, X_3]^a = 2\lambda i X_3^a \\
[X_1, X_4]^a = 0 , & [X_2, X_5]^a = 0 \\
[X_2, X_4]^a = -2X_2^a , & [X_3, X_4]^a = 2X_3^a \\
[X_2, X_3]^a = X_4^a , & [X_3, X_5]^a = 0 \\
[X_4, X_5]^a = \text{? unknown}, & \\
\text{with } [X_i, X_j]^a = -[X_j, X_i]^a . &
\end{array}\right\} \quad (8)$$

This is a partial Lie algebra as it is not complete (closed).

Fortunately, from equations (8), using the Jacobi identity, we have $[X_4, X_5]^a = 0$, so that

$$\left.\begin{array}{ll}
[X_1, X_2]^a = -2\lambda i X_2^a , & [X_1, X_3]^a = 2\lambda i X_3^a \\
[X_1, X_4]^a = 0 , & [X_1, X_5]^a = 0 \\
[X_2, X_3]^a = X_4^a , & [X_2, X_4]^a = -2X_2^a \\
[X_2, X_5]^a = 0 , & [X_3, X_4]^a = 2X_3^a \\
[X_3, X_5]^a = 0 , & [X_4, X_5]^a = 0
\end{array}\right\} \quad (9)$$

gives a Lie algebra generated by the functions X_i^a, $i = 1,2, \ldots, 5$, $a = 1, \ldots, m$.

We introduce the notation:

$$A_i = X_i^a \frac{\partial}{\partial y^a} \equiv A_i^a \frac{\partial}{\partial y^a} ,$$

where A_i^a are functions of y^a's alone. Hence the vector fields A_i appear as if they are among a basis of vector fields on the fiber N. Using this notation in equation (9), we get

$$\left.\begin{aligned}
&[A_1, A_2] = -2\lambda i A_2 , && [A_1, A_3] = 2\lambda i A_3 \\
&[A_1, A_4] = 0 , && [A_1, A_5] = 0 \\
&[A_2, A_3] = A_4 && [A_2, A_4] = -2A_2 \\
&[A_2, A_5] = 0 , && [A_3, A_4] = 2A_3 \\
&[A_3, A_5] = 0 , && [A_4, A_5] = 0
\end{aligned}\right\} \quad (10)$$

Equations (10) represent a complete Lie algebra generated by the vector fields A_1, A_2, A_3, A_4 and A_5. Fortunately, this finite Lie algebra has been obtained without imposing any conditions for closure, which is not the case in general.

Now, let us define:

$$\left.\begin{aligned}
Y_1 &= \frac{1}{2} A_4 , \\
Y_2 &= A_2 , \\
Y_3 &= A_3 , \\
Y_4 &= A_5 , \\
Y_5 &= \epsilon(A_1 + \lambda i A_4)
\end{aligned}\right\} \quad (11)$$

with ϵ as a parameter.

Then equations (10) become

$$\left.\begin{aligned}
&[Y_1, Y_2] = Y_2 , \\
&[Y_1, Y_3] = -Y_3 , \\
&[Y_2, Y_3] = 2Y_1 , \\
&[Y_1, Y_4] = [Y_1, Y_5] = 0 , \\
&[Y_2, Y_4] = [Y_2, Y_5] = 0, \\
&[Y_3, Y_4] = [Y_3, Y_5] = 0,
\end{aligned}\right\} \quad (12)$$

which is a closed Lie algebra generated by $\{Y_1, Y_2, Y_3, Y_4, Y_5\}$, with $\{Y_1, Y_2, Y_3\}$ as basis for $sL(2,R)$ and $\{Y_4, Y_5\}$ as its centre. Calling this Lie algebra by L and its centre by $\mathbb{C}$, then $L/\mathbb{C} = sL(2,R)$. This result covers the results for all soliton prolongations which have been done before in references [30, 31], [9] and [4].

Clearly the parameter ϵ appearing in (11) gives a 1-parameter family of such Lie algebra L which are homomorphic to the one generated by $\{A_1, A_2, A_3, A_4, A_5\}$ and satisfying $L/C = sL(2,R)$.

Remarks:

(i) The Lie algebra L is now acting as a Lie algebra of vector fields on the fibre space $\pi : M \times N \longrightarrow M$.

(ii) We have obtained sL(2,R) independent of the fiber dimension, m of N.

(iii) Any other finite Lie algebra L' with centre C' will also satisfy the relation $L'/C' = sL(2,R)$ for real fibre manifold, or $L'/C' = sL(2,\mathbb{C}) \oplus sL(2,\mathbb{C})$ for complex fibre manifold as in reference [9] of Dodd and Gibbon for some individual equations from the soliton class.

(iv) If dim N = m = 1, then a 1-dimensional representation of L is taken as

$$Y_1 = y \frac{\partial}{\partial y} \quad , \qquad Y_2 = y^2 \frac{\partial}{\partial y}$$

$$Y_3 = - \frac{\partial}{\partial y} \quad , \qquad Y_4 = Y_5 = 0.$$

(v) Clearly from equations (10) the set of vector fields A_1, A_2, A_3, A_4 and A_5 generate a Lie algebra homomorphic to L by the homomorphism

$$A_1 = Y_5 - 2\lambda i Y_1$$

$$A_2 = Y_2$$

$$A_3 = Y_3$$

$$A_4 = 2Y_1$$

$$A_5 = Y_4 \quad , \quad \text{with } \epsilon = 1.$$

(vi) Also, as we have λ appearing in equations (10), we have a 1-parameter family of homomorphic Lie algebras appearing with the prolongation equations of the A.K.N.S. system.

2 The soliton connection

Writing the 1-forms

$$\eta^a = dy^a + F^a\, dx + G^a\, dt \quad , \qquad a = 1, \ldots, m$$

in the form

$$\eta^a = dy^a - \omega^a \quad , \text{ we get}$$

$$\omega^a = -F^a\, dx - G^a\, dt.$$

Hence for the A.K.N.S. system (1), we have

$$\omega^a = -(A_1^a + A_2^a\, q + A_3^a\, r)dx - (A_2^a\, B + A_3^a\, C + A_4^a\, A + A_5^a)dt \,. \qquad (13)$$

If we write $\omega^a = A_i^a\, \beta^i$, then from equation (13), we get the 1-forms β^i's. They are

$$\beta^1 = -dx \quad ,$$

$$\beta^2 = -q\, dx - B\, dt \quad ,$$

$$\beta^3 = -C\, dt - r\, dx \quad ,$$

$$\beta^4 = -A\, dt \quad ,$$

$$\beta^5 = -dt \quad .$$

These β^i's are independent of the number of prolongation variables y^a, i.e. independent of the fiber dimension, and they define a Cartan-Ehresmann connection on the fibre space, $\pi : M \times N \longrightarrow M$.

Since we have the prolongation Lie algebra of $(A_1, A_2, A_3, A_4, A_5)$, acting on N and because it is homomorphic to L, or simply to the factor algebra $L/\mathcal{C} \cong sL(2,R)$, the connection is an SL(2,R)-connection.

To calculate the curvature of this connection, we make use of the relation given before for the curvature of Cartan-Ehresmann connections, that is

$$\Omega^i = d\beta^i + \frac{1}{2} C^i_{jk} \beta^j \wedge \beta^k \qquad , \; i,j,k = 1,2, \ldots, 5$$

where

$$[A_i, A_j]^a = C^k_{ij} A^a_k \qquad ,$$

and C^i_{jk} are the structure constants of the Lie algebra of $(A_1, A_2, \ldots, A_5)$. In fact, we have

$$\Omega^1 = 0, \quad \text{since } C^1_{ij} = 0 \text{ for all } i \text{ and } j.$$

$$\Omega^2 = -dq \wedge dx - dB \wedge dt + \frac{1}{2} C^2_{ij} \beta^i \wedge \beta^j ,$$

but $C^2_{12} = -2\lambda i$, $C^2_{24} = -2$ and $C^2_{ij} = 0$ for the rest, therefore

$$\Omega^2 = -dq \wedge dx - dB \wedge dt - 2\lambda i \; \beta^1 \wedge \beta^2 - 2 \; \beta^2 \wedge \beta^4$$

i.e.

$$\Omega^2 = -dq \wedge dx - dB \wedge dt - 2\lambda iB \; dx \wedge dt - 2Aq \; dx \wedge dt.$$

Also

$$\Omega^3 = -dr \wedge dx - dC \wedge dt + \frac{1}{2} C_{ij} \beta^i \wedge \beta^j,$$

but $C^3_{13} = 2\lambda i$, $C^3_{34} = 2$ and $C^3_{ij} = 0$ for the rest of i,j, so that

$$\Omega^3 = -dr \wedge dx - dC \wedge dt + 2\lambda iC \; dx \wedge dt + 2Ar \; dx \wedge dt,$$

Similarly

$$\Omega^4 = -dA \wedge dt + (qC - rB) \; dx \wedge dt,$$

and

$$\Omega^5 = 0, \text{ for } C^5_{ij} = 0 \text{ for all } i,j.$$

This connection with curvature components Ω^1, Ω^2, Ω^3, Ω^4 and Ω^5 is the soliton connection obtained by F. Pirani and two others via the inverse scattering equations of the A.K.N.S. system, [7].

Remarks:

(i) Feeding with an equivalent set of 1-forms β^i's, one gets another Lie algebra homomorphic to that of $(A_1, A_2, A_3, A_4, A_5)$ where by equivalent, it is meant that they generate the same ideal.

(ii) As λ is appearing in Ω^i's, a 1-parameter family of connections is associated to the A.K.N.S. system via its prolongation.

(iii) The vanishing of the curvature components Ω^i's, gives rise to the A.K.N.S. system again.

(iv) All such connections are SL(2,R)-connections since all prolongation Lie algebras are homomorphic to sL(2,R).

We continue making use of the soliton connection to obtain the scattering equations for the A.K.N.S. system as will be shown in the next section.

3 Inverse scattering equations for the A.K.N.S. system

In fact, the scattering method plays the role of a technique for solving non-linear systems by associating with it linear systems whose solutions correspond to the solutions of the non-linear one.

For the A.K.N.S. system, the inverse scattering equations are obtainable via prolongation processes using linear representations of the prolongation Lie algebras.

To get such linear representations for $\{A_1, A_2, A_3, A_4, A_5\}$, we make use of its relation to L. A linear representation of L gives a linear representation of $(A_1, A_2, \ldots, A_5)$. So we work first with L, or in other words with the factor algebra $L/\mathbb{C} \equiv sL(2,R)$.

We note that for the existence of m-dimensional linear representation of sL(2,R), m must be even and the representation space , which we will take to be N, is looked at as a product $N^m = P^{2n}$, with coordinates presentation as $(y^1, y^2, \ldots, y^n, y^{n+1}, \ldots, y^m)$.

In this case, an m-dimensional linear representation of sL(2,R) could be:

$$Y_1 = \sum_{a=1}^{n=m/2} \left(y^{n+a} \frac{\partial}{\partial y^{n+a}} - y^a \frac{\partial}{\partial y^a} \right),$$

$$Y_2 = \sum_{a=1}^{n} y^a \frac{\partial}{\partial y^{n+a}} ,$$

$$Y_3 = \sum_{a=1}^{n} y^{n+a} \frac{\partial}{\partial y^a} .$$

We extend this to a linear representation of L by just defining

$Y_4 = 0$, and

$Y_5 = 0$.

The homomorphism between L and $(A_1, A_2, \ldots, A_5)$ gives rise to the following m-dimensional linear representation of $(A_1, \ldots, A_5)$:

$$A_1 = -2\lambda i \sum_{a=1}^{n} \left(y^{n+a} \frac{\partial}{\partial y^{n+a}} - y^a \frac{\partial}{\partial y^a} \right) ,$$

$$A_2 = \sum_{a=1}^{n} y^a \frac{\partial}{\partial y^{n+a}} ,$$

$$A_3 = \sum_{a=1}^{n} y^{n+a} \frac{\partial}{\partial y^a} ,$$

$$A_4 = 2 \sum_{a=1}^{n} \left(y^{n+a} \frac{\partial}{\partial y^{n+a}} - y^a \frac{\partial}{\partial y^a} \right) ,$$

$$A_5 = 0$$

and this gives, by writing $A^a_i = {}_i\alpha^a_b\, y^b$, the matrices

$$_1(\alpha^a_b) = -2\lambda i \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},$$

$$_2(\alpha^a_b) = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

$$_3(\alpha^a_b) = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},$$

$$_4(\alpha^a_b) = 2 \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \text{ and}$$

$$_5(\alpha^a_b) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where I is the unit n x n matrix.

By the results of Chapter III, section 4 , the inverse scattering equations, i.e.

$$dy^a = {}_i\alpha^a_b\, y^b\, \beta^i$$

are now in the form

$$\begin{pmatrix} dy^1 \\ dy^2 \\ \cdot \\ \cdot \\ \cdot \\ dy^m \end{pmatrix} = \begin{pmatrix} (-2\lambda i dx + 2Adt)I & (-qdx - Bdt)I \\ & \\ (-rdx - Cdt)I & (2\lambda i dx - 2Adt)I \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ \cdot \\ \cdot \\ \cdot \\ y^m \end{pmatrix}$$

As the A.K.N.S. system is solvable via the 2-component scattering, a 2-dimensional version of the above equation is exactly the scattering equation of the A.K.N.S. system. That is

$$\begin{pmatrix} dy^1 \\ dy^2 \end{pmatrix} = \begin{pmatrix} -2\lambda i dx + 2Adt & -qdx - Bdt \\ -rdx - Cdt & 2\lambda i dx - 2Adt \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} .$$

In short notation, this could be written as

$$dY = \Omega Y \tag{14}$$

with

$$\Omega = \begin{pmatrix} -2\lambda i dx + 2Adt & -qdx - Bdt \\ -rdx - Cdt & 2\lambda i dx - 2Adt \end{pmatrix}$$

is a 2 x 2 matrix of 1-forms, with $\operatorname{tr} \Omega = 0$, and $Y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$.

Remarks:

(i) The scattering equations are not unique, and we have a 1-parameter family of such matrix Ω which fits the scattering scheme.

(ii) As equations (14) are integrable, their integrability conditions are, in matrix notation,

$\Theta = d\Omega - \Omega \wedge \Omega = 0,$

and all it amounts to is the A.K.N.S. system so that the integrability conditions are satisfied if, and only if, the A.K.N.S. system is satisfied.

(iii) The matrix Ω is now written in the form, using prolongation notation,

$\Omega = ({}_i\alpha^a_b \, \beta^i), \quad i = 1,2, \ldots,5$

i.e.

$$\Omega = \begin{pmatrix} {}_i\alpha^1_1 \, \beta^i & {}_i\alpha^2_1 \, \beta^i \\ {}_i\alpha^1_2 \, \beta^i & {}_i\alpha^2_2 \, \beta^i \end{pmatrix} ,$$

where β^i's are the set of 1-forms we got before, and $A_i \longrightarrow {}_i(\alpha^a_b)$ is a matrix representation of the prolongation linear Lie algebra.

Furthermore, we have the following:

Proposition:

With each choice of the 1-forms β^i's and the corresponding prolongation Lie algebra $(A_1, \ldots, A_5)$, each Ω in the above form satisfies the relation tr $\Omega = 0$.

Proof: Clearly, with the following 2-dimension representation of $(A_1, A_2, \ldots, A_5)$, i.e.

$$A_1 = -2\lambda i\left(y^2 \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial y^1}\right),$$

$$A_2 = y^1 \frac{\partial}{\partial y^2},$$

$$A_3 = y^2 \frac{\partial}{\partial y^1},$$

$$A_4 = 2\left(y^2 \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial y^1}\right), \text{ and}$$

$$A_5 = 0$$

so that

$$_1(\alpha^a_b) = -2\lambda i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad _2(\alpha^a_b) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$_3(\alpha^a_b) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad _4(\alpha^a_b) = 2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$_5(\alpha^a_b) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If Ω is in the form $\Omega = ({}_i\alpha^a_b\, \beta^i)$, with β^i's given in this case by

$\beta^1 = -dx$, $\beta^2 = -qdx - Bdt$

$\beta^3 = -rdx - Cdt$, $\beta^4 = -Adt$

$\beta^5 = -dt$,

then

$$tr\ \Omega = {}_i\alpha_1^1\ \beta^i + {}_i\alpha_2^2\ \beta^i$$

$$= -2\lambda i dx + 2Adt + 2\lambda i dx - 2Adt$$

$$= 0$$

which proves the prop. in this case.

We continue making use of these results to geometrize solitons. This will be done in the following section.

4 Geometrization of 2-dimension soliton equations

Let us write the matrix Ω, simply in the form

$$\Omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & -\omega_1 \end{pmatrix} ,$$

with

$$\omega_1 = {}_i\alpha_1^1\ \beta^i = -2\lambda i dx + 2Adt,$$

$$\omega_2 = {}_i\alpha_1^2\ \beta^i = -qdx - Bdt ,$$

$$\omega_3 = {}_i\alpha_2^1\ \beta^i = -rdx - Cdt ,$$

for the A.K.N.S. system.

Now to geometrize solitons (the A.K.N.S. system) we first mention some facts and notation from the theory of surfaces. It is well known that a 2-dimensional surface is described by the equations

$$d\sigma^1 = \omega \wedge \sigma^1 ,$$

$$d\sigma^2 = -\omega \wedge \sigma^2$$

where σ^1, σ^2 are orthogonal 1-forms with respect to the Riemannian metric on tangent planes, and ω is the curvature 1-form which satisfies

$$d\omega = -K\,\sigma^1 \wedge \sigma^2 ,$$

where K is the curvature of the surface. Moreover, if K is negative constant, then the surface is called pseudospherical surface. So, if K = -1, the equations of such a pseudospherical surface are:

$$\left.\begin{aligned} d\sigma^1 &= \omega \wedge \sigma^2 \\ d\sigma^2 &= -\omega \wedge \sigma^1 \\ d\omega &= \sigma^1 \wedge \sigma^2 \end{aligned}\right\} \quad , K = -1. \tag{15}$$

Now, by transforming the 1-forms $(\omega_1, \omega_2, \omega_3)$ in Ω to the set of 1-forms $(\sigma^1, \sigma^2, \omega)$, pseudospherical surfaces could be associated via solutions of the A.K.N.S system. So, we introduce the correspondence:

$$\left.\begin{aligned} \sigma^1 &= {}_i\alpha^2_1\,\beta^i + {}_i\alpha^1_2\,\beta^i = \omega_2 + \omega_3 \\ \sigma^2 &= -2\,{}_i\alpha^1_1\,\beta^i = -2\omega_1 \\ \omega &= {}_i\alpha^2_1\,\beta^i - {}_i\alpha^1_2\,\beta^i = \omega_2 - \omega_3 \end{aligned}\right\} \tag{16}$$

so that the matrix Ω becomes

$$\Omega = \begin{pmatrix} -\frac{1}{2}\sigma^2 & \frac{1}{2}(\omega + \sigma^1) \\ \frac{1}{2}(\sigma^1 - \omega) & \frac{1}{2}\sigma^2 \end{pmatrix}$$

and $(\sigma^1, \sigma^2, \omega)$ are now describing a pseudospherical surface of constant negative curvature -1, [24].

The equations of the pseudospherical surface (15) are described locally by the A.K.N.S system equations, since

$$d\sigma^1 - \omega \wedge \sigma^2 = 0 \Rightarrow$$

$$0 = (r_t + q_t - B_x - C_x - 2\lambda iB + 2\lambda iC - 2Aq + 2Ar)\; dx \wedge dt$$

Also,

$d\sigma^2 + \omega \wedge \sigma^1 = 0 \Rightarrow$

$0 = -(r_t + q_t - B_x - C_x - 2\lambda iB + 2\lambda iC - 2Aq + 2Ar)dx \wedge dt$ and

$d\omega - \sigma^1 \wedge \sigma^2 = 0 \Rightarrow$

$(A_x - qC + rB)\ dx \wedge dt = 0$

so that equations (15) are satisfied if, and only if,

$r_t = C_x - 2Ar - 2\lambda iC,$

$q_t = B_x + 2Aq + 2\lambda iB,$

$A_x = qC - rB,$

are satisfied. That is if, and only if, the A.K.N.S. system is satisfied, and in this case we have

$$\left.\begin{aligned} \sigma^1 &= -(r + q)dx - (B + C)dt \\ \sigma^2 &= 4\lambda idx - 4Adt \\ \omega &= (r - q)dx - (C - B)dt \end{aligned}\right\} \qquad (17)$$

Now, precisely, with each solution of the A.K.N.S. system, one associates a 1-parameter family of pseudospherical surfaces. Accordingly, a 1-parameter family of metrics are associated on these surfaces. Such metrics are called soliton metrics. In fact we have

$dS^2 = (\sigma^1)^2 + (\sigma^2)^2$

i.e. $dS^2 = (r^2 + q^2 + 2rq - 16\lambda^2)\ dx^2 + (rB + qC + Bq + rC - 16i\lambda A)\ dxdt$
$\qquad + (B^2 + C^2 + 2BC + 16A^2)\ dt^2.$

Remarks:

(i) The metric has constant curvature -1 if, and only if, the A.K.N.S. system is satisfied.

(ii) Clearly, we have a 1-parameter family of metrics for each solution of the A.K.N.S. system because of the appearance of λ in dS^2.

(iii) As the forms ω_1, ω_2 and ω_3 in Ω are specified by the prolongation via the set of 1-forms η^a, the metric depends upon the prolongation Lie algebra (A_1, A_2, ..., A_5).

(iv) The connection 1-form ω on the pseudospherical surface is compatible with the metric by the structure equations (15) of the surface so that it defines a metric connection on the pseudospherical surface.

(v) The pseudospherical surfaces are integral submanifolds of I, the exterior differential system defining the A.K.N.S. system, so that if

$$U : S \longrightarrow M$$

is a submanifold map with S pseudospherical surface, then

$$U^*(I) = 0,$$

and U is a solution of the A.K.N.S. system.

(vi) The isometry group of the soliton metric is related to the Bäcklund transformations of the A.K.N.S. system, as transformations between the pseudospherical surfaces described locally by the A.K.N.S. system. This point will be shown in the next chapter.

Now, looking closely at the 1-forms ω_1, ω_2 and ω_3 in Ω , one finds that

$$\left.\begin{aligned} d\omega_1 &= 2\ \omega_2 \wedge \omega_3 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \\ d\omega_3 &= -\ \omega_1 \wedge \omega_3 \end{aligned}\right\} \qquad (18)$$

Equations (18) are the set of equations satisfied by the set of left invariant 1-forms of sL(2,R). Moreover, they are satisfied if, and only if, the A.K.N.S. system is satisfied.

Thus the underlying sL(2,R)-structure of the 2-dimension solitons is now proved, as it was conjectured in [6].

Furthermore, one can say that there is a strong relation between solitons and some matrix groups as Lie groups. This point will be useful and will be exploited in the next chapters in more detail.

CHAPTER V

A.K.N.S. SYSTEM PROLONGATIONS AND BACKLUND TRANSFORMATIONS

In this chapter we direct our attention mainly towards purely theoretical descriptions of Bäcklund transformations for soliton equations. In particular we make use of the results obtained by prolongations of the A.K.N.S. system to describe algebraic and geometric formulations of its Bäcklund transformations.

We start as follows:

1 Geometric formulation

In the last section of Chapter IV, the pseudospherical surface property was studied, in particular to the A.K.N.S. system, from the matrix Ω, of 1-forms,

$$\Omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_1 \end{pmatrix} \quad , \quad \text{tr}\ \Omega = 0,$$

via differential geometric prolongation of the system on the fibre space $\pi : M \times N \to M$, where N is a 2-dimensional fiber with coordinates $\{y^a\}$, a = 1,2.

Now, as mentioned before, a change of the coordinates of the fiber, say

$$Y \to Y' = AY$$

with A a 2×2 matrix with det A = 1, changes the matrix Ω as follows:

$$\Omega \to \Omega' = A^{-1}\, dA + A^{-1}\, \Omega\, A,$$

and its integrability conditions, the matrix Θ,

$$\Theta = d\Omega - \Omega \wedge \Omega \quad ,$$

changes according to $\Theta \longrightarrow \Theta' = A^{-1} \Theta A$.

Let Ω' be in the form

$$\Omega' = \begin{pmatrix} \omega_1' & \omega_2' \\ \omega_3' & -\omega_1' \end{pmatrix} \quad , \text{ as } \operatorname{tr} \Omega' = 0$$

because the equations

$$dY = \Omega Y, \ \operatorname{tr} \Omega = 0$$

and $\Theta = 0$,

are form invariant under such transformation. Assume Ω' is obtained from Ω by the transformation A,

$$A = \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix} \quad ,$$

so that

$$\Omega' = \begin{pmatrix} \omega_1 - \frac{i}{2}\, d\alpha & e^{i\alpha}\omega_2 \\ e^{-i\alpha}\omega_3 & -\omega_1 + \frac{i}{2}\, d\alpha \end{pmatrix}$$

with α as the rotation angle in A.

Now, with Ω and each solution of the A.K.N.S. system there is an associated 1-parameter family of pseudospherical surfaces with the correspondence

$$\left.\begin{aligned} \sigma^1 &= \omega_2 + \omega_3 \\ \sigma^2 &= -2\omega_1 \\ \omega &= \omega_2 - \omega_3 \end{aligned}\right\} \qquad (1)$$

between $(\omega_1, \omega_2, \omega_3)$ and $(\sigma^1, \sigma^2, \omega)$. Similarly, with Ω' and each solution of the system, there is associated a 1-parameter family of pseudospherical surfaces by the correspondence

$$\sigma'^1 = \omega_2' + \omega_3'$$
$$\sigma'^2 = -2\omega_1'$$
$$\omega' = \omega_2' - \omega_3'$$

between $(\omega_1', \omega_2', \omega_3')$ and $(\sigma'^1, \sigma'^2, \omega')$. Denote this correspondence by T.

A Bäcklund transformation now, denoted by BT, is the transformation which maps one pseudospherical surface defined by $(\sigma^1, \sigma^2, \omega)$ to another pseudospherical surface defined by $(\sigma'^1, \sigma'^2, \omega')$ i.e.

$$BT : \Omega \longrightarrow \Omega' = BT(\Omega).$$

That is:

$$BT : \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & -\omega_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \omega_1' & \omega_2' \\ \omega_3' & -\omega_1' \end{pmatrix} = \begin{pmatrix} \omega_1 - \frac{i}{2}d\alpha & e^{i\alpha}\omega_2 \\ e^{-i\alpha}\omega_3 & -\omega_1 + \frac{i}{2}d\alpha \end{pmatrix}$$

so that the rotation angle α is determined by the following completely integrable pfaffian system:

$$0 = \beta \equiv \frac{2}{i}\omega_1 - d\alpha + \xi\,\omega_2\, e^{i\alpha} + \xi^{-1}\,\omega_3\, e^{-i\alpha} \qquad (2)$$

where ξ is a Bäcklund transformation parameter. Equation (2) may be written in the form

$$0 = \beta' = \frac{2}{i}\omega_1' + \xi\,\omega_2' + \xi^{-1}\omega_3' \quad ,$$

in terms of Ω'.

Thus, for the A.K.N.S. system, or the 2-dimension solitons, we have the following:

$$\begin{array}{ccc} \Omega_{\lambda,u} & \xrightarrow{\ A\ } & \Omega'_{\lambda,u'} \\ T\downarrow & & \downarrow T \\ P\cdot S & \xrightarrow[BT]{} & P'\cdot S \end{array} \quad \text{(pseudospherical surface)}$$

where u and u' denote solutions of the A.K.N.S. system, A is a gauge transformation, T is the correspondence (1) and BT is the Bäcklund transformation.

Accordingly BT will look like

$BT = T\ A\ T^{-1}$,

and as the gauge transformations A form a group, also such BT form a group. We call this group the Bäcklund group. Clearly, successive application of two BT's gives a BT:

$$BT_{\xi_2}\ BT_{\xi_1} = BT_{\xi_{21}}$$

with $\alpha_{21} = \alpha_2 + \alpha_1$, $\quad \xi_{21} = \xi_2 \xi_1$

as was pointed out in [23]. The Bäcklund group is conjugate to the gauge group which is sL(2,R) in this case.

In fact Bäcklund transformations in this formulism appear as the original Bäcklund concept for his transformations as transformations between surfaces of constant negative curvatures. A similar geometric formulation of the Bäcklund transformations of the sine-Gordon equation has been made by S. Chern and C. Treng [29], where it appeared as lines of congruence between surfaces of constant negative curvature -1 in R^3. In this formulation, the soliton equation (sine-Gordon) is identical to the Gauss equation of the surface as a submanifold of R^3. But their Bäcklund transformations do not form a group and they are just a 1-parameter family of lines of congruence which satisfy the premutability theory, that is $(BT_{\xi_1})_{\xi_2} = (BT_{\xi_2})_{\xi_1}$.

Because we have constructed the matrix Ω for the A.K.N.S. system via prolongations, the Bäcklund transformations also depend on the prolongation Lie algebras (A_1, A_2, ..., A_5), which are homomorphic to sL(2,R).

This point will be clarified and studied with another formulation of Bäcklund transformations in the following section.

2 Bäcklund maps and prolongations for soliton equations

We continue with same notations as before and work with the A.K.N.S. system,

$$A_x = qC - rB$$
$$q_t = B_x + 2Aq + 2\lambda iB$$
$$r_t = C_x - 2Ar - 2\lambda iC$$

to be defined on a manifold $M = J^1(X, N_1)$, with

$X \equiv$ the space of (x,t) and

$N_1 \equiv$ the space of (A,B,C,q,r).

Assume a prolongation structure is given as before on the fibre space

$$\pi : J^o(X, N_1) \times N \longrightarrow J^o(X, N_1)$$

with N, the fiber of π, having coordinates $\{y^a\}$.

This prolongation structure is given by the Lie algebra like, equations (9) in section 1 , Chapter IV. We write it here as

$$\left.\begin{aligned}
[A_1, A_2]^a &= -2\lambda i A_2^a \quad ,\\
[A_1, A_3]^a &= 2\lambda i A_3^a \quad ,\\
[A_1, A_4]^a &= 0 = [A_1, A_5]^a \quad ,\\
[A_2, A_3]^a &= A_4^a \quad ,\\
[A_2, A_4]^a &= -2A_2^a \quad ,\\
[A_2, A_5]^a &= 0 \quad ,\\
[A_3, A_4]^a &= 2A_3 \quad ,\\
[A_3, A_5]^a &= 0.
\end{aligned}\right\} \qquad (3)$$

where, as we have shown, $A_i = A_i^a \dfrac{\partial}{\partial y^a}$ are vector fields on N and

$$[A_i, A_j]^a = (A_i^b \frac{\partial A_j^a}{\partial y^b} - A_j^b \frac{\partial A_i^a}{\partial y^b}), \text{ by definition.}$$

Now, define the map

$$\psi : J^o(X, N_1) \times N \longrightarrow J^1(X, N),$$

by

$$\left.\begin{aligned}
y_x^a &\equiv \psi_1^a = A_1^a + A_2^a\, q + A_3^a\, r\\
y_t^a &\equiv \psi_2^a = A_2^a\, B + A_3^a\, C + A_4^a\, A + A_5^a
\end{aligned}\right\} \qquad (4)$$

where the A_i^a's are functions of y's alone which satisfy the commutation relations (3). Then we have the following result:

Theorem:

The above map ψ is an ordinary Bäcklund map for the A.K.N.S. system.

Proof: We need to show that the integrability conditions of the map ψ could be written as

$Z' = Z \times N$, and Z is the A.K.N.S. system.

For that, the extended hash (total derivative) operator on $J^1(X, N_1) \times N$ is given by

$$\tilde{\partial}_1^1 = \frac{\partial}{\partial x} + A_x \frac{\partial}{\partial A} + B_x \frac{\partial}{\partial B} + C_x \frac{\partial}{\partial C} + q_x \frac{\partial}{\partial q} + r_x \frac{\partial}{\partial r} + \psi_1^a \frac{\partial}{\partial y^a},$$

and

$$\tilde{\partial}_2^1 = \frac{\partial}{\partial t} + A_t \frac{\partial}{\partial A} + B_t \frac{\partial}{\partial B} + C_t \frac{\partial}{\partial C} + q_t \frac{\partial}{\partial q} + r_t \frac{\partial}{\partial r} + \psi_2^a \frac{\partial}{\partial y^a}.$$

The integrability conditions of ψ are:

$$\tilde{\partial}_1^1 \psi_2^a - \tilde{\partial}_2^1 \psi_1^a = 0 \tag{5}$$

That is

$$\left(\frac{\partial}{\partial x} + A_x \frac{\partial}{\partial A} + \dots + r_x \frac{\partial}{\partial r} + \psi_1^a \frac{\partial}{\partial y^b}\right)(\psi_2^a) -$$

$$\left(\frac{\partial}{\partial t} + A_t \frac{\partial}{\partial A} + \dots + r_t \frac{\partial}{\partial r} + \psi_2^a \frac{\partial}{\partial y^b}\right)(\psi_1^a) = 0.$$

Substituting for ψ_1^a and ψ_2^a from the formula (4), in equation (5), we get:

$$\begin{aligned} 0 = {} & (B_x A_2^a + C_x A_3^a + A_x A_4^a + B_x A_2^a + C_x A_3^a + A_x A_4^a) \\ & + (A_1^b + A_2^b\, q\, A_3^b\, r)\left(B \frac{\partial A_2^a}{\partial y^b} - C \frac{\partial A_3^a}{\partial y^b} + A \frac{\partial A_4^a}{\partial y^b} + \frac{\partial A_5^a}{\partial y^b}\right) \\ & - (q_t A_2^a + r_t A_3^a + q_t A_2^a + r_t A_3^a) \\ & - (A_5^b + A A_4^a + C A_3^a + B A_2^a)\left(\frac{\partial A_1^a}{\partial y^b} + q \frac{\partial A_2^a}{\partial y^b} + r \frac{\partial A_3^a}{\partial y^b}\right). \end{aligned}$$

This equation can be written in the form:

$$\begin{aligned} & 2B_x A_2^a + 2C_x A_3^a + 2A_x A_4^a - 2q_t A_2^a - 2r_t A_3^a + \left(A_1^b \frac{\partial A_5^a}{\partial y^b} - A_5^b \frac{\partial A_1^a}{\partial y^b}\right) \\ & + B\left(A_1^b \frac{\partial A_2^a}{\partial y^b} - A_2^a \frac{\partial A_1^a}{\partial y^b}\right) + C\left(A_1^b \frac{\partial A_3^a}{\partial y^b} - A_3^b \frac{\partial A_1^a}{\partial y^b}\right) + q\left(A_2^b \frac{\partial A_5^a}{\partial y^b} - A_5^b \frac{\partial A_2^a}{\partial y^b}\right) \\ & + qC\left(A_2^b \frac{\partial A_3^a}{\partial y^b} - A_3^b \frac{\partial A_2^a}{\partial y^b}\right) + A\left(A_1^b \frac{\partial A_4^a}{\partial y^b} - A_4^b \frac{\partial A_1^b}{\partial y^b}\right) + qA\left(A_2^b \frac{\partial A_4^a}{\partial y^b} - A_4^b \frac{\partial A_2^a}{\partial y^b}\right) \\ & + r\left(A_3^b \frac{\partial A_5^a}{\partial y^b} - A_5^b \frac{\partial A_3^a}{\partial y^b}\right) + rB\left(A_3^b \frac{\partial A_2^a}{\partial y^b} - A_2^a \frac{\partial A_3^a}{\partial y^b}\right) + rA\left(A_3^b \frac{\partial A_4^a}{\partial y^b} - A_4^b \frac{\partial A_3^a}{\partial y^b}\right) = 0 \end{aligned}$$

Now, using the fact that

$$[A_i, A_j]^a = (A_i^b \frac{\partial A_j^a}{\partial y^b} - A_j^b \frac{\partial A_i^a}{\partial y^b}),$$

the above equation becomes:

$$2B_x A_2^a + 2C_x A_3^a + 2A_x A_4^a - 2q_t A_2^a - 2r_t A_3^a + [A_1, A_5]^a + B[A_1, A_2]^a$$

$$+ C[A_1, A_3]^a + q[A_2, A_5]^a + A[A_1, A_4]^a + qC[A_2, A_3]^a + qA[A_2, A_4]^a$$

$$+ r[A_3, A_5]^a + rB[A_3, A_2]^a + rA[A_3, A_4]^a = 0.$$

Using the commutation relations (3), we get:

$$2B_x A_2^a + 2C_x A_3^a + 2A_x A_4^a - 2q_t A_2^a - 2r_t A_3^a - 2\lambda i B A_2^a + 2\lambda i C A_3^a + qCA_4^a$$

$$- 2qAA_4^a - rBA_4^a + 2rAA_3^a = 0.$$

This could be written in the form:

$$\begin{aligned} &A_2^a(2B_x - 2q_t - 2\lambda iB - 2qA) + \\ &A_3^a(2C_x - 2r_t + 2\lambda iC + 2rA) + \\ &A_4^a(2A_x + qC - rB) \qquad = 0. \end{aligned}$$

Now, because the vector fields A_2, A_3 and A_4 are linearly independent, so are the functions A_2^a, A_3^a and A_4^a as functions of y. So we are led to the integrability conditions $Z' = Z \times N$, and Z is in the form

$$\left.\begin{aligned} 2A_x &= rB - qC , \\ q_t &= B_x - qA - \lambda iB , \\ r_t &= C_x + rA + \lambda iC , \end{aligned}\right\} \longrightarrow Z.$$

This system Z could be written in the form

$$\begin{aligned} \overline{A}_x &= qC - rB \\ q_t &= B_x + 2q\overline{A} + 2\overline{\lambda}iB \\ r_t &= C_x - 2r\overline{A} - 2\overline{\lambda}iC \end{aligned}$$

with just taking

$$\bar{A} = -\frac{1}{2} A \quad \text{and} \quad \bar{\lambda} = -\frac{1}{2} \lambda$$

so that Z is the A.K.N.S. system, which proves the theorem.

Another way of proving the theorem is to write the functions ψ_1^a and ψ_2^a in terms of the Lie algebra L generated by $(Y_1, \ldots, Y_5)$, given before in Chapter IV.

That is

$$\psi_1^a = (Y_5^a - 2\lambda i Y_1^a + qY_2^a + rY_3^a) \quad ,$$

and $$\psi_2^a = (Y_4^a + 2AY_1^a + CY_3^a + BY_2^a) \quad ,$$

and by the same way, using the commutation relations, equations (12) in Chapter IV, we get the A.K.N.S. system appearing in the integrability conditions of ψ.

Remarks:

(i) Clearly the integrability conditions Z' of the map ψ are satisfied if, and only if, the A.K.N.S. system is satisfied.

(ii) Because ψ contains λ, we have a 1-parameter family of Bäcklund maps for the A.K.N.S. system.

(iii) It is hard to know, with this general form of ψ, whether ψ is self-Bäcklund transformation for the A.K.N.S. system. But with specifications to equations from the soliton class covered by the A.K.N.S. system, one may be able to obtain Bäcklund transformations via the map ψ.

So, let us consider a 2-dimensional linear representation of the prolongation Lie algebra of $(A_1, A_2, \ldots, A_5)$, given by:

$$A_1 = -2\lambda i\left(y^2 \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial y^1}\right) ,$$

$$A_2 = y^1 \frac{\partial}{\partial y^2} ,$$

$$A_3 = y^2 \frac{\partial}{\partial y^1} ,$$

$$A_4 = 2\left(y^2 \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial y^1}\right) ,$$

$$A_5 = 0$$

so that the functions A_i^a's are given as follows

$$A_1^1 = 2\lambda i y^1 , \qquad A_1^2 = -2\lambda i y^2$$

$$A_2^1 = 0 , \qquad A_2^2 = y^1$$

$$A_3^1 = y^2 , \qquad A_3^2 = 0$$

$$A_4^1 = -2y^1 , \qquad A_4^2 = 2y^2$$

$$A_5^1 = 0 , \qquad A_5^2 = 0.$$

Thus

$$\left.\begin{array}{l} \psi_1^a = \begin{pmatrix} \psi_1^1 \\ \psi_1^2 \end{pmatrix} = \begin{pmatrix} 2\lambda i & r \\ q & -2\lambda i \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} , \\ \psi_2^a = \begin{pmatrix} \psi_2^1 \\ \psi_2^2 \end{pmatrix} = \begin{pmatrix} -A & C \\ B & A \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \end{array}\right\} \qquad (6)$$

This is a 2-dimensional version of the map ψ, which is very useful for the 2-dimension solitons solvable by the 2-component scattering scheme.

In fact equations (6) are the scattering equations for the A.K.N.S. system, which again proves the interrelation between Bäcklund transformations and the scattering method for soliton equations as it was conjectured by many people and we have just shown it clearly via prolongations.

Formula (6) could be linearized to a 1-dimension version of ψ by using 1-dimensional representations of the prolongation Lie algebra. That is

$$A_1 = -2\lambda i y \frac{\partial}{\partial y} ,$$

$$A_2 = y^2 \frac{\partial}{\partial y} ,$$

$$A_3 = - \frac{\partial}{\partial y} ,$$

$$A_4 = 2y \frac{\partial}{\partial y} ,$$

$$A_5 = 0 ,$$

which is a result of using the 1-dimensional representation of L given by

$$Y_1 = y \frac{\partial}{\partial y} , \qquad Y_4 = 0$$

$$Y_2 = y^2 \frac{\partial}{\partial y} , \qquad Y_5 = 0$$

$$Y_3 = - \frac{\partial}{\partial y} .$$

Hence

$$A_1^1 = -2\lambda i y , \qquad A_2^1 = y^2$$

$$A_3^1 = -1 , \qquad A_4^1 = 2y$$

$$A_5^1 = 0,$$

so that

$$\left.\begin{aligned} y_x &= \psi_1^1 = -2\lambda i y + y^2 q - r \\ y_t &= \psi_2^1 = By^2 - C + 2Ay \end{aligned}\right\} \qquad (7)$$

The above equations (7) are of Riccati type which is the standard form of the Bäcklund transformation equations. In fact these equations may be used to derive Bäcklund transformations for non-linear evolution equations constructed within the A.K.N.S. scheme, see the Appendix.

Also, it was used by Chen [3] to calculate Bäcklund transformations for some individual equations from the soliton class. Moreover he classified the resulting transformations into three classes as follows:

Class I: q = constant = -2 and $i\lambda = k$.

This class contains the Bäcklund transformation for the K.dV. equation:

$$u_t + 12uu_x + u_{xxx} = 0$$

which is

$$(W + W')_x = k^2 - (W' - W)^2 ,$$

$$(W - W')_t = A(W - W' + k) - \frac{C}{2}(W - W' + k) + 2B$$

between W and W' which are solutions of the K.dV. equation

Class II: with $q = -r$ and $i\lambda = \frac{1}{2}k$.

This class contains the Bäcklund transformations for the modified K.dV. equation:

$$u_t + 6u^2u_x + u_{xxx} = 0,$$

as well as the sine-Gordon equation:

$$2u_{xt} = \sin 2u,$$

which has two equivalent Bäcklund transformations in the form:

$$(W \pm W')_x = k \sin(W \pm W') ,$$

$$(W \pm W')_t = \frac{1}{k} \sin(W \pm W'),$$

with W and W' solutions of the sine-Gordon equation.

<u>Class III</u>: with $q = -r^*$ and $\lambda = \xi + i\eta$.

This class contains the Bäcklund transformations for the non-linear Schördinger equation:

$$iu_t + u_{xx} + 2u^2u^* = 0,$$

where u^* is the complex conjugation of u.

Now, the λ-dependence of ψ justifies the saying that one can have a 1-parameter family of Bäcklund maps deriving to Bäcklund transformations.

Moreover, so far we have just used a 1-dimensional representation for the prolongation Lie algebra to get these Bäcklund transformations where it is easy to compute it. Thus the dependence of Bäcklund transformations on the representations of the prolongation Lie algebra is strongly conjectured. This conjecture is also made by Atiyah, Dodd and Gibbon in [9]. They said that one may be able to obtain Bäcklund transformations from the invariance properties of the prolongation Lie algebra together with some roots for this Lie algebra to be searched for.

It is hoped that this point will be clear in the next section through application to the A.K.N.S. system Bäcklund transformations in another formulation.

3 Bundle formulation of Backlund transformations

Let us continue with the previous notations and results of prolongations of the A.K.N.S. system.

We had sL(2,R) appearing as a homomorphic Lie algebra to the prolongation Lie algebra of $(A_1, A_2, \ldots, A_5)$ and independent of the fiber dimension $m = \dim N$, so that one can say sL(2,R) is acting on the fiber of the fibre space $\pi : M \times N \longrightarrow M$,
and we have a fibre bundle $(M \times N, M, \pi)$ with structure group sL(2,R).

Now, as the soliton connection represents a 1-parameter family of connections, we assume instead, having a 1-parameter family of fibre bundles, denote it by E_λ, λ is the parameter.

On each E_λ, we have an ideal, say J_λ, generated by $\left\{\eta^a, \pi^* I_\lambda = I_\lambda\right\}$ which is closed by the definition of prolongation, where I_λ is the exterior differential system generated by $\{\alpha_1, \alpha_2, \alpha_3\}$ on M and representing the A.K.N.S. system.

Let $E_{\bar\lambda}$ denote another fibre bundle, with $J_{\bar\lambda}$ the differential ideal on $E_{\bar\lambda}$ defined by

$$J_{\bar\lambda} = \left\{\bar\eta^a, \pi^* I_{\bar\lambda}\right\},$$

with $J_{\bar\lambda}$ as a prolongation of $I_{\bar\lambda}$ on $E_{\bar\lambda}$ together with another set of prolongation 1-forms $\bar\eta^a$.

Now, on each E_λ the connection is defined via the set of 1-forms η^a, simply, by the vectorial 1-form

$$\eta = \eta^a \frac{\partial}{\partial y^a},$$

so that

$$\eta = dy^1 \frac{\partial}{\partial y^1} + dy^2 \frac{\partial}{\partial y^2} + \ldots + dy^m \frac{\partial}{\partial y^m} + (A_1 + A_2 q + A_3 r)dx$$
$$+ (A_5 + A_2 B + A_3 C + A_4 A)dt,$$

where $(A_1, \ldots, A_5)$ are the generators of the prolongation Lie algebra, $A_i = A_i^a \frac{\partial}{\partial y^a}$, given by equations (3).

Let $\bar{\eta}$ be the vectorial 1-form on $E_{\bar{\lambda}}$ defining the connection on this fibre bundle. Because the soliton connection defines the A.K.N.S. system as the submanifold of M where the curvature vanishes, and since the 1-parameter family of connections all have this property, the connections defined by η on E_λ and $\bar{\eta}$ on $E_{\bar{\lambda}}$, define the same set of equations for the integral submanifolds of I_λ and $I_{\bar{\lambda}}$, that is the A.K.N.S. system.

Let us denote by e, a point in E_λ, say

$e = (u,y) \in E_\lambda$ with $u \in M$ and $y \in N$.

Similarly, a point $\bar{e} = (\bar{u},\bar{y}) \in E_{\bar{\lambda}}$

Define a C^∞-map $\phi : E_\lambda \rightarrow E_{\bar{\lambda}}$,

by $e \longrightarrow (\phi_1(e), \phi_2(y))$,

where ϕ_1 and ϕ_2 are given by

$$\bar{A} = \phi_{1A}(x, t, A, B, C, q, r, y^1, \ldots, y^m),$$

$$\bar{B} = \phi_{1B}(x, t, A, B, C, q, r, y^1, \ldots, y^m)$$

$$\bar{r} = \phi_{1r}(x, t, A, B, C, q, r, y^1, \ldots, y^m),$$

and

$$\bar{y}^a = \phi_2^a (y^1, y^2, \ldots, y^m), \; a = 1, 2, \ldots, m.$$

We note that if the map ϕ is such that

$\phi^* I_{\bar{\lambda}} \subset J_\lambda$, then ϕ clearly defines a Bäcklund transformation, in the Wahlquist-Estabrook sense, via prolongation.

Suppose that ϕ is such a map, then we have the following diagram

$$\begin{array}{ccc} E_{\lambda} & \stackrel{\phi}{\longleftrightarrow} & E_{\bar{\lambda}} \\ g_{\lambda}\Big\updownarrow & & \Big\updownarrow g_{\bar{\lambda}} \\ E_{\lambda} & \stackrel{\phi}{\longleftrightarrow} & E_{\bar{\lambda}} \end{array}$$

with the action of $g \in SL(2,R)$ defined by

$$g(e) = (u,y)g = (u, yg),$$

and $\longleftrightarrow$ denotes both directions, [9].

Now, from the diagram, we have conditions on the map ϕ to be compatible with the action of $SL(2,R)$ on N. These conditions are

(i) $\phi g_{\lambda} = g_{\bar{\lambda}}\phi$,

(ii) $\phi^{-1}g_{\bar{\lambda}} = g_{\lambda}\phi^{-1}$,

(iii) $g_{\bar{\lambda}}^{-1}\phi = \phi g_{\lambda}^{-1}$.

These conditions characterize the map ϕ. Moreover they classify Bäcklund transformations in such a way that it may go with Chen classes for the A.K.N.S. system Bäcklund transformations.

The map ϕ_1 in ϕ is to be taken from the invariance properties of the prolongation Lie algebra of $(A_1, \ldots, A_5)$, for example, setting $\lambda = \xi^2$, then $\xi \longrightarrow -\xi$ keeps the Lie algebra $(A_1, \ldots, A_5)$ invariant, hence ϕ_1 could be taken as $\phi_1 : \xi \longrightarrow -\xi$.

But the problem is still unsolved, because ϕ_2 is not specified and there is not a unique way to do that. For example one may require linear vector fields to be mapped into linear vector fields, and so he may choose ϕ_2, as one possibility of many, in the form

$$\bar{y}^a = h^a_b y^b \ , \quad a,b = 1,2, \ldots, m$$

where h^a_b are functions of (x,t) only when pulled back to a solution submanifold.

In fact, when considering conditions (i), (ii) and (iii) on ϕ, different specifications of h^a_b together with g_λ and $g_{\bar{\lambda}}$ give three classes of such map ϕ for the A.K.N.S. system and the processes will lead to the Riccati form of the Bäcklund equations of the system, equations (7).

Remark:

We have just used SL(2,R) as the structure group of E_λ. But one can also use the underlying Lie group of the prolongation Lie algebra $(A_1, \ldots, A_5)$, say G, and in this case g_λ becomes an element of G.

In fact there is no loss, as they are homomorphic and SL(2,R) is a unifying one, as we have shown before.

CHAPTER VI

A POSSIBLE MODEL FOR n-DIMENSION SOLITON EQUATIONS

1 The model construction

In the study of 2-dimensional solitons, it was shown that they share the following properties:

(i) They have a large number of conservation laws.
(ii) They are solvable via the inverse scattering method.
(iii) They have Bäcklund transformations.
(iv) They describe locally pseudospherical surfaces.

Moreover, the relationship between the Lie group SL(2,R) and the 2-dimension solitons has shown its strength in interpreting these properties. This relationship is clear in all different ways with which solitons could be handled.

Because there is not, yet, any higher dimension soliton known, it is quite difficult to imagine what it could be, or what it will look like. Despite of that, let us try to make use of the results we have obtained for the 2-dimension solitons, and assume that n-dimension solitons may be characterized by similar properties as in the 2-dimension case. Namely:

(i) They have a large number of conservation laws.
(ii) They are solvable via the scattering method (n-dimensional scheme).
(iii) They have Bäcklund transformations.
(iv) They describe n-dimensional pseudospherical surfaces.

In fact property (iii) is the backbone of these properties as it leads to them all.

In the 2-dimension case, it results in three 1-forms θ^1, θ^2, θ^3 in the matrix form Ω,

$$\Omega = \begin{pmatrix} \theta^1 & \theta^2 \\ \theta^3 & -\theta^1 \end{pmatrix} \quad , \quad tr\Omega = 0$$

which gives the soliton equation in the form

$$d\Omega - \Omega \wedge \Omega = 0.$$

These three forms θ^1, θ^2 and θ^3 satisfy the same Maurer-Cartan equations for the left invariant 1-forms of sL(2,R), say ω_1, ω_2 and ω_3 as was shown in Chapter 5.

Now, accordingly, we shall suggest the Lie group SL(n,R) , for n-dimensional solitons, to play the role played by SL(2,R) in the 2-dimension case. In fact from the group theoretic point of view and its relation to soliton equations, [5], any semisimple Lie group G might work. But we shall stick to sL(n,R) as it is a generalization to SL(2,R).

Let us assume we have a differential equations system Z which is of soliton type. Simply, we require Z to have Bäcklund transformations.

Assume Z has n independent variables and m dependent variables. Moreover, it is of order (h + 1), h is an integer ≥ 0. Hence Z could be defined on $J^{h+1}(M, N_1)$ with the coordinate presentations:

$\{x^a\}$, a = 1,2, ..., n, n = dim M, for the independent variables,

$\{z^\mu\}$, = 1,2, ..., m, m = dim N_1, for the dependent variables.

Now, define the Bäcklund map ψ,

$$\psi : J^h(M, N_1) \times N_2 \longrightarrow J^1(M, N_2),$$

where N_2 is another space of new dependent variables with coordinates presentation $\{y^A\}$, A = 1,2, ..., dim N_2.

Assume that ψ is an ordinary Bäcklund map for the system Z, i.e. its integrability conditions Z' are in the form

$Z' = Z \times N_2$.

Moreover, on the fibre space

$pr_1 : J^h(M, N_1) \times N_2 \longrightarrow J^h(M, N_1)$

let SL(n,R) be the group acting on N_2, the fiber of pr_1.

The Bäcklund connection associated to the system Z via the map ψ is given by the set of 1-forms $\{\psi^* \theta^A\}$, with

$\theta^A = dy^A - y^A_a \, dx^a$,

i.e. simply, by the set of 1-forms

$\theta^A = dy^A - \psi^A_a \, dx^a$,

where ψ^A_a are the functions defining the map ψ.

Now, because $J^h(M, N_1) \times N_2 \longrightarrow J^h(M, N_1)$ is a fibre bundle, the action of SL(n,R) must be compatible with the local product diffeomorphisms. This means that if

$P_U : U \times N_2 \longrightarrow pr_1^{-1}(U)$, and

$P_U' : U' \times N_2 \longrightarrow pr_1^{-1}(U')$

are two such diffeomorphisms, with U and U' subsets of $J^h(M, N_1)$ and $U \wedge U'$ is not empty, and if $\xi \in U \wedge U'$, with

$P_\xi : N_2 \longrightarrow pr_1^{-1}(\xi)$

by $\quad \eta \longrightarrow P_U(\xi, \eta)$

and $P'_\xi : N_2 \longrightarrow pr_1^{-1}(\xi)$

by $\quad \eta \rightarrow P_U'(\xi, \eta)$

then

$P_\xi^{-1} \circ P'_\xi : N_2 \longrightarrow N_2$, is an action of some $g_\xi \in SL(n,R)$, on N_2 depending smoothly on ξ.

In other words, SL(n,R) is the structure group of the fibre bundle.

Furthermore the action of SL(n,R) is also compatible with parallel transport by the connection. This means that parallel transport, suitably composed with the local product diffeomorphisms above, should yield an action of SL(n,R). In more detail let ξ_1, ξ_2 be points of $J^h(M, N_1)$, and γ a curve joining them, with $\gamma(t_1) = \xi_1$ and $\gamma(t_2) = \xi_2$. For each $\eta \in pr_1{}^{-1}(\xi_1)$, let $\bar{\gamma}_\eta$ be the horizontal lift of γ through η. Then the parallel transport along γ is given by

$$\tau_\gamma : pr_1{}^{-1}(\xi_1) \longrightarrow pr_1{}^{-1}(\xi_2),$$

by $$\eta \longrightarrow \bar{\gamma}_\xi(t_2).$$

Let U_1 and U_2 be neighbourhoods of ξ_1 and ξ_2 respectively for which the product diffeomorphisms are defined and let P_1 and P_2 be the maps

$$P_1 : N_2 \longrightarrow pr_1{}^{-1}(\xi_1), \text{ by } y \longrightarrow P_{U_1}(\xi_1, y)$$

$$P_2 : N_2 \longrightarrow pr_1{}^{-1}(\xi_2), \text{ by } y \longrightarrow P_{U_2}(\xi_2, y).$$

Then compatibility implies that

$$P_2^{-1} \circ \tau_\gamma \circ P_1 : N_2 \longrightarrow N_2$$

is an action of $g \in SL(n,R)$ on N_2, i.e. the connection is an SL(2,R)-connection.

These compatibility conditions imply that the connection coefficients and hence the functions ψ_a^A, must admit a factorization in the form

$$\psi_a^A = X_i^A\ (y)\ \omega_a^i\ (\xi)$$

with the index i ranging and summing over 1,2,3, ..., dim sL(n,R) = $n^2 - 1$, and X_i^A are functions of y's alone but $\omega_a^i\ (\xi)$ are functions on the base manifold $J^h(M, N_1)$.

Thus, the connection 1-forms may be written as

$$\theta^A = dy^a - X_i^A\,(y)\,\omega^i$$

with $\omega^i = \omega_a^i(\xi)\,dx^a$, are 1-forms to be lifted from M to $J^h(M, N_1)$.

The functions $X_i^A\,(y)$ are such that

$A_i = X_i^A \dfrac{\partial}{\partial y^A}$ are among a basis for the Lie algebra sL(n,R) and

$\omega^i = \omega_a^i\,dx^a$ are determined by the same equations satisfied by the left invariant 1-forms of sL(n,R). These equations are

$$d\omega^i = \frac{1}{2}\sum_{j,k=1}^{n^2-1} C_{jk}^i\,\omega^j \wedge \omega^k, \qquad (1)$$

where C_{jk}^i are the structure constants of sL(n,R).

The integrability conditions of the Bäcklund map

$\psi : J^h(M, N_1) \times N_2 \longrightarrow J^1(M, N_2)$

are

$$\tilde{\partial}_a^{h+1}\,(\psi_b^A) - \tilde{\partial}_b^{h+1}\,(\psi_a^A) = 0\ , \qquad (2)$$

where the total derivative operator $\tilde{\partial}_a^h$ is given on $J^h(M, N_1) \times N_2$ by

$$\tilde{\partial}_a^{h+1} = \partial_a^{h+1\#} + \psi_a^A \frac{\partial}{\partial y^A}\ ,$$

i.e. $$\tilde{\partial}_a^{h+1} = \frac{\partial}{\partial x^a} + z_a^\mu \frac{\partial}{\partial z^\mu} + z_{ab}^\mu \frac{\partial}{\partial z_b^\mu} + \ldots + z_{aa_1a_2\ldots a_h} \frac{\partial}{\partial z_{a_1a_2\ldots a_h}^\mu} + \psi_a^A \frac{\partial}{\partial y^A}\ .$$

Then with ψ in the form

$$\psi_a^A = X_i^A\,\omega_a^i\ ,$$

equations (2) become

$$0 = \left(\frac{\partial}{\partial x^a} + z_a^\mu \frac{\partial}{\partial z^\mu} + \ldots + z_{aa_1\ldots a_h}^\mu \frac{\partial}{\partial z_{a_1\ldots a_h}^\mu} + \psi_a^B \frac{\partial}{\partial y^B}\right)(X_i^A\,\omega_b^i)$$

$$- \left(\frac{\partial}{\partial x^b} + z_b^\mu \frac{\partial}{\partial z^\mu} + \ldots + z_{ba_1\ldots a_h}^\mu \frac{\partial}{\partial z_{a_1\ldots a_h}^\mu} + \psi_b^B \frac{\partial}{\partial y^B}\right)(X_i^A\,\omega_a^i)$$

for $a \neq b = 1, 2, \ldots, n$.

i.e.

$$X_i^A \left\{ \frac{\partial \omega^i_b}{\partial x^a} - \frac{\partial \omega^i_a}{\partial x^b} + z^\mu_a \frac{\partial \omega^i_b}{\partial z^\mu} - z^\mu_b \frac{\partial \omega^i_b}{\partial z^\mu} - z^\mu_{aa_1} \frac{\partial \omega^i_b}{\partial z^\mu_{a_1}} - z^\mu_{ba_1} \frac{\partial \omega^i_a}{\partial z^\mu_{a_1}} \right.$$

$$\left. + \ldots + z^\mu_{aa_1a_2 \ldots a_h} \frac{\partial \omega^i_b}{\partial z^\mu_{a_1 \ldots a_h}} - z^\mu_{ba_1 \ldots a_h} \frac{\partial \omega^i_a}{\partial z^\mu_{a_1 \ldots a_h}} \right\}$$

$$+ X_j^B \omega^j_a \omega^i_b \frac{\partial X^A_j}{\partial y^B} - X^B_j \omega^j_b \omega^i_a \frac{\partial X^A_i}{\partial y^B} = 0$$

which turns out to be

$$X_i^A \left\{ \frac{\partial \omega^i_b}{\partial x^a} - \frac{\partial \omega^i_a}{\partial x^b} + z^\mu_a \frac{\partial \omega^i_b}{\partial z^\mu} - z^\mu_b \frac{\partial \omega^i_a}{\partial z^\mu} + \ldots + z^\mu_{aa_1 \ldots a_h} \frac{\partial \omega^i_b}{\partial z^\mu_{aa_1 \ldots a_h}} \right.$$

$$\left. - z^\mu_{ba_1 \ldots a_h} \frac{\partial \omega^i_a}{\partial z^\mu_{a_1 \ldots a_h}} + \omega^i_a \omega^j_b \frac{\partial X_j^B}{\partial y^A} - \omega^i_b \omega^j_a \frac{\partial X_j^B}{\partial y^A} \right\} = 0 \qquad (3)$$

But if $X_i = X_i^A \frac{\partial}{\partial y^A}$ are basis for sL(n,R) then by definition

$$X_i^A \frac{\partial X_j^B}{\partial y^A} - X_j^A \frac{\partial X_i^B}{\partial y^A} = [X_i^A , X_j^A]^B$$
$$= C^k_{ij} X^B_k$$

so that equations (3) become

$$X_i^A \left\{ \frac{\partial \omega^i_b}{\partial x^a} - \frac{\partial \omega^i_a}{\partial x^b} + z^\mu_a \frac{\partial \omega^i_b}{\partial z^\mu} - z^\mu_b \frac{\partial \omega^i_a}{\partial z^\mu} + \ldots + z_{aa_1 \ldots a_h} \frac{\partial \omega^i_b}{\partial z^\mu_{a_1 \ldots a_h}} \right.$$

$$\left. - z^\mu_{ba_1 \ldots a_h} \frac{\partial \omega^i_a}{\partial z^\mu_{a_1 \ldots a_h}} \right\} + C^k_{ij} X^A_k \omega^i_a \omega^j_b = 0$$

where C^k_{ij} are the structure constants of SL(n,R).

This equation can be written as

$$X_i^A \left\{ \frac{\partial \omega^i_b}{\partial x^a} - \frac{\partial \omega^i_a}{\partial x^b} + z^\mu_a \frac{\partial \omega^i_b}{\partial z^\mu} - z^\mu_b \frac{\partial \omega^i_a}{\partial z^\mu} + \dots + z^\mu_{aa_1 \dots a_h} \frac{\partial \omega^i_b}{\partial z^\mu_{a_1 \dots a_h}} - z^\mu_{ba_1 \dots a_h} \frac{\partial \omega^i_a}{\partial z^\mu_{a_1 \dots a_h}} + C^i_{jk}\, \omega^j_a\, \omega^k_b \right\} = 0 \qquad (4)$$

Equations (4) are in the form $Z' = Z \times N_2$ where Z is generated by the equations

$$\left\{ \frac{\partial \omega^i_b}{\partial x^a} - \frac{\partial \omega^i_a}{\partial x^b} + z^\mu_a \frac{\partial \omega^i_b}{\partial z^\mu} - z^\mu_b \frac{\partial \omega^i_a}{\partial z^\mu} + \dots + z^\mu_{aa_1 \dots a_h} \frac{\partial \omega^i_b}{\partial z^\mu_{a_1 \dots a_h}} - z^\mu_{ba_1 \dots a_h} \frac{\partial \omega^i_a}{\partial z^\mu_{a_1 \dots a_h}} + C^i_{jk}\, \omega^j_a\, \omega^k_b \right\} = 0$$

for $a \neq b = 1, 2, \dots, n$.

We should say that it is not always the case that the functions X_i^A which actually occur in ψ_a^A are themselves a basis for the Lie algebra $sL(n,R)$, since some of the $\omega^i{}_a$ might vanish. It could be a basis for some Lie algebra homomorphic to $sL(n,R)$ as is the case in the 2-dimension solitons.

Now, because of the special choice of the functions $\omega^i{}_a$, where $\omega^i = \omega^i{}_a\, dx^a$ satisfy equations (1), i.e.

$$d\omega^i = \frac{-1}{2} \sum_{k,j=1}^{n^2-1} C^i_{jk}\, \omega^j \wedge \omega^k$$

so that

$$d\omega^i{}_a \wedge dx^a = \frac{1}{2} \sum_{k,j} C^i_{jk}\, \omega^j{}_a\, \omega^k{}_b\, dx^a \wedge dx^b.$$

Hence

$$\frac{\partial \omega^i{}_a}{\partial x^b} = -\frac{1}{2} \sum_{j,k} C^i_{jk}\, \omega^j{}_a\, \omega^k{}_b\ .$$

Also

$$\frac{\partial \omega^i_b}{\partial x^a} = \frac{1}{2} \sum_{j,k} C^i_{jk}\, \omega^j_a\, \omega^k_b$$

Thus, we have

$$\frac{\partial \omega^i_b}{\partial x^a} - \frac{\partial \omega^i_a}{\partial x^b} + \sum_{j,k} C^i_{jk}\, \omega^j_a\, \omega^k_b = 0 \qquad (5)$$

The system of equations (4) and (5) is now appearing as the integrability conditions for the map ψ. It is analogous to a system obtained by J. Corones, B. Markovski and A. Rizov in their Lie group framework for solitons, [5].

This system is called the soliton system.

By combining equations (4) and (5), we get a system of equations in the form

$$\sum_{a\neq b=1}^{n} \left\{ z^\mu_a \frac{\partial \omega^i_b}{\partial z^\mu} - z^\mu_b \frac{\partial \omega^i_a}{\partial z^\mu} + z^\mu_{aa_1} \frac{\partial \omega^i_b}{\partial z^\mu_a} - z^\mu_{ba_1} \frac{\partial \omega^i_a}{\partial z^\mu_a} + \ldots - z^\mu_{ba_1 \ldots a_h} \frac{\partial \omega^i_a}{\partial z^\mu_{a_1 \ldots a}} \right.$$

$$= 0 \qquad \text{for } i = 1,2, \ldots, \dim sL(n,R) \qquad (6)$$

We call the system of equations (6) the soliton condition, that is the condition for the system Z to represent an n-dimension soliton.

Before commenting on this system, we consider a special case, when n = 2, i.e. the 2-dimension solitons, with the Lie group SL(2,R) and m = 1, A = 1.

A basis for sL(2,R) is $\{X_1, X_2, X_3\}$ where

$[X_1, X_2] = X_2$,

$[X_1, X_3] = -X_3$,

$[X_2, X_3] = 2X_1$.

The map ψ is then given by

$$\psi_1 = A_1 + A_2 q + A_3 r \quad ,$$

$$\psi_2 = A_5 + A_2 B + A_3 C + A_4 A$$

for the general A.K.N.S. system, with $(A_1, A_2, \ldots, A_5)$ homomorphic to sL(2,R) by

$$A_1 = -2\lambda i X_1 \quad ,$$

$$A_2 = X_2 \quad ,$$

$$A_3 = X_3 \quad ,$$

$$A_4 = 2X_1 \quad ,$$

$$A_5 = 0$$

and the 1-forms ω^i, i = 1,2,3 are given by

$$\omega^1 = \omega^1_a \, dx^a = -2\lambda i \, dx^1 + 2A \, dx^2 \quad ,$$

$$\omega^2 = \omega^2_a \, dx^a = -q \, dx^1 - B \, dx^2 \quad ,$$

$$\omega^3 = \omega^3_a \, dx^a = -r \, dx^1 - C \, dx^2 \quad ,$$

so that

$$\omega^1_1 = -2\lambda i \quad , \qquad \omega^1_2 = 2A$$

$$\omega^2_1 = -q \quad , \qquad \omega^2_2 = -B$$

$$\omega^3_1 = -r \quad , \qquad \omega^3_2 = -C.$$

Substituting in equations (5), we have:

for i = 1, a = 1, b = 2,

$$\frac{\partial \omega^1_2}{\partial x^1} - \frac{\partial \omega^1_1}{\partial x^2} + \sum_{j,k=1}^{3} C^1_{jk} \, \omega^j_1 \, \omega^k_2 = 0,$$

which gives

$$2A_{x^1} + 2qC - 2rB = 0 \qquad \text{(i)}$$

Also, for i = 2, a = 1, b = 2, we get

$$-B_{x^1} + q_{x^2} + C^2_{12}\,\omega^1_1\,\omega^2_2 + C^2_{21}\,\omega^2_1\,\omega^1_2 = 0$$

i.e.

$$-B_{x^1} + q_{x^2} + 2\lambda iB + 2Aq = 0 \qquad \text{(ii)}$$

For $i = 3$, $a = 1$, $b = 2$, we have

$$\frac{\partial\omega^3_2}{\partial x^1} - \frac{\partial\omega^3_1}{\partial x^2} + \sum_{j,k=1}^{3} C^3_{jk}\,\omega^j_1\,\omega^k_2 = 0$$

i.e.

$$-C_{x^1} + r_{x^2} - 2\lambda iC - 2Ar = 0 \qquad \text{(iii)}$$

Equations (i), (ii) and (iii) are the A.K.N.S. system:

$$A_{x^1} = qC - rB,$$

$$q_{x^2} = B_{x^1} + 2\lambda iB + 2Aq \quad ,$$

$$r_{x^2} = C_{x^1} - 2\lambda iC - 2Ar,$$

with $A \longrightarrow -A$ and $\lambda \longrightarrow -\lambda$, representing the 2-dimension solitons.

Now, let us see what about the soliton condition, equations (6), in this case, with $i = 1$, $A = 1$, $a = 1$, $b = 2$ and $h = 0$, so that we have a 1-dimensional representation of $sL(2,R)$ given by

$$X_1 = y\,\frac{\partial}{\partial y} \quad ,$$

$$X_2 = y^2\,\frac{\partial}{\partial y},$$

$$X_3 = -\,\frac{\partial}{\partial y} \quad .$$

The map ψ is now given in terms of $sL(2,R)$ basis as:

$$\psi^1_1 = -2\lambda iy + qy^2 - r$$

$$\psi^1_2 = By^2 - C - 2Ay$$

which are the Riccati form of ψ for the A.K.N.S. system.

Equations (6), give, in this case:

$$2A_{x^1} - 2A_{x^1} = 0 \quad ,$$

$$B_{x^1} - B_{x^1} + q_{x^2} - q_{x^2} = 0 \quad ,$$

$$C_{x^1} - C_{x^1} + r_{x^2} - r_{x^2} = 0 \quad ,$$

which means that equations (6) are already satisfied for the case of 2-dimension solitons (the A.K.N.S. system), as they are solitons.

2 Remarks about the model

The advantage of this approach is that it determines at the same time the Bäcklund transformations via the map ψ as was shown in previous chapters, i.e. if for an integer s, the image of the integrability conditions $\tilde{Z}^s$ of ψ^s, the s^{th} prolongation of ψ:

$$\psi^s : J^{h+s}(M, N_1) \times N_2 \longrightarrow J^{s+1}(M, N_2),$$

is a differential equations system Z_1 on $J^{s+1}(M, N_2)$, then ψ determines a Bäcklund transformation between Z and Z_1. Also self-Bäcklund transformations may be obtained in this way (see the Appendix). Moreover, linear systems, similar to the scattering equations, could be associated by choosing linear representations for the Lie algebra of X_i^A in the equations defining the Bäcklund map ψ. This is as follows:

$$y_a^A \equiv \psi_a^A = X_i^A \, \omega^i{}_a$$

and with a linear representation of X_i^A's (or sL(n,R)), say

$$X_i^A = {}_i\alpha_B^A \, y^B$$

$$A,B = 1, \ldots, \dim N_2$$

$$i,j = 1, \ldots, \dim SL(n,R) = n^2 - 1$$

where ${}_i\alpha_B^A$ are constants and $X_i \longrightarrow {}_i(\alpha_B^A)$ is a matrix representation of the Lie algebra of X_i's.

Hence, we have

$$y^A_a = {}_i\alpha^A_B \, \omega^i_a \, y^B$$

which could be another Bäcklund map solving the Bäcklund problem for Z. Thus, we have

$$dy^A = {}_i\alpha^A_B \, y^B \, \omega^i_a \, dx^a$$

or

$$dy^A = \Omega^A_B \, y^B, \quad \text{with}$$

$\Omega^A_B = {}_i\alpha^A_B \, \omega^i_a \, dx^a$, is a matrix of 1-forms. In other words, as in our case

$$\Omega^A_B = {}_i\alpha^A_B \, \omega^i \tag{7}$$

and the linear equations are, in matrix notation,

$$dY = \Omega Y, \tag{8}$$

which is an integrable system.

Because of equation (7), the matrix Ω is not unique. It depends upon the linear representation we choose for the Lie algebra and also upon the set of 1-forms ω^i which represent the set of left invariant 1-forms of sL(n,R) pulled back to M by some function

$$F : M \longrightarrow SL(n,R).$$

This F exists by Frobenius theorem. The variables z^μ and their derivatives may be considered as functions on M. In fact, if ω'^j_a are another set of functions with

$E^i_j \, \omega'^j_a = \omega^i_a$ and E^i_j are constants,

then ω'^j_a will satisfy the same equation (5) if

$$E^i_\ell \, E^j_h \, C^k_{ij} = C^j_{\ell h} \, E^k_j \tag{*}$$

with i, j, k, ℓ, h = 1,2, ..., dim SL(n,R), and C^k_{ij} are its structure constants.

Conditions (*) mean that E^i_j form a matrix belonging to the adjoint representation of SL(n,R). Therefore, linear transformations of $\omega^i{}_a$ through the adjoint representation do not yield essentially new solutions of equations (5) and we may say that two solutions of (5) differing by a transformation from the adjoint representation of SL(n,R) belong to the same class.

Clearly, the matrix Ω is a dim $N_2 \times$ dim N_2 matrix, with dim N_2 arbitrary. A possible choice, similar to that in the case of 2-dimension solitons, is to choose dim $N_2 = n$ so that Ω becomes an $n \times n$ matrix of 1-forms with trace zero which is sL(n,R)-valued.

A fact is that, for each $G \in SL(n,R)$, the matrix of forms $G^{-1}\,dG$ takes its value in the Lie algebra SL(n,R) and there is a local SL(n,R)-valued function F on M,

$$F : M \longrightarrow SL(n,R)$$

such that

$$\Omega = (\Omega^A_B)^n_{A,B=1} = F^{-1}\,dF.$$

This function F specifies, and is specified by, the Bäcklund map ψ as the matrix Ω is obtained via ψ. Also it helps getting linear systems to be associated to the non-linear one (equations (5)), namely

$$dF = F\Omega ,$$

where for each row of F, one may be led to linear partial differential equations which may be closely related to that of equation (8).

Now, with this set up, equations (5) look like conditions for

$$\Theta \equiv d\Omega - \Omega \wedge \Omega$$

to be zero on submanifolds, where $dx^a \wedge dx^b \neq 0$ for $a,b = 1,2, \ldots, n$ i.e. on which $dx^1 \wedge dx^2 \ldots \wedge dx^n \neq 0$.

A geometric interpretation of this is that the matrix of 1-forms Ω, defines an SL(n,R)-connection on a principal SL(n,R)-bundle over M (which can be taken to be R^n). The n-dimension soliton, equations (5), express the fact that the curvature of the connection vanishes, $\Theta = 0$, and equation (8) are the parallel transport equations of this connection. This suggests that there may be a relation between solitons in their general form and gauge fields, Yang-Mills theories and instantons which physicists have interpreted geometrically as connections on principal bundles with some Lie groups as structure groups. In fact, such a relation can be formulated by defining the Yang-Mills tensor τ^i_{ab},

$$\tau^i_{ab} \equiv \frac{\partial \omega^i_a}{\partial x^b} - \frac{\partial \omega^i_b}{\partial x^a} - C^i_{jk}\, \omega^j_a\, \omega^k_b$$

through the "vector potentials" ω^i_a.

The soliton system is then seen to be conditions for the vanishing of the Yang-Mills tensor τ^i_{ab} constructed for the gauge group SL(n,R).

However, as the theory of n-dimensional solitons is still in its infancy and much work remains to be done, the precise relation is not yet clear.

The concept of a conservation law for the n-dimension soliton system may also make sense. In fact the system could be written as a set of differential forms with maximum degree n, namely

$$\begin{aligned}\theta^i_{ab} &= d\omega^i_b \wedge dx^1 \wedge dx^2 \wedge \dots \wedge d\hat{x}^a \wedge dx^{a+1} \wedge \dots \wedge dx^n \\ &\quad - d\omega^i_a \wedge dx^1 \wedge dx^2 \wedge \dots \wedge d\hat{x}^b \wedge dx^{b+1} \wedge \dots \wedge dx^n \\ &\quad - C^i_{jk}\, \omega^j_a\, \omega^k_b\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n\end{aligned}$$

where $d\hat{x}^a$ means that dx^a is excluded.

A differential ideal representing the system can be chosen as $I = \{\theta^i_{ab}\}$, since I is closed.

The conservation laws will be given via a set of closed n-forms ϕ^A on $J^{h+1}(M, N_1) \times N_2$ which satisfy $\phi^A \subset \{\theta^i_{ab}\}$ and $\phi^A(z^\mu(x^a)) = 0$ for every $z^\mu(x^a)$ a solution of the system Z.

Therefore, as well known, each ϕ^A is in the form

$$\phi^A = d\chi^A, \text{ for } \chi^A \text{ an } (n-1)\text{-form.}$$

Thus, from the general version of Stokes theorem, i.e.

$$\int_{M_1} \chi^A = \int_{M_2} d\chi^A$$

where M_2 is an n-dimensional manifold with a closed (n-1)-dimensional manifold M_1 as boundary, conservation laws may be obtainable. The forms χ^A may be chosen as

$$\chi^A = dy^1 \wedge \ldots \wedge d\hat{y}^A \wedge \ldots \wedge dy^n + \sum_{I+J=n-1} F^A_{IJ}\, dx^I \wedge dy^J$$

with F^A_{IJ} functions on $J^{h+1}(M,N_1) \times N_2$, and

$$dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_i} \quad ,$$

$$dy^J = dy^{j_1} \wedge \ldots \wedge dy^{j_{n-i-1}} \quad ,$$

and $I + J = n-1$ as sets of indices, $\dim N_2 \geq n-1$

Now, knowing in advance that the prolongation Lie algebra is sL(n,R) (or any homomorphic Lie algebra) by the construction of the system, the prolongation conditions, namely

$$d\chi^A \subset \{I, \chi^A\} \quad ,$$

may be looked at as conditions on the functions F^A_{IJ} to calculate χ^A and hence the conservation laws. But the calculations look complicated with this general set up.

A final remark is that, for the pseudospherical n-dimensional surface property of the n-dimensional soliton system, we should mention that such surface cannot be isometrically immersed in R^N with $N < 2n-1$, as was shown by T. Willmore for the case $n = 2$; i.e. 2-dimensional surface in R^3, and by E. Cartan for an n-dimensional hyperbolic space form to be immersed in another hyperbolic space form with greater curvature.

A suggestion for what may be done, similar to the 2-dimension case, is to make use of the matrix of 1-forms Ω and look for a correspondence by means of which these 1-forms in Ω describe an n-dimensional surface with constant negative curvatire in R^N, $N \geqslant 2n-1$.

In such a case, Bäcklund transformations have a very significant interpretation as "n-dimension pseudospherical line congruences" between these n-dimensional pseudospherical surfaces, a concept generalized in [29], where a generalization of the sine-Gordon equation is defined via the Gauss and Codazzi equations of these n-dimensional surfaces as submanifolds of R^{2n-1}. In fact solutions of this generalized sine-Gordon equation are defined via maps $A : R^n \longrightarrow O(n)$, the group of $n \times n$ orthogonal matrices, which is a subgroup of $SL(n,R)$, and that makes the choice of $SL(n,R)$ for general n-dimensional solitons appears right.

But still not much can be said about the pseudospherical surface property for the n-dimensional solitons with this generalization.

This, together with other properties mentioned in the above remarks, are topics for future investigation.

Moreover, the soliton condition, equation (6), as conditions on the functions $\omega^i{}_a$, requires further investigation.

APPENDIX

BÄCKLUND AUTOMORPHISMS

In fact, as mentioned before, not every Bäcklund map which solves the Bäcklund problem for a given system of differential equations is itself a Bäcklund automorphism, i.e. it defines a self-Bäcklund transformation. However by means of this Bäcklund map, Bäcklund automorphisms are obtainable. Its construction is as follows:

Let ψ be the Bäcklund map

$$\psi : J^h(M, N_1) \times N_2 \longrightarrow J^1(M, N_1)$$

for a system Z on $J^{h+1}(M, N_1)$, where as usual M, N_1 and N_2 are manifolds, with coordinate presentations x^a, z^μ and y^A respectively and

$a,b = 1,2, \ldots, \dim M,$

$\mu,\nu = 1,2, \ldots, \dim N_1,$

$A,B = 1,2, \ldots, \dim N_2.$

Introduce the product

$$J^\ell(M, N_1) \times_M J^k(M, N_2) = \left\{(\xi_1, \xi_2) \in J^\ell(M, N_1) \times J^k(M, N_2) \,|\, \alpha(\xi_1) = \alpha(\xi_2)\right\}$$

In particular

$$J^\ell(M, N_1) \times_M J^\ell(M, N_2) \cong J^\ell(M, N_1 \times N_2)$$

is an isomorphism, [22].

However, for simplicity, F. Pirani considers diffeomorphisms which leave M and N_1 pointwise fixed. Thus, we suppose given (or to be sought for) a diffeomorphism

$$\tilde{X}^o : J^o(M, N_1 \times N_2) \longrightarrow J^o(M, N_1 \times N_2)$$

compatible with the identity map on $J^o(M, N_1)$, i.e.

$$\begin{array}{ccc} J^o(M, N_1 \times N_2) & \xrightarrow{\tilde{X}^o} & J^o(M, N_1 \times N_2) \\ {\scriptstyle P_1}\downarrow & & \downarrow \\ J^o(M, N_1) & \xrightarrow[id_{J^o(M,N_1)}]{} & J^o(M, N_1) \end{array}$$

commutes, where

$$P_1 : J^o(M, N_1) \; X_M \; J^o(M, N_2) \cong J^o(M, N_1 \times N_2) \longrightarrow J^o(M, N_1).$$

If x'^a, z'^μ, y'^A are also standard local coordinates on the codomain of $\tilde{X}^o$, then $\tilde{X}^o$ has the presentation

$$x'^a = x^a$$

$$z'^\mu = z^\mu$$

$$y'^A = \zeta^A(x^a, z^\mu, y^B),$$

where ζ^A are functions on $J^o(M, N_1 \times N_2)$.

Also, let P_2 denote the natural projection,

$$P_2 : J^o(M, N_1 \times N_2) \longrightarrow J^o(M, N_2).$$

Then

$$X^o = P_2 \circ \tilde{X}^o : J^o(M, N_1 \times N_2) \longrightarrow J^o(M, N_2)$$

also leaves M pointwise fixed.

Now, because of the case that

$$\begin{array}{ccc} J^o(M, N_1) & \xrightarrow{id} & J^o(M, N_1) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ M & \xrightarrow[id_M]{} & M \end{array}$$

commutes, the $\tilde{X}^o$ is compatible with id_M. i.e.

$$\begin{array}{ccc} J^o(M, N_1 \times N_2) & \xrightarrow{\tilde{X}^o} & J^o(M, N_1 \times N_2) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ M & \xrightarrow[id_M]{} & M \end{array}$$

commutes.

The action of $\tilde{X}^o$ may be extended to $C^\infty(M, N_1 \times N_2)$:

if $f \in C^\infty(M, N_1 \times N_2)$, the map f^{id_M} induced from f by $\tilde{X}^o$ is

$$f^{id_M} = \beta \circ \tilde{X}^o \circ J^o f \circ (id_M)^{-1},$$

where β is the target map.

Consequently the action may be extended to

$J^h(M, N_1 \times N_2)$: let $\xi \in J^h(M, N_1 \times N_2)$ and f be any map in the equivalence class ξ so that

$$\xi = J^h_{\alpha(\xi)} f.$$

Then define the h^{th} prolongation of $\tilde{X}^o$,

$$\tilde{X}^h : J^h(M, N_1 \times N_2) \longrightarrow J^h(M, N_1 \times N_2),$$

by

$$\tilde{X}^h(\xi) = J^h_{id_M \circ \alpha(\xi)} f^{id_M} .$$

This is a unique map which leaves $J^h(M, N_1)$ pointwise fixed, since $\tilde{X}^o$ leaves $J^o(M, N_1)$ pointwise fixed.

Therefore, this map passes to the quotient and projects to define a unique map, denoted by $\overline{X}^h$:

$$\overline{X}^h : J^h(M, N_1) \times N_2 \longrightarrow J^h(M, N_1) \times N_2$$

which leaves $J^h(M, N_1)$ pointwise fixed. The local coordinate presentation of $\overline{X}^h$ is

$$x'^a = x^a ,$$

$$z'^\mu = z^\mu,$$

$$z'^\mu_a = z^\mu_a,$$

$$\vdots$$

$$z'^\mu_{a_1 \ldots a_h} = z^\mu_{a_1 \ldots a_h},$$

$$y'^A = \zeta^A(x^a, z^\mu, y^B).$$

Moreover, X^o may be prolonged to a unique map X^h,

$$X^h : J^h(M, N_1 \times N_2) \longrightarrow J^h(M, N_2)$$

by the same way. In fact we have the diagram

$$\begin{array}{ccc} J^h(M, N_1 \times N_2) & \xrightarrow{X^h} & J^h(M, N_2) \\ {\scriptstyle \pi^h_o}\downarrow & & \downarrow{\scriptstyle \pi^h_o} \\ J^o(M, N_1 \times N_2) & \xrightarrow[X^o]{} & J^o(M, N_2) \end{array}$$

Now, the map

$$((\pi^h_1 \circ Pr_1) \times \psi) \circ \Delta_{J^h(M, N_1) \times N_2} : J^h(M, N_1) \times N_2 \longrightarrow J^1(M, N_1 \times N_2)$$
$$\cong J^1(M, N_1) \times_M J^1(M, N_2)$$

passes to the quotient to define a map

$$\tilde{\psi} : J^h(M, N_1) \times N_2 \longrightarrow J^1(M, N_1) \times_M J^1(M, N_2) \cong J^1(M, N_1 \times N_2).$$

The coordinate presentation of $\tilde{\psi}$ is

$$x'^a = x^a \quad ,$$
$$z'^\mu = z^\mu \quad ,$$
$$z'^\mu_a = z^\mu_a \quad ,$$
$$y'^A = y^A \quad ,$$
$$y'^A_a = \psi^A_a \quad ,$$

where ψ^A_a are the functions on $J^h(M, N_1) \times N_2$ determining the Bäcklund map ψ and $\Delta_{J^h(M, N_1) \times N_2}$ is the diagonal map :

$$J^h(M, N_1) \times N_2 \longrightarrow (J^h(M, N_1) \times N_2) \times (J^h(M, N_1) \times N_2).$$

Finally, let

$$\psi^X = X^1 \circ \tilde{\psi} \circ (\bar{X}^h)^{-1} : J^h(M, N_1) \times N_2 \rightarrow J^1(M, N_2).$$

This map has the coordinate presentation

$$x''^a = x'^a$$
$$y''^A = y'^A$$
$$y''^A_a = \psi^{XA}_a$$

(the one primed coordinates in the domain of ψ^X and the two primed in its codomain) where $\psi_a^{XA} = f_a^A \circ (\bar{X}^h)^{-1}$ and f_a^A are functions on the codomain of $(\bar{X}^h)^{-1}$ which have the presentation

$$f_a^A = \frac{\partial \zeta^A}{\partial y^B} \psi_a^B + \frac{\partial \zeta^A}{\partial z^\mu} z_a^\mu + \frac{\partial \zeta^A}{\partial x^a} .$$

In fact $\psi^{X*}(d\Omega^1(M, N_2) \subset I(\tilde{\Omega}^{h,\psi})$ (see equation (4) of section 6, Chapter I) hence ψ^X, like ψ, solves the Bäcklund problem for the system Z.

In general, ψ^X will be distinct from ψ. It is this that makes it possible to construct Bäcklund automorphisms by the above procedure.

Clearly, ψ^X depends upon the given diffeomorphism

$$\tilde{X}^o : J^o(M, N_1 \times N_2) \longrightarrow J^o(M, N_1 \times N_2),$$

and to yield Bäcklund self-transformation, one may impose the condition that $J^o(M, N_1)$ and $J^o(M, N_2)$ are tied by the identity diffeomorphism in addition to $\tilde{X}^o$ such that

$$(id)^{h+1} : J^{h+1}(M, N_1) \longrightarrow J^{h+1}(M, N_2)$$

satisfies that

$$(id)^{h+1*}(Z') = Z$$

where Z' is the system of differential equations on $J^{h+1}(M, N_2)$ which is the image of the integrability conditions of the h^{th} prolongation of the Bäcklund map ψ.

It may be conjectured that this diffeomorphism $\tilde{X}^o$ has something to do with the invariance properties of the prolongation Lie algebra. This deserves further study.

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