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GASDYNAMIC SHOCK WAVES

by

R.D. ANDREWS.
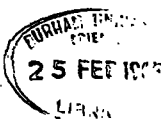
CHAPTER 1.   Introduction.

In this dissertation, we discuss briefly the phenomena leading to discontinuities in one-dimensional gasdynamic flows, and then go on to discuss the end states across a stationary wave. The functions of the Rayleigh and Fanno lines are examined to give a qualitative description of the relations across the wave and, in particular the direction of flow as governed by the Second Law of Thermodynamics is extracted. The entropy increase is also briefly demonstrated analytically across a compressive shock for a polytropic gas. A brief consideration of the stability of a general gas also confirms the expectation that a steady flow is only possible in the case of a compressive shock wave. The final chapter presents a discussion of the structure of a steady standing plane shock wave. The predictions of the Navier-Stokes equation are contrasted with those of a promising model from the realms of the Kinetic Theory. Attention is drawn to the need for experimental investigation into the totality of variables giving rise to viscosity and, especially, the immediate need for a clear picture of the dependence of it upon the velocity gradient.

Throughout this work, we have ignored the phenomena associated with hot gases, namely, dissociation,

ionization and chemical reactions at any temperature. This, of course, limits us to weak and moderately strong shock waves, the maximum strength of which is determined by the chemical nature of the gas passing through it. Heat transfer is only allowed by conduction and by convection. We consider no radiative effects at all.

CHAPTER 2. Discontinuous solutions of the Equations for inviscid gas flow.

It is well known that r and s, the Riemann invariants given by

$$r = a/(\gamma-1) + \tfrac{1}{2}u \quad ; \quad s = a/(\gamma-1) - \tfrac{1}{2}u$$

represent the propagation of small disturbances through a one-dimensional gas flow in which a is the local speed of sound and u is its velocity, which is parallel to the direction of variation of the gas variables spatially, namely, the x-direction. γ is the ratio of the specific heats. Along a disturbance given by r (say) = constant, all the gas variables remain unchanged in time. Such a wavelet propogates with a velocity $(a + u)$, which remains constant in it. This leads to the long-known phenomenon of the steepening of sound waves of finite amplitude, illustrated in Fig. 2.1. Thus, a discontinuous solution arises in an inviscid gas.

Another way to describe this process is to plot the disturbances on the x-t plane, where t is the time. This again gives rise to discontinuities (Fig. 2.2).

We note that compressive waves tend to steepen, whereas expansion waves level out.

From an experimental point of view, there is no doubt that regions do exist where very rapid changes

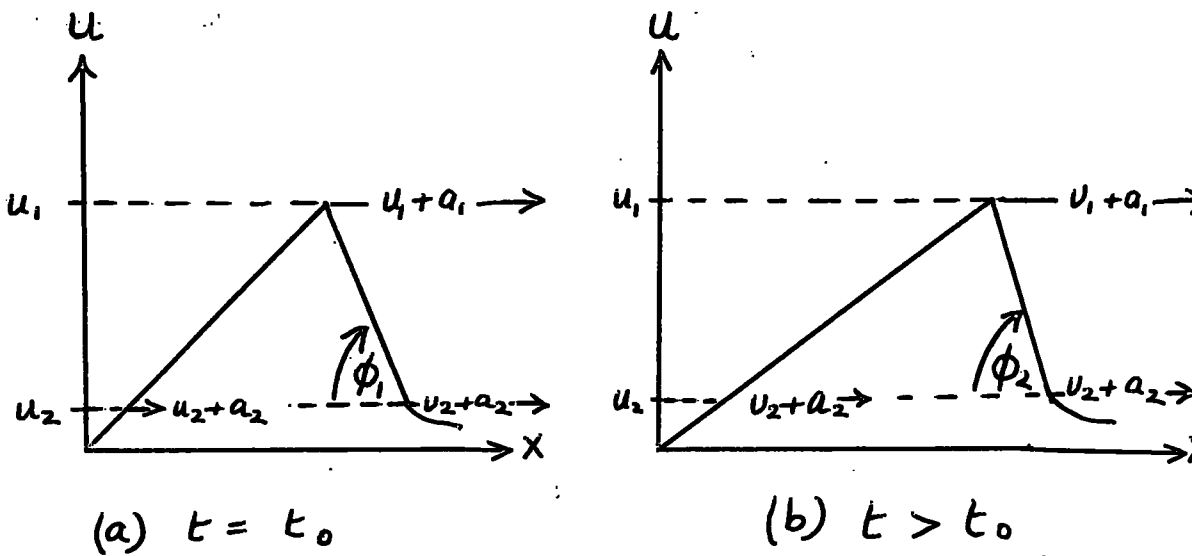Fig. 2.1. Change of waveform of sound waves of
finite amplitude with time.
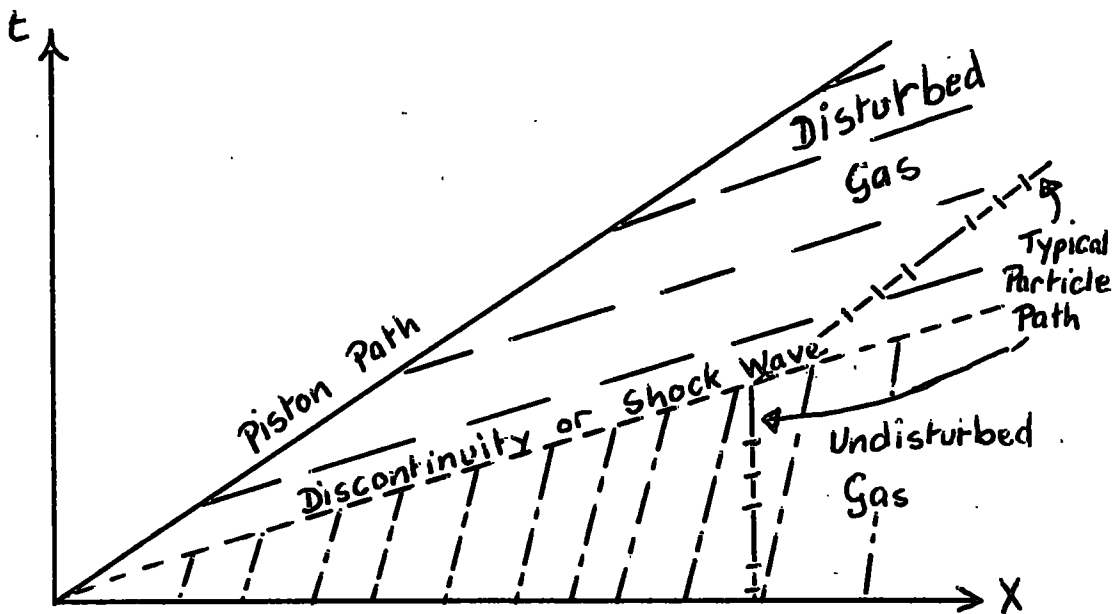$u_1 + a_1 > v_2 + a_2$ so that $\phi_2 > \phi_1$.

Fig. 2.2. $r-$ characteristics of flow in a cylinder due
to a piston moving impulsively from rest into
it. Note the discontinuity.

take place in the gas in some flows. In particular, shock
waves are observed in shock tubes in the laboratory with
unfailing regularity.

CHAPTER 3.   Conditions across a shock wave and elementary properties.

The equations of motion for one-dimensional gas flow are

$$\operatorname{div}(\rho \underline{u}) = 0 \tag{3.1}$$

$$\frac{D}{Dt}(u_i) = \frac{1}{\rho}\frac{\partial}{\partial x}(\tau_{ix}) \tag{3.2}$$

$$\frac{D}{Dt}\left(e + \tfrac{1}{2}u^2\right) = \frac{1}{\rho\partial x}(\tau_{ix}u_i) - \frac{1}{\rho}\operatorname{div}(q_i) \tag{3.3}$$

where "one-dimensional" means that all quantities depend upon x and t only. $\underline{u}$ is the velocity, ρ the density, e the internal energy, τ_{ij} the stress tensor and $\underline{q}$ is the heat flow. A suffix "x" indicates the component of the vector represented by the symbol so suffixed, in the x-direction. In the steady state these have the first integrals, in the one-dimensional case :—

$$\rho u_x = \text{constant} = m \tag{3.4}$$

$$\rho u_x u_i = \tau_{ix} + \text{constant} \tag{3.5}$$

$$\rho u_x\left(e + \tfrac{1}{2}u^2\right) = u_i \tau_{ix} - q_x + \text{constant} \tag{3.6}$$

We assume the usual definitions for τ_{ix} and $\underline{q}$, i.e.

$$\tau_{ix} = p\,\delta_{ix} + \lambda_i \frac{du_i}{dx} \quad \text{and} \quad q_i = K\frac{\partial T}{\partial x_i}, \quad \text{where } \lambda_i$$

is a viscous coefficient taking into account, not only the classical compression viscosity, but also the effects

arising from relaxation phenomena, which we discuss in a later chapter. δ_{ix} is the Kronecker delta, p the pressure and K the thermal conductivity. We suppose that the conditions on either side of a shock, or at least at some distance on each side from it, are uniform, so that we may determine relations between these states by putting $\partial/\partial x \equiv 0$. We get, using the coordinate system moving in the x-direction with the shock wave :-

$$[\rho u_x] = 0 \qquad (3.7)$$

$$[u_y] = [u_z] = 0 \qquad (3.8)$$

$$[\rho u_x^2 + p] = 0 \qquad (3.9)$$

$$[e + \tfrac{1}{2} u^2 + p/\rho] = 0 \qquad (3.10)$$

$[A]$ is the difference in A across the shock, $A_1 - A_2$, A_1 being the value of A upstream of the discontinuity, and A_2 is that downstream. For a polytropic gas,

$$e = C_v T \qquad (3.11)$$

where C_v is the specific heat at constant volume and T is the temperature. Using equations (3.7) to (3.11) we obtain

$$M_2^2 = \frac{M_1^2 + 2/(\gamma-1)}{2\gamma M_1^2/(\gamma-1) - 1} \qquad (3.12)$$

where M is the local Mach number of the flow. $M_2^2 \geqslant 0$ and so

$$M_1^2 \geqslant \frac{\gamma-1}{2\gamma} \qquad (3.13)$$

We may obtain the well known Rankine-Hugoniot equation :-

$$\frac{P_2}{P_1} = \left\{ \left(\frac{\gamma+1}{\gamma-1}\right) \frac{\rho_2}{\rho_1} - 1 \right\} \bigg/ \left\{ \frac{\gamma+1}{\gamma-1} - \frac{\rho_2}{\rho_1} \right\} \qquad (3.14)$$

from which we deduce that the density ratio ρ_2/ρ_1 never exceeds 4 for a monatomic gas, 6 for a diatomic one or 7 in the case of a general polyatomic gas (Fig. 3.1).

We now turn to an important alternative method of determining the end states across a shock wave, namely, the concepts of the Rayleigh and Fanno lines. A Rayleigh process is one in which the mass flux, m, and the momentum flux, P, remain constant, but in which the energy flux, E, changes, for example as a result of heat conduction or radiation. This is one of several processes investigated by Rayleigh (1910). A Rayleigh line is the locus of points in a suitable plane which are met with in the Rayleigh process with particular associated constants. Suitable planes are p - v, T - s (Saunders 1953; Anderson, 1963) and h - s (Crocco, 1958) planes, s being the entropy.

A Fanno process is one in which H and m remain unaltered, but in which momentum is not conserved. Further, the processes are carried out in an inviscid fluid, so
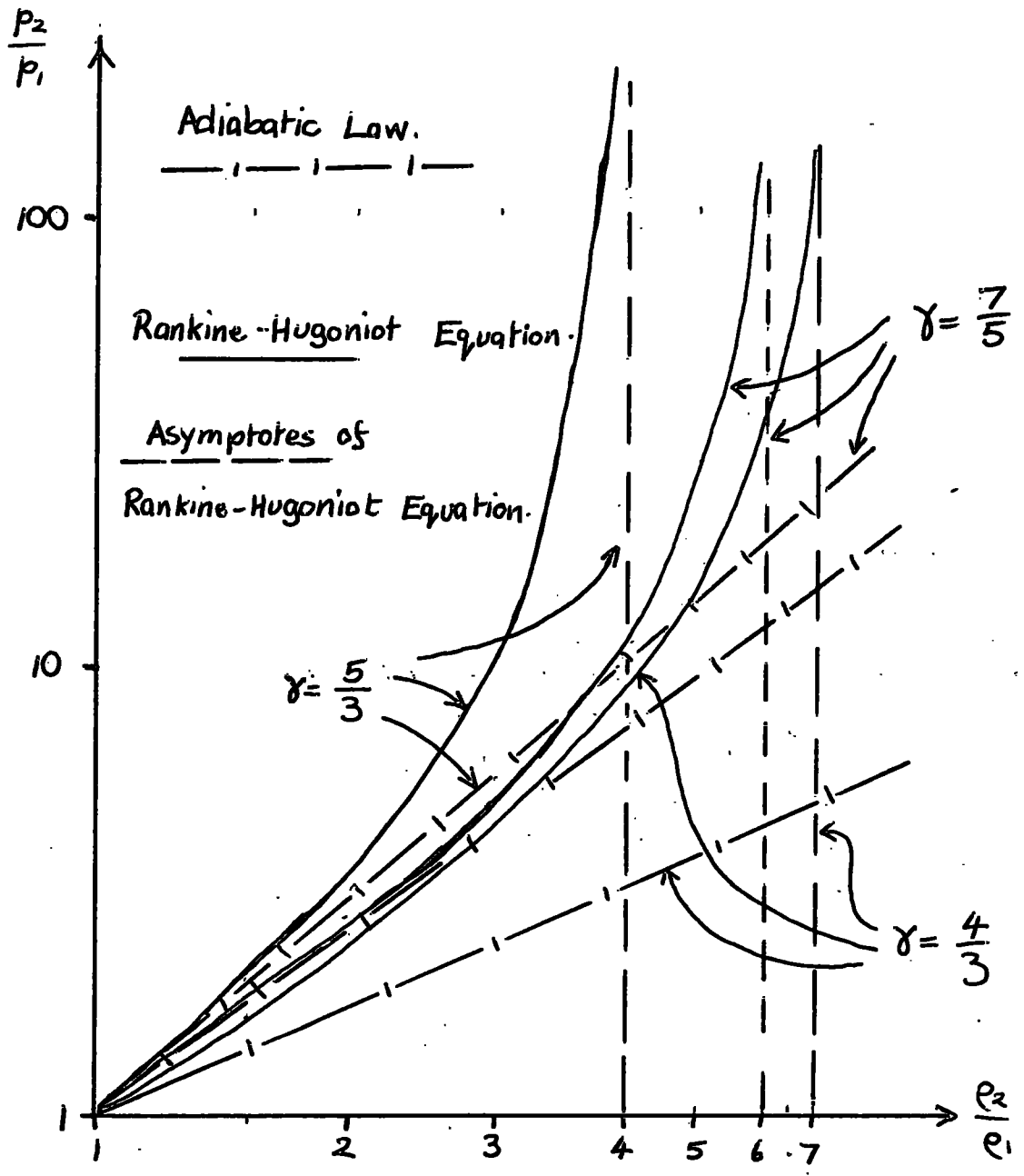
Fig. 3.1. The Rankine - Hugoniot and Adiabatic Relations.

that we have -

$$P + m^2 v = P \qquad (3.15)$$

$$h + \tfrac{1}{2} m^2 v^2 = E/\rho = H \qquad (3.16)$$

h being the specific enthalpy, v the specific volume,
$1/\rho$, and H the stagnation enthalpy of the gas. The
Rayleigh and Fanno lines are derived from equations
(3.15) and (3.16) respectively and take the forms shown
in Fig. 3.2.

In the initial and final states across a
shock wave, the equations for inviscid flow are valid,
and so (3.15) and (3.16) apply. Also, H, m, and P are
conserved across the shock. Therefore, the upstream and
downstream conditions are given by the intersections of
the Rayleigh and Fanno lines.

We consider the lines to have two parts, viz.,
the part on which the corresponding velocity is supersonic
and the part on which it is subsonic. They are positioned
as shown in the figures. Separating them is a sonic point,
at which the entropy is the maximum on that line. We
here follow Crocco's argument (1958) to show this for the
Rayleigh line. A similar kind of argument also applies in
the case of the Fanno line. We may re-write (3.4) and
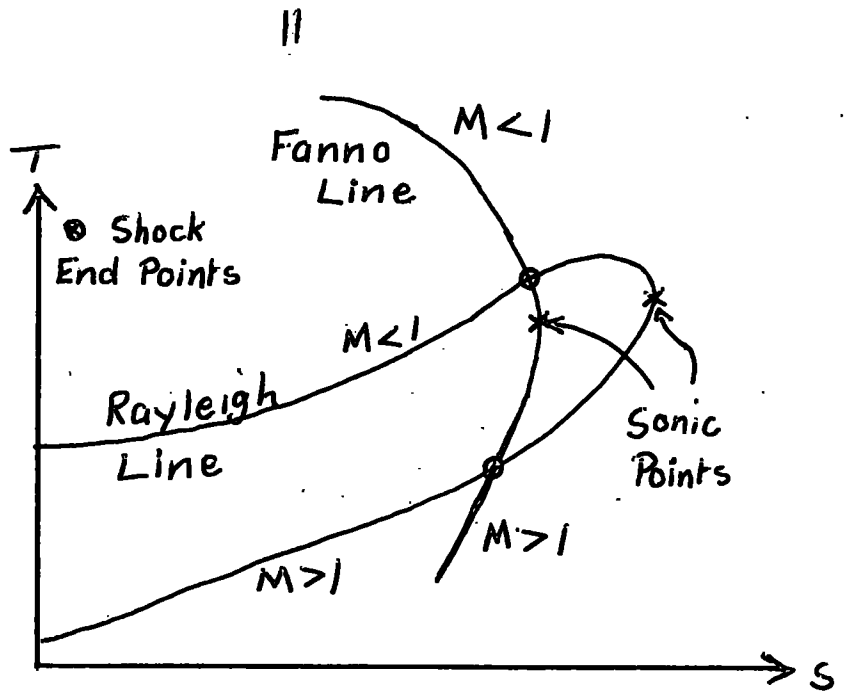(3.15) - subject to the restrictions of the Rayleigh process-
in the form :-

Fig. 3.2. Fanno and Rayleigh lines in T-s plane.

$$d\rho/\rho \;+\; du/u \;=\; 0 \qquad (3.18)$$

$$dp \;+\; \rho u\, du \;=\; 0 \qquad (3.19)$$

Eliminating du we get :-

$$dp \;=\; u^2\, d\rho \qquad (3.20)$$

Now, we may express any thermodynamic quantity as a function of any other appropriate pair. In particular, the pressure is a function of density and entropy only, so that :-

$$dp \;=\; \left(\frac{\partial p}{\partial \rho}\right)_s d\rho \;+\; \left(\frac{\partial p}{\partial s}\right)_\rho ds$$

i.e.
$$dp \;=\; a^2\, d\rho \;+\; \left(\frac{\partial p}{\partial s}\right)_\rho ds \qquad (3.21)$$

Eliminating $d\rho$ between (3.20) and (3.21) we get :-

$$\frac{M^2-1}{M^2}\, dp \;=\; \rho^2 \left(\frac{\partial T}{\partial \rho}\right)_s ds \qquad (3.22)$$

On introducing (3.22) into the thermodynamic relationship

$$T\, ds \;=\; dh \;-\; dp/\rho \qquad (3.23)$$

one obtains

$$\left(\frac{ds}{dh}\right)_{\substack{Rayleigh\\Line}} = \frac{M^2-1}{T} \Bigg/ \left\{ \left[1 + \frac{\rho}{T}\left(\frac{\partial T}{\partial \rho}\right)_s\right] M^2 - 1 \right\}$$

i.e.
$$\left(\frac{ds}{dh}\right)_{\substack{Rayleigh\\Line}} = \frac{M^2-1}{T(\gamma M^2-1)} \qquad (3.24)$$

which shows that s is a maximum when M = 1. Thus a Rayleigh line has as its point of maximum entropy the sonic point,

in whatever plane we choose to construct the line.

We may write (3.16) in the form :-

$$dh + m^2 v \, dv = dH$$

and (3.19) as :- $\quad dp + m^2 \, dv = 0 \qquad \Bigg\} \quad (3.25)$

along a Rayleigh line. Eliminating dh and dp between (3.23) and (3.25) we obtain :-

$$(T ds)_{Rayleigh \, Line} = (dH)_{Rayleigh \, Line} \qquad (3.26)$$

Suppose that P_1 and P_2 are the two end points for a particular shock in the T - s plane. Now H is conserved across a shock. Therefore,

$$\oint_{P_1}^{P_2} dH = 0$$

where the integration is performed over any suitable contour. In particular, we may integrate along the Rayleigh line, so that

$$\oint_{\substack{P_1 \\ R-Line}}^{P_2} dH = \oint_{\substack{P_1 \\ R-Line}}^{P_2} T ds = 0 \qquad (3.27)$$

Thus, if we are given the conditions in state 1, we construct the corresponding Rayleigh line and then determine P_2 by means of (3.27). However, the primary use of the Rayleigh and Fanno lines is in a qualitative investigation of shock end states.

From the shape of the two lines in the h - s plane (Fig. 3.3.) it is easy to see that the flow
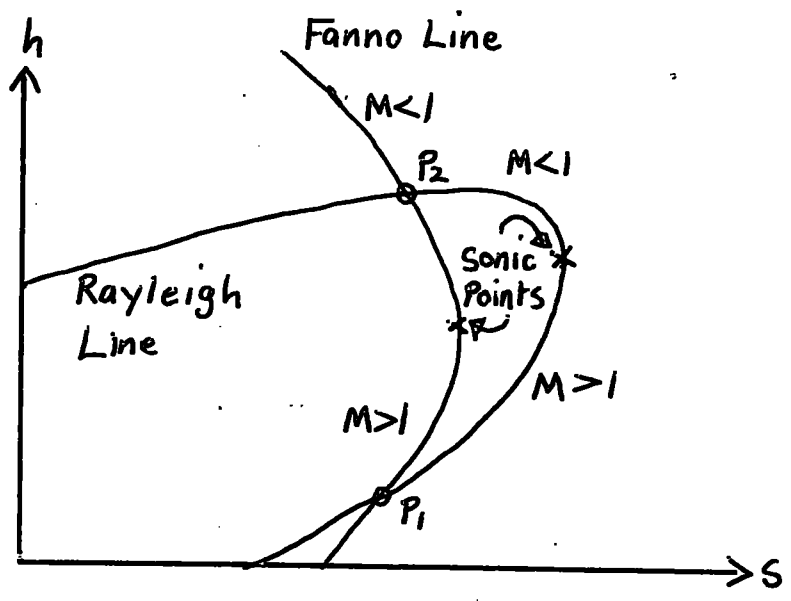
Fig. 3.3. Rayleigh and Fanno lines in h-s plane show that the normal component of flow is supersonic on one side of a shock and subsonic on the other.
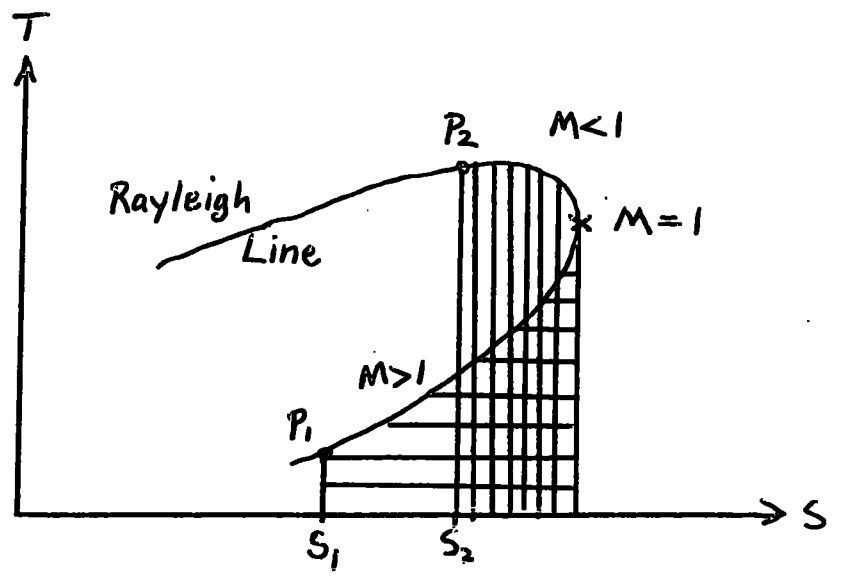
Fig. 3.4. Increase in entropy across a compressive shock.

is supersonic on one side and subsonic on the other. For
a rigorous proof of this, from Rayleigh and Fanno line
considerations, see Crocco (1958). This fact is also
evident from (3.12).

We may also deduce from (3.27) in the T - s
plane that, for a polytropic gas, s increases only if
$M_1 > 1$ and $M_2 < 1$ since (3.27) implies that the
vertically and horizontally shaded areas in Fig. 3.4 are
equal, and this can only be so if $M_1 > 1$ and $M_2 < 1$
under the demand of the Second Law of Thermodynamics that
s should increase across a shock.

CHAPTER 4.   The Direction of variation of quantities
              across a shock.

          We may easily show analytically that entropy
increases across a shock if and only if $M_1 > 1$ and $M_2 < 1$
following Illingworth (1953). We may show from the results
of Chapter 3 :-

$$-[s] = C_v \left\{ \log \left(\frac{2\gamma}{\gamma+1} M_1^2 - \frac{\gamma-1}{\gamma+1} \right) - \gamma \log \left[\frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2} \right] \right\} \quad (4\cdot1)$$

Differentiating this with respect to M_1^2 we get

$$-\frac{d}{dM_1^2} [s] = \frac{2\gamma C_v (\gamma-1)(M_1^2-1)^2}{M_1^2 \left\{ 2\gamma M_1^2 - (\gamma-1) \right\} \left\{ 2 + (\gamma-1) M_1^2 \right\}} \quad (4\cdot2)$$

which is seen to be positive for $\gamma > 1$, so long as (3.13)
holds, which it invariably does. When $M_1^2 = 1$, (4.1)
shows that $[s] = 0$. This completes the analytical proof.

          In the above argument it is assumed that C_v
and γ are constant throughout the shock. This is not
necessarily so, for they are both temperature dependent.
In showing the same result using the Rayleigh line, we
made assumptions on the gas variables and so also on the
shape of the line. We therefore turn to a simple
consideration of the stability of gasdynamic shocks, due
to Landau and Lifshitz (1953). We note (1) In a one-
dimensional gas flow plane sound waves propogate in both
directions relative to the gas at rest, so that disturbance

of the gas produces, in general, two acoustic disturbances.
(ii) If we disturb a shock wave slightly, we may expect an
entropy wave to be produced in addition to acoustic waves.
An entropy wave is a portion of gas which has a different
entropy from the gas flowing with it in the large. Such a
wave is simply carried with the rest of fluid at the fluid
velocity. (iii) There are three conservation equations
relating quantities across a shock, viz., those of mass,
momentum and energy fluxes.

Bearing in mind (i) and (ii), if we count the
number of waves emitted by a shock when it is impinged upon
by a sound wave, and add one for the movement of the shock
layer, then we have the number of amplitudes to be determined
by the three conservation equations of (iii). If this
equals three, then we say that the shock is stable, for the
amplitudes are determined uniquely. If there are more waves
emitted than there are equations to determine them and the
movement of the shock wave, then at least one of the
amplitudes may be chosen arbitrarily. In particular, when
the incident xxxx sound wave has zero amplitude, the wave
of arbitrary amplitude may have a non-zero value. Thus,
the shock may emit disturbances of arbitrary amplitude
spontaneously, and the shock layer itself may even move
about. This is obviously an unstable situation.

The count is shown in Fig. 4.1. We see that

U_1

a_1

O

a_2

U_2

Downstream:

1 Entropy Wave
1 Acoustic Wave

Upstream:

Nil.

No. Waves = ②
‖STABLE‖

Downstream:

1 Entropy Wave
2 Acoustic Waves

Upstream:

Nil.

No. Waves = ③
‖UNSTABLE‖

Downstream:

1 Entropy Wave
1 Acoustic Wave

Upstream:

1 Acoustic Wave.

No. Waves = ③
‖UNSTABLE‖

Downstream:

1 Entropy Wave
2 Acoustic Waves

Upstream:
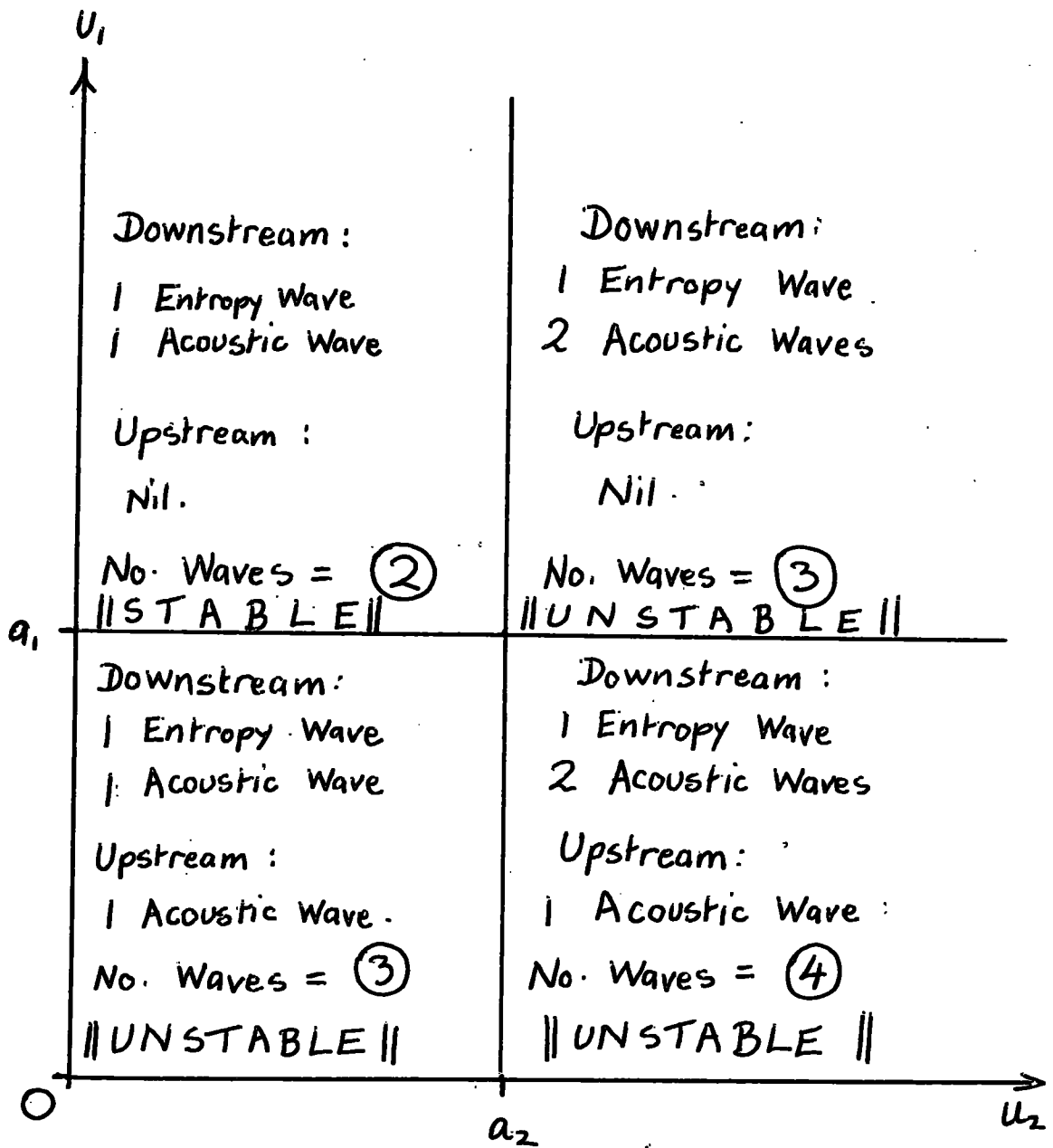
1 Acoustic Wave

No. Waves = ④
‖UNSTABLE‖

Fig. 4.1. Number of Waves and their nature emitted by a stationary shock wave encountering a small disturbance, and the stability of the shock.

so long as $M_1 > 1$ and $M_2 < 1$ then the shock is stable. In every other case, it is unstable.

Other forms of disturbance should, of course, be considered. Landau and Lifshitz (1953) report that for instability to occur in the cases then investigated, it was required that $(\partial^2 v / \partial p^2)_S$ should change sign, which is extremely unlikely to occur in nature.

We conclude that the only type of shock which can exist in a steady state, and obeying the thermodynamic demands, is one in which the velocity normal to the plane of the shock is supersonic relative to its upstream and subsonic downstream of it, i.e. a compressive shock.

CHAPTER 5.   Shock Structure.

(i) <u>Preliminary Remarks.</u> In Chapter 2 we saw that certain solutions for compressible, thermally opaque, inviscid fluids contain discontinuities. At such discontinuities, temperature and velocity gradients become infinite and so, however small they might be in continuous solutions, viscosity and thermal conduction may be expected to become important in the regions where these discontinuities are expected to form. In particular, these diffusities may be expected to smooth the discontinuities. Nevertheless, we still expect to find regions where very rapid changes do take place, because these are observed experimentally.

(ii) <u>Validity of the Equations.</u> The rapid changes in, for example, the velocity of the gas take place over distances of the order of the mean free molecular path of the gas. That is, molecules of the gas undergo only a few collisions when passing from one steady state to another. The question therefore arises, is the continuum approach to shock structure still valid in view of this fact? The primary objection is that over the small distances involved, the continuum concepts, such as density, pressure, etc., are said to be meaningless. In one-dimensional flow, it is assumed that all quantities are functions of x alone, so

that an element ϵ of fluid whose thickness, δx , say, is small, may be considered to be of very great extent in the y and z directions. This means that our usual element of fluid in gasdynamics has an unusual shape, but in ϵ we expect to find conditions to be nearly uniform throughout, so long as we look at adequately large areas of gas parallel to the y - z plane and so long as δx is large compared to molecular dimensions. From this point of view, density is clearly meaningful in a steady shock, as is internal energy. Because of (3.8) we may choose axes so as to reduce the transverse bulk velocities to zero through the shock. Since we have a large number of molecules in our extensive element, we can in principle find the root mean square of the transverse molecular velocity, U_t, say. Then we may define the pressure by means of an appropriate kinetic theory equation of the form:-

$$P \propto \rho U_t^2 \qquad (5.1)$$

where ρ is defined as above. We have, of course, assumed isotropy in defining this. There is no objection to doing this. We show that T is meaningful by the same sort of argument, or else define it by means of a suitable equation of state, such as $P = \rho RT$, if the conditions are such as to allow this to be true. We define T_x the stress in the x-direction as follows :-

$$\frac{\delta T_x}{\delta x} = \text{ -(Momentum entering } \in \text{ from upstream per unit time)}$$

$$+\text{(Momentum leaving } \in \text{ downstream per unit time)} \quad (5\cdot2)$$

Assuming that $\delta T_x / \delta x$ tends to a limit as $\delta x \to 0$, which is nearly reached before δx becomes comparable with molecular dimensions, which is a reasonable assumption, then we may replace $\delta T_x / \delta x$ by $d T_x / dx$, and determine T_x by quadrature. We may define the viscous stress, V_x , to be :-

$$V_x = T_x + p \qquad\qquad (5\cdot3)$$

The problem arises of how to determine V_x without having to work backwards from a known gas flow, which we would have to do under the above definition. We prefer to determine V_x in terms of, say, the velocity gradient du/dx, so that we are in a position to be able to calculate flows, as is our custom in larger scale hydrodynamics.

Thus, there is no a priori objection to the continuum approach to shock structure. However, the problem of determining V_x does remain very important.

In a localised shock rather than an extensive one described above, "steady" presumably means that when the situation is looked at over long periods of time, then the average conditions in the shock are the same in each period.

Gilbarg and Paolucci (1953) pointed out that the equations of the Navier - Stokes theory do not

necessarily yield the wrong results when applied over small distances. The justification for this remark is that in the field of ultrasonic absorption, where the same objection was raised, the continuum theory gave very good agreement with experiment.

An alternative approach to the problem of shock structure is via the Kinetic Theory of Gases. The important equation in this approach is the Boltzmann Equation (5.4),

$$\left(\frac{\partial}{\partial t} + \underline{u} \cdot \frac{\partial}{\partial \underline{x}} \right) f(\underline{x}, t; \underline{v}) = \mathcal{G}(f) + f \mathcal{L}(f) \quad (5.4)$$

where f is the number density of molecules in the $\underline{x} - \underline{v}$ phase-space at time t, where ($\underline{x}$) is physical space and ($\underline{v}$) is the molecular velocity space. $\mathcal{G}$ (f) and $\mathcal{L}$ (f) are non-linear integral operators on f, giving respectively the numbers of molecules gained and lost per unit time at ($\underline{x}$,t;$\underline{v}$) . $\mathcal{G}$ and $\mathcal{L}$ depend for their forms upon the inter-molecular force fields. Unfortunately, little is known for certain about these, and so various models need to be postulated and the one that gives results best in agreement with experiment chosen.

Several objections may be brought to bear upon the Boltzmann Equation. Any argument against the continuum approach from the point of view of meaningfulness of the quantities at some point in a shock wave applies a fortiori to the kinetic theory methods, for f is not only obtained

by an averaging process in **x**- space, but also by one in **y**- space. The validity of the Boltzmann equation in non-equilibrium states remains somewhat in doubt (Hayes, 1958). Choice of a particular model for $\mathcal{G}$ and $\mathcal{L}$ is necessarily somewhat arbitrary. However, we shall discuss later in this dissertation one model which seems to be quite promising but which is, nevertheless, due to a somewhat arbitrary choice of molecular behaviour, namely, the Bhatnagar, Gross and Krook (or B-G-K) model. For weak shocks, i.e. when $(p_2/p_1 - 1)$ is small, it gives complete agreement with the Navier - Stokes solution, but departs from it for stronger shocks. Another major drawback of the Boltzmann equation is that it is so highly intractable that all the work to date using it has been limited to monatomic gases.

Experimental investigation of shock structure is rendered difficult by the small thickness of the shock. By far the most successful to date was that published in 1955 by Sherman, and reported by Pain and Rogers (1962), Liepmann and Roshko in their "Elements of Gasdynamics", Bradley (1962) and others. Sherman increased the thickness of the shock wave by running his wind tunnel at low pressure, thereby increasing the molecular mean free path. To produce plane shocks he introduced an open cylinder into the flow. Measurements were made by means of a hot wire technique.

Up to Mach numbers of about 2 in the upstream flow, excellent agreement was obtained with the Navier-Stokes predictions for shock structure but, at higher speeds, discrepancies do arise (see Fig. 5.5).

(iii) <u>Viscosity</u>. We shall first of all discuss the shock structure given by the Navier-Stokes equation. However, this must be preceded by a short discussion on viscosity. We only discuss those effects due to compression, i.e., we discuss the significance of the coefficient λ in Chapter 3.

In a monatomic gas, λ has the classical value $(4/3)\mu$, where μ is the coefficient of shear viscosity. This is a result of the fact that a monatomic gas molecule has only three degrees of freedom, viz., those of translation.

In gases other than monatomic ones, the molecules possess degrees of freedom other than translational ones, for example, vibrational and rotational degrees of freedom. On changing the state of a gas, the energy in these various modes will also change, in general. The translational energy reaches its new level after a few molecular collisions (Lighthill, 1956). However, the other modes may need rather more collisions before they finally become settled into their new state in the molecules. One effect of this lag in attaining equilibrium is to alter the thermodynamic quantities from their equilibrium values at points where the fluid

particles are undergoing changes. In particular, the pressure is changed, and this in turn alters the stress T_x. We may allow for this effect to a first approximation by altering λ from its classical value. Henceforth, we always allow for this bulk viscosity in λ, unless otherwise pointed out.

(iv) <u>Weak Shocks.</u> The values of the quantities c_p, c_v, λ and K (see Chapter 3) are dependent upon the temperature. In many circumstances, they are virtually constant. In the case of all but the weakest shocks, however, the temperature varies considerably across them, so that we must take these temperature dependences into account. In the case of weak shocks, we may neglect the variation in these quantities. We follow Taylor (1910) in obtaining an analytic solution for the structure of a weak shock. More detail is given in Taylor and Maccoll (1935).

We simplify the situation by choosing axes moving in the shock layer so that $u_y = u_z = 0$, which we may do by (3.8), and so that the solution is independent of time. We omit the suffix x henceforth in this chapter.

The equations of motion are :-

$$\rho u = m \qquad (3.4)$$

$$p + mu - \lambda \frac{du}{dx} = P = mB \qquad (3.5)$$

$$pu + \lambda u \frac{du}{dx} + \tfrac{1}{2} mu^2 + \frac{m}{\gamma-1} \cdot \frac{p}{\rho} - K \frac{dT}{dx} = E = mA \quad (3.6)$$

where we have specialized to a polytropic gas. The equation of state is

$$p = \rho RT \qquad (5.5)$$

From (3.5) we obtain

$$pu = -mu^2 + \lambda u \frac{du}{dx} + (p_i + mu_i) u \qquad (5.6)$$

and substituting this into (3.6), using (3.4) and (5.5), get

$$y \left\{ \left(\frac{2K}{R} + \frac{\lambda}{\gamma-1} \right) u - \frac{KB}{R} \right\} - \frac{\lambda K}{mR} y \frac{dy}{du}$$

$$= \frac{\gamma+1}{2(\gamma-1)} mu^3 - \frac{m\gamma}{\gamma-1} Bu^2 + mAu \qquad (5.7)$$

where $y = u\, du/dx$. Now the velocities u_1 and u_2 of the end states are given by the roots of

$$\frac{\gamma+1}{2(\gamma-1)} \xi^2 - \frac{\gamma}{\gamma-1} B \xi + A = 0 \qquad (5.8)$$

as may quickly be seen from (3.5) and (3.6), and so the right-hand side of (5.7) vanishes if $u = u_1$ or u_2. Thus,

$y/u = du/dx$ vanishes when $u = u_1$ or u_2, and so equation
(5.7) represents the transition between two regions where
the velocities, u_1 and u_2, do not vary with x. For a weak
shock, $u_1 \doteq u_2$ so that $M_1^2 \doteq M_2^2$ and so $M_1^2 \doteq 1$.
This implies that $u_1^2 \doteq \gamma p_1/\rho_1$. Then the right-hand
side of (5.7) becomes of order $\delta^2 m u_1$, where $\delta = u_1 - u$
is assumed small. Assuming $(\lambda K y \, dy/du)/mR$ to be small
compared with the rest of the right-hand side of (5.7) and
K to be of the same order of magnitude as λ, we find
y to be of order $\delta^2 m/\lambda$ and we deduce that we may indeed
neglect the term $\lambda K y \, dy/(mR \, du)$. Thus equation (5.7)
reduces to

$$\frac{du}{dx}\left\{\frac{K}{R}\left(1-\frac{1}{\gamma}\right) + \frac{\lambda}{\gamma-1}\right\} u_1 = -\frac{(\gamma+1)\,m}{2(\gamma-1)}(u_1-u)(u-u_2) \quad (5.9)$$

which has the solution

$$x = \frac{D}{u_1 - u} \log \frac{u_1 - u}{u - u_2} + \text{constant} \qquad (5.10)$$

where

$$D = \left\{\frac{K}{R}\left(1-\frac{1}{\gamma}\right) + \frac{\lambda}{\gamma-1}\right\} \frac{2(\gamma-1)}{\rho_1(\gamma+1)} \qquad (5.11)$$

and $u_1 > u_2$. We take the constant in (5.10) to be zero,
which just fixes the axes so that $u = \frac{1}{2}(u_1 + u_2)$ at $x = 0$.
The shock then has the profile shown in Fig. 5.1.

One noteworthy feature of the shock profile is
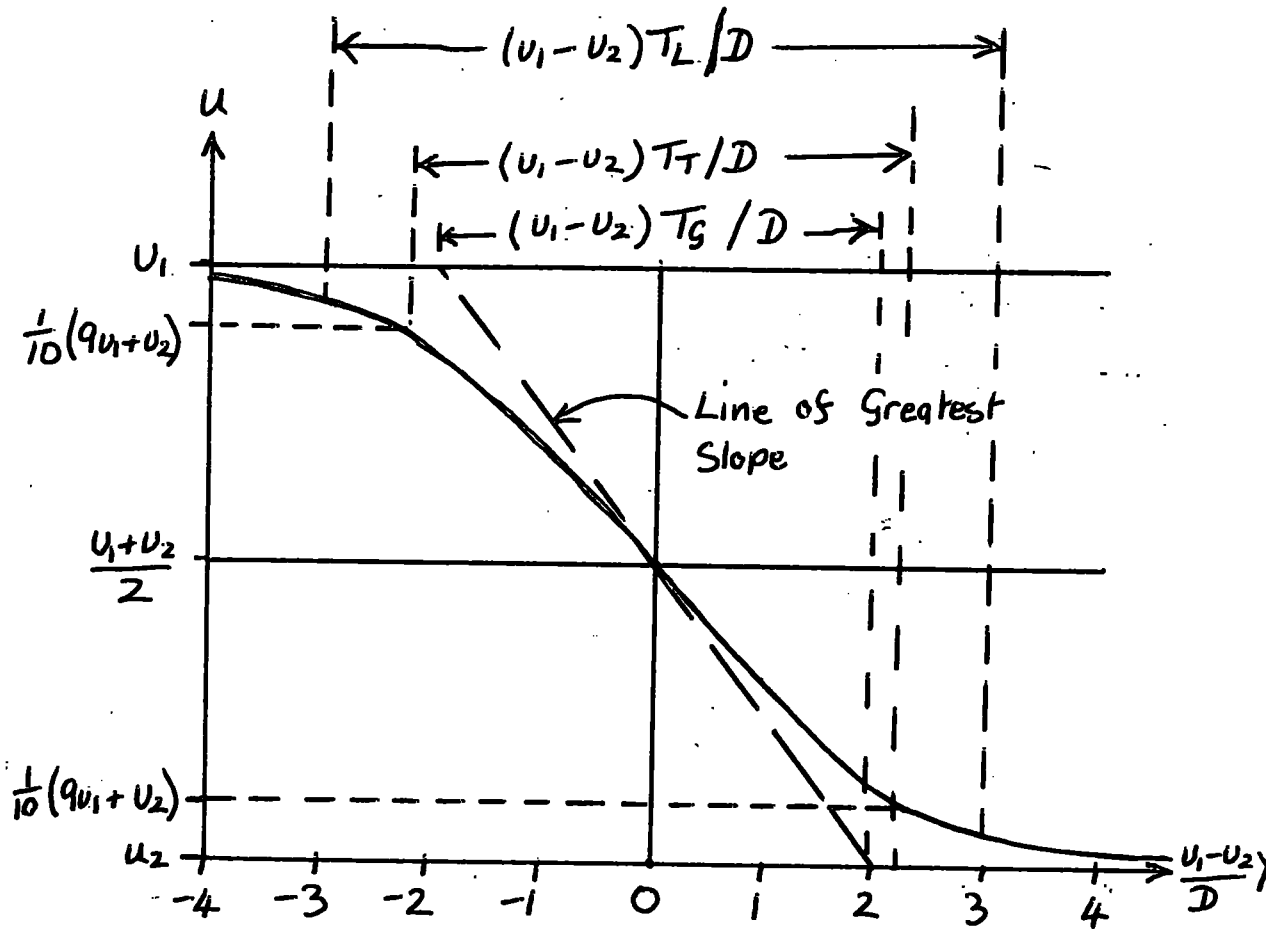that the velocity is asymptotic to the end values for large

Fig. 5.1. The structure of a weak shock wave, showing the various definitions of the thickness.

(Taylor and Maccoll, 1935)

values of $|x|$. One quantity of great interest to experimental workers is the thickness of the shock wave. Because of the profile's form, any definition of shock wave thickness is necessarily somewhat arbitrary. Taylor defines it to be the distance over which the middle 80% of the change in velocity takes place, viz., between $u = (9u_1 + u_2)/10$ and $u = (u_1 + 9u_2)/10$. Therefore the Taylor thickness, T_T, is

$$T_T = \frac{4 \cdot 4}{u_1 - u_2} D \qquad (5.12)$$

Lighthill (1956) defines it to be the length of the region where the middle 90% of the velocity change takes place, so that the Lighthill thickness, T_L, is given by

$$T_L = \frac{6}{u_1 - u_2} D \qquad (5.13).$$

where he takes $\log_e 19$ (= 2.9445) to be 3. Gilbarg and Paolucci take as their shock thickness, T_G, the length given by

$$T_G = \frac{u_1 - u_2}{|du/dx|_{max}} \qquad (5.14)$$

In the profile for weak shocks this is easily seen to be

$$T_G = \frac{4D}{u_1 - u_2} \qquad (5.15)$$

A further definition of shock thickness is that given by Liepmann et al. (1962). This is defined as :-

$$T_s = \frac{u_1 - u_2}{V^*} \mu^* \qquad (5.16)$$

where V^* is the maximum value of the viscous stress which, in fact, occurs at the sonic point, and μ^* is the value of the classical coefficient of shear viscosity at the sonic point. They then show (numerically) that this definition gives very nearly the same thicknesses as Gilbarg's definition, under various dependences of viscosity upon temperature, and with different upstream Mach numbers.

Lighthill (1956) extends his definition of shock thickness from weak shocks to moderately weak shocks by assuming that in the front and rear outskirts of a moderately weak shock, the values of D remain virtually constant and equal to the appropriate end values of D, D_1 and D_2. The thickness thus obtained is therefore

$$T_L' = 3 \frac{D_1 + D_2}{u_1 - u_2} \qquad (5.17)$$

One would expect the thickness of the shock to be over-estimated in the front part and under-estimated in the rear, since, for weak shocks D_2/D_1 is less than unity leading to a (qualitative) profile as shown in Fig. 5.2 by the broken line. Values for D_2/D_1 are shown in Table 1, where it is assumed that $\gamma = 1.4$, and that viscosity, thermal conductivity and bulk viscosity all vary as $T^{0.8}$,
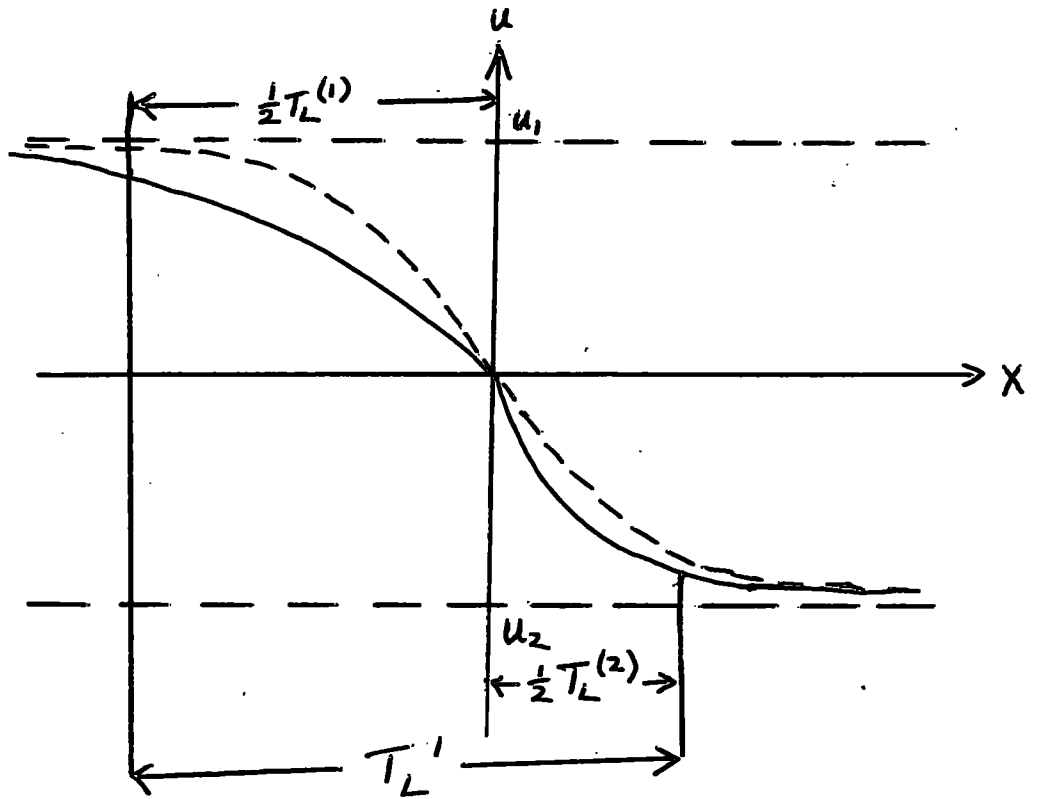
Fig. 5·2. Lighthill's model for a moderately weak shock wave

(~ Lighthill's Model ; ⌒ Expected Real Profile)

$\dfrac{P_2 - P_1}{P_1}$	0	0.5	1	2	3	5	10	15	20
$\dfrac{D_2}{D_1}$	1	0.82	0.73	0.63	0.58	0.55	0.58	0.64	0.70

TABLE I. The ratio of diffusivities across shocks of various strengths.

(Lighthill, 1956)

as a rough guide to what happens in nature.

(v) <u>Stronger Shocks.</u> We shall now turn our attention away from the somewhat limited solution for weak shocks to stronger shocks, where allowance is made for the variation of λ and K with temperature. This problem was first successfully tackled by Gilbarg in 1951, but we shall quote results from the later paper by Gilbarg and Paolucci (1953).

We have the following equations :-

$$\rho u = m \tag{3.4}$$

$$p + \rho u^2 - \lambda \frac{du}{dx} = P \tag{3.5}$$

$$\rho u \left(e + \tfrac{1}{2} u^2 + \frac{p}{\rho}\right) - u \lambda \frac{du}{dx} - K \frac{dT}{dx} = E \tag{3.6}$$

Eliminating ρ , we obtain

$$\left.\begin{aligned} \lambda \frac{du}{dx} &= p + b(u-a) \\[2ex] K \frac{dT}{dx} &= b\left[e - \tfrac{1}{2}(u-a)^2 - c\right] \end{aligned}\right\} \tag{5.18}$$

where

$$a = \frac{P}{m} \quad ; \quad b = m \quad ; \quad c = \frac{E}{m} - \frac{P^2}{2m^2}$$

We specialize immediately to polytropic gases, where

$$p = \rho R T \quad ; \quad e = C_v T \quad ; \quad R = C_p - C_v$$

We define $\delta = \frac{1}{2}(\gamma - 1)$ and note that δ only depends upon the particular gas being considered, and not upon its state. We now reduce equations (5.18) to the non-dimensional forms :-

$$\bar{\lambda} \cdot \frac{dw}{dx} = w + \frac{\theta}{w} - 1 \equiv N(w, \theta) \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \quad (5.19)$$

$$\bar{K} \frac{d\theta}{dx} = \theta - \delta\left[(1-w)^2 + \alpha\right] \equiv L(w, \theta)$$

by means of the substitutions

$$w = \frac{mu}{P} \quad ; \quad \theta = \frac{RTm^2}{P^2} \quad (5.20)$$

and where

$$\alpha = \frac{2Em}{P^2} - 1 \quad ; \quad \bar{\lambda} = \frac{\lambda}{m} \quad ; \quad \bar{K} = \frac{K}{C_v m} \quad (5.21)$$

Now, the curves $N(w, \theta) = 0$ and $L(w, \theta) = 0$ in the $w - \theta$ plane are the Rayleigh and Fanno lines, respectively. They are the parabolas

$$\theta = \delta\left\{(1-w)^2 + \alpha\right\} \quad ; \quad \theta = \frac{1}{4} - \left(\frac{1}{2} - w\right)^2$$

and therefore have only two intersections in the complex projective plane. If we start from a real situation where

we know, say, the upstream state, then we have two real
intersections at which it is obvious from the equations
and the definitions of α and E that $w > 0$ at both
end points. If $\theta < 0$ at the intersections, then
$M_1^2 < (\gamma - 1)/2\gamma$, in violation of (3.13), in which
case we cannot set up a standing shock wave. Thus if the
Rayleigh and Fanno lines intersect at one physically
possible point, then they will intersect again at one and
only one other such point. This is so in any other plane
suitable for constructing the lines in, for we can set up
a 1 - 1 correspondence between the planes.

We suppose the intersections to be meaningful
and at $Z_1 = (w_1, \theta_1)$ and $Z_2 = (w_2, \theta_2)$ in the w - θ plane.
They have the relative positions shown in Fig. 5.3 from
the thermodynamic considerations of Chapters 3 and 4. Then

$$
\left.\begin{aligned}
\left.\begin{array}{l} w_1 \\ w_2 \end{array}\right\} &= \frac{1}{2(\delta+1)}\left\{2\delta+1 \pm \sqrt{1-4\delta(\delta+1)\alpha}\right\} \\[2em]
\left.\begin{array}{l} \theta_1 \\ \theta_2 \end{array}\right\} &= \frac{1}{2(\delta+1)^2}\left\{1 + 2(\delta+1)\alpha \mp \sqrt{1-4\delta(\delta+1)\alpha}\right\}
\end{aligned}\right\} \quad (5.22)
$$

Thus, the end states depend only upon the parameter α ,
which, in turn, depends only upon the upstream Mach number
of the flow, thus :-

$$\alpha = \frac{1}{\delta(\delta+1)} \cdot \frac{\eta}{(\eta+1)^2} \qquad \text{where}$$

$$\eta = \frac{P_2}{P_1} = \frac{2\delta+1}{\delta+1} M_1^2 - \frac{\delta}{\delta+1} \qquad \left(\text{from (3.7) to (3.10)}\right)$$

The next stage, which is crucial, is to prove that the equations (5.19) are, in fact, integrable, subject to the conditions that $(w(x),\theta(x)) \longrightarrow (w_1,\theta_1)$ as $x \longrightarrow -\infty$ and $(W(x),\theta(x)) \longrightarrow (w_2, \theta_2)$ as $x \longrightarrow +\infty$.

From (5.19) and the definitions of Z_1 and Z_2 as the intersections of $L = 0$ and $N = 0$, we see that Z_1 and Z_2 are singular points of the system. We therefore investigate the solution trajectories of (5.19) in the neighbourhoods of Z_1 and Z_2. The characteristic equation of (5.19) is

$$0 = \begin{vmatrix} \dfrac{N_\theta}{\bar{\lambda}} & \dfrac{N_w}{\bar{\lambda}} - \zeta \\[3mm] \dfrac{L_\theta}{\bar{K}} - \zeta & \dfrac{L_w}{\bar{K}} \end{vmatrix} \qquad (5.23)$$

$$= \zeta^2 - \left(\frac{N_w}{\bar{\lambda}} + \frac{L_\theta}{\bar{K}}\right)\zeta + \left(L_\theta N_w - L_w N_\theta\right)/\bar{\lambda}\bar{K}$$

where the subscripts w and θ denote partial differentiation with respect to those variables. After some algebra, we find that the roots of the characteristic equation are both real and positive at Z_1 and real and of opposite signs at Z_2. Thus, from the theory of solution trajectories, we

have an unstable node at Z_1 and a saddle point at Z_2.
For a proof of this, see Appendix A.

The parabolas divide the w - θ plane into the
five regions ①(L) to ⑤(V) in Fig. 5.3. In regions ①(1) and
⑪(11), dθ/dw > 0 and in ⑪⑫(111) , ⑭(1V) and ⑤(V) , dθ/dw
< 0. On the parabola L = 0, dθ/dw = 0, whereas on N = 0,
dw/dθ = 0. Now, there are two integral curves which
approach Z_2 as x → +∞ , and two which approach it as
x → -∞ (which can be seen by considering (5.24) and
by considering the signs of L(w,θ) and N(w,θ)). The
members of each pair of these curves have the same slopes
at Z_2, but approach it from opposite directions. The slopes
are given by (see Appendix B)

$$\frac{d\theta}{dw} = \frac{-Lw}{L_\theta - \xi K} (Z_2) = -\frac{2\delta(1-w_2)}{1- \xi \bar{K}_2} \quad (5.24)$$

and since w < 1, dθ/dw < 0 for ξ negative, so that
one solution trajectory runs into region ⑭(1V) , the finite
region bounded by the parabolas. On this trajectory,
dθ/dx > 0 (since N > 0 in ⑭(1V)) and so Z_2 is approached
as x → +∞ , since θ increases as this happens. From
sign considerations, we may show that no solution
trajectories enter ⑭(1V) across L = 0 and N = 0 with x
increasing, between Z_1 and Z_2. Therefore, the integral
curve through Z_2 which we have just considered must pass
through Z_1 , and can only do so as x → -∞ . Thus the
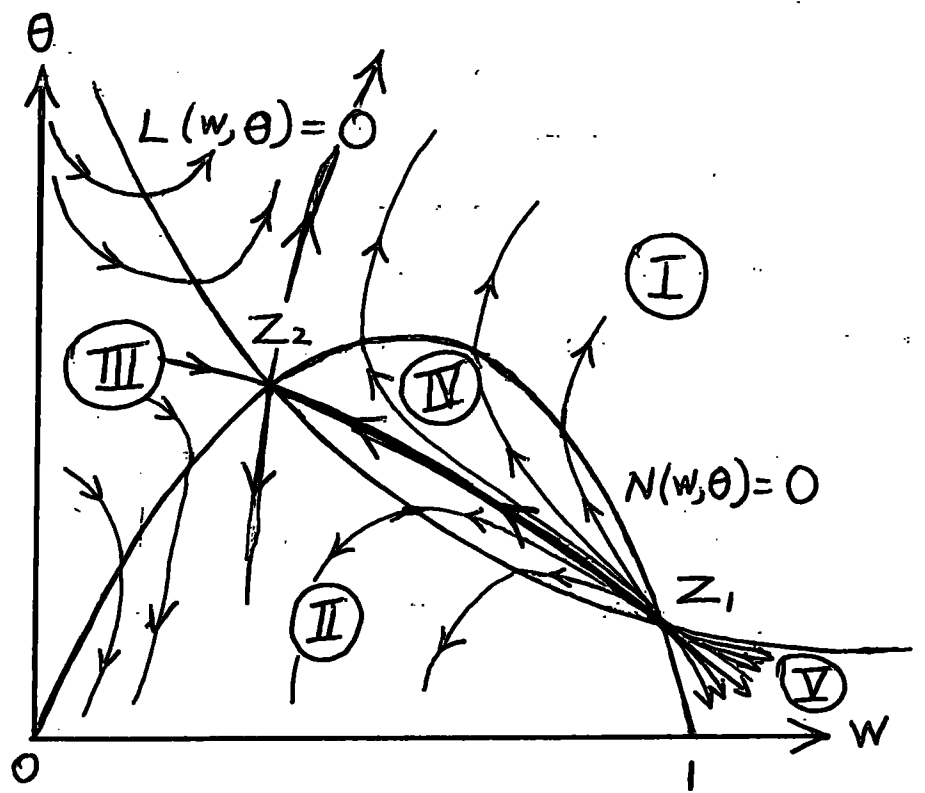
Fig. 5·3. Solution Trajectories of Equation (5·19) in the (w-θ) plane.

The arrows indicate the direction of X increasing

(Gilbarg and Paolucci, 1953)

solution trajectories do take the form shown in Fig. 5.3.

This completes the proof of the integrability of the equations. It is of interest to note that Taylor (1910) proved that the integral curve, if it exists, only intersects the Rayleigh and Fanno lines at the end points and, further that in the $p - \rho$ plane it lies in the finite region bounded by these two lines and, therefore, by topological argument, in the region (IV) in the w - θ plane. If the equations were not integrable, then we would have to abandon the Navier-Stokes equation as an accurate description of gasdynamic flow, although it gives an adequate description of many less violent phenomena.

We now have at our disposal a procedure for computing the shock profile. We start by choosing a point close to Z_2 in (IV) laying on the line through Z_2 having the slope given by (5.24), and integrate the equation

$$\frac{d\theta}{dw} = \frac{\overline{\lambda}}{\kappa} \cdot \frac{w\{\theta - \delta[(1-w)^2 + \alpha\}}{w^2 - w + \theta} \qquad (5.25)$$

numerically. Having thus computed the solution trajectory in the w - θ plane, we may obtain the shock profile in the u - x, T - x planes or any other suitable ones, by quadrature of (5.19) and, finally, by the substitution of (5.20) and (5.21).

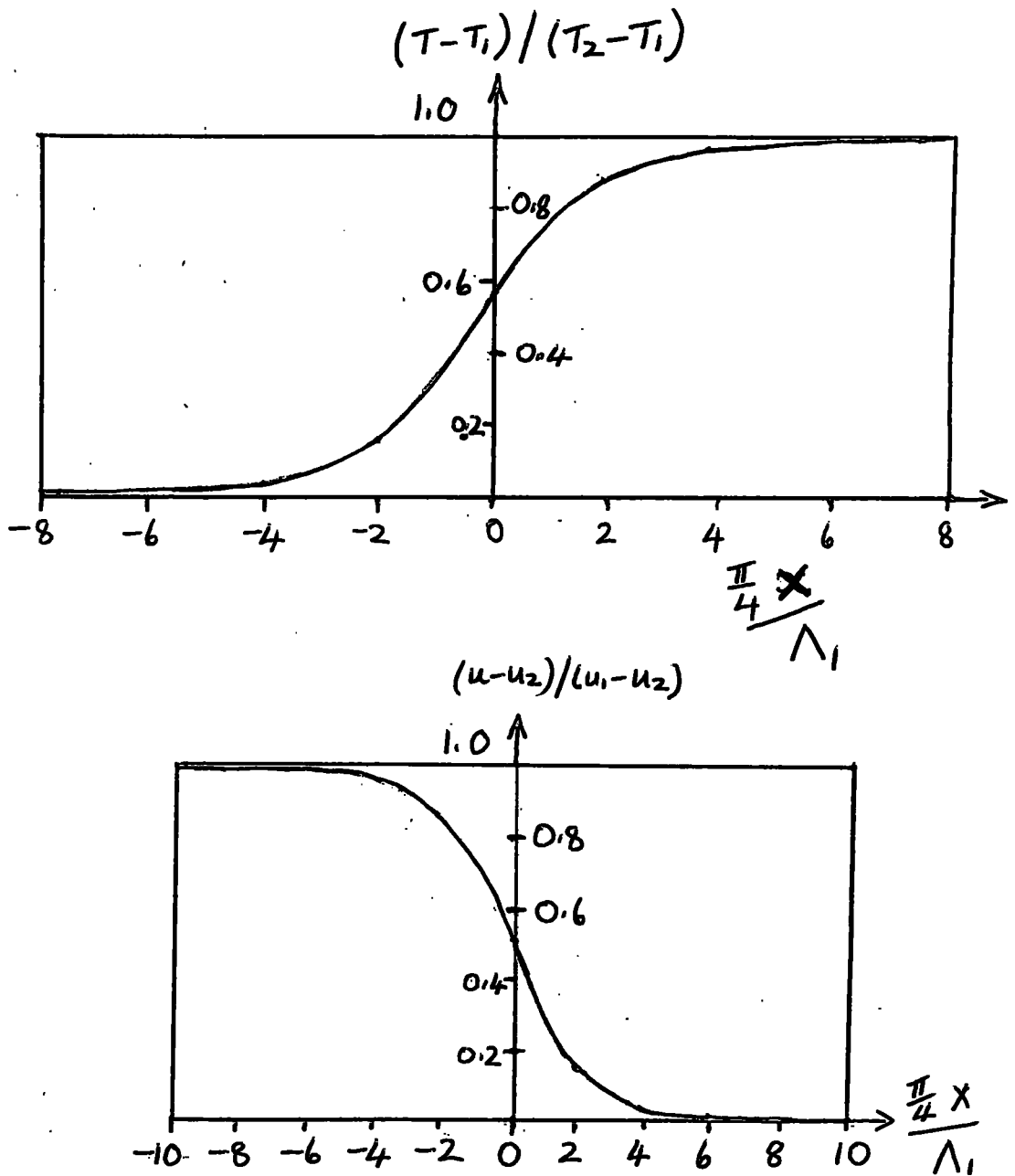Typical profiles are shown in Fig. 5.4, for

$(T-T_1)/(T_2-T_1)$

$(u-u_2)/(u_1-u_2)$

Fig. 5.4. Shock profiles according to the continuum theory at $M_1 = 1.5$. Λ_1 is the molecular mean free path upstream of the shock.

(Liepmann et al., 1962)

which the quantities used are the viscosity and molecular
weight of argon, but with a Prandtl Number of 1 instead of
$\frac{2}{3}$, with an upstream Mach number of 1.5. We note that the
profile is of the same form as that for a weak shock wave
in that the end values are not quite attained in the finite
part of the fluid, even though they are, for all practical
purposes, reached very quickly.

Thus, we are in a position to determine the
shock thickness from our profile and to compare the results
with experiment, so long as the experimenters know exactly
which thickness they are measuring. Sherman's results are
compared with calculated values of shock thickness in
Fig. 5.5. Sherman's method was sufficiently fine to allow
him to know which thickness he measured, so that this
particular comparison is properly meaningful. Bulk viscosity
was fully allowed for in the computations, where a diatomic
gas is assumed. It is seen that agreement between theory
and experiment is very good up to incident Mach numbers
of two, but it is not so good at higher speeds, as we
have remarked before.

(vi) <u>The Mechanism of Viscosity.</u> In this section
we discuss the
classical explanation of gaseous viscosity, and the
implications of the molecular basis of it in the case of
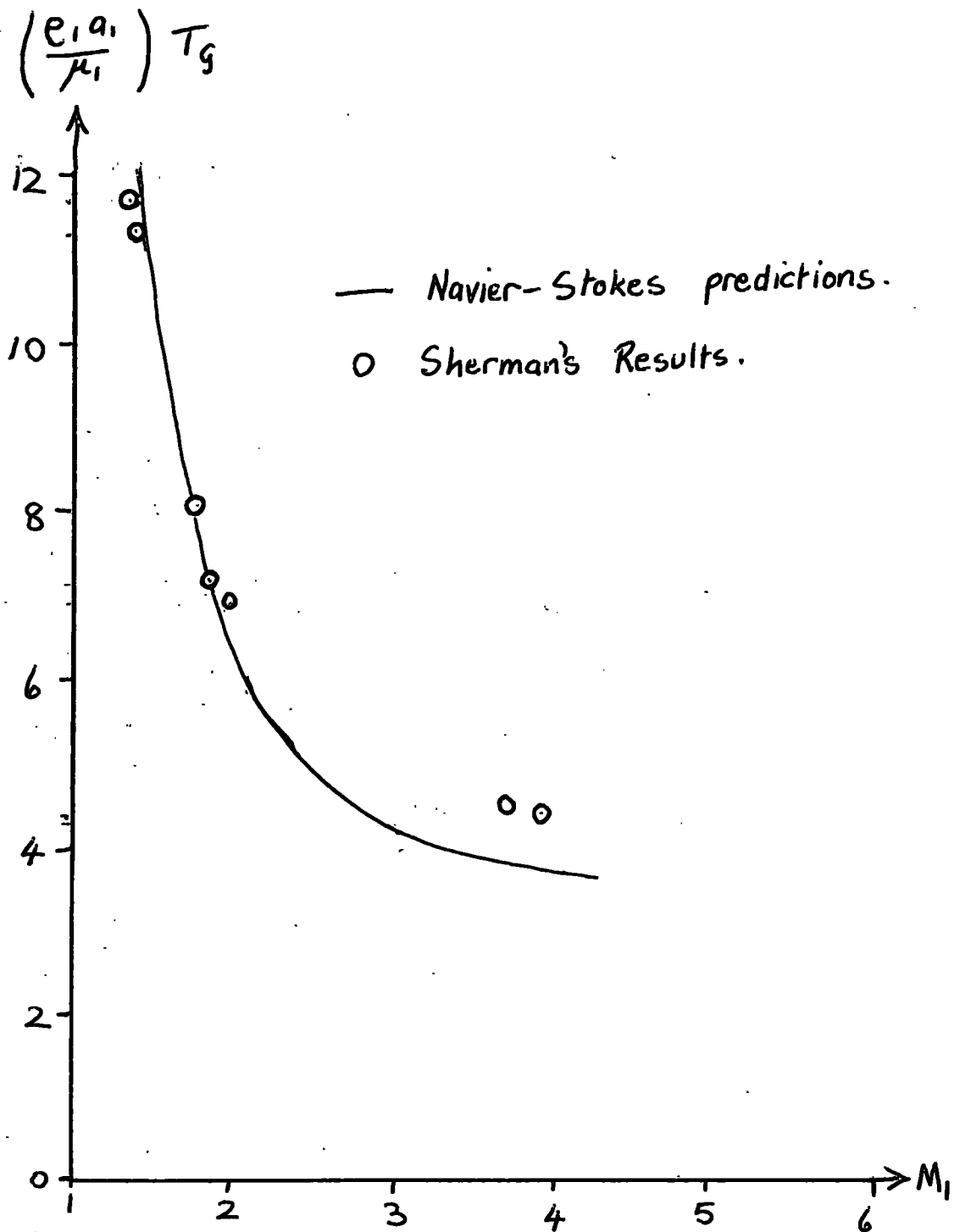very great velocity gradients, such as are only met with

Fig. 5.5. Comparison of Navier-Stokes and Experimental Shock Thicknesses of a Diatomic Gas.

(Pain and Rogers, 1962)

in shocks. We discuss the classical explanation, which
limits the remarks to monatomic gases and we limit ourselves
to compressive viscosity.

- The reason for the existence of viscosity is
that the thermal motion of the molecules in a fluid allows
them to transport momentum from one part of a fluid to
another. In a uniform flow, the momentum carried in one
direction is exactly matched by that carried in the
opposite one. In non-uniform flows, this is no longer so.
In deriving the linear relation between viscosity and
velocity gradient, it is assumed that the molecular mean
velocity does not change very much over distances of the
order of the mean free path, so that when a molecule
traverses the boundary between two layers of fluid moving
at different bulk velocities, then after one collision it
is effectively part of the new fluid, having a typical
thermal velocity with respect to the rest of the molecules.
In this case, the distribution of path-lengths of molecules
about the mean is not important, and neither is the pattern
in which the particles rebound on collision critical. ~~in~~
~~most.~~

In the case of very great velocity gradients,
then this condition is not fulfilled, and so a molecule
penetrating into, say, some fluid moving very much more
slowly than that from which it came, will usually need

more than one collision to relieve it of its excess momentum. This evidently is a mechanism which <u>might</u> cause the linear law to break down. In short, the momentum carried by a faster molecule into a slower region is likely to end up where the linear law does not expect to find it, when there is a very large velocity gradient. The same remarks also apply to the deficit of momentum carried into a fast portion of fluid by a slow molecule.

Molecules in a monatomic gas do not actually collide. When they pass close to one another, the forces acting between them deflect their paths in general. Thus, the distance one of our fast molecules is likely to penetrate the slower gas will depend upon the nature of the force field. If this is such that small deflections are most likely to occur, and large ones happen only when the molecules try to pass through one another, then more encounters will obviously be required to slow our particle down than if most encounters gave it a pretty large deflection. Thus, we see that there are many considerations to be taken into account in determining the law of viscosity, even in the case of a monatomic gas, and so the more likely it is that the law is ultimately non-linear in a complete scheme.

(vii) <u>The Range of Applicability of the Navier-</u>
<u>Stokes Equation to Shock Structure.</u> A

result of the kinetic theory is that the Navier-Stokes equation might no longer apply to monatomic gases when V_x/p is not small in one-dimensional flow (Liepmann et al., 1962) presumably as the result of the sort of effects suggested above. When V/p (we omit the suffix "x" from V_x from now on) is small, then the fluid is definitely Newtonian, but what happens when it is not is not clear. Where V/p is not small, we would not, therefore, be surprised to find the flow departing from the Navier-Stokes flow. Liepmann et al. (1962) examine the variation of this quantity through the shock as follows.

A major simplification is at once introduced in the case of a monatomic gas. The stagnation enthalpy, H, does not depart from its upstream value by more than 1.4% of that value throughout the shock wave.

(3.6) may be re-written thus :-

$$m\left(H - H_1\right) = u\,\lambda\frac{du}{dx} + \frac{K}{C_p}\,\frac{dh}{dx} \qquad (5.26)$$

where h is the specific enthalpy. We define a modified Prandtl number, Pr' , thus,

$$\text{Pr}' = \lambda\,c_p/K$$

where λ , it is recalled, is the total compression viscous coefficient. If $\Delta H = (H - H_1)$, ΔH has a maximum when H has one too. At such a point,

$$dH = dh + u\,du = 0$$

so that (5.26) may be written as

$$\frac{\Delta H_{max}}{H_1} = \frac{P_r' - 1}{P_r'} \cdot \frac{Vu}{H_1 m} \tag{5.27}$$

Since the right-hand side of (5.27) involves the small parameter $(Pr' - 1)$, for $Pr' \doteq 1$, we may evaluate Vu for $Pr' = 1$, since any closer evaluation of $\Delta H/H_1$ merely includes terms of order $(Pr' - 1)^2$. To this order of accuracy, (5.26) gives H = constant, and the maximum value of $|Vu|$ is thus given by

$$|Vu|_{max} = \left\{ (\gamma+1)/2\gamma \right\} mu_1^2 \left(1 - \frac{a^{*2}}{u_1^2} \right)^2 \tag{5.28}$$

(after some algebra) where a^* is the velocity of the sonic stream with the stagnation enthalpy H_1, so that

$$a^{*2} = 2\left\{ (\gamma-1)/(\gamma+1) \right\} H_1 \tag{5.29}$$

Since M_1 varies between 1 and $+\infty$,

$$\frac{\gamma+1}{\gamma-1} \geqslant \frac{u_1^2}{a^{*2}} \geqslant 1$$

so that (5.27) may be written, using (5.29)

$$\frac{\Delta H}{H_1} < \frac{P_r' - 1}{P_r'} \cdot \frac{(\gamma+1)}{8\gamma} \left(1 - \frac{a^{*2}}{u_1^2} \right)^2$$

So,
$$\frac{\Delta H}{H_1} < \frac{Pr' - 1}{Pr' \cdot 2\gamma(\gamma+1)}$$

For a monatomic gas, $Pr' = 8/9$, and so

$$\Delta H / H_1 < 0.014$$

Therefore, we may take H to be constant throughout the shock. Using this, we find that V has its maximum value at the sonic point, using (3.5). Re-writing (3.5) in the form

$$m(u - u_1) + p - p_1 = V \qquad (5.30)$$

and defining $\eta^2 = (\gamma + 1)/(\gamma - 1)$ and $W = u/a$, we manipulate (3.7) to give

$$\frac{V}{P} = \frac{-\eta^2(W_1 - W)(W - W_2)}{\eta^2 - W^2} \qquad (5.31),$$

remembering that $W_1 W_2 = 1$ (which may be obtained from (3.12), using $M^2 = 2W^2/\{(\gamma + 1) - (\gamma - 1)W^2\}$). ~~and defining~~ ~~= (+ 1)/(- 1).~~

Sample plots of (5.31) are shown in Fig. 5.6, together with the positions of the sonic points in the flows which, as we shall see, are the points of maximum stress. (5.30) may be written as

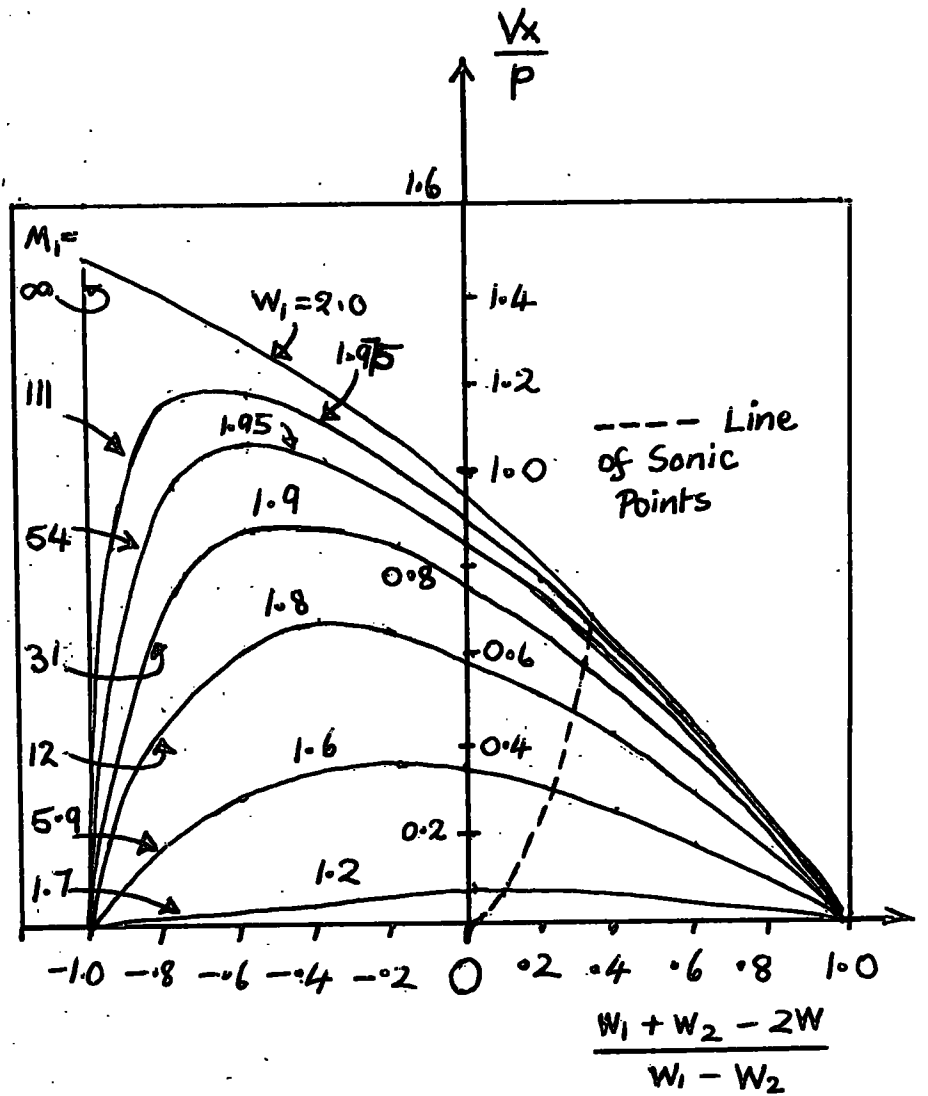$$V + p_1 + mu_1 = \frac{\gamma+1}{2\gamma} m\left(u + \frac{a^{*2}}{u}\right)$$

Fig. 5.6 $\frac{V_x}{P}$ as a function of velocity in a shock wave, according to the Navier-Stokes Equation.

(Liepmann et al., 1962)

and the left-hand side, and therefore V, thus has a

maximum at the sonic point u = a * so that

$$V_{max} = V^* = \frac{\eta^2}{\eta^2+1} \, ma^* \, \frac{(w_1-1)^2}{w_1}$$

Thus, we see from Fig. 5.6 that if the Navier-Stokes

equations do break down somewhere in a shock wave, they

will first of all do so in the low pressure (upstream)

portion of it.

(viii) The Bhatnagar - Gross - Krook Model's
Shock Structure. We saw in Section vi
above that the roots

of viscosity lay in the molecular nature of fluids. It

therefore seems that the Boltzmann equation offers most

scope in determining the laws governing viscous effects.

We saw in Section ii that the great drawback in the

continuum approach to gasdynamics is that this law is not

yet completely framed. For this reason, we must not be too

surprised if real shock structure differs from the Navier-

Stokes structure for strong shocks, but we expect the weak

structures to agree. The problem with all kinetic theory

approaches is, as we have seen, that the inter-molecular

forces are incompletely known, and this has resulted in

most cases with a disagreement with the continuum results

at values of M_1 not very close to unity. However, the B-G-K model

gives the required closeness of its predictions to those

of the Navier-Stokes method for weak shocks, and shows the deviation in the upstream part of the shock shown to be acceptable in the previous section.

One possible model for monatomic gas molecules is that they behave like Maxwellian molecules, dynamically speaking, when they are close to one another as in a molecular "collision" (i.e., they might have force fields that are such that the dominant force between molecules very close to each other is a repulsion proportional to r^{-5}, where r is the distance between the centres). However, at larger distances we are at liberty to assume that other forces become important as well, and to a first approximation all the forces cancel each other out. Therefore, the molecules behave in between collisions as if they are spheres of radius R, where R is the distance between molecular centres at which the Maxwellian force first becomes dominant. On collision, the deflection or scattering laws are determined by this force law. This is the motivation behind the B-G-K model. Thus, the collisions are determined by it as if the molecules were spheres of radius R and they are scattered as if they were Maxwellian molecules in this model. Nevertheless, it must be stressed that the model is just as arbitrarily selected as any other one.

Liepmann et al. (1962) apply the B-G-K model

to the problem of shock structure. They assume that the
fact that this gives a Prandtl number equal to 1, whereas
if it is $\frac{2}{3}$ in the case of a real monatomic gas, is not
important. The reason for this is that the difference
between the results given for Prandtl numbers of 1 and $\frac{2}{3}$
in the case of a weak shock according to the continuum
approach is small, as they show by a sample computation
(Fig. 5.7).

When introduced into the Boltzmann equation,
the B-G-K model gives

$$\frac{\partial \mathcal{f}}{\partial t} + \underline{v} \cdot \frac{\partial \mathcal{f}}{\partial \underline{x}} = An\left\{ n\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} exp\left[-\beta(\underline{v}-\underline{u})^{2}\right] - \mathcal{f} \right\} \quad (5.32)$$

where f, $\underline{v}$ and $\underline{x}$ are defined as above, n is the local
number density, β = 2T/R and A = (mean thermal speed)$/$
(n X mean free path). Liepmann et al. then compute
solutions to (5.32) for various values of M_1 by means of
an iterative process. This necessitated the use of a
rough solution to start the process. Liepmann et al.
found that the use of the discontinous solution
u(x) = u_1H(-x) + u_2H(x), where H(x) is the well-known
Heavidide unit function was unsatisfactory, because of
the infinite slope at x = 0, but the Navier-Stokes
solution proved to be adequate. The quantities used were
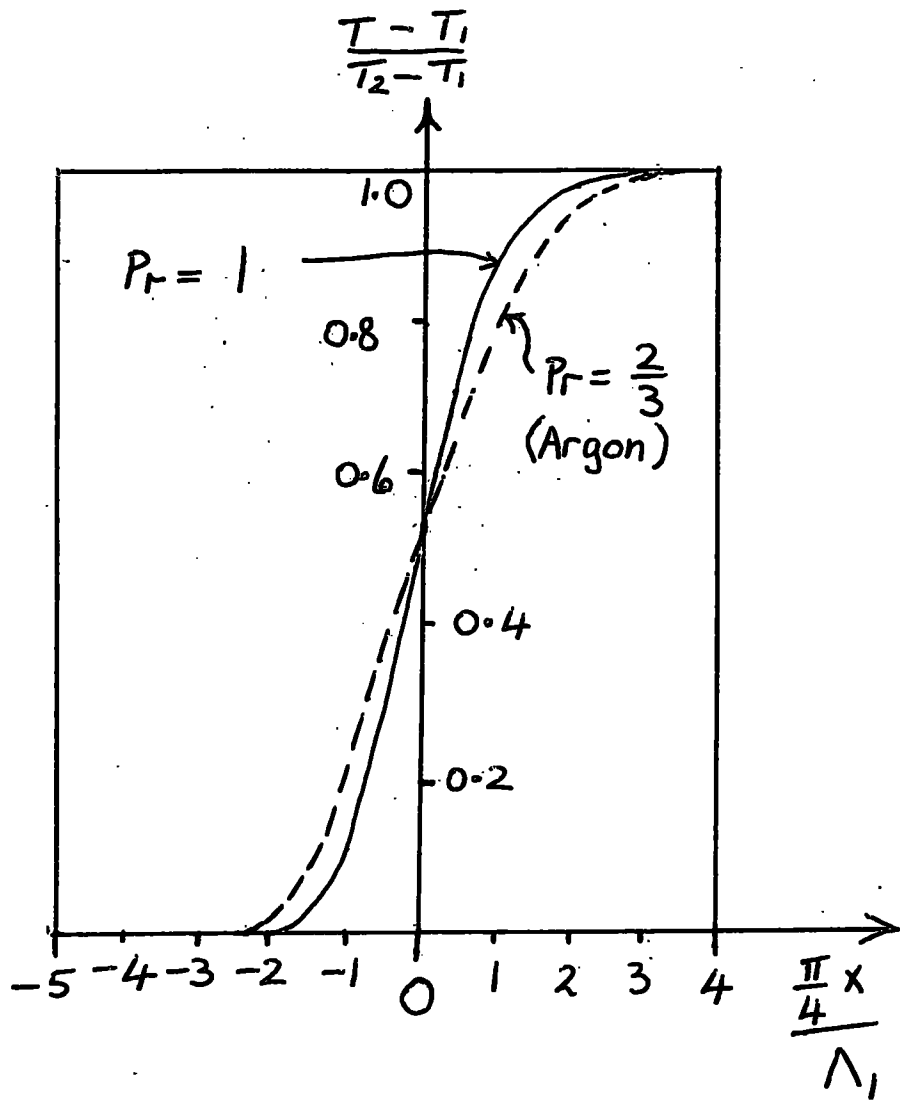the molecular weight and viscosity law of Argon but with

Fig. 5.7   Effect of Prandtl Number, Pr, upon Navier-Stokes shock wave profile.

(Liepmann et al, 1962)

a Prandtl number equal to unity, in the continuum solution, and where needed in the B-G-K problem. Their results are illustrated in Figs. 5.8, 5.9, 5.10 and 5.11. Two important facts immediately stand out. One is that we have a very good agreement between the two approaches to the problem of shock structure at low upstream Mach numbers, which is essential to a satisfactory kinetic theory, and the other is that the possible deviation from the Navier-Stokes solution at higher speeds in the part of the wave anticipated, and not elsewhere, is exhibited. Thus, the B-G-K model might be a suitable one for use in predicting gasdynamic flows but, until experimental evidence of the structure of stronger shocks becomes available, we can only say that it looks promising. The problem of the Prandtl number must also be borne in mind when assessing the value of this model. This alone must throw some suspicion on the extent of the usefulness of the B-G-K model in the prediction of shock structure.

(ix) <u>Non-linear viscosity in the Continuum Theory.</u> Since it is likely that viscosity is a non-linear function of velocity gradient, it is valuable to examine the integrability of the equations of the continuum approach with allowance made for non-linear viscosity. We follow Gilbarg and Paolucci (1953) in the particular case where it is
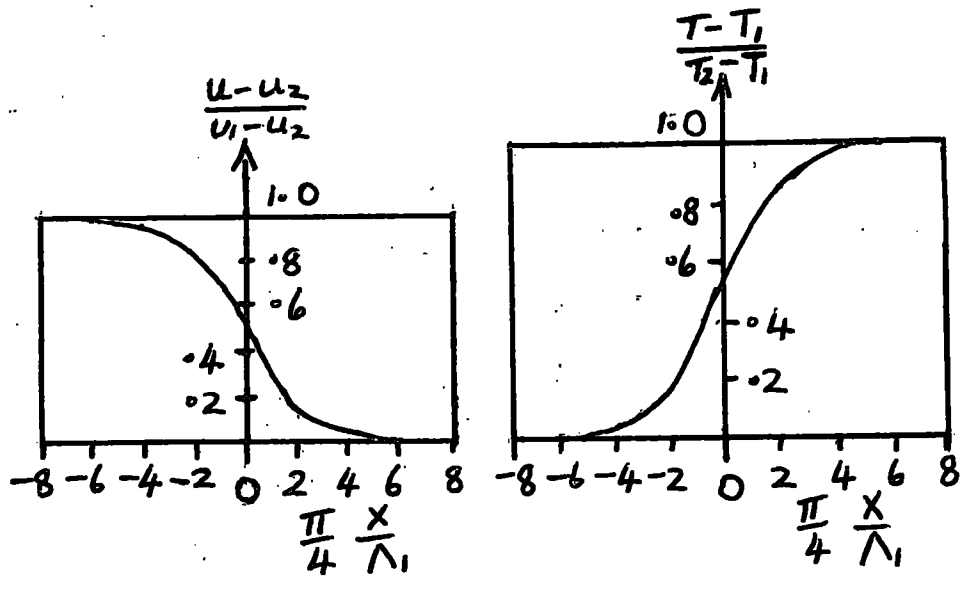
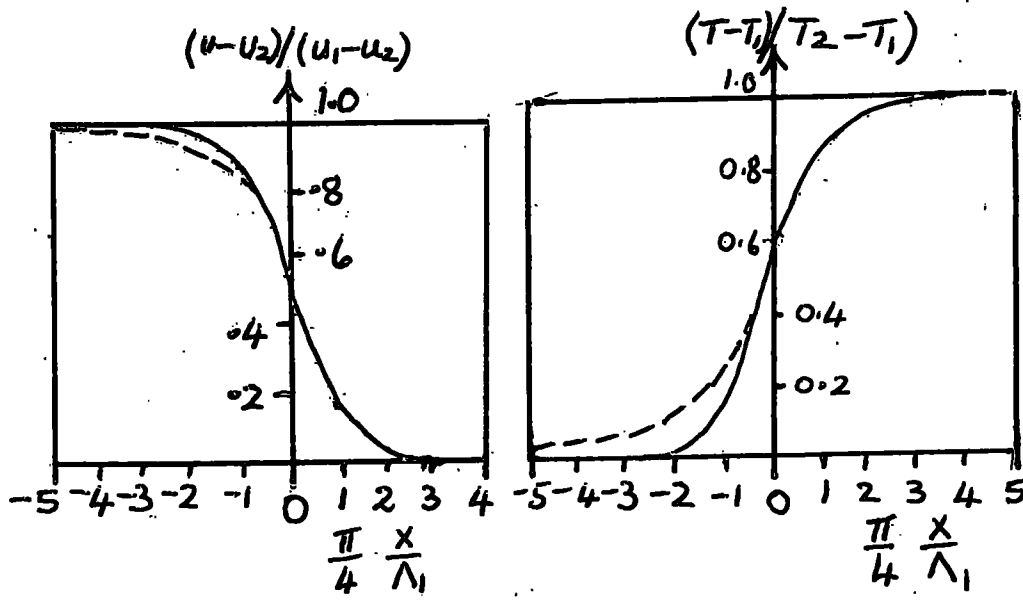Fig. 5.8 Shock Structure at $M_1 = 1.5$. N-S and B-G-K solutions are indistinguishable. (L. et al, 1962)

Fig. 5.9 Shock Structure at $M_1 = 3.0$
— N-S structure
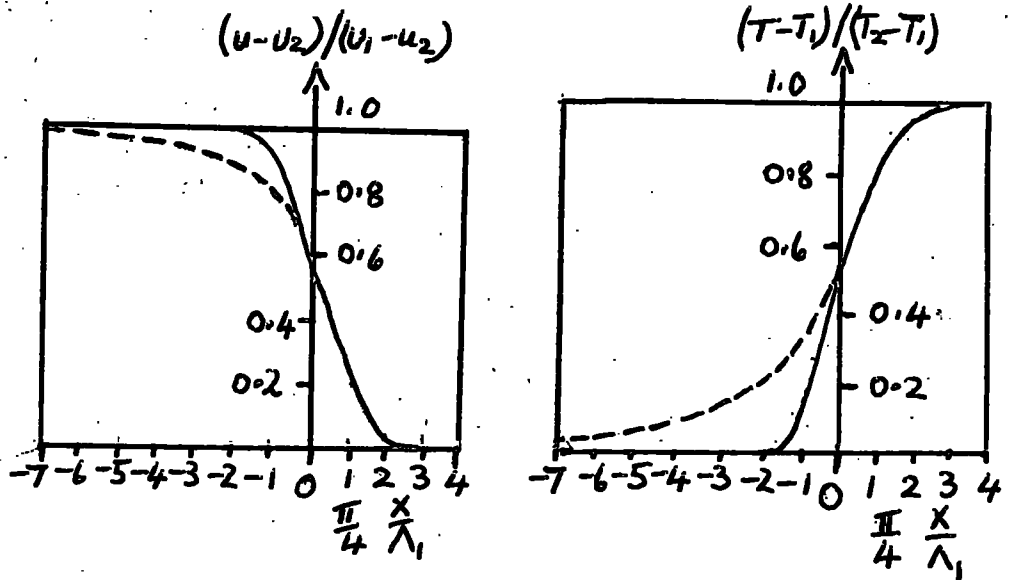--- B-G-K structure
(Liepmann et al., 1962)

Fig. 5.10 Shock Structure at $M_1 = 5.0$
— N-S Structure
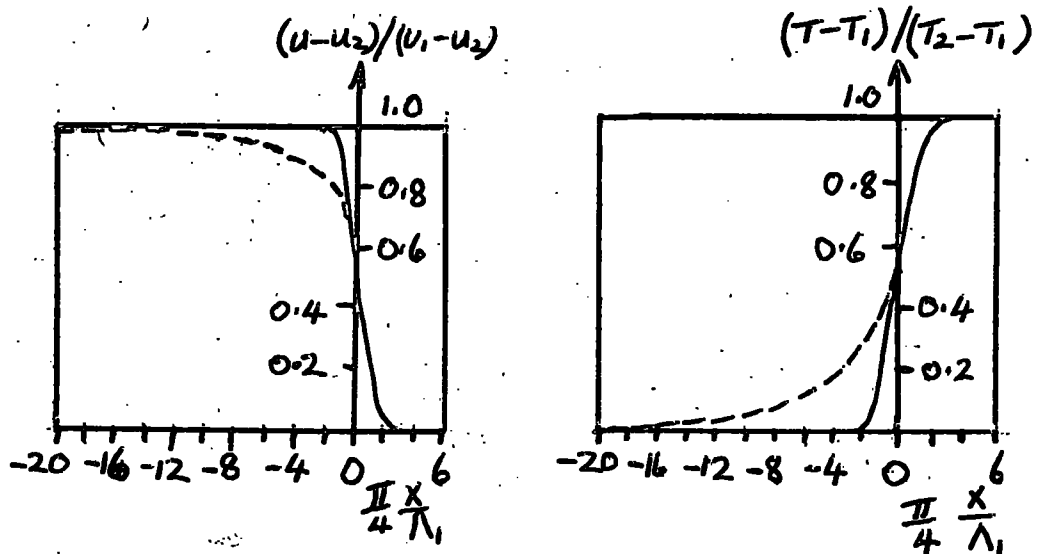- - - B-G-K Structure
(Liepmann et al., 1962)

Fig. 5.11 Shock Structure at $M_1 = 10.0$
— N-S structure
- - - B-G-K structure
(Liepmann et al., 1962)

assumed that the viscous stress depends only upon the velocity gradient. The argument readily extends to the case of dependence upon the temperature gradient.

In the non-linear case, V takes the form

$$V = g(u', \rho, T) \qquad (5.33)$$

where u' = du/dx and g is subject to the restrictions

$$(a) \qquad g(0, \rho, T) = 0$$
$$(b) \quad \frac{\partial g}{\partial u'}(0, \rho, T) = \lambda(\rho, T) > 0 \qquad \text{all } \rho \text{ and T.}$$

(a) indicates that so long as there is no velocity gradient, there are no viscous forces and (b) shows that for small velocity gradients, g behaves like $u'\lambda$, the usual definition of V. Further, we identify for a particular flow, $g(u', \rho, T) = g(u', m/u, T)$ with $g(u', u, T)$. Equations (5.19) now take the form

$$g(du/dx, u, T) = p + b(u - a)$$
$$K \, dT/dx = b\left\{ e - \tfrac{1}{2}(u - a)^2 - c \right\} \qquad (5.34)$$

Equations (5.19) now become

$$\emptyset(dw/dx, w, \Theta) = W + \Theta/w - 1 \equiv N(w, \Theta)$$
$$d\Theta/dx = \Theta - \delta\left\{ (1 - w)^2 + \alpha \right\} \equiv L(w, \Theta) \qquad (5.35)$$

where $\emptyset(w', w, \Theta)$ is the non-dimensional function in the w – Θ plane corresponding to $g(u', u, T)$. $\emptyset$ essentially has the same properties (a) and (b) namely

$$\phi(0,w,\theta) \equiv 0$$

$$\frac{\partial \phi}{\partial w'} (0, w, \theta) = \overline{\lambda} (w,\theta) > 0$$

In some neighbourhood of $N(w,\theta) = 0$,

$$w' = w' \left[N(w,\theta), w, \theta \right] \equiv \psi (w,\theta)$$

such that $\psi = 0$ on $N = 0$. We assume further that ψ is defined throughout Region ④ⓥ in Fig. 5.3. Now,

$$\psi(w,\theta) = 0 \Rightarrow w'(\psi,\theta) = 0 \Longrightarrow N(w,\theta) = 0$$

so that $\psi(w,\theta) = 0$ only on $N(w,\theta) = 0$ and so ψ is of constant sign throughout ④ⓥ , and is in fact negative, for

$$\frac{\partial \psi}{\partial \theta}\bigg|_{N=0} = - \frac{\phi_\theta - N_\theta}{\phi_{w'}}\bigg|_{w'=0, N=0} = \frac{N_\theta}{\phi_{w'}}\bigg|_{w'=0, N=0} > 0$$

If we can show the existence of an integral curve for

$$\left.\begin{aligned}
\frac{dw}{dx} &= \psi (w, \theta) \\[2mm]
\frac{d\theta}{dx} &= \frac{L(w,\theta)}{\overline{K}}
\end{aligned}\right\} \tag{5.36}$$

then, clearly, we have also shown the existence of one for system (5.35).

We use a proof similar to that used in the linear case. The characteristic equation of (5.36) is

$$
0 = \begin{vmatrix} \dfrac{\partial \psi}{\partial w} - \xi & \dfrac{\partial \psi}{\partial \theta} \\[3mm] \dfrac{\partial}{\partial w}\left(\dfrac{L}{K}\right) & \dfrac{\partial}{\partial \theta}\left(\dfrac{L}{K}\right) - \xi \end{vmatrix}_{Z=Z_i \ (i=1,}
$$

Now,

$$
\frac{\partial \psi}{\partial w}(Z_i) = - \frac{\phi_w(0, Z_i) - M_w(Z_i)}{\phi_{w'}(0, Z_i)} = \frac{M_w(Z_i)}{\bar{\lambda}(Z_i)}
$$

$$
(i=1,2,
$$

and

$$
\frac{\partial \psi}{\partial \theta}(Z_i) = \frac{M_\theta(Z_i)}{\bar{\lambda}(Z_i)} \qquad (i=1,2)
$$

Also,

$$
\frac{\partial}{\partial w}\left(\frac{L}{K}\right)_{z=z_i} = \frac{L_w(Z_i)}{\bar{K}(Z_i)} \qquad (i=1,2)
$$

and

$$
\frac{\partial}{\partial \theta}\left(\frac{L}{K}\right)_{z=z_i} = \frac{L_\theta(Z_i)}{\bar{K}(Z_i)} \qquad (i=1,2)
$$

Thus, we have precisely the same characteristic equation
as before and so, similarly,

(i) Z_1 and Z_2 are, respectively, node and saddle point
      of (5.36);

(ii) there is a unique solution trajectory that approaches
      Z_2 from within (IV) as x — $+\infty$. This curve
      cannot intersect the boundary of (IV) for the same
      reasons as before, and so approaches Z_1 as $x \rightarrow -\infty$.

Thus, as before, we have a method at our
disposal for integrating the equations numerically. However,
little work has been performed to determine shock profiles
in the non-linear case. The primary reason is that no
adequate experimental techniques have yet been devised
to measure the non-linearity of compression viscosity.
Indeed, it may well be that observation of shock structure
will be the means of measuring it. It does, in fact, seem
most likely that quantitative investigation of relaxation
phenomena, particularly, will benefit from a detailed
examination of shock structure in the laboratory once we
have solved the viscosity problem, including the role
played by these phenomena.

APPENDIX A. <u>Proof of the forms of the solution</u>
<u>trajectories near to the singular points</u>
<u>in the w - θ plane.</u>

At Z_i, L and N are both zero. So, at
$Z_i + \delta z$, where δz is small

$$L\left(Z_i + \delta Z\right) = \left.\frac{\partial L}{\partial \theta}\right|_{Z_i} \cdot \delta\theta \;+\; \left.\frac{\partial L}{\partial w}\right|_{Z_i} \cdot \delta w \quad (A.1)$$

$$N\left(Z_i + \delta Z\right) = \left.\frac{\partial N}{\partial \theta}\right|_{Z_i} \cdot \delta\theta \;+\; \left.\frac{\partial N}{\partial w}\right|_{Z_i} \cdot \delta w \quad (A.2)$$

Therefore, close to Z_i

$$\frac{dw}{dx} = \frac{N}{\bar{\lambda}} = A\,\delta\theta \;+\; B\,\delta w \qquad (A.3)$$

$$\frac{d\theta}{dx} = \frac{L}{\bar{K}} = C\,\delta\theta \;+\; D\,\delta w \qquad (A.4)$$

where $A = \left.\frac{\partial N}{\partial \theta}\right|_{Z_i}$, $B = \left.\frac{\partial N}{\partial w}\right|_{Z_i}$, $C = \left.\frac{\partial L}{\partial \theta}\right|_{Z_i}$ and $D = \left.\frac{\partial L}{\partial w}\right|_{Z_i}$.

We omit Z_i henceforth, it being understood. We translate
the axes parallel to themselves so that Z_i lays on the
origin. We call the new coordinates w' and θ' and henceforth
omit the dashes. Thus, in the neighbourhood of the origin,

$$\frac{dw}{d\theta} = \frac{A\theta + Bw}{C\theta + Dw} \qquad\qquad (A.5)$$

We introduce new variables ψ and ϕ such that

$$\psi = \alpha\theta + \beta w \quad \text{and} \quad \phi = \gamma\theta + \delta w, \qquad (A.6)$$
and

$$\left. \begin{array}{l} \frac{d\psi}{dx} = \alpha\left(C\theta + Dw\right) + \beta\left(A\theta + Bw\right) = \lambda\psi \\[4mm] \frac{d\phi}{dx} = \gamma\left(C\theta + Dw\right) + \delta\left(A\theta + Bw\right) = \mu\phi \end{array} \right\} \qquad (A.7)$$

Where $\alpha, \beta, \gamma, \delta, \lambda$ and μ are constants. For this to hold for an arbitrary starting point on a solution trajectory, which it must do, in the θ - w plane,

$$\alpha(C - \lambda) + A\beta = 0; \quad \alpha D + \beta(B - \lambda) = 0$$

We may obtain a non-trivial solution for α and β if and only if

$$\begin{vmatrix} A & B - \lambda \\[4mm] C - \lambda & D \end{vmatrix} = 0 \qquad (A.8)$$

Similarly, for γ and δ,

$$\begin{vmatrix} A & B - \mu \\[4mm] C - \mu & D \end{vmatrix} = 0 \qquad (A.9)$$

Our solution is reducible if the λ's are real and distinct. If λ_1 and λ_2 are the roots, with $\lambda_1 > \lambda_2$, then we may choose

$$\alpha = A \; ; \quad \beta = \lambda_1 - C \; ; \quad \gamma = A \; ; \quad \delta = \lambda_2 - C$$

<div align="right">(A.10)</div>

which gives

$$\frac{d\psi}{dx} = \lambda_1 \psi \qquad ; \qquad \frac{d\phi}{dx} = \lambda_2 \phi \qquad \text{(A.11)}$$

so that

$$\frac{d\phi}{d\psi} = \frac{\lambda_2}{\lambda_1} \frac{\phi}{\psi} \qquad \text{(A.12)}$$

λ_j are real and distinct if and only if $(B + C)^2 - 4(BC - AD) > 0$. The λ_j have the same sign when $(BC - AD) > 0$, and opposite signs when $(BC - AD) < 0$.

(A.12) gives :-

$$\phi = G \psi^{\lambda_2/\lambda_1} \qquad \text{(A.13)}$$

where G is any constant. Thus, when the λ_j have the same sign, Z_i is a node. When they have opposite signs, then Z_i is a saddle point. Both these remarks apply first of all in the $(\phi - \psi)$ plane. From the form of the transformation between the $(\phi - \psi)$ plane and the (θ, w) plane, (A.6), we deduce that saddle points go into saddle

points and nodes go into nodes, since α, β, γ and δ are not all zero.

This concludes the proof of that when the roots of the characteristic equation

$$\begin{vmatrix} \dfrac{1}{\lambda} \dfrac{\partial N}{\partial \theta} & \dfrac{1}{\lambda} \dfrac{\partial N}{\partial w} - \zeta \\[4mm] \dfrac{1}{K} \dfrac{\partial L}{\partial \theta} - \zeta & \dfrac{1}{K} \dfrac{\partial L}{\partial w} \end{vmatrix} = 0$$

are real and both of the same sign, then we have a node at the singular point and, when they have different signs, then we have a saddle point where the derivatives are evaluated at the singularity. We do not investigate the case of complex roots here.

Some of this proof was derived from notes taken at lectures delivered to M.Sc students at the University of Newcastle upon Tyne in February, 1964, by Dr. Mitchell.

APPENDIX B.   Slopes of Solution Trajectories through the Saddle Point.

We suppose that the slope of the solution curve through the saddle point may be written in the form, where we have transformed axes as in Appendix A,

$$\frac{d\theta}{dw} = \frac{-L_w}{L_\theta + \nu \bar{K}} \qquad (B.1)$$

which is always possible for $L_w = 0$ at Z_2 only in the trivial case where $Z_1 = Z_2$. (A.5) may be re-written as

$$\theta' = \frac{\bar{\lambda}}{\bar{K}} \cdot \frac{L_\theta \theta' + L_w}{N_\theta \theta' + N_w} \quad \text{where} \quad \theta' = \frac{d\theta}{dw}\bigg/_{Z_2} \qquad (B.2)$$

$$\text{evaluated along a solution trajectory}$$

and eliminating $d\theta/dw$ between (B.1) and (B.2) we obtain

$$\nu^2 + \left(\frac{L_\theta}{\bar{K}} + \frac{N_w}{\bar{\lambda}}\right)\nu + \frac{(L_\theta N_w - L_w N_\theta)}{\bar{\lambda} \; \bar{K}} \qquad (B.3)$$

which is the characteristic equation (5.23) of the system, with ν as the eigen-number. Thus the slope of the solution trajectory through Z_2 is that quoted in (5.24).

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