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# Reflecting Boundaries and Massless Factorized Scattering in Two Dimensions

by

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# ABSTRACT


**Reflecting Boundaries and Massless Factorized Scattering in Two Dimensions**

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This thesis is concerned with two-dimensional models that are integrable in the presence of a boundary and whose spectrum in the bulk is constituted of massless particles. Although there is already a vast literature on the subject (e.g. Kondo and Callan-Rubakov models), the common minimal denominator in all these situations is the fact that the bulk theory is conformally invariant and it is the boundary that is responsible for the broken scale invariance. Here, our purpose is to consider the alternative situation, where the boundary respects the conformal invariance of the theory and the renormalization group trajectory is controlled by a bulk perturbation. The model in question is the principal chiral model at level $k = 1$. We propose the set of permissible boundary conditions suggested by the symmetries of the problem and compute the corresponding minimal reflection matrices. For one of the boundary conditions we compute the boundary ground state energy and the boundary entropy using the technique of boundary thermodynamic Bethe ansatz. In the infrared limit our results are shown to be in complete agreement with the predictions of the boundary conformal field theory approach. Finally, we consider the classical supersymmetric Liouville theory on the half-line and compute the boundary conditions compatible with the superconformal invariance. We construct an infinite set of commuting integrals of motion using Lax-pair techniques and discuss some aspects of the quantum theory as well as its relation to the super Korteweg-de Vries equation.

# Declaration

This thesis is the result of research carried out between May 1995 and February 1998. The work presented in this thesis has not been submitted in fulfilment of any other degree or professional qualification.

No claim of originality is made for chapters $1-7$. Chapter 8 arose from useful discussions with Dr. Patrick Dorey. The content of chapter 9 resulted in the publication *The Super-Liouville Equation on the Half-Line*, Phys. Lett. B405 (1997) 271. I have benefited greatly from the advise of Drs. Anne Taormina, Peter Bowcock, Patrick Dorey and Prof. Edward Corrigan.

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*To my brother Pedro*

# Contents

# Chapter 1

# Introduction

Quantum Field Theory (QFT) provides the framework to unify such disparate concepts as quantum mechanics and special relativity. The principles of relativity allow for particle production, according to Einstein's celebrated formula, $E = mc^2$, and this prevents the description in terms of elastic collisions between isolated particles. Rather, a co-operative variable - a field - is introduced, representing all the particles of a given type. The action has to be relativistic and respect other known symmetries of the theory. Quantum mechanics then provides a set of rules for computing the allowed values of a given physical quantity (observable) and the probabilities for measuring them. The object of primary importance is the S-matrix. Formally, it can be seen as a map between the sets of allowed initial and final states of the system. The matrix element $S_{if}$ corresponds to the probability amplitude for the system to evolve from the initial state $i$ to the final state $f$ after the physical process has occurred.

Since the discovery of relativity and quantum mechanics, the main driving force in the quest to understand the laws of nature has been the introduction of more complex and richer symmetry structures. Space-time is believed to be invariant under the Poincaré group and generally covariant in the presence of gravitational fields. Moreover, we can supplement these with other symmetries which may or may not be linked to the space-time medium. It has proven fruitful to generalize quantum electrodynamics (QED) by the introduction of the principle of gauge invariance for non-abelian groups (electroweak model, quantum chromodynamics). The "discovery" of the quark model permitted physicists to classify hadrons (strongly interacting particles) into multiplets of equal mass - a sort of periodic table at the subatomic level. More recently, physicists have tried to amalgamate fermions and bosons into a single description by the ingenious principle of *supersymmetry*. For each fermion there is a corresponding bosonic superpartner with equal mass. Although to this date no superpartner of the known particles has been detected experimentally, most physicists are reluctant to the prospect of abandoning the idea of supersymmetry. This is because supersymmetric field theories exhibit many desirable features [1]. Besides their intrinsic beauty in terms of rich mathematical structures, supersymmetric theories obey a number of nonrenormalization theorems that make the analysis of Feynman diagrams much more attractive. Equally they seem to solve the hierarchy problem of GUT theories [2] and eliminate undesirable particles such as tachyons from the spectra of string theories [1], [6].

The recent developments in quantum chromodynamics (QCD) and more generally the attempts to go beyond the standard model (e.g. string theory) have shown compelling arguments for a nonperturbative approach to quantum field theory. Indeed, to study quark confinement in the asymptotically free QCD, one has to work in the strong coupling regime where perturbative methods fail. Similarly, if string theory is to be a theory of quantum gravity, it has to resolve distances of the order of the Planck scale, where perturbation theory can hardly make any sense. These considerations have led to the search for solutions of the field theories, many of which are topologic in nature - e.g. instantons, solitons, vortices, monopoles.

Our experience with classical mechanics tells us that whenever we have a conservation law (e.g. energy, momentum, angular momentum) we can integrate out a degree of freedom, thus effectively reducing their number. If the number of independent symmetries equals the number of degrees of freedom then we can in principle completely solve the problem. Quantum mechanically, the above picture still holds. We can obtain all the information about the system by simultaneously diagonalizing all the observable mutually commuting integrals of motion. A system with these properties is said to be "*integrable*".

In field theory the number of degrees of freedom diverges in view of the fact that every field is defined in each space point. An integrable field theory would therefore be one with an infinite number of conservation laws or equivalently - by Noether's theorem - with some infinite dimensional symmetry. However this contradicts a remarkable theorem by Coleman and Mandula [26]. This theorem states that provided the S matrix satisfies a number of technical assumptions (unitarity, analyticity) the symmetries of the theory can be shown to be locally isomorphic to the direct product of the Poincaré group and a *finite* number of internal symmetries. An integrable quantum field theory would necessarily be free (i.e. with a trivial S matrix). Quantum field theories in (1+1)-dimensions are the exception. Indeed many two-dimensional theories are known where the existence of an infinite number of involutive integrals of motion is compatible with a non trivial scattering [25]. What is it that makes two dimensions so special? To start with, in one spatial dimension, any two particles with different velocities are bound to meet at some point in time. Peculiarities of two-dimensional kinematics then show that there is a permissible type of scattering that is elastic and factorizable. The precise meaning of these statements will be made clear in subsequent chapters. Mathematically, there are known cases of symmetry groups that are finite dimensional in a generic dimension but become infinite dimensional in two dimensions. One of the main topics of this thesis - *conformal invariance* - provides such an example [12]. Using the striking similarity between the generating functional in a QFT and the partition function of a statistical system, one can establish a correspondence between (d+1)-dimensional euclidean statistical models and (d,1)-dimensional quantum field theories [10]. This correspondence only makes sense in the neighbourhood of a second order phase transition of the statistical system. Near such a phase transition the fluctuations of the fields are correlated over long distances and occur equally on all scales [5]. The system loses memory of the details of the underlying microscopic structure (e.g. lattice) and may effectively be described by a continuous field theory. The system is then said to be scale invariant, i.e. invariant under the global rescalings of the form:

$$x \to \rho x, \tag{1.1}$$

where $\rho$ is some real constant. If in addition to being scale invariant the system is also homogeneous and isotropic then it becomes conformally invariant, i.e. invariant under dilatations of the form (1.1) where now $\rho$ may depend upon the position. There are good arguments to believe that each universality class in two dimensional statistical mechanics can be described by a conformal field theory. By universality class one understands the identical behaviour (e.g. critical exponents) of *a priori* distinct models near a second order phase transition [18].

The prototype for such a theory is the two dimensional Ising model, which has been exactly solved in refs.[11], [9]. The Hamiltonian for this system is:

$$H = -\sum_{n=1}^{n_t}\sum_{m=1}^{m_s}\left(\beta_s s_{n,m}s_{n,m+1} + \beta_t s_{n,m}s_{n+1,m}\right), \tag{1.2}$$

where the n- and m-indices label the rows and columns of a two-dimensional lattice. The spin variables $s_{n,m}$ take the values $\pm 1$ and $\beta_s$, $\beta_t$ are coupling constants. We are considering periodic boundary conditions in the horizontal direction. We notice immediately that (1.2) is invariant under the simultaneous reversal of all spins. We will call this a $\mathcal{Z}_2$ symmetry. This Hamiltonian models the ferromagnetic transition between a high-temperature disordered phase and a low-temperature ordered one. The symmetry breaking occurs at the *Curie temperature* $T_C$ also called the critical point. If we denote the configuration of the spins in the nth row by $S_n = \{s_{n,1}, s_{n,2}, \cdots, s_{n,m_s}\}$, the total energy of the configuration $\{S_1, \cdots, S_{n_t}\}$ is given by:

$$E\{S_1, \cdots, S_{n_t}\} = \sum_{n=1}^{n_t}\left[\epsilon(S_n, S_{n+1}) + \epsilon(S_n)\right], \tag{1.3}$$

where

$$\epsilon(S_n) = -\sum_{m=1}^{m_s}\beta_s s_{n,m}s_{n,m+1}, \tag{1.4}$$

represents the energy of the interaction amongst the spins in the nth row and

$$\epsilon(S_n, S_{n+1}) = -\sum_{m=1}^{m_s}\beta_t s_{n,m}s_{n+1,m}, \tag{1.5}$$

is the energy between the spins of the nth row and their neighbours on the $(n+1)$th row. The partition function is thus:

$$\mathcal{Z} = \sum_{\{S_n\}} e^{-E\{S_1,\cdots,S_{n_t}\}}. \tag{1.6}$$

We also introduce a matrix $T$ called the *transfer matrix* with the following entries:

$$< S_n|T|S_{n+1} >= exp\left\{-\epsilon(S_n, S_{n+1}) - \epsilon(S_n)\right\}. \tag{1.7}$$

This allows us to rewrite the partition function in the form:

$$\mathcal{Z} = \sum_{S_1}\cdots\sum_{S_{n_t}} < S_1|T|S_2 > \cdots < S_{n_t}|T|S_1 >= Tr\left(T^{n_t}\right), \tag{1.8}$$

where we used the periodicity condition, $S_{n_t+1} = S_1$. As we approach the phase transition where the continuum description becomes adequate, we can start by considering the limit when the lattice spacing $a_t$ in the '*time*'-direction (vertical) becomes infinitesimal and simultaneously $n_t \to \infty$ (the ratio of the two remaining constant), [14], [10]. We will then be able to interpret the transfer matrix as an evolution operator $e^{-\tau H}$ between two consecutive rows, infinitely close to each other ($\tau \to 0$). $H$ is interpreted as a Hamiltonian and $\tau = e^{-2\beta_t}$ as a time parameter. One can show that:

$$H = \sum_{m=1}^{m_s} \left(\sigma_m^x - \lambda \sigma_m^z \sigma_{m+1}^z\right), \tag{1.9}$$

where $\sigma^i$ are Pauli's spin matrices and $\sigma_{m_s+1}^i = \sigma_1^i$. $\lambda = \beta_s/\tau$ is a coupling constant. This Hamiltonian describes a one dimensional Ising model interacting with a transverse magnetic field. Let us now consider the new set of operators:

$$\begin{cases} \mu_j^x = \sigma_j^z \sigma_{j+1}^z \\ \mu_j^z = \sigma_1^x \sigma_2^x \cdots \sigma_j^x . \end{cases} \quad (j = 1, \cdots, m_s) \tag{1.10}$$

One can show that these new operators satisfy the same algebra as the Pauli matrices. If we rewrite the Hamiltonian (1.9) in terms of these, we get:

$$H = \sum_{m=1}^{m_s} \left(\mu_m^z \mu_{m+1}^z - \lambda \mu_m^x\right) = -\lambda \sum_{m=1}^{m_s} \left(\mu_m^x - \frac{1}{\lambda} \mu_m^z \mu_{m+1}^z\right). \tag{1.11}$$

In particular, we obtain for the energy of a given configuration :

$$E^\sigma(\lambda) = -\lambda E^\mu \left(\frac{1}{\lambda}\right). \tag{1.12}$$

The transformation (1.10) is called a *duality transformation* because it relates the high temperature ($\lambda << 1$) phase with the low temperature ($\lambda >> 1$) phase of the system. The two spectra (1.12) become equal at $\lambda = 1$, the *self-dual* point [10].

In the continuum limit, when the lattice spacing $a_s$ in the '*space*'-direction (horizontal) becomes infinitesimal and $m_s \to \infty$ ($m_s/a_s = const$), the resulting continuum field theory describing the system (after a Wick rotation) is:

$$H = \frac{1}{2} \int dx \left\{ \lambda \psi^\dagger(x) \left(-i\gamma^5 \frac{d}{dx} \psi(x)\right) - m\psi^\dagger(x)\gamma^0\psi(x) \right\}, \tag{1.13}$$

where $\gamma^\mu$ ($\mu = 0, 1$) are the Dirac matrices in the Majorana representation:

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.14}$$

The operator (1.13) is the Hamiltonian for a free massive Majorana spinor. It is striking that an *interacting* theory such as the Ising model (1.2) is described by a *free* theory near the phase transition. This is misleading because the theory at the critical point is actually strongly interacting (with diverging correlation length) as can be checked from the anomalous (non canonical) dimensions defined by the two-point Green's functions for basic operators such as the spin density. The continuum field theory is called a *free field realization of the conformal symmetry.* We shall come back to this later.

The mass $m$ in (1.13) can be expressed in terms of the coupling constant as:

$$m = \lim_{a_s \to 0} \left( \frac{1-\lambda}{a_s} \right). \tag{1.15}$$

We notice that the theory becomes massless at the self-dual point. Intuitively, this should correspond to the critical point, otherwise the scale invariance would be broken by this mass scale. The theory (1.13) with $m \neq 0$ is therefore interpreted as a thermal ($\lambda = T/T_C$) perturbation of the conformal theory induced by the energy density operator $\bar{\psi}\psi$. The resulting theory being free is obviously integrable. It is a first example of a wide range of massive integrable theories that arise from perturbing conformal field theories with appropriate relevant scalar fields in their spectra [38]. From the renormalization group (RG) point of view conformal field theories correspond to fixed points of the RG iterations [25]. An interesting topic in relativistic field theory (RFT) is the complete classification of all possible trajectories flowing from these critical points [83]. Similar behaviour of the RG for two different systems near a fixed point implies that the two systems belong to the same universality class [5]. Each distinct trajectory corresponds to a different RFT. In two dimensions there are three typical patterns. The trajectory may stay at the fixed point forever, being described by the corresponding conformal field theory at all distances. Alternatively, as we discussed earlier, it can be drawn out in some relevant direction and described at large distances by some massive RFT [38]. Finally, the third situation, consists of the trajectory terminating at another fixed point [83], [57]. Both at large and short distances these theories are characterized by conformal field theories. From the statistical mechanics point of view they can be regarded as crossovers between different universality classes of critical phenomena.

A good starting point would be a complete classification of all conformal theories. This would be equally interesting in the context of string theory. One of these conformal theories is the vacuum of the string theory after compactifying the redundant space-time dimensions [7]. A subclass of these theories called *rational conformal field theories* is believed to have been completely classified using a technique known as *coset construction*

[55]. Rational theories have only a finite number of operators with which they can be perturbed in the fashion described above, thus rendering the enumeration of all trajectories flowing from it a more tractable task.

In real applications one usually has to deal with bounded systems having some interface with the external world. *Boundary quantum models* have a wide range of applications. Some famous examples are the monopole-catalysed baryon decay [22], the Kondo model [94], dissipative quantum mechanics [23] and quantum Hall liquids with constriction [24]. They are also interesting in the context of open string theories [66], [64]. From a theoretical point of view one has to construct boundary interactions that preserve part of the integrability properties of the bulk theory. Cardy made some important contributions to the study of boundary conformal theories [60], [61]. More recently, a work by Ghoshal and Zamolodchikov [68] has led to many new developments.

The main purpose of this thesis is to study massless theories in the presence of reflecting boundaries. Massless scattering has a number of conceptual difficulties related to the very notion of asymptotic particle moving at the speed of light [93]. Notwithstanding this, the successes of this approach have been outstanding, especially in the study of the Kondo effect [96] and of the massless flows between distinct universality classes [93].

The layout of this thesis is as follows. In the first part (chapters 2-7), I present all the necessary background material on integrable and boundary integrable models as well as some useful results concerning the Kondo problem and the principal chiral model. I have tried whenever possible to illustrate the results with the simplest example available - the Ising model. No claim of originality is made for this part. Also it should not be expected to be a thorough and comprehensive review of each topic. Rather, I have tried to put emphasis only on those aspects that I find relevant to understand the last part of this dissertation. Chapters 8 and 9 encompass the research that I carried out. In chapter 8 I study the principal chiral model in the presence of a boundary with scale invariant boundary conditions. The model itself is massless but not conformally invariant. The technique of thermodynamic Bethe ansatz for this problem is carried out in great detail. In chapter 9, I consider the $N = 1$ supersymmetric extension of the Liouville theory on the half-line. This theory is superconformally invariant and therefore massless. I will determine the boundary conditions compatible with integrability at the classical level and discuss some aspects of the quantum theory and its relation to the super Korteweg-de Vries equation. Finally, in chapter 10 I present the conclusions.

# Chapter 2

# Conformal Field Theory

*In this chapter I give an overview of some aspects of conformal field theories in two dimensions. As we shall see both this and the factorized scattering approach to integrable theories in the next chapter do not require a Lagrangian formulation. Rather the data are encoded into the particular representation of the conformal algebra.*

## 2.1 Conformal field theory in d dimensions

The universality classes of critical systems near the second order phase transition are classified by a set of basic operators $\{\phi_i\}$ [18] having anomalous dimensions $\{\Delta_i\}$[1]. This is the spectrum of the theory. Under scaling transformations of the space (1.1) the basic operators transform like:

$$\phi_i(x) \to \rho^{\Delta_i} \phi_i(\rho x). \tag{2.1}$$

By *anomalous dimension* we mean that it is different from canonical dimensions for free noninteracting fields. The general situation for the critical phenomena is that the basic fields (operators) will be strongly interacting. And this is reflected in the fact that their dimensions, defined by two-point functions will be anomalous. In practical terms, this means that near a second order phase transition no scale is preferred and if we perform a transformation (1.1) then the correlation functions will remain unchanged. For an isotropic and homogeneous model, we would also expect translational and rotational invariances to be symmetries of the system. According to Belavin et al.[12], these two assumptions together with the global scaling symmetry are sufficient to ensure that the system be conformally invariant.

[1] which in turn are connected to the set of critical exponents of the theory.

Let us now try to describe the above concepts in a more systematic way ([13], [14]). Under a generalized coordinate transformation, $x^\mu \to x'^\mu(x)$ ($\mu = 1, \cdots, d$) the metric transforms according to the tensorial rule:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x). \tag{2.2}$$

A conformal transformation corresponds to performing local rescalings of distances, [13], $g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$, where $\Omega(x)$ is an arbitrary scalar function. For an infinitesimal transformation,

$$\begin{cases} x'^\mu & = x^\mu + \zeta^\mu, \\ \Omega(x) & = 1 - 2\sigma(x), \end{cases} \tag{2.3}$$

we get the set of Killing-Cartan equations:

$$\partial_\nu \zeta_\mu + \partial_\mu \zeta_\nu = 2\sigma(x)\eta_{\mu\nu}, \tag{2.4}$$

where we assume the constant metric (Euclidean or Minkowski space) $g_{\mu\nu}(x) = \eta_{\mu\nu}$. In arbitrary dimension $d$, we have four types of solutions [14]:

(i) Translations: $x'^\mu = x^\mu + a^\mu$, where $a^\mu$ are constants. This symmetry is generated by the momentum operator, $P^\lambda = i\partial^\lambda$.

(ii) Lorentz transformations (Minkowski) or rotations (Euclidean): $x'^\mu = \Lambda^\mu_\nu x^\nu$, where $\Lambda^\mu_\nu$ satisfies $\eta_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$. These are generated by the angular momentum tensor: $M_{\rho\sigma} = i(x_\rho \partial_\sigma - x_\sigma \partial_\rho)$.

(iii) Dilatations: $x'^\mu = \rho x^\mu$, where $\rho$ is a constant. Generator : $D = i x_\nu \partial^\nu$.

(iv) Special conformal transformations: $x'^\mu = \frac{x^\mu - c^\mu x^2}{1 - 2c\cdot x + c^2 x^2}$, where $c^\mu$ are constants and $c \cdot x = c_\mu x^\mu$. Generator: $K^\rho = i(2x^\rho x_\nu \partial^\nu - x^2 \partial^\rho)$.

If we compute the commutation relations for the generators of these transformations, we realize that they form an algebra. In particular, we can show that [14]:

$$\left[D, P^2\right] = -2iP^2. \tag{2.5}$$

The elements $M^{\mu\nu}$, $P^\rho$ constitute a subalgebra which is associated with the group of Poincaré transformations.

From eq.(2.5) and the Baker-Hausdorff-Campbell relation we have:

$$\begin{aligned} e^{i\rho D} P^2 e^{-i\rho D} &= P^2 + i\rho\left[D, P^2\right] + \tfrac{1}{2!}(i\rho)^2\left[D, \left[D, P^2\right]\right] + \cdots = \\ &= \left[\textstyle\sum_{n=0}^{\infty} \frac{(2\rho)^n}{n!}\right] P^2 = e^{2\rho} P^2, \end{aligned} \tag{2.6}$$

where $\rho$ is an arbitrary (nonzero) constant. From the above formula we see that $e^{i\rho D}P^2e^{-i\rho D}$ and $P^2$ cannot have identical spectra unless all the eigenvalues of $P^2$ vanish. This implies that only massless theories can be invariant under the conformal group. Because of (1.15) and of the fact that the system at the critical point is conformally invariant, we conclude that the self-dual point $\lambda = 1$ of the Ising model corresponds to the critical point in agreement with what we said before.

Conformal invariance imposes orthogonality of the 2-point function for basic operators [19], in the sense that:

$$< \phi_i(x)\phi_j(0) > \propto \frac{\delta_{\Delta_i,\Delta_j}}{|x|^{2\Delta_i}}, \tag{2.7}$$

and it also fixes the 3-point function:

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) >= \frac{const.}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}, \tag{2.8}$$

where $x_{ij} = x_i - x_j$. The higher-point functions in arbitrary dimensions are not fixed but will obey certain constraints.

Before we proceed, we have to introduce an additional operator, the stress tensor $T_{\mu\nu}$, which generates the conformal transformations. If the theory is translationally invariant, then $T^{\mu\nu}$ is conserved. Rotational invariance implies that $T_{\mu\nu}$ be symmetric. Finally invariance under dilatations requires a traceless stress tensor. These properties can be summarized in the following set of equations:

$$\partial_\mu T^{\mu\nu} = 0, \qquad T_{\mu\nu} = T_{\nu\mu}, \qquad \eta^{\mu\nu}T_{\mu\nu} = 0. \tag{2.9}$$

As we discussed before, these conditions are necessary and sufficient for conformal invariance.

## 2.2 Conformal invariance in two dimensions

In arbitrary dimension the conformal group is finite and the best we can hope for is a finite number of constraints to place on the correlation functions. In two dimensions, however, the situation changes drastically. This is because the conformal group becomes infinite dimensional [12], as can be readily verified from the Killing-Cartan equations (2.4):

$$\begin{cases} \partial_{\bar{z}}\zeta(z,\bar{z}) = 0, \\ \partial_z\bar{\zeta}(z,\bar{z}) = 0, \end{cases} \tag{2.10}$$

in terms of[2] $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ and $\zeta = \zeta_1 + i\zeta_2$, $\bar{\zeta} = \zeta_1 - i\zeta_2$. We also get an extra bonus: our symmetry seems to decouple into two identical sectors: analytic ($z$-dependence) and antianalytic ($\bar{z}$-dependence), which can be treated independently. At the end of the day, when we have computed all the physical quantities we are interested in, we can go back to the '*real*' surface: $\bar{z} = z^*$.

The infinitesimal conformal transformation is then:

$$\begin{cases} z \to z + \zeta(z), \\ \bar{z} \to \bar{z} + \bar{\zeta}(\bar{z}). \end{cases} \tag{2.11}$$

By performing a succession of infinitesimal transformations we obtain the finite transformation:

$$\begin{cases} z \to f(z), \\ \bar{z} \to \bar{f}(\bar{z}). \end{cases} \tag{2.12}$$

The conformal group in two dimensions is thus the direct product:

$$G = \Gamma \otimes \bar{\Gamma}, \tag{2.13}$$

of analytic transformations $\Gamma$ of the variable $z$ and antianalytic transformations $\bar{\Gamma}$ of the variable $\bar{z}$. If we go back to the infinitesimal transformations (2.11), because of the analyticity conditions (2.10) we can perform a Laurent expansion:

$$\zeta(z) = \sum_{n=-\infty}^{+\infty} \zeta_n z^{n+1}, \tag{2.14}$$

and we get immediately a differential representation for the generators of the Lie algebra associated with the group $\Gamma$ [14]:

$$l_n = -z^{n+1}\frac{\partial}{\partial z}, \qquad n = 0, \pm 1, \pm 2, \cdots \tag{2.15}$$

They satisfy the commutation relations,

$$[l_n, l_m] = (n - m) l_{n+m}. \tag{2.16}$$

This is called a '*classical or centreless Virasoro algebra*'. Because $\Gamma$ and $\bar{\Gamma}$ are independent, we have $[l_n, \bar{l}_m] = 0$. Also from (2.9), we conclude that the stress tensor in two dimensions for a conformally invariant theory has two nonvanishing components:

$$\begin{cases} T = T_{zz} = \frac{1}{2}(T_{11} - iT_{12}), \\ \bar{T} = T_{\bar{z}\bar{z}} = \frac{1}{2}(T_{11} + iT_{12}), \end{cases} \tag{2.17}$$

[2] At this stage we assume that we are in Euclidean space.

that satisfy the conservation law:

$$\partial_{\bar{z}} T = \partial_z \bar{T} = 0. \tag{2.18}$$

Notice that the Virasoro algebra (2.16) contains the $sl(2, C)$ subalgebra:

$$[l_1, l_{-1}] = 2l_0, \qquad [l_0, l_{\pm 1}] = \mp l_{\pm 1}. \tag{2.19}$$

The corresponding conformal transformation is given by:

$$z' = \frac{az+b}{cz+d}, \qquad \text{with } ad - bc = 1. \tag{2.20}$$

Of course there is a similar subalgebra in the antianalytic sector. They consist of the conformal transformations that exist in arbitrary dimension. In fact,

$$\begin{cases} l_{-1}, \bar{l}_{-1} & \text{generate translations,} \\ l_0 + \bar{l}_0 & \text{generates dilatations,} \\ i(l_0 - \bar{l}_0) & \text{generates rotations,} \\ l_1, \bar{l}_1 & \text{generate special conformal transformations.} \end{cases} \tag{2.21}$$

These are called *Möbius* or *projection* transformations[3],

$$PSL(2, C) \approx SL(2, C)/\mathcal{Z}_2. \tag{2.22}$$

The basic fields of the theory (also called *primary operators*) are field operators $\left\{\phi_{h,\bar{h}}(z, \bar{z})\right\}$ which transform under (2.12) as [13]:

$$\phi_{h,\bar{h}}(z, \bar{z}) \to (f'(z))^h \left(\bar{f}'(\bar{z})\right)^{\bar{h}} \phi_{h,\bar{h}}\left(f(z), \bar{f}(\bar{z})\right). \tag{2.23}$$

This expression can be seen as a generalization of the definition of a tensor with $h$ $z$-indices and $\bar{h}$ $\bar{z}$-indices. $h$, $\bar{h}$ are called the conformal weights (not necessarily integers) of $\phi_{h,\bar{h}}$. $\Delta = h + \bar{h}$ is the scaling dimension that describes the behaviour under global scale transformations. Similarly, the spin $s = h - \bar{h}$ describes the behaviour under rotations. Any operator that transforms like (2.23) under the subgroup (2.22) of Möbius transformations is called *quasi-primary*. Clearly every primary operator is quasi-primary but the converse is not true.

From (2.23), we have for the infinitesimal transformation (2.11):

$$\delta_{\zeta,\bar{\zeta}} \phi_{h,\bar{h}} = \left[\zeta(z)\partial_z + \bar{\zeta}(\bar{z})\partial_{\bar{z}} + h\partial_z \zeta(z) + \bar{h}\partial_{\bar{z}}\bar{\zeta}(\bar{z})\right] \phi_{h,\bar{h}}(z, \bar{z}). \tag{2.24}$$

[3]The quotient group $SL(2, C)/\mathcal{Z}_2$ arises because the set of transformations (2.20) remain unchanged under $a \to -a$, $b \to -b$, $c \to -c$, $d \to -d$.

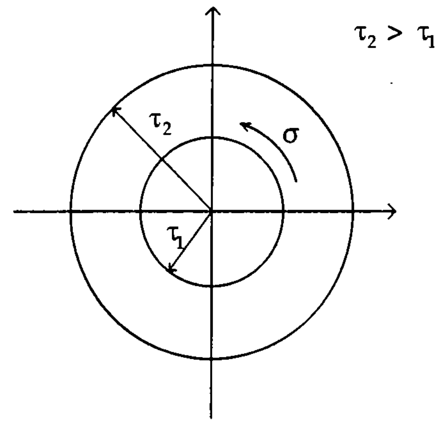


Figure 2.1: Radial Quantization

Before we proceed, a few considerations are in order. For each conserved quantity like the stress tensor there is, from Noether's theorem, an associated conserved charge $Q$. Explicitly, if $\partial_\mu j^\mu = 0$, then $Q = \int dx_1 j^2$ satisfies $\partial_{x_2} Q = 0$. From the fact that we are working on the complex plane, we can take full advantage of contour integrals and Cauchy's theorem if we can find a quantization scheme such that integrals like the previous one ($Q$) can be performed over closed contours. This is called the '*radial quantization*', [13]. We define $\tau$ and $\sigma$ as the 'time' and 'space' variables according to (see fig. 2.1):

$$z = e^{\tau + i\sigma}, \qquad \bar{z} = e^{\tau - i\sigma}, \tag{2.25}$$

where $\tau \in [0, +\infty]$ and $\sigma \in [0, 2\pi[$.

In this picture the remote past corresponds to the origin and the distant future to $|z| \to +\infty$. The fact that the space is compactified to the circle $S^1$ removes any infrared divergences. The Hamiltonian (i.e. the time evolution operator) is now the dilatation operator $D = l_0 + \bar{l}_0$ and the operator $M = i(l_0 - \bar{l}_0)$ responsible for rotations, now implements the spatial translations along the circle $S^1$.

We also define the chronological radial ordering with respect to $\tau$ for any two operators $A_1$ and $A_2$ by:

$$T_\tau \left\{ A_1(z_1, \bar{z}_1) A_2(z_2, \bar{z}_2) \right\} = \begin{cases} A_1(z_1, \bar{z}_1) A_2(z_2, \bar{z}_2), & \text{if } |z_1| > |z_2|, \\ \\ A_2(z_2, \bar{z}_2) A_1(z_1, \bar{z}_1), & \text{if } |z_2| > |z_1|. \end{cases} \tag{2.26}$$

Given that the stress tensor generates conformal transformations, we can write the follow-

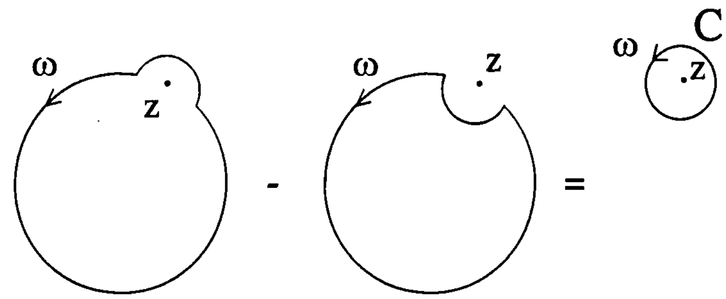


Figure 2.2: contours of integration

ing identity[4]:

$$\frac{1}{2\pi i}\left[\oint_{C'} dw\zeta(w)T(w), X(z,\bar{z})\right] = \delta_\zeta X(z,\bar{z}), \tag{2.27}$$

where $X(z,\bar{z})$ is some local operator. The above expression has to be dealt with some care. Products of operators $A(w)B(z)$ in Euclidean space radial quantization are only defined for $|w| > |z|$. Consequently, (2.27) can be written as:

$$\delta_\zeta X(z,\bar{z}) = \frac{1}{2i\pi}\left(\oint_{|w|>|z|} - \oint_{|z|>|w|}\right) dw\zeta(w)T_\tau\left(T(w)X(z,\bar{z})\right). \tag{2.28}$$

The contours are depicted in fig.2.2.

And we get:

$$\delta_\zeta X(z,\bar{z}) = \frac{1}{2i\pi}\oint_C dw\zeta(w)T_\tau\left(T(w)X(z,\bar{z})\right). \tag{2.29}$$

From (2.24), we get the following Ward identity:

$$\begin{aligned}&\tfrac{1}{2i\pi}\oint_C dw\zeta(w) < T(w)\Pi_{i=1}^N\phi_i(z_i,\bar{z}_i) > -\tfrac{1}{2i\pi}\oint_C d\bar{w}\bar{\zeta}(\bar{w}) < \bar{T}(\bar{w})\Pi_{i=1}^N\phi_i(z_i,\bar{z}_i) >= \\ &= \textstyle\sum_{i=1}^N\left[\zeta(z_i)\partial_{z_i} + \bar{\zeta}(\bar{z}_i)\partial_{\bar{z}_i} + h_i\partial_{z_i}\zeta(z_i) + \bar{h}_i\partial_{\bar{z}_i}\bar{\zeta}(\bar{z}_i)\right] < \Pi_{i=1}^N\phi_i(z_i,\bar{z}_i) >,\end{aligned} \tag{2.30}$$

where $C$ is now a contour enclosing all the points $(z_1,\bar{z}_1),\cdots,(z_N,\bar{z}_N)$ and $\phi_1,\cdots,\phi_N$ are primary operators with conformal weights $(h_1,\bar{h}_1),\cdots,(h_N,\bar{h}_N)$. We have also dropped the notation $T_\tau$ for simplicity. From the above equation we see that we can decouple the $z$ and $\bar{z}$ parts. Using Cauchy's theorem, we get say for the analytic sector:

$$< T(z)\phi_1(z_1)\cdots\phi_N(z_N) >= \sum_{i=1}^N\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_{z_i}\right) < \phi_1(z_1)\cdots\phi_N(z_N) > . \tag{2.31}$$

The following short-distance expansion therefore holds for any primary operator $\phi_h$:

$$T(z)\phi_h(w) = \frac{h}{(z-w)^2}\phi_h(w) + \frac{1}{z-w}\partial_w\phi_h(w) + \cdots \tag{2.32}$$

[4]We redefined the stress tensor $T \to \frac{1}{2\pi i}T$ for later convenience.

The dots stand for regular terms. Also it can be shown that:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \cdots, \tag{2.33}$$

where $c$ is a (for now) arbitrary constant called the '*central charge*'. Comparing the two previous equations we conclude that the stress tensor $T$ has dimension $h = 2$ as we would expect for a rank 2 tensor. Furthermore, we see that $T$ is not a primary operator because of the 4th order pole. Under an infinitesimal conformal transformation, we get:

$$\delta_\zeta T(z) = \zeta(z)\partial_z T(z) + 2\partial_z\zeta(z)T(z) + \frac{c}{12}\partial_z^3\zeta(z). \tag{2.34}$$

This can be integrated, yielding:

$$T(z) \to (\partial_z f)^2 T(f(z)) + \frac{c}{12}S(f,z), \tag{2.35}$$

under $z \to f(z)$, where the quantity,

$$S(f,z) = \frac{\partial_z f \partial_z^3 f - 3/2(\partial_z^2 f)^2}{(\partial_z f)^2}, \tag{2.36}$$

is known as the "*Schwartzian derivative*", [12]. For Möbius transformations we have $S(f,z) = 0$. Consequently, $T$ is a quasi-primary operator. It is also convenient to define a Laurent expansion of the energy-momentum tensor,

$$T(z) = \sum_{n\in \mathcal{Z}} L_n z^{-n-2}, \tag{2.37}$$

in terms of modes $L_n$, satisfying

$$L_n^\dagger = L_{-n}. \tag{2.38}$$

It can be shown from (2.33) and the method described above to go from commutators to operator product expansions (OPE) that they satisfy the following algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{2.39}$$

This is called a *Virasoro algebra with central charge c*. It can be regarded as a sort of quantum correction to the classical algebra (2.16) introduced by the second term on the r.h.s. The name of the central charge stems from the fact that it is a central element of the algebra, i.e. an element that commutes with all the generators of the algebra. If we construct an irreducible representation of the algebra (2.39) then after Schur's lemma we can regard c for all practical purposes as being just a number. The meaning of it will become clear in what follows.

Let us now try to construct the representations of (2.39). If we require $T(z)|0>$ to be regular at $z = 0$, we get:

$$\begin{array}{ll} L_m|0>= 0, & m \geq -1, \\ <0|L_m = 0, & m \leq 1, \end{array} \tag{2.40}$$

where $|0>$ is the vacuum state of the theory. We also define the state,

$$|h> \equiv \phi_h(0)|0>, \tag{2.41}$$

where $\phi_h$ is a primary operator. This state satisfies:

$$L_0|h>= h|h>, \qquad L_n|h>= 0, \qquad (n > 0). \tag{2.42}$$

The operators $L_n$ with $n > 0$ act on this state as annihilation operators. From (2.38) the operators $L_n$ with $n < 0$ can be considered to be creation operators. The representations of the Virasoro algebra are constructed by acting with the latter on the *highest weight states* $|h>$[5]:

$$\begin{array}{ccl} \underline{\text{level}} & \underline{\text{dimension}} & \underline{\text{state}} \\ & & \\ 0 & h & |h> \\ 1 & h+1 & L_{-1}|h> \\ 2 & h+2 & L_{-2}|h>, L_{-1}^2|h> \\ 3 & h+3 & L_{-3}|h>, L_{-1}L_{-2}|h>, L_{-1}^3|h> \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \tag{2.43}$$

It was shown that the above representation will be unitary whenever, [16]:

$$\begin{array}{ll} (i) & c \geq 1, h \geq 0, \\ & \\ (ii) & c(m) = 1 - \frac{6}{m(m+1)}, \qquad m = 3, 4, \cdots \\ & \\ & h = h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \qquad 1 \leq p + q \leq m. \end{array} \tag{2.44}$$

In fact these conditions are both necessary and sufficient. The sequence in (ii) is called the minimal series[6] and it corresponds to conformal theories with $c < 1$. The associated set of conformal weights is given by the *Kac formula* $h_{p,q}$.

One can also think of the descendant states above as being generated by operators appearing in OPEs with the stress tensor. For instance consider for some primary field $\phi_{h,\bar{h}}$:

$$\begin{array}{c} T(z)\phi_{h,\bar{h}}(w, \bar{w}) = \sum_{n \geq 0}(z - w)^{n-2} L_{-n}\phi_{h,\bar{h}}(w, \bar{w}) = \\ = \frac{1}{(z-w)^2} L_o\phi + \frac{1}{z-w} L_{-1}\phi + L_{-2}\phi + (z - w)L_{-3}\phi + \cdots \end{array} \tag{2.45}$$

---

[5] Although it is called highest weight state it actually has the lowest eigenvalue of $L_0$.

[6] These conformal theories have a finite number of primary operators and there is no additional continuous symmetry besides the conformal invariance. Hence the name *minimal*.

The fields $L_{-n}\phi_{h,\bar{h}}(w,\bar{w}) = \oint \frac{dz}{2i\pi}\frac{1}{(z-w)^{n-1}}T(z)\phi_{h,\bar{h}}(w,\bar{w})$ also denoted as $\phi^{(-n)}_{h,\bar{h}}$ are called the *Virasoro descendants*. Their conformal weights are $(h+n,\bar{h})$. Also from the OPE (2.32) we know that:

$$\phi^{(0)}_{h,\bar{h}} = L_0\phi_{h,\bar{h}} = h\phi_{h,\bar{h}}, \qquad \phi^{(-1)}_{h,\bar{h}} = L_{-1}\phi_{h,\bar{h}} = \partial_z\phi_{h,\bar{h}}. \tag{2.46}$$

Moreover we can define further descendants (or *secondary fields*):

$$\phi^{(-n_1,\cdots,-n_p;-m_1,\cdots,-m_q)}_{h,\bar{h}} \equiv L_{-n_1}\cdots L_{-n_p}\bar{L}_{-m_1}\cdots\bar{L}_{-m_q}\phi_{h,\bar{h}}, \tag{2.47}$$

with conformal weights $(h+\sum_{i=1}^{p} n_i, \bar{h}+\sum_{j=1}^{q} m_j)$, for $n_i, m_j \geq 0$. All the correlation functions of the secondary fields are given by differential operators acting on those with primary operators only. For example:

$$< \phi_1(w_1,\bar{w}_1)\cdots\phi_n(w_n,\bar{w}_n)(L_{-k}\phi)(z,\bar{z}) >= \mathcal{L}_{-k} < \phi_1(w_1,\bar{w}_1)\cdots\phi_n(w_n,\bar{w}_n)\phi(z,\bar{z}) >, \tag{2.48}$$

where $\mathcal{L}_{-k}$ is the differential operator ($k \geq 2$):

$$\mathcal{L}_{-k} = -\sum_{j=1}^{n}\left(\frac{(1-k)h_j}{(w_j-z)^k} + \frac{1}{(w_j-z)^{k-1}}\frac{\partial}{\partial w_j}\right). \tag{2.49}$$

Now we can think of the states $L_{-n}|h>$ defined previously as being created by the operators $\phi^{(-n)}_h$:

$$L_{-n}|h >= \phi^{(-n)}_h(0)|0> . \tag{2.50}$$

So for a given conformal field theory in two dimensions, its operators are classed into conformal blocks each labeled by a primary field. They satisfy the following set of *fusion rules* (OPEs):

$$[\phi_i]\times[\phi_j] = \sum_k N^k_{ij}\,[\phi_k]\,, \tag{2.51}$$

where $[\phi_i]$ stands for some operator belonging to the highest weight representation indexed by the primary field $\phi_i$. This operator algebra is assumed to be closed and associative. From (2.48) and (2.49) we see that we can in principle compute all the correlation functions from the knowledge of the basic correlators involving only primary operators. Similarly all the structure constants $N^k_{ij}$ in (2.51) can be obtained from the structure constants describing the fusions of the primary fields only, [12].

The unitary representations (2.44) are not necessarily irreducible. In fact there exist *null vectors* $|\chi>$ satisfying:

$$< \lambda|\chi >= 0, \tag{2.52}$$

for all $|\lambda>$ in the representation $(c, h)$. It can be shown that every null vector is also a highest weight vector, thus implying a reducible representation. Consider, for instance, the level 2 vector,

$$|\chi>=(L_{-2}+aL_{-1}^2)|h>. \tag{2.53}$$

If it is a highest weight vector, then in particular:

$$L_1|\chi>=L_2|\chi>=0. \tag{2.54}$$

These constraints yield:

$$\begin{aligned} a &= -\tfrac{3}{2(2h+1)}, \\ h &= \tfrac{1}{16}\left(5-c\pm\sqrt{(1-c)(25-c)}\right). \end{aligned} \tag{2.55}$$

Notice that $h_{1,2}$ and $h_{2,1}$ in (2.44) satisfy the second equation in (2.55).

We obtain irreducible representations by projecting out the null states, i.e. by setting $\chi=0$. This is consistent given that $|\chi>$ is orthogonal to all vectors in the conformal theory (cf.(2.52)). In particular, any correlator involving null vectors must vanish. For $|\chi>$ given by (2.53)-(2.55), we have:

$$\begin{aligned} \chi(z) &= \left(L_{-2}(z)-\tfrac{3}{2(2h+1)}L_{-1}^2(z)\right)\phi_h(z)= \\ &=\phi_h^{(-2)}(z)-\tfrac{3}{2(2h+1)}\partial_z^2\phi_h(z). \end{aligned} \tag{2.56}$$

Consequently:

$$\begin{aligned} 0 &=<\chi(z)\phi_1(z_1)\cdots\phi_n(z_n)>= \\ &=<\left\{\phi_h^{(-2)}(z)-\tfrac{3}{2(2h+1)}\partial_z^2\phi_h(z)\right\}\phi_1(z_1)\cdots\phi_n(z_n)>= \\ &=\left\{\mathcal{L}_{-2}(z,z_1,\cdots,z_n)-\tfrac{3}{2(2h+1)}\partial_z^2\right\}<\phi_h(z)\phi_1(z_1)\cdots\phi_n(z_n)>. \end{aligned}$$

Explicitly, we get the following partial differential equation:

$$\left\{\frac{3}{2(2h+1)}\partial_z^2-\sum_{i=1}^{n}\left(\frac{h_i}{(z-z_i)^2}+\frac{1}{z-z_i}\partial_{z_i}\right)\right\}<\phi_h(z)\phi_1(z_1)\cdots\phi_n(z_n)>=0. \tag{2.57}$$

In summary, if a theory is degenerate (i.e. if it has null vectors), then we are able to write down partial differential equations for genus zero correlators of primary fields. The minimal series (2.44) describes such theories.

## 2.3 Modular invariance and the Casimir effect

As we mentioned in the introduction it is by now widely accepted that each universality class in two dimensional statistical systems is described by a conformal field theory with a particular central charge. The correspondence between certain statistical models and some elements in the Friedan, Qiu and Shenker (FQS) classification (2.44) of $c < 1$ conformal field theories has been established by various authors. For example, the Ising model and the 3-state Potts model have been identified with the elements $m = 3$ and $m = 5$, respectively, [18]. However, this correspondence is by no means one-to-one. For instance, both the universality class of the 3-state Potts model and that of a generic tetracritical point have been identified with $m = 5$. Notwithstanding this, not all scaling dimensions allowed by the Kac formula appear in the Potts model, and some appear twice. In the tetracritical model, on the other hand, it seems as though all values are present.

We now describe a method to determine the operator content of a given minimal theory, solely on the basis of conformal and modular invariance, [20]. As a by-product we inherit an interpretation of the central charge as a measure for the Casimir effect that arises in a finite geometry.

Conformal invariance allows us to relate a theory formulated on a finite strip of width $R$ with periodic boundary conditions with that on the infinite complex plane, via the conformal mapping[7]:

$$w = \frac{R}{2\pi} ln z. \tag{2.58}$$

Since this transformation is not a Möbius transformation, its Schwartzian derivative (2.36) is non-vanishing and there will appear an anomaly term in the transformed stress tensor:

$$T_{strip}(w) = \left(\frac{2\pi z}{R}\right)^2 T_{plane}(z) - \frac{c\pi^2}{6R^2}. \tag{2.59}$$

We note in particular that:

$$< T_{strip}(w) > \neq 0. \tag{2.60}$$

This allows us to interpret the anomaly term as a Casimir effect, that is, a shift in the free energy of the system due to finite size effects. In fact it can be shown that the free

[7] On the strip , we interpret the coordinate $\sigma \in [0, 2\pi[$ as the space direction and $t \in [-\infty, +\infty]$ as the time, where $w = t + i\sigma$

energy per unit length will be given by [15], [17]:

$$F(R) \sim -\frac{\pi c}{6R} + fR, \tag{2.61}$$

where $f$ is a universal term. Notice that the first term vanishes when $c \to 0$ or $R \to \infty$. Using (2.59) and a similar expression for the antianalytic part, we have the following correspondence:

$$L_0 + \bar{L}_0 \to \frac{R}{2\pi}\left(H_P + \frac{\pi c}{6R}\right), \tag{2.62}$$

where $H_P$ is the Hamiltonian of the system on the strip with periodic boundary conditions. So $H_P$ and $(L_0 + \bar{L}_0)$ have the same eigenstates. We also stress the fact that we restrict our analysis to the unitary theories with a finite number of primary operators falling into the classification of FQS.

Now we consider a toroidal configuration, so that different values of $t$ are identified modulo $L$. We also define the *modular parameter* $q$ as:

$$q \equiv exp\left(-2\pi\frac{L}{R}\right) = exp(-2\pi\tau). \tag{2.63}$$

The partition function in such a geometry is be given by:

$$Z_{PP}(R,L) = Tr\left(e^{-LH_p}\right) = \sum_n e^{\pi c\tau/6} e^{-2\pi\tau x_n}, \tag{2.64}$$

where we used (2.62). We are considering the vertical L-axis to be the time direction. The sum is over all simultaneous eigenstates $\{|n>\}$ of $H_P$ and $(L_0 + \bar{L}_0)$. $x_n$ is the eigenvalue of $(L_0 + \bar{L}_0)$ corresponding to the eigenstate $|n>$.

But this is not the whole story. Since we are dealing with minimal conformal field theories, the above sum splits into sums over conformal blocks. Let us consider the contribution of a conformal block, whose primary operator has dimensions $(h_{p,q}, h_{\bar{p},\bar{q}})$ given by the Kac formula:

$$e^{\pi c\tau/6} \sum_{N,\bar{N}=0}^{\infty} d_{p,q}(N) d_{\bar{p},\bar{q}}(\bar{N}) exp\left[-2\pi\tau(h_{p,q} + h_{\bar{p},\bar{q}} + N + \bar{N})\right], \tag{2.65}$$

where $d_{p,q}(N)$ and $d_{\bar{p},\bar{q}}(\bar{N})$ are the degeneracies of the levels $N$ and $\bar{N}$, respectively. The above expression can be decoupled, as expected, into analytic and antianalytic parts, yielding:

$$Z_{PP}(q) = \sum_{pq,\bar{p}\bar{q}} \mathcal{N}_P(pq,\bar{p}\bar{q}) \chi_{pq}(q) \chi_{\bar{p}\bar{q}}(q), \tag{2.66}$$

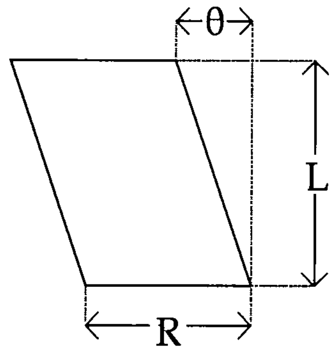


Figure 2.3: Torus

where $\mathcal{N}_P(pq, \bar{p}\bar{q})$ denotes the number of times the representation $(pq, \bar{p}\bar{q})$ appears in the partition function and

$$\chi_{pq}(q) \equiv q^{-c/24} Tr_{pq} q^{L_0} = q^{-c/24+h_{pq}} \sum_{N=0}^{\infty} d_{pq}(N) q^N, \tag{2.67}$$

is called the character of the representation $(p, q)$ of the Virasoro algebra. These objects have been thoroughly studied by Rocha-Caridi, [21].

We recall that our aim is to compute the objects $\mathcal{N}_P$. We are now in a position to invoke the argument of modular invariance. If the partition function is defined on a torus, then it had better not break the symmetries of this topology. As we are all aware, we can visualize a torus as a parallelogram with opposite ends identified. The *modular group* is just the group of global reparametrisations of the torus. Let us try to understand what this means at the level of the partition function. The operator which implements the translations in Euclidean time is $exp(-LH_P)$ whereas the translations in Euclidean space are implemented by $exp(i\theta P)$, where $P$ is the momentum operator. Hence the zero-point one-loop amplitude (twisted at an angle $\theta$) is given by (see fig.2.3), [42]:

$$Z_{PP}(\tau, \theta) = Tr\left(e^{i\theta P} e^{-LH_P}\right). \tag{2.68}$$

Since the direction of time on the torus was chosen by us arbitrarily, we could have instead considered a system of length $L$ propagating for a lapse of time $R$. The answer should be unaltered. Namely:

$$Z_{PP}(\tau, 0) = Z_{PP}\left(-\frac{1}{\tau}, 0\right). \tag{2.69}$$

Also a twist at an angle $2\pi$ should leave the partition function unchanged:

$$Z_{PP}(\tau,0) = Z_{PP}(\tau,2\pi). \tag{2.70}$$

Cardy proved that the conformal characters (2.67) transform linearly under the modular transformation (2.69), [20]:

$$\chi_{pq}(q) = \sum_{p'q'} M_{pq}^{p'q'} \chi_{p'q'}(\tilde{q}), \tag{2.71}$$

where $\tilde{q} \equiv q(-1/\tau) = exp(2\pi/\tau)$ and:

$$M_{pq}^{p'q'} = \left(\frac{8}{m(m+1)}\right)^{1/2} (-1)^{(p+q)(p'+q')} sin\left(\frac{\pi pp'}{m}\right) sin\left(\frac{\pi qq'}{m+1}\right). \tag{2.72}$$

Imposing (2.69) and assuming that all the characters are distinct, linearly independent and finite in number, we obtain the following set of *inversion sum rules*:

$$[M \otimes M]\mathcal{N}_P = \mathcal{N}_P. \tag{2.73}$$

Now remember that we want $Z_{PP}(q)$ to be invariant under the whole modular group. Therefore, only the $\mathcal{N}_P(pq,\bar{p}\bar{q})$ such that the spin $h_{p,q} - h_{\bar{p},\bar{q}}$ is an integer (bosons) are permitted. A further constraint resides in the reality of the partition function:

$$\mathcal{N}_P(pq,\bar{p}\bar{q}) = \mathcal{N}_P(\bar{p}\bar{q},pq). \tag{2.74}$$

Finally we impose that the identity operator (with dimensions $(0,0)$) should appear exactly once in the theory:

$$\mathcal{N}_P(11,11) = 1. \tag{2.75}$$

From these constraints we see that

$$\mathcal{N}_P(pq,\bar{p}\bar{q}) = \delta_{p,\bar{p}}\delta_{q,\bar{q}} \tag{2.76}$$

is always a solution of the inversion sum rules (2.73). This is actually the unique solution for $m = 3$ which corresponds to the Ising model. The allowed operators are the identity (1), the energy density ($\epsilon$) and the spin density ($\sigma$) operators with conformal weights $(0,0)$, $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(\frac{1}{16},\frac{1}{16}\right)$, respectively.

It is important to note that the existence of a solution to the inversion sum rules by itself does not imply the existence of a corresponding model, since the sum rules are only a necessary condition for the model to be consistent.

Also, eq.(2.71) is more general than claimed. For any *rational* conformal theory (i.e with a finite number of primary operators), we have:

$$\chi_i(q) = \sum_j S_i^j \chi_j(\tilde{q}), \tag{2.77}$$

where $\chi_i(q)$ is the character of the representation $i$. Verlinde conjectured that modular invariance is so powerful a constraint as to completely fix the structure constants $N_{ij}^k$ of the fusion rules (2.51) according to the *Verlinde formula*[8] [65]:

$$\sum_i S_i^j N_{pl}^i = \frac{S_p^j S_l^j}{S_0^j}. \tag{2.78}$$

From this formula we can derive the following fusion rules for the Ising model:

$$\begin{aligned} &\epsilon\epsilon \sim 1 \\ &\sigma\sigma \sim 1 + \epsilon \\ &\epsilon\sigma \sim \sigma \end{aligned} \tag{2.79}$$

Notice that this operator algebra closes with the operators 1 and $\epsilon$ only. However the sum rules tell us that the magnetization $\sigma$ must be included to get a consistent theory.

[8]The symbol '0' corresponds to the identity representation.

# Chapter 3

# Integrable Models

*In this chapter we review some of the main results of two-dimensional integrable quantum field theories and the bootstrap approach.*

The study of integrable theories in two dimensions started a long time ago with the analysis of non-relativistic models with a $\delta$-type interaction, [8], [75]. The existence of an infinite number of involutive (mutually commuting) integrals of motion (IM) transforming according to representations of increasing "*spin*" of the group of rotations played a decisive rôle. These models were solved using the technique of *algebraic Bethe Ansatz* [8] whereby one would assume a localized interaction (with an interaction range of the order $R_c$) so that particles far away from each other (distances a lot greater than $R_c$) could be described by asymptotic free states. These states, also called the *Bethe wave functions*, are simultaneous eigenstates of the IM. The corresponding Schrödinger equation can be regarded as being just the first in an infinite series of eigenvalue problems - one for each IM. In these models, particle production is excluded by the fact that they are non-relativistic. Solving the hierarchy of eigenvalue problems for an arbitrary number $N$ of particles yields not only the $N$-particle Bethe wave function but also the exact elastic scattering matrix. The S-matrix on the other hand can be shown to decompose into the product of two-particle scattering amplitudes. This is called *factorized scattering* and is another consequence of the existence of nonlocal IM, [36]. Once the Bethe wave function is determined we can place our system in a finite periodic box of length $L$, thus quantizing the momenta of the $N$ particles according to a set of transcendental equations, also known as the *Bethe equations.* By taking the thermodynamic limit $N, L \to \infty$ with $D = N/L = constant$ and taking into account the previous quantization condition for the momenta, we can subsequently extract a number of thermodynamic properties of the system.

Integrability is also compatible with Lorentz invariance. Some relativistic models such as the sine-Gordon theory, [37] or more generally two-dimensional conformal field theories are integrable. The deep connections between integrable models and conformal theories have started to be unravelled. It is by now widely accepted that many integrable models are perturbations of conformal field theories, [25].

## 3.1 Integrable perturbed conformal theories and conservation laws

Suppose that we have a conformal minimal unitary model (cf(2.44)) denoted by $\mathcal{M}(m)$ (where $m = 3, 4, \cdots$) and perturb it with a relevant primary scalar field $\Phi_{pq}(z, \bar{z})$ with conformal dimensions $(h_{pq}, h_{pq})$, given by the Kac formula (2.44). The fact that it is relevant means that $\Delta = h + \bar{h} = 2h_{pq} < 2$. The action is :

$$\mathcal{S} = \mathcal{S}_0 + \lambda \int \Phi_{pq}(z, \bar{z}) d^2 z, \tag{3.1}$$

where $\mathcal{S}_0$ is the action of the minimal non-perturbed conformal theory. From the above equation we can infer the conformal dimensions of the constant $\lambda$: $(1 - h_{pq}, 1 - h_{pq})$. Alternatively we can say that $\lambda$ has the scaling dimension, $\lambda \sim (length)^{-2(1-h_{pq})}$. Because the primary field is relevant we conclude that the exponent in the above expression is negative. Also the fact that $\lambda$ is a dimensional quantity means that the scale invariance is broken.

Suppose that $T_{s+1}$ is some conserved quantity of dimensions $(s+1, 0)$ in the conformal theory, i.e.

$$\partial_{\bar{z}} T_{s+1} = 0. \tag{3.2}$$

From the closure of the operator algebra (2.51) we have:

$$T_{s+1}(z)\Phi_{pq}(w, \bar{w}) = \sum_{n=2}^{m} \frac{d_{pq}^{(n)}}{(z-w)^n} \Phi_{pq}^{(n)}(w, \bar{w}) + \frac{1}{z-w} B_{pq}(w, \bar{w}) + \cdots \tag{3.3}$$

where $\Phi_{pq}^{(n)}$, $B_{pq}$ are some local fields in the operator algebra, $m$ is some integer and $d_{pq}^{(n)}$ are dimensionless constants. The integer $m$ has to be finite. To see this notice that the first term on the r.h.s of eq.(3.3) yields the conformal dimensions $(l_n, \bar{l}_n)$ for the fields $\Phi_{pq}^{(n)}$:

$(l_n, \bar{l}_n) = (s + h_{pq} + 1 - n, h_{pq})$. Since these have to be non-negative for a unitary theory, we conclude that:

$$m \leq h_{pq} + s + 1. \tag{3.4}$$

When we perturb the theory according to (3.1), the conformal symmetry is broken and (3.2) no longer holds. We have instead:

$$\partial_{\bar{z}} T_{s+1} = \lambda R_s + \cdots + \lambda^n A_s^{(n)} + \cdots \tag{3.5}$$

The local field $R_s$ has dimensions $(s + h_{pq}, h_{pq})$. The subscript "$s$" means that $R_s$ and $A_s^{(n)}$ have spin $s$. The field $A_s^{(n)}$ has dimensions $(s + 1 - n + nh, 1 - n + nh)$. Again from the unitarity of the theory, we conclude that the sum in (3.5) is finite. Terms for which $n \geq \frac{1}{1-h}$ should therefore be excluded. In most cases only the first one survives. We will assume that this is the case.

Because of (3.5), $T_{s+1}$ depends on both $z$ and $\bar{z}$. The Ward identity is:

$$< T_{s+1}(z, \bar{z}) \cdots > = < T_{s+1}(z) \cdots >_0 + \lambda \int dw d\bar{w} < T_{s+1}(z) \Phi_{pq}(w, \bar{w}) \cdots >_0 + \mathcal{O}(\lambda^2), \tag{3.6}$$

where $<>_0$ denotes the correlations in the nonperturbed conformal theory and we assumed that $< \Phi_{pq}(z, \bar{z}) >_0 = 0$. From eq.(3.3), Cauchy's theorem and the identity,

$$\partial_{\bar{z}} \frac{1}{z - w + i\epsilon} = \delta(z - w)\delta(\bar{z} - \bar{w}), \tag{3.7}$$

we get to first order in $\lambda$:

$$\partial_{\bar{z}} T_{s+1}(z, \bar{z}) = \lambda \left[ \sum_{n=2}^{m} \frac{(-1)^{n-1}}{(n-1)!} d_{pq}^{(n)} \partial_z^{n-1} \Phi_{pq}^{(n)}(z, \bar{z}) + B_{pq}(z, \bar{z}) \right]. \tag{3.8}$$

We conclude that we get an off-critical conservation law,

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}, \tag{3.9}$$

provided $B_{pq}$ is a total z-derivative.

It is easy to show that the stress tensor always yields a conservation law for a minimal theory perturbed by a relevant primary operator (cf.(3.1)). In fact:

$$T(z)\Phi_{pq}(w, \bar{w}) = \frac{h_{pq}}{(z - w)^2} \Phi_{pq}(w, \bar{w}) + \frac{1}{z - w} \partial_w \Phi_{pq}(w, \bar{w}) + \cdots \tag{3.10}$$

Comparing (3.3) with (3.10):

$$\Theta_0(z, \bar{z}) = \lambda(1 - h_{pq})\Phi_{pq}(z, \bar{z}). \tag{3.11}$$

This method is suited to understand which chiral conserved currents in the conformal theory are deformed into off-critical laws of the form (3.9). Zamolodchikov suggested a "*counting argument*", [29], that allows us in principle to compute the spins of the conserved charges in the integrable theory, solely from the knowledge of the critical conformal theory and the relevant perturbing primary operator. In practice, the computations are quite involved and we are able to compute only the first few spins.

We start by defining the irreducible Virasoro module $\Lambda$ of the identity operator. $\Lambda$ admits the decomposition,

$$\Lambda = \oplus_{s=0}^{\infty} \Lambda_s, \tag{3.12}$$

into subspaces $\Lambda_s$ of dimensions $(s, 0)$. We also introduce the factor space,

$$\hat{\Lambda}_{s+1} = \Lambda_{s+1}/L_{-1}\Lambda_s. \tag{3.13}$$

The elements of (3.13) fall into equivalence classes, so that two elements $A_{s+1}, B_{s+1} \in \Lambda_{s+1}$ are represented by the same class provided there exists some element $C_s \in \Lambda_s$, such that (cf.(2.46)):

$$B_{s+1} = A_{s+1} + \partial_z C_s. \tag{3.14}$$

Similarly, if $\Omega$ denotes the irreducible module of the relevant primary operator $\Phi_{pq}$ and,

$$\Omega = \oplus_{s=0}^{\infty} \Omega_s, \tag{3.15}$$

is its decomposition into subspaces of conformal dimensions $(h_{pq} + s, h_{pq})$, then we can also define the factor space:

$$\hat{\Omega}_{s+1} = \Omega_{s+1}/L_{-1}\Omega_s. \tag{3.16}$$

From eq.(3.5) we see that the symbol $\partial_{\bar{z}}$ can be seen as an operator:

$$(\partial_{\bar{z}})_{s+1} : \hat{\Lambda}_{s+1} \rightarrow \Omega_s. \tag{3.17}$$

We also introduce the operator $Z_{s+1} \equiv \Pi_s \circ (\partial_{\bar{z}})_{s+1} : \hat{\Lambda}_{s+1} \rightarrow \hat{\Omega}_s$, where $\Pi_s : \Omega_s \rightarrow \hat{\Omega}_s$ is the operator that projects any element in $\Omega_s$ onto its equivalence class in $\hat{\Omega}$. If $dim(\hat{\Lambda}_{s+1}) > dim(\hat{\Omega}_s)$, then this means that the kernel of $Z_{s+1}$ is nontrivial, i.e. there are at least two elements $A_{s+1}, B_{s+1} \in \hat{\Lambda}_{s+1}$ that are mapped onto the same equivalence class in $\hat{\Omega}_s$. In other words, there exists some $\Theta_{s-1} \in \Omega_{s-1}$ such that (cf.(3.14)):

$$\partial_{\bar{z}} A_{s+1} - \partial_{\bar{z}} B_{s+1} = \partial_z \Theta_{s-1}. \tag{3.18}$$

If we define $T_{s+1} = A_{s+1} - B_{s+1}$, we recover eq.(3.9). In summary, whenever $dim(\hat{\Lambda}_{s+1}) > dim(\hat{\Omega}_s)$, there exist conserved charges $P_s$ of spin $s$, given by:

$$P_s = \int [T_{s+1} dz + \Theta_{s-1} d\bar{z}] . \tag{3.19}$$

Of course by parity, we also have:

$$\begin{aligned} \partial_z \bar{T}_{s+1} &= \partial_{\bar{z}} \bar{\Theta}_{s-1} , \\ \bar{P}_s &= \int \left[\bar{T}_{s+1} d\bar{z} + \bar{\Theta}_{s-1} dz\right] . \end{aligned} \tag{3.20}$$

The dimensions of $\hat{\Lambda}_{s+1}$ and $\hat{\Omega}_s$ can actually be computed by equating powers of $q$ (hence the name "*counting argument*") in the following formulas involving the Virasoro characters of the representations corresponding to the identity operator ($\chi_{1,1}$) and the operator $\Phi_{pq}$ ($\chi_{p,q}$):

$$\begin{aligned} &\textstyle\sum_{s=0}^{\infty} q^s dim(\hat{\Lambda}_s) = (1-q)\chi_{1,1}(q) + q , \\ &\textstyle\sum_{s=0}^{\infty} q^{s+h_{pq}} dim(\hat{\Omega}_s) = (1-q) q^{(c-1)/24 - h_{pq}} \chi_{p,q}(q) . \end{aligned} \tag{3.21}$$

As an illustration, we consider the critical Ising model perturbed by the magnetization operator, [29]. The central charge is 1/2 and the magnetization operator $\sigma$ has dimensions $\left(\frac{1}{16}, \frac{1}{16}\right)$. The dimensional perturbation parameter $\lambda$ has conformal dimensions $\left(\frac{15}{16}, \frac{15}{16}\right)$. The dimensionalities of $\hat{\Lambda}_{s+1}$ and $\hat{\Omega}_s$ can be found for odd $s \leq 21$ in table I. Even spins $s$ are not considered because in that case $dim(\hat{\Lambda}_{s+1}) < dim(\hat{\Omega}_s)$.

Table I

| $s$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $dim(\hat{\Lambda}_{s+1})$ | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 | 11 |
| $dim(\hat{\Omega}_s)$ | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 5 | 6 | 8 | 12 |

These dimensionalities have been computed using eq.(3.21) and:

$$\begin{aligned} \chi_{1,1}(q) &= \tfrac{1}{2}\left[\Pi_{n=0}^{\infty}(1+q^{n+1/2}) + \Pi_{n=0}^{\infty}(1-q^{n+1/2})\right] , \\ \chi_{1,2}(q) &= q^{1/16}\Pi_{n=1}^{\infty}(1+q^n) = q^{1/16}\Pi_{n=0}^{\infty}(1-q^{2n+1})^{-1} . \end{aligned} \tag{3.22}$$

From table I, we conclude that the spins of the conserved charges are: s=1,7, 11,13,17,19.

Zamolodchikov found[30] that by perturbing a minimal model $\mathcal{M}(m)$ ($m \geq 5$) with the relevant primary operators (i) $\Phi_{(1,3)}$, (ii) $\Phi_{(1,2)}$ or (iii) $\Phi_{(2,1)}$ one would obtain integrable theories with one conservation law for every odd value of the spin $s$ in the case (i) and one for every $s = 1, 6n \pm 1$ ($n = 1, 2, \cdots$) for (ii) and (iii). We have already checked from the previous analysis that the stress tensor yields the spin $s = 1$ conserved charge in the off-critical theory. Other candidates have been suggested according to the particular perturbation. For instance, the spins in (i) coincide with those of the IM of the quantum Korteweg-de Vries (qKdV) equation, [120]. This equation is defined in the canonical form, [28]:

$$\begin{aligned} \dot{T} &= [T, H], \\ H &= \oint \tfrac{dz}{2\pi i} : T^2(z) :, \end{aligned} \tag{3.23}$$

where the dot indicates the time derivative. Using the technique to transform commutators into OPEs, we get:

$$\dot{T} = \frac{1}{6}(1-c)\partial_z^3 T - 3\partial_z(: T^2 :). \tag{3.24}$$

The first few conserved charges are, [28]

$$\begin{aligned} H_1 &= \oint dz T, \\ H_3 &= \oint dz : T^2 :, \\ H_5 &= \oint dz \left[ : T^3 : - \tfrac{1}{12}(c+2) : (\partial_z T)^2 : \right], \\ H_7 &= \oint dz \left[ : T^4 : + \tfrac{1}{6}(c+8) : (T^2 \partial_z^2 T) : + \tfrac{1}{180}(c^2 + 14c - 21) : (\partial_z^2 T)^2 : \right]. \end{aligned} \tag{3.25}$$

It is straightforward to show that the above IM commute mutually. We already know that the first IM yields a conservation law in the off-critical theory whatever the particular choice of the primary operator. One can then show that the first order pole in the OPE $T_{s+1}(z)\Phi_{(1,3)}(w, \bar{w})$ $(s = 3, 5, 7)$ is also a total derivative, which means that the KdV hierarchy (3.25) are deformed into off-critical conservation laws yielding the correct spins in (i).

However, this is no longer true in cases (ii) and (iii). In ref.[31], Zamolodchikov analyzed the case (iii) for the 3-state Potts model (corresponding to $m = 5$). This theory possesses in addition to the stress tensor a conserved spin-3 field $W$, satisfying the following OPEs:

$$\begin{aligned} T(z)W(w) &= \tfrac{3}{(z-w)^2} W(w) + \tfrac{1}{z-w}\partial_w W(w) + \cdots, \\ W(z)W(w) &= \tfrac{c/3}{(z-w)^6} + \tfrac{2}{(z-w)^4} T(w) + \tfrac{1}{(z-w)^3}\partial_w T(w) + \\ &+ \tfrac{1}{(z-w)^2}\left[\tfrac{3}{10}(1-b^2)\partial_w^2 T(w) + 2b^2 : T^2(w) :\right] + \\ &+ \tfrac{1}{z-w}\left[\tfrac{1}{30}(2-9b^2)\partial_w^3 T(w) + b^2 : (\partial_w T(w))^2 :\right] + \cdots, \end{aligned} \tag{3.26}$$

where $b^2 = 16/(22+5c)$. We see that because of nonlinear terms like $: T^2(w) :$ they do not form a Lie algebra. Rather, they are known in the literature as a $W_3$-algebra, [31], [39]. Also we notice that the quantities,

$$P_1 = \oint dz T, \qquad P_2 = \oint dz W, \tag{3.27}$$

commute. They are the first IM of the quantum Boussinesq equation (qB):

$$\begin{aligned} \dot{T} &= -2\partial_z W, \\ \dot{W} &= -\tfrac{1}{30}(2-9b^2)\partial_z^3 T - b^2 \partial_z (: T^2 :). \end{aligned} \tag{3.28}$$

The next few IM can be found in ref.[28]:

$$\begin{aligned} P_4 &= \oint dz : TW :, \\ P_5 &= \oint dz \left[ : W^2 : + \tfrac{1}{3} b^2 : T^3 : - \tfrac{i}{30}(2-9b^2) : (\partial_z T)^2 : \right], \\ P_7 &= \oint dz \left[ : (\partial_z W)^2 : + \beta : TW^2 : + \left( -\tfrac{1}{30} + \tfrac{9}{40} b^2 \right) : (\partial_z^2 T)^2 : + \right. \\ &\quad \left. + \tfrac{1}{6}\beta b^2 : T^4 : - \tfrac{3}{8} b^2 (\beta+2) : T^2 \partial_z^2 T : \right], \end{aligned} \tag{3.29}$$

with $\beta = -15b^2/(1+8b^2)$. There are two consistent reductions of the $W$-algebra obtained by setting either $W = 0$ or $W = b\sqrt{\frac{2-c}{8}}\partial_z T$. These two reduced sets were then shown to yield the spins of the IM in (ii) and (iii) in ref.[28].

## 3.2 Zamolodchikov's c-theorem

For a given perturbed conformal field theory, Zamolodchikov showed that there exists some function $c(\lambda)$ (usually called the $c$-function) of the coupling constant(s) $\lambda$ that decreases along the renormalization group (RG) trajectory in such a way that it is stationary at the fixed points, where its value coincides with that of the central charge describing the UV or IR limiting behaviour of the theory, [32]. The idea behind the proof is actually quite simple and amounts to dimensional analysis, [25]. The components of the stress tensor $(T, \bar{T}, \Theta)$ satisfy the following conservation laws:

$$\partial_{\bar{z}} T = \partial_z \Theta, \qquad \partial_z \bar{T} = \partial_{\bar{z}} \Theta. \tag{3.30}$$

They have dimensions $(2,0)$, $(0,2)$ and $(1,1)$, respectively. We are implicitly assuming rotational invariance, so that their correlators take the following form:

$$
\begin{aligned}
&< T(z,\bar{z})T(0,0) >= \tfrac{F(mz\bar{z})}{z^4}, \\
&< T(z,\bar{z})\Theta(0,0) >= \tfrac{G(mz\bar{z})}{z^3\bar{z}}, \\
&< \Theta(z,\bar{z})\Theta(0,0) >= \tfrac{H(mz\bar{z})}{z^2\bar{z}^2}.
\end{aligned}
\tag{3.31}
$$

Here, $m$ is a mass scale associated with the coupling constant $\lambda$. If we define the logarithmic derivatives, e.g. $\dot{F}(x) = \frac{d}{dlogx}F(x)$, we get from (3.30), (3.31) the following ordinary differential equations for $F$, $G$ and $H$:

$$\dot{F} - \dot{G} + 3G = 0, \qquad \dot{G} - G - \dot{H} + 2H = 0. \tag{3.32}$$

Defining,

$$C = 2F + 4G - 6H, \tag{3.33}$$

we have:

$$\dot{C} = -12H. \tag{3.34}$$

We now assume that the quantum field theory (QFT) satisfies the condition of reflection positivity. This means that $H$ is a positive quantity and therefore either $C$ decreases along the RG trajectory or takes stationary values. At the critical point, the QFT is scale invariant and therefore the stress tensor becomes traceless, i.e. $\Theta = 0$. Consequently $G = H = 0$. Also at this point $F = c/2$, where $c$ is the central charge of the conformal theory. As we claimed before the c-function is stationary at this point (cf.(3.34)) and its value (3.33) is equal to the central charge of the corresponding conformal theory.

## 3.3 Massless flows

In most cases the IR fixed point is trivial, in the sense that Zamolodchikov's $c$-function takes the value $c_{IR} = 0$. However, there known cases where this is not true. The best known examples are those of RG flows induced by the $\Phi_{(1,3)}$ perturbations of the unitary minimal models with diagonal modular invariant partition functions, [34], [33], [35]. From the statistical mechanics viewpoint they can be regarded as crossovers between different universality classes of critical phenomena. Both at large and short distances these theories are conformally invariant and the point where the flow crosses over from the region of one fixed point to the other introduces a mass scale (associated with the coupling constant of the perturbation) that breaks the scale invariance. The corresponding QFT is thus massless but not scale invariant. The RG flow is attracted to the non-trivial IR conformal field theory along the direction defined by some irrelevant operator $\Phi(z,\bar{z})$, i.e. an operator with scaling dimension $\Delta > 2$. At large distances we can describe the theory by the following IR-effective action, [91], [92]:

$$\mathcal{S} = \mathcal{S}_{IR} + \lambda \int d^2z \Phi(z,\bar{z}) + h.t., \tag{3.35}$$

where $\mathcal{S}_{IR}$ denotes the action of the IR conformal field theory and $h.t.$ stands for an infinite number of operators of higher dimension. These terms can in principle contribute as counterterms each one with its independent coupling constant, which means that this effective theory is nonrenormalizable. It can be best thought of as an asymptotic series.

## 3.4 Kinematics in 2D and factorized scattering

In two dimensions for any scattering process there is only one independent Lorentz scalar, [36]. This is because the system is parametrised by two components of the total momentum, $p^0 = e, p^1 = p$, that satisfy the dispersion relation, $e^2 - p^2 = m^2$ ($m$ is the total mass of the system). Consequently, only one of these quantities is independent.

Consider the Mandelstam variable $s = (p_1 + p_2)^2$ for the scattering of two particles with momenta $p_1, p_2$. If we parametrise the momenta in terms of the rapidities $(\theta_1, \theta_2)$,

$$(p_i^0, p_i^1) = m_i(cosh\theta_i, sinh\theta_i), \qquad (i = 1, 2) \tag{3.36}$$

we have:

$$s(\theta) = m_1^2 + m_2^2 + 2m_1 m_2 cosh\theta, \tag{3.37}$$

where $\theta = \theta_1 - \theta_2$. As we have discussed before, we can assume in the Lehman, Zimmermann, Symanzik approach to QFT that in the remote past and distant future the spectrum of the theory consists of free stable states of say $N$ particle types. A type $a$ ($a = 1, \cdots, N$) particle state $|A_a(\theta)>$ of rapidity $\theta$ is created by acting on the vacuum with the creation operator $A_a(\theta)$:

$$|A_a(\theta)> = A_a(\theta)|0> . \tag{3.38}$$

These asymptotic states are chosen to be simultaneous eigenstates of all the IM $P_s$:

$$P_s|A_a(\theta)>=\omega_s^a(\theta)|A_a(\theta)> . \tag{3.39}$$

Lorentz invariance requires that,

$$\omega_s^a(\theta)=\chi_s^a e^{s\theta}, \tag{3.40}$$

where $\chi_s^a$ are some constants. In particular $\chi_1^a$ are just the masses $m_a$ of the particles, because $s=1$ corresponds to the momentum operator. The multiparticle states are defined by multiple action of the creation operators,

$$|A_{a_1}(\theta_1)\cdots A_{a_M}(\theta_M)>=A_{a_1}(\theta_1)\cdots A_{a_M}(\theta_M)|0> . \tag{3.41}$$

By locality, we have:

$$P_s|A_{a_1}(\theta_1)\cdots A_{a_M}(\theta_M)>=\left(e^{s\theta_1}\chi_s^{a_1}+\cdots+e^{s\theta_M}\chi_s^{a_M}\right)|A_{a_1}(\theta_1)\cdots A_{a_M}(\theta_M)> . \tag{3.42}$$

Imposing the conservation of all the $P_s$ (for any $s$), acting on any multiparticle state (with any number of particles of arbitrary rapidities) poses stringent constraints on the scattering, so that only those processes are allowed for which the mass spectrum of the theory (i.e. the number of particles and their masses) is conserved as well as the set of two-momenta, [36]. Notice also that the states (3.41) do not form a basis for $M$ particle states. This is because they are not all linearly independent. If we define (3.41) with $\theta_1>\theta_2>\cdots>\theta_M$, then these states will be linearly independent. They are interpreted as a set of incoming particles (in-states). Outgoing particles (out-states) would correspond to $\theta_1<\theta_2<\cdots<\theta_M$. The S-matrix is therefore defined by:

$$|A_{b_M}(\theta_M)\cdots A_{b_1}(\theta_1)>=S_{b_1\cdots b_M}^{a_1\cdots a_M}(\theta_1,\cdots,\theta_M)|A_{a_1}(\theta_1)\cdots A_{a_M}(\theta_M)>, \tag{3.43}$$

where we have used the fact that the individual momenta are conserved and we assume a summation over repeated indices. The above scattering matrix factorizes into the product of two-particle processes defined by the following non-commutative algebra, also known as the *Zamolodchikov-Fateev algebra*:

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2)=S_{a_1a_2}^{b_1b_2}(\theta_1,\theta_2)A_{b_2}(\theta_2)A_{b_1}(\theta_1). \tag{3.44}$$

However, from (3.37), we see that Lorentz invariance implies,

$$S_{a_1a_2}^{b_1b_2}(\theta_1,\theta_2)=S_{a_1a_2}^{b_1b_2}(\theta_1-\theta_2). \tag{3.45}$$

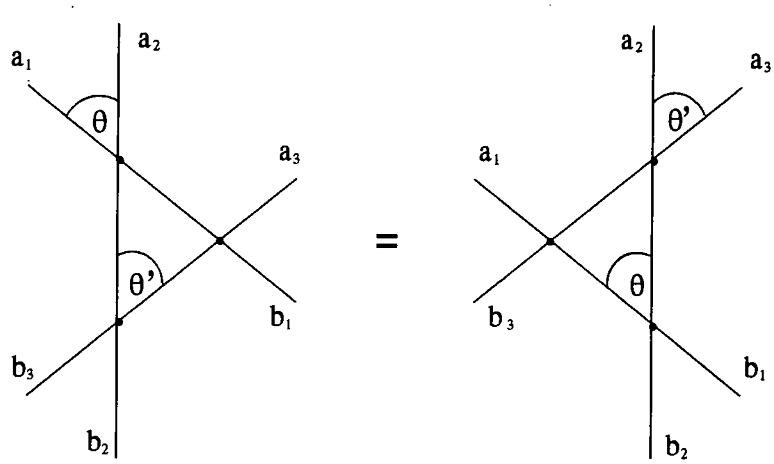


Figure 3.1: Yang-Baxter equation

In all known cases the S-matrix is an analytic function of the centre of mass energy, $s$, and invariant momentum transfer, $t$, in some neighbourhood of the physical region, except at normal thresholds, [26]. Given that in two dimensions $s$ is the only invariant Lorentz scalar, we can exploit the crossing invariance of the scattering processes by using relation (3.37) and imposing the following *analyticity-crossing condition*:

$$S_{ab}^{cd}(\theta) = S_{a\bar{d}}^{c\bar{b}}(i\pi - \theta), \tag{3.46}$$

in terms of the rapidity $\theta$. We also require the two-particle S-matrix to be *unitary*:

$$S_{ab}^{a'b'}(\theta) S_{b'a'}^{cd}(-\theta) = \delta_a^d \delta_b^c. \tag{3.47}$$

This relation can be seen as the compatibility condition for applying twice the commutation relations (3.44). The associativity of the algebra implies the following cubic relation,

$$S_{a_1 a_2}^{b_1 b_2}(\theta) S_{b_1 b_3}^{c_1 c_3}(\theta + \theta') S_{b_2 a_3}^{c_2 b_3}(\theta') = S_{a_1 a_3}^{b_1 b_3}(\theta + \theta') S_{b_1 b_2}^{c_1 c_2}(\theta) S_{a_2 b_3}^{b_2 c_3}(\theta'), \tag{3.48}$$

also known as the *star-triangle or Yang-Baxter equation.* Physically, it encodes the equality between two alternative factorization schemes of a three-particle process into two-particle ones as depicted in the diagram of fig.3.1.

Let us now consider briefly the analytic structure of the S matrix. The inverse transformation of (3.37),

$$\theta = ln \left[ \frac{s - m_1^2 - m_2^2 + \sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{2 m_1 m_2} \right], \tag{3.49}$$

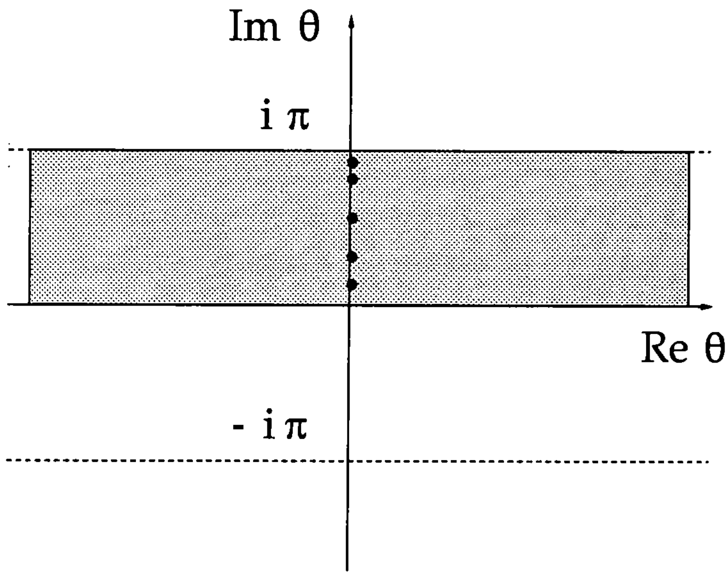


Figure 3.2: analytic structure of the S matrix

maps the physical sheet on the $s$ plane onto the the strip $0 \leq Im\theta \leq \pi$. The second sheet is mapped onto the strip $-\pi \leq Im\theta \leq 0$ and this structure is repeated with period $2i\pi$. The two square branch cut singularities at $(m_1 - m_2)^2$ and $(m_1 + m_2)^2$ in the complex plane of the Mandelstam variable $s$ are mapped onto $\theta = i\pi$ and $\theta = 0$, respectively. Simple poles on the imaginary axis of the physical sheet are associated with the bound states of the particles (see fig.3.2), [36].

In fig.3.2 the shaded area represents the physical sheet and the dots stand for the bound states. Suppose that $iu^c_{a_1 a_2}$ is the position of the pole of $S^{b_1 b_2}_{a_1 a_2}(\theta)$ associated with the bound state $A_c$ of the direct channel (see fig.3.3), i.e.

$$S^{b_1 b_2}_{a_1 a_2}(\theta) \simeq \frac{i f^c_{a_1 a_2} f^{b_1 b_2}_c}{\theta - iu^c_{a_1 a_2}}. \tag{3.50}$$

The objects $f^c_{a_1 a_2}$ represent the three-point couplings for the "fusion" of the particles $A_{a_1}, A_{a_2}$ and the bound state $A_c$ (fig.3.4).

The mass of the bound state satisfies the equation (cf.(3.37)):

$$m^2_c = m^2_{a_1} + m^2_{a_2} + 2m_{a_1} m_{a_2} cosu^c_{a_1 a_2}. \tag{3.51}$$

The bootstrap principle consists of assuming that all the bound states are stable massive states belonging to the spectrum of the theory, [25]. Consequently, the factorized scattering

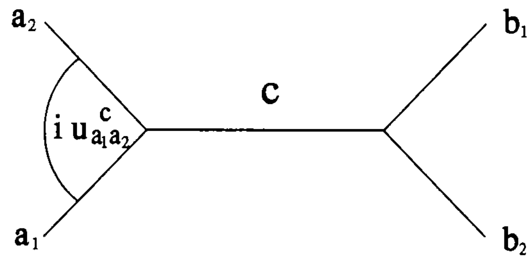


Figure 3.3: bound states

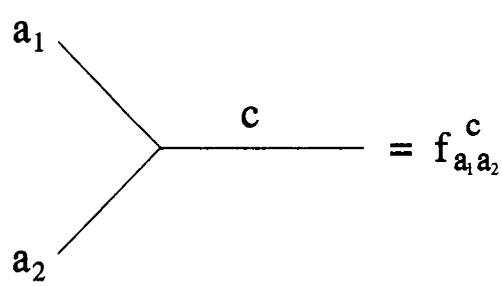


Figure 3.4: 3-point coupling

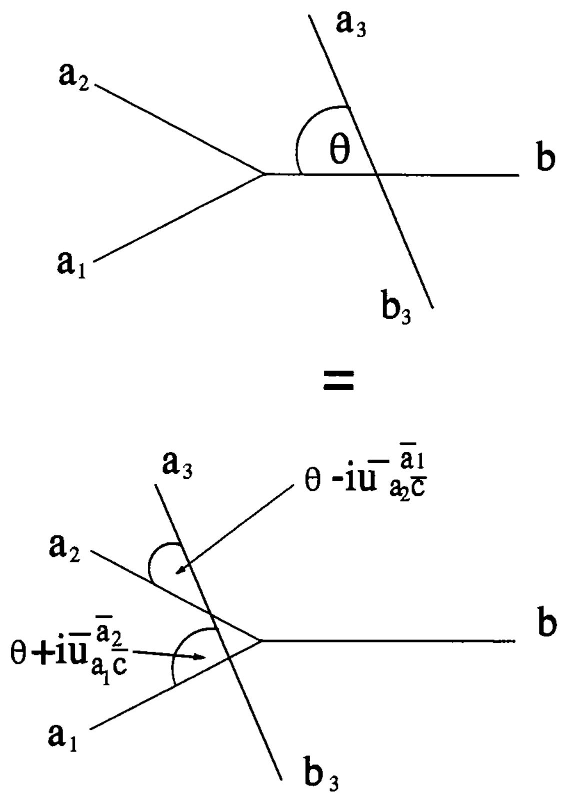


Figure 3.5: boostrap equations

applies to these states as well, yielding a set of bootstrap equations,

$$f^{c}_{a_1 a_2} S^{bb_3}_{ca_3}(\theta) = f^{b}_{c_1 c_2} S^{c_1 b_3}_{a_1 c_3}(\theta + i\bar{u}^{\bar{a}_2}_{a_1 \bar{c}}) S^{c_2 c_3}_{a_2 a_3}(\theta - i\bar{u}^{\bar{a}_1}_{a_2 \bar{c}}), \tag{3.52}$$

where $\bar{u} \equiv i\pi - u$. They are depicted in fig.3.5.

This set of conditions (eqs.(3.46), (3.47), (3.48), (3.52)) is sufficient to pin down the S-matrix up to a so-called "*CDD ambiguity*", [36],

$$S^{cd}_{ab}(\theta) \to S^{cd}_{ab}(\theta)\Phi(\theta), \tag{3.53}$$

where the "*CDD factor*" $\Phi(\theta)$ is an arbitrary function of the rapidity satisfying the equations:

$$\Phi(\theta) = \Phi(i\pi - \theta), \qquad \Phi(\theta)\Phi(-\theta) = 1. \tag{3.54}$$

The bootstrap equations may impose further restrictions on this function. Eliminating completely the CDD ambiguity involves a lot of guesswork. In most cases, the minimal solution (i.e. with the smallest number of zeroes and singularities) is usually the correct one.

## 3.5 Thermodynamic Bethe Ansatz

The analysis of higher order poles is more subtle and plays an important rôle in the classification program of bootstrap systems. This program consists of finding solutions to the constraints of unitarity, crossing symmetry, factorizability as well as all the constraints associated with the bootstrap principle, without any mention to any off-shell action. Although this alternative description of the QFT in terms of asymptotic states and S-matrix is necessarily on-shell, there are good arguments to believe that it actually contains all the information about the QFT. Some thermodynamic properties of the system can be extracted by the technique of *thermodynamic Bethe Ansatz* (TBA) and correlation functions computed using the form factor approach of Smirnov and Kirillov [27] or quantum determinants by Korepin et al.[8].

The technique of TBA consists of placing our system on the torus of fig.(2.3) with $\theta = 0$, [80]. The partition function is given by:

$$Z(R,L) = Trexp(-R\mathcal{H}_L)\,. \tag{3.55}$$

Using the argument of modular invariance, we can equally write:

$$Z(R,L) = Trexp(-L\mathcal{H}_R)\,. \tag{3.56}$$

$\mathcal{H}_R$ and $\mathcal{H}_L$ are the Hamiltonians for the system quantized along the $R$- and $L$-axes, respectively. In the limit $L \to \infty$, the partition function (3.56) is dominated by the ground state energy $E_0(R)$ of $\mathcal{H}_R$ and hence:

$$Z(R,L) \simeq exp[-LE_0(R)]\,. \tag{3.57}$$

Similarly, (3.55) is controlled by the bulk free energy $f(R)$ of the system at the finite temperature $1/R$:

$$Z(R,L) \simeq exp[-LRf(R)]\,. \tag{3.58}$$

Equating (3.57) and (3.58), we get:

$$E_0(R) = Rf(R). \tag{3.59}$$

The fact that we are placing our system in a finite box with periodic boundary conditions will impose certain quantization conditions upon the momenta of the particles. Assuming (for simplicity) that we have a system with $N$ bosons obeying the diagonal algebra[1],

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2) = S_{a_1a_2}(\theta_1 - \theta_2)A_{a_2}(\theta_2)A_{a_1}(\theta_1), \tag{3.60}$$

we obtain the following set of *Bethe equations*:

$$e^{ip_kL}\Pi^N_{(l\neq k)}S_{a_ka_l}(\theta_k - \theta_l) = 1, \qquad (k = 1, 2, \cdots, N), \tag{3.61}$$

where $p_k = m_k cosh\theta_k$. Notice that if the scattering is trivial, we get the well known formula for a set of $N$ free bosons in a periodic box of length $L$:

$$p_k = \frac{2\pi n_k}{L}, \tag{3.62}$$

where $n_k$ are arbitrary integers. More generally, we have to specify the selection rules obeyed by the particle states. For simplicity we assume that there is only one type of particle with the scalar S matrix $S(\theta)$ satisfying the unitarity condition $S(\theta)S(-\theta) = 1$. Obviously, $S(0) = \pm 1$. Suppose first that $S(0) = -1$. If the particle is a boson then its wave function should be symmetric. We thus conclude that states with particles of the same rapidity should be excluded. If, on the other hand, it is a fermion, then these states are allowed. Conversely, if $S(0) = +1$, then it is the fermions that obey an exclusion principle.

We now define the dimensionless quantity $r = m_1 R$. $m_1$ is the smallest mass in the spectrum of the theory and can be best thought of as the inverse of the correlation length. The limits $r \to 0$ and $r \to \infty$ determine the UV and IR behaviours of the theory in the thermodynamic limit. The UV limit is described by a conformal theory, where conformal invariance predicts the following scaling behaviour (cf.(2.62), (2.64)):

$$E_0(R) = \frac{2\pi}{R}\left(h_{min} + \bar{h}_{min} - \frac{c}{12}\right), \tag{3.63}$$

where $(h_{min}, \bar{h}_{min})$ are the smallest dimensions in the spectrum of the UV conformal theory and $c$ is its central charge. Zamolodchikov's $c$-function is defined by:

$$\tilde{c}(r) = -\frac{6R}{\pi}E_0(R). \tag{3.64}$$

Consequently:

$$\lim_{r\to 0}\tilde{c}(r) = c - 12(h_{min} + \bar{h}_{min}). \tag{3.65}$$

In chapter 8 we will perform this computation for the principal chiral model explicitly and in great detail. This case has some additional complications stemming from the fact that it is not a diagonal theory in the sense of eq.(3.60), [57].

[1] There is no summation over repeated indices.

# Chapter 4

# The principal chiral model

*This chapter is intended as an overview of the principal chiral model. I will focus on the integrability and background scattering as well as the limiting infrared Wess-Zumino-Witten conformal field theory. Some aspects of Kac-Moody algebras and coset constructions are explained. Particular emphasis is placed on the group $SU(2) \times SU(2)$. The bulk spectrum coincides with those of the Kondo model and of the Callan-Rubakov effect.*

The principal chiral model $PCM_k$ is defined by the action:

$$S_{PCM_k}[g] = \frac{1}{2\lambda^2} \int_{\partial\mathcal{B}} Tr \left\{ (g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g) \right\} d^2x + ik\Gamma(g), \tag{4.1}$$

where $\Gamma[g]$ is the Wess-Zumino-Witten (WZW) term [40]:

$$\Gamma[g] = \frac{1}{24\pi} \int_{\mathcal{B}} Tr \left\{ (g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\lambda g) \right\} \epsilon^{\mu\nu\lambda} d^3x. \tag{4.2}$$

In eq.(4.1) $g$ is a Lie group-valued field defined on a two-dimensional compact spacetime surface $\partial\mathcal{B}$. The region of integration $\mathcal{B}$ in eq.(4.2) is a three-dimensional simply connected manifold whose boundary is $\partial\mathcal{B}$. Topological arguments show that the ambiguity in this definition amounts to the WZW functional being determined up to a positive integer which can be reabsorbed into the constant $k$ in eq.(4.1) [41]. If the Lie group $G$ is simple[1] this ensures the positivity of the action [4].

For $k = 0$ the theory corresponds to a nonlinear sigma model. Its behaviour is massive. If $k \neq 0$ the renormalization group (RG) analysis reveals that it interpolates between two fixed points, [57]. The ultraviolet (UV) fixed point is controlled essentially by the first term in eq.(4.1). The RG flow of the coupling $\lambda^2$ terminates at the infrared (IR) fixed

[1] If it is semi-simple then it is the direct sum of simple components and the argument still holds.

point $\lambda^2 = 8\pi/k$, where the theory becomes massless at all distances (conformal). For a generic $k$ the RG trajectory arrives at the IR fixed point along the direction defined by the irrelevant field $Tr(g^{-1}\bar{\partial}gg^{-1}\partial g)$ of dimension $1+2/(k+2)$. For $k=1$, this field does not exist in the conformal theory and the incoming direction is defined by the operator $T\bar{T}$ composed from the components of the stress tensor of the IR conformal field theory. The point where the model crosses over from the region of one fixed point to the other introduces a mass scale that breaks the conformal invariance.

The $PCM_k$ was argued to be massless and integrable in refs. [47]-[51] and its thermodynamic Bethe Ansatz (TBA) equations proposed in [85] for $G = SU(2)$. Zamolodchikov's c-function was then shown to take the values $c_{UV} = 3$ and $c_{IR} = 3k/(k+2)$ at the fixed points. Al.B.Zamolodchikov and A.B.Zamolodchikov subsequently proposed the background scattering in terms of massless particles that leads to the correct TBA equations for $k = 1$ [57]. Following a prescription developed by Smirnov and Kirillov [52] in the context of the $SU(2)$-invariant Thirring model, they also showed that the form factors associated with the chiral currents obey the correct commutation relations. However, there is no known method to deal with the central term. Notwithstanding this, it can be shown to take the correct value by TBA analysis, [57]. Mejean and Smirnov [46] derived the form factors for the trace of the stress tensor.

## 4.1 The WZW model and Kac-Moody algebras

The action (4.1) satisfies the remarkable property [49],

$$S[gh^{-1}] = S[g] + S[h^{-1}] - \int_{\partial B} Tr\left\{\frac{1}{\lambda^2}g^{-1}(\partial_\mu g)h^{-1}(\partial^\mu h) - \frac{ik}{8\pi}\epsilon^{\mu\nu}g^{-1}(\partial_\mu g)h^{-1}(\partial_\nu h)\right\}, \tag{4.3}$$

which can be proved using Stokes' theorem. It is also worth noting that it enjoys invariance under the global transformation:

$$g(x) \to \Omega g(x)\bar{\Omega}^{-1}, \tag{4.4}$$

where $\Omega$, $\bar{\Omega}$ are arbitrary x-independent elements in the group $G$. The classical field equations arising from minimizing the action (4.1) with respect to the field $g$ are:

$$-\frac{1}{\lambda^2}\partial_\mu(g^{-1}\partial^\mu g) + \frac{ik}{8\pi}\epsilon^{\mu\nu}g^{-1}(\partial_\mu g)g^{-1}\partial_\nu g = 0. \tag{4.5}$$

At the IR stable fixed point, we have, [40], [44]:

$$S_{WZW,k}[g] = kW[g], \tag{4.6}$$

where

$$W[g] = \frac{1}{16\pi}\int_{\partial\mathcal{B}} Tr\left\{(g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g)\right\} d^2x + i\Gamma[g]. \tag{4.7}$$

Also formula (4.3) becomes:

$$S[gh^{-1}] = S[g] + S[h^{-1}] - \frac{k}{2\pi}\int_{\partial\mathcal{B}} d^2x Tr\left\{g^{-1}\partial_{\bar{z}}g)(h^{-1}\partial_z h)\right\}, \tag{4.8}$$

where $z$, $\bar{z}$ have the usual meaning. Using identity (4.8) it is straightforward to show that the invariance (4.4) is elevated to the infinite dimensional symmetry[2]:

$$g(x) \to \Omega(z)g(x)\bar{\Omega}^{-1}(\bar{z}), \tag{4.9}$$

at the IR fixed point. And the equations of motion (4.5) yield:

$$\partial_z(g^{-1}\partial_{\bar{z}}g) = 0. \tag{4.10}$$

Clearly from (4.10):

$$\partial_{\bar{z}}(\partial_z g g^{-1}) = g\partial_z(g^{-1}\partial_{\bar{z}}g)g^{-1} = 0. \tag{4.11}$$

We therefore define the basic currents:

$$\begin{cases} J = J^a t_a = -\frac{1}{2}k\partial_z g g^{-1}, \\ \bar{J} = \bar{J}^a t_a = -\frac{1}{2}k g^{-1}\partial_{\bar{z}} g, \end{cases} \tag{4.12}$$

where $t^a$ are the antihermitean generators of the Lie algebra $\mathcal{G}$ associated with the Lie group $G$:

$$[t^a, t^b] = f^{ab}{}_c t^c. \tag{4.13}$$

The currents (4.12) are the generators of the left and right infinite dimensional symmetries (4.9). Henceforth we shall only consider one of the sectors, say the analytic sector, as all the results that follow are easily transcribed into the antianalytic sector.

For infinitesimal current symmetry transformations,

$$\Omega(z) = I + \omega_a(z)t^a, \tag{4.14}$$

we have, [44]:

$$\delta_\omega J^a(z) = f^{ab}{}_c\omega_b(z)J^c(z) + \frac{1}{2}k\partial_z\omega^a(z). \tag{4.15}$$

---

[2]Actually it is the direct product of two symmetries: left and right gauge transformations.

Since the theory is conformal there is a traceless operator with components $T$, $\bar{T}$, that generates the conformal transformations (2.12) and satisfies $\partial_{\bar{z}}T = \partial_z\bar{T} = 0$. Assuming that the fundamental field $g$ is a primary operator with respect to the conformal algebra, then it can be shown that $J$ given by eq.(4.12) is also a primary operator with dimension 1. Under (2.11) we have:

$$\delta_\epsilon J^a(z) = \epsilon(z)\partial_z J^a(z) + \partial_z\epsilon(z)J^a(z). \tag{4.16}$$

The previous identities can be cast in the form of operator product expansions (OPE), [44], [42]:

$$\begin{cases} T(z)T(w) & = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \cdots \\ T(z)J^a(w) & = \frac{1}{(z-w)^2}J^a(w) + \frac{1}{z-w}\partial_w J^a(w) + \cdots \\ J^a(z)J^b(w) & = \frac{k\delta^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c}{z-w}J^c(w) + \cdots \end{cases} \tag{4.17}$$

The analyticity properties of $J$ and $T$ allow for the Laurent expansions:

$$\begin{cases} J^a(z) & = \sum_{n=-\infty}^{+\infty} J^a_n z^{-n-1}, \\ T(z) & = \sum_{-\infty}^{+\infty} L_n z^{-n-2}. \end{cases} \tag{4.18}$$

Equations (4.17) entail the following commutation relations for the modes:

$$\begin{cases} [L_n, L_m] & = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}, \\ [L_n, J^a_m] & = -mJ^a_{n+m}, \\ \left[J^a_n, J^b_m\right] & = f^{ab}{}_c J^c_{n+m} + \frac{1}{2}kn\delta^{ab}\delta_{n+m,0}. \end{cases} \tag{4.19}$$

The last set are called a *Kac-Moody algebra* $\hat{\mathcal{G}}$ at level $k$ based on the Lie algebra $\mathcal{G}$ of the group $G$. Every Kac-Moody algebra contains a subalgebra of the zero modes, which is isomorphic to the algebra $\mathcal{G}$ (cf.(4.13)):

$$\left[J^a_0, J^b_0\right] = f^{ab}{}_c J^c_0. \tag{4.20}$$

The definition of a primary operator requires some caution in this context. Remember that a primary operator in a conformal theory was defined to be an operator that transforms covariantly (i.e. as a tensor) under any conformal transformation. In particular its transformation rules are dictated by the subgroup $PSL(2, C)$ of projective transformations. In the framework of the WZW model we shall have to impose some additional requirement for an operator to be primary. This is because the conformal invariance is extended by the additional Kac-Moody symmetry (4.19). By analogy we require that a primary operator transform under a current symmetry transformation as it does under the subgroup of global gauge transformations generated by (4.20).

Specifically $\phi_l(z)$ is a primary operator of dimension $h_l$ with respect to (4.17), provided, [44]:

$$\begin{cases} T(z)\phi_l(w) &= \frac{h_l}{(z-w)^2}\phi_l(w) + \frac{1}{z-w}\partial_w\phi_l(w) + \cdots \\ J^a(z)\phi_l(w) &= \frac{t_l^a}{z-w}\phi_l(w) + \cdots \end{cases} \tag{4.21}$$

Here $t_l^a$ is the 'left' representation of the generators (4.13). With this definition neither $J$ nor $T$ are primary operators. On the other hand the fundamental field $g$ is a primary operator.

These equations lead to the set of Ward identities:

$$\begin{aligned} < T(z)\phi_1(z_1)\cdots\phi_N(z_N) > &= \textstyle\sum_{j=1}^N \left\{\frac{h_j}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right\} < \phi_1(z_1)\cdots\phi_N(z_N) >, \\ < J^a(z)\phi_1(z_1)\cdots\phi_N(z_N) > &= \textstyle\sum_{j=1}^N \frac{t_j^a}{z-z_j} < \phi_1(z_1)\cdots\phi_N(z_N) > . \end{aligned} \tag{4.22}$$

Hitherto I have not made any reference to the particular form of the stress tensor or how it is related to the fundamental field $g$ and the Kac-Moody currents for that matter. All the results listed above therefore apply to any conformal field theory with current algebra symmetry.

For the WZW model, the energy-momentum tensor is given by the Sugawara form [53]:

$$T(z) = \frac{1}{c_v + k} : J^a(z)J^a(z) :, \tag{4.23}$$

where $c_v$ is the second Casimir of the adjoint representation, defined by $f^a{}_{cd}f^{bcd} = c_v\delta^{ab}$. We thus get for the Virasoro generators:

$$L_n = \frac{1}{c_v + k}\sum_{m=-\infty}^{+\infty} : J_m^a J_{n-m}^a :, \tag{4.24}$$

where the normal ordering has the usual meaning[3]. It is interesting to note that the operators $J_n^a$ play the same rôle as do the oscillator modes $\alpha_n^\mu$ in the mode expansion of the flat space closed bosonic string theory. In fact the Virasoro modes have a form similar to (4.24) in terms of the $\alpha_n^\mu$, [6]:

$$L_n = \frac{1}{2}\sum_{\mu=1}^{d}\sum_{m=-\infty}^{+\infty} : \alpha_m^\mu\alpha_{n-m}^\mu : . \tag{4.25}$$

[3] Modes with $n > 0$ are placed to the right of the product.

When the group $G$ is abelian (e.g.$U(1)$) we actually get the Heisenberg algebras,

$$\left[J_n^a, J_m^b\right] = \frac{1}{2}kn\delta^{ab}\delta_{n+m,0}, \tag{4.26}$$

obeyed by the oscillator modes $\alpha_n^\mu$.

Let us now go back to definition (4.24). It is easy to show that the central charge is given by:

$$c = \frac{kD}{c_v + k}, \tag{4.27}$$

where $D$ is the dimension of the group $G$.

To construct irreducible representations, we proceed as before. We start by looking for the null vectors of the theory. In this case there are three types of null vectors possible:

$$\begin{array}{rl} (i) & \text{purely Virasoro algebra,} \\ (ii) & \text{combined Virasoro and current algebra,} \\ (iii) & \text{purely current algebra.} \end{array} \tag{4.28}$$

The first case was already discussed in chapter 2 and corresponds to minimal conformal theories with $c < 1$.

Let us consider the following type (ii) null vector, [44]:

$$\chi = (\alpha L_{-1} - J_{-1}^a J_0^a)\phi = (\alpha L_{-1} - J_{-1}^a t^a)\phi = 0, \tag{4.29}$$

where $\phi$ is a primary operator with dimension $h$. Since $\chi$ is a null vector then in particular:

$$L_1\chi = J_1^a\chi = 0.$$

These constraints fix the anomalous dimension and the value of $\alpha$:

$$\begin{aligned} h &= \tfrac{c_\phi}{c_v+k}, \\ \alpha &= -\tfrac{1}{2}(c_v + k), \end{aligned} \tag{4.30}$$

where $c_\phi$ is the second Casimir of the representation R, $t^a t^a = -c_\phi I$.

Moreover, given that $\chi$ is a null vector, any correlator where it appears must vanish. Using the definition of the modes together with the Ward identities, we obtain the following set of partial differential equations (*Knizhnik-Zamolodchikov equations*):

$$\left[\alpha\frac{\partial}{\partial z_i} - \sum_{j\neq i}^{n} \frac{t_i^a t_j^a}{z_i - z_j}\right] < \phi_1(z_1)\phi_2(z_2)\cdots\phi_n(z_n) >= 0, \tag{4.31}$$

following the same procedure as in section 2.2 for degenerate conformal theories. By investigating type (iii) null vectors a remarkable set of selection rules emerge, [42]. These state that most of the *a priori* conceivable representations lead to vanishing correlators and thus effectively decouple from the theory. In fact we will be left with only a finite number of representations. The only allowed ones are the highest weight irreducible integrable representations. By integrable we mean that the highest weight representation of the pseudospin associated with the highest root is finite.

To understand this better, consider $\theta$ to be the highest root of $\mathcal{G}$, associated with the element $-\tau^\theta$ of $\mathcal{G}$. We also define the element $h_\theta = [\tau^\theta, \tau^{-\theta}]$ of the Cartan subalgebra and choose a normalization such that $[h_\theta, \tau^\theta] = 2\tau^\theta$. Then we have the following $SU(2)$ pseudospin commutation relations:

$$\begin{aligned} [P^+, P^-] &= P^3, \\ [P^3, P^\pm] &= \pm 2P^\pm, \end{aligned} \tag{4.32}$$

where $P^- = J^\theta_{-1}$, $P^+ = J^{-\theta}_{+1}$, $P^3 = m - J^h_0$ and $m = 2k/(\theta,\theta)$ is called the level of the representation. Thus any representation of the current algebra decouples into a sum of representations of $SU(2)$ pseudospin. The representation of the Kac-Moody algebra is integrable if the representation ,

$$\left\{\phi_\lambda, P^-\phi_\lambda, (P^-)^2\phi_\lambda, \cdots\right\}, \tag{4.33}$$

containing the highest weight component $\phi_\lambda$ of the primary field $\phi$ with weight $\lambda$ is finite. In particular:

$$M = m - 2\frac{(\lambda,\theta)}{(\theta,\theta)} \geq 0, \qquad \text{M is integer}, \tag{4.34}$$

for an integrable representation. The fundamentals of these selection rules lie on strictly group theoretical arguments. They therefore hold not only for WZW type models but for any current algebra invariant theory. They place stringent constraints on the correlators so that many of them (even some involving integrable fields) vanish.

Let us consider our main example $SU(2)$. The highest weight root $\theta$ is the simple root $\alpha$, normalized so that $(\alpha,\alpha) = 2$. For spin $l$ we have $\lambda = l\alpha$ and so $2(\lambda,\theta)/(\theta,\theta) = 2l$. From (4.34) we conclude that this representation is integrable provided,

$$2l \leq k. \tag{4.35}$$

The allowed representations are thus:

$$l = 0, \frac{1}{2}, 1, \cdots, \frac{k}{2}, \tag{4.36}$$

and there are $(k+1)$ of them. From eq.(4.30) we infer the following conformal dimension for these representations:

$$h_l = \frac{l(l+1)}{k+2}. \tag{4.37}$$

The central charge (cf.(4.27)),

$$c = \frac{3k}{k+2}, \tag{4.38}$$

is in agreement with the value $c_{IR}$ of Zamolodchikov's c-function at the infrared stable fixed point.

An interesting question concerns the modular properties of the theory and can be stated as follows. Given the representation theory discussed above that holds for the left and right sectors, which of the allowed integrable representations do actually appear in a given theory, in what (analytic-antianalytic) combinations and how often? Gepner and Witten [42] found that if the coefficient $k$ of the action (4.1) is properly quantized and the group is simply connected then the partition function on the torus is modular invariant whenever the spectrum consists of *all* the left-right symmetric integrable representations, each appearing exactly once. Actually their result is more general than that as they also analyzed non-simply connected groups using orbifold ideas. But that need not concern us here.

A theory that satisfies these properties is called diagonal, because the partition function on the torus takes the diagonal form:

$$Z_{torus}(q) = \sum_{j\in J} \chi_j^*(q)\chi_j(q), \tag{4.39}$$

where $\chi_j(q) = Tr_j q^{L_0 - c/24}$ is the character of the representation $j$ and $q$ is the modular parameter. $J$ is the set of allowed representations.

Given the fact that $SU(2)$ is simply connected, we conclude that the corresponding conformal theory is diagonal. The Kac-Moody characters $\chi_l^{(k)}$ for the isospin $l$, level $k$ representation of the affine $A_1^{(1)}$ algebra are given by [43]:

$$\chi_l^{(k)}(\tau) = \frac{1}{\eta^3(\tau)} \sum_{n=-\infty}^{+\infty} [2n(k+2)+2l+1]\, exp\left\{\frac{i\pi\tau\,[2n(k+2)+2l+1]^2}{2(k+2)}\right\}, \tag{4.40}$$

where $0 \le l \le k/2$ and $\eta$ is Dedekind's function:

$$\eta(\tau) = exp\left(\frac{i\pi\tau}{12}\right) \Pi_{n=1}^{\infty}\left(1 - exp(2i\pi n\tau)\right). \tag{4.41}$$

For a given level $k$, the $(k+1)$ characters $\chi_l^{(k)}$ are all distinct and linearly independent. Their behaviour under the modular transformation (2.69), is encoded in the modular matrix (2.77), [43]:

$$S_l^{(k)l'} = \sqrt{\frac{2}{k+2}} sin \left\{ \frac{\pi(2l+1)(2l'+1)}{k+2} \right\}. \tag{4.42}$$

As an illustration, we consider the case $k = 1$. The central charge is equal to 1 and there are two integrable representations corresponding to isospins $l = 0$ and $l = 1/2$. Their conformal weights are (cf.(4.37)) $h_0 = \bar{h}_0 = 0$ and $h_{1/2} = \bar{h}_{1/2} = 1/4$. They are identified with the identity operator and the fundamental operator $g$ of the WZW action respectively[45]. The modular matrix $S^{(1)}$ reads:

$$S^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.43}$$

Using Verlinde's formula, we can also compute the structure constants of the fusion rules. They are found to be:

$$\begin{cases} N_{0l}^i = N_{l0}^i = \delta_l^i, \\ \\ N_{\frac{1}{2}\frac{1}{2}}^0 = 1, N_{\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}} = 0. \end{cases} \tag{4.44}$$

More generally, the structure constants for the $\widehat{SU(2)}_k$ conformal field theory can be shown to be of the form, [96]:

$$N_{jp}^l = \begin{cases} 1, & if|j-p| \leq l \leq min\{j+p, k-j-p\}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.45}$$

## 4.2 GKO coset construction

The Goddard, Kent and Olive (GKO) coset construction, [53]-[55] is a method for constructing representations of the Virasoro algebra out of representations of affine Kac-Moody algebras. It is widely believed that all the rational conformal field theories can be obtained in this fashion.

One of the key elements of this construction is the Sugawara form of the stress tensor. For a group $G$ with algebra $\mathcal{G}$ we have (cf.(4.24)):

$$L_n^{\mathcal{G}} = \frac{1}{k + c_v^{\mathcal{G}}} \sum_{m=-\infty}^{+\infty} : J_{n+m}^a J_{-m}^a : . \tag{4.46}$$

The $L_n^{\mathcal{G}}$ satisfy the Virasoro algebra:

$$\left[L_n^{\mathcal{G}}, L_m^{\mathcal{G}}\right] = (n-m)L_{n+m}^{\mathcal{G}} + \frac{c^{\mathcal{G}}}{12}(n^3 - n)\delta_{n+m,0}, \tag{4.47}$$

with central charge:

$$c^{\mathcal{G}} = \frac{kD_{\mathcal{G}}}{c_v^{\mathcal{G}} + k}. \tag{4.48}$$

$D_{\mathcal{G}}$ is as before the dimension of the group. If $\mathcal{G}$ is semi-simple there are generators $\left\{L_n^{\mathcal{G}_i}\right\}$ for each simple factor $\mathcal{G}_i$ $(1 \leq i \leq N)$ of $\mathcal{G}$. Then

$$L_n^{\mathcal{G}} = \sum_{i=1}^{N} L_n^{\mathcal{G}_i} \tag{4.49}$$

constitute a Virasoro algebra with central element:

$$c^{\mathcal{G}} = \sum_{i=1}^{N} c^{\mathcal{G}_i} = \sum_{i=1}^{N} \frac{k_i D_{\mathcal{G}_i}}{c_v^{\mathcal{G}_i} + k_i}. \tag{4.50}$$

We will henceforth assume that the representation of the Kac-Moody algebra is such that the following hermiticity property holds:

$$(J_n^a)^\dagger = J_{-n}^a, \qquad n \in \mathcal{Z}. \tag{4.51}$$

which implies (2.38) for the Virasoro generators. Friedan, Qiu and Shenker [16] have shown that this is a necessary condition for a Virasoro representation to be unitary.

Now consider a subalgebra $\mathcal{H} \subset \mathcal{G}$. The basis of $\mathcal{G}$ is chosen in such a way that the first $dim\mathcal{H}$ generators, $J_n^a$ $(a = 1, \cdots, dim\mathcal{H})$ form a basis of $\mathcal{H}$. By construction, there will be two Virasoso algebras $L_n^{\mathcal{G}}$, $L_n^{\mathcal{H}}$ and their difference

$$K_n = L_n^{\mathcal{G}} - L_n^{\mathcal{H}} \tag{4.52}$$

satisfies the Virasoro algebra with central charge,

$$c = c^{\mathcal{G}} - c^{\mathcal{H}}, \tag{4.53}$$

and commutes with $\mathcal{H}$:

$$[K_m, J_n^a] = 0, \qquad 1 \leq a \leq dim\mathcal{H}, \qquad m, n \in \mathcal{Z}. \tag{4.54}$$

Now we take $\mathcal{G} = su(2) \oplus su(2)$ and $\mathcal{H}$ to be the diagonal $su(2)$ subalgebra. We will generate the minimal sequence of FQS [16] by taking a level $k$ representation for the first $su(2)$ factor and a level 1 representation for the second one. Thus, the central charge is (cf.(4.50)):

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)}, \tag{4.55}$$

where we have used the fact that $D = 3$ for $su(2)$ and that the representation for the diagonal $su(2)$ has level equal to $k+1$. In eq.(4.55), we get the minimal series by taking $k = 1, 2, \cdots$

GKO have also shown [55] that the representation obtained by taking the $(k, l)$ representation with level $k$ and isospin $l$ of the first $\hat{su}(2)$ factor and the $(1, \epsilon)$, $\epsilon = 0$ or $1/2$, representation of the second $\hat{su}(2)$ factor decomposes into the direct sum of representations $(k', l') \times (c, h)$ of $\hat{\mathcal{H}} \times V$ (where $V$ denotes the Virasoro algebra $\{K_n\}$),

$$(k, l) \times (1, \epsilon) \approx \oplus_q \left(k+1, \frac{1}{2}(q-1)\right) \times (c, h_{p,q}(c)), \tag{4.56}$$

where $c$ is given by eq.(4.55), $p = 2l+1$ and the sum is taken over $1 \leq q \leq k+2$ such that p-q is even or odd, depending on whether $\epsilon = 0$ or $1/2$. In this way we generate *all* the values of $h_{p,q}$ given by the Kac formula (2.44). The representations of the antianalytic Virasoro algebra $\{\bar{K}_n\}$ will obviously be generated in identical fashion out of the representations of $\{\bar{J}_n^a\}$. As an example consider the universality class of the Ising model which has been identified with $c = 1/2$. In (4.55) this corresponds to $k = 1$. Its representations are thus generated by the coset group $\hat{SU}(2)_1 \times \hat{SU}(2)_1 / \hat{SU}(2)_2$. From (4.56) we have:

$$\begin{cases} (1,0) \oplus (1,0) & \to (2,0) \times \left(\frac{1}{2}, h_{1,1}\left(\frac{1}{2}\right)\right) \oplus (2,1) \times \left(1, h_{2,1}\left(\frac{1}{2}\right)\right) \\ (1,0) \oplus \left(1, \frac{1}{2}\right) & \to \left(2, \frac{1}{2}\right) \times \left(\frac{1}{2}, h_{2,2}\left(\frac{1}{2}\right)\right) \\ \left(1, \frac{1}{2}\right) \oplus (1,0) & \to \left(2, \frac{1}{2}\right) \times \left(\frac{1}{2}, h_{2,2}\left(\frac{1}{2}\right)\right) \\ \left(1, \frac{1}{2}\right) \oplus \left(1, \frac{1}{2}\right) & \to (2,0) \times \left(\frac{1}{2}, h_{2,1}\left(\frac{1}{2}\right)\right) \oplus (2,1) \times \left(\frac{1}{2}, h_{1,1}\left(\frac{1}{2}\right)\right) \end{cases} \tag{4.57}$$

We conclude that all the representations are indeed generated in this fashion.

Now consider the coset group $\hat{SU}(2)_k \otimes \hat{SU}(2)_2 / \hat{SU}(2)_{k+2}$. Its central charge is:

$$c = \frac{3k}{k+2} + \frac{3}{2} - \frac{3(k+2)}{k+4} = \frac{3}{2}\left\{1 - \frac{8}{(k+2)(k+4)}\right\}. \tag{4.58}$$

So, for $k = 1, 2, \cdots$ we recover the so called superconformal minimal series, [105]. We also have the following decomposition of representations:

$$\left(k, \frac{1}{2}[p-1]\right) \times (F, 2) \approx \oplus_q \left(k+2, \frac{1}{2}[q-1]\right) \times (c, h_{p,q}(c))_F, \tag{4.59}$$

where $c$ is given by (4.58), $1 \leq p \leq k+1$, and $F$ denotes the appropriate representation of Neveu-Schwarz (NS) or Ramond (R) super Virasoro algebra. The sum on the r.h.s. is over all $q$ such that $1 \leq q \leq k+3$ and $p - q \in \mathcal{Z} + 2\epsilon$, with $\epsilon = 0$ in the NS case and $\epsilon = 1/2$ in the R case. Here $h_{p,q}$ are the superconformal dimensions in either sector.

## 4.3 Background factorized scattering for k=1

Here we assemble a few results about the factorized scattering in the $PCM$, [57]. As we have discussed before the RG flow is massless but the scale invariance is broken at the crossover between the two fixed points. The spectrum of the theory therefore consists of stable massless particles: left-movers and right-movers. It is convenient to parametrise the on-mass-shell 2-momenta $(e,p)$ of the particles in terms of the rapidity variables $-\infty \leq \beta, \beta' \leq \infty$:

$$\begin{cases} e = p & = \frac{M}{2} e^{\beta}, \quad \text{for right-movers,} \\ e = -p & = \frac{M}{2} e^{-\beta'}, \quad \text{for left-movers.} \end{cases} \tag{4.60}$$

With this parametrisation opposite momenta still correspond to opposite rapidities, [93]. This situation however poses a few conceptual difficulties. For example it is difficult to imagine how two right-movers both travelling at the speed of light in the same direction are ever going to meet and interact. The very notion of asymptotic massless state is not very clear in two dimensions. However, by computing the massless limit of certain theories one can obtain perfectly sensible results. By ignoring these difficulties, one is able to derive the correct properties of massless flows by TBA, [93]. In any case if one wants to construct an inverse scattering program for conformal field theories, [89], [90], in order to understand how these theories are perturbed in a integrability preserving fashion then this approach in terms of massless states diagonalizing an infinite set of integrals of motion is inevitable.

For left-left and right-right scattering all Mandelstam variables vanish and since the scattering depends only on the dimensionless ratios of the momenta, the mass scale $M$ is arbitrary. The right-left scattering on the other hand distinguishes a preferable scale normalization $M$. The Mandelstam variable is now:

$$s = M^2 exp(\beta_1 - \beta_2), \tag{4.61}$$

for the scattering of a right-mover of rapidity $\beta_1$ and a left-mover of rapidity $\beta_2$. The soft scattering corresponds not to $\beta_1 \sim \beta_2$ but rather to $\beta_1 - \beta_2 \to -\infty$. The mass scale $M$ can thus be chosen so that the crossover occurs at $\beta_1 \sim \beta_2$.

Besides being massless these particles form doublets $(u,d)$ under the global $SU(2)$ symmetry (4.4). However there is an additional structure: each particle is also a kink, [98]. The $\hat{SU}(2)_k$ WZW model has $(k+1)$ degenerate vacua. The allowed kinks interpolate between adjacent vacua. So, for instance, each left-moving particle doublet can be labeled by $(u^{(L)}_{c,c\pm1}, d^{(L)}_{c,c\pm1})$, where $c$ is an index referring to the vacua ($c = 1, 2, \cdots, k+1$). In the simplest case ($k = 1$), which is the one we are interested in, the only nontrivial structure is that of a $(u, d)$ doublet. We then represent the $SU(2)_R$ doublet of right-movers by the symbol $R_a(\beta)$ ($a = \pm$ is the right isotopic index) and the $SU(2)_L$ doublet of left-movers by $L_{\bar{a}}(\beta')$ ($\bar{a} = \pm$ is the left isotopic index) with energy spectra given by (4.60).

The charge conjugation operator $C$ is defined with respect to the $SU(2)$ symmetry by:

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.62}$$

We shall denote the antiparticles of $R_a(\beta)$ and $L_{\bar{a}}(\beta')$ by $\bar{R}_{\underline{a}}(\beta)$ and $\bar{L}_{\underline{\bar{a}}}(\beta')$, respectively. Let us now consider the general $2 \to 2$ scattering of a particle $R_a(\beta_1)$ with its antiparticle $\bar{R}_{\underline{b}}(\beta_2)$. The S-matrix element is given by [56][4]:

$$\begin{aligned} {}^{out}< R_c(\beta_1')\bar{R}_{\underline{d}}(\beta_2')|R_a(\beta_1)\bar{R}_{\underline{b}}(\beta_2)>^{in} &= \delta(p_1' - p_1)\delta(p_2' - p_2)F^{cd}_{ab}(\beta) \\ &-\delta(p_1' - p_2)\delta(p_2' - p_1)B^{dc}_{ab}(\beta), \end{aligned} \tag{4.63}$$

where $\beta \equiv \beta_1 - \beta_2$. For the "forward" and "backward" amplitudes we choose the following isospin preserving expressions:

$$\begin{cases} F^{cd}_{ab}(\beta) = \delta^c_a\delta^d_b u_1(\beta) + \delta_{ab}\delta^{cd}v_1(\beta), \\ B^{dc}_{ab}(\beta) = \delta^c_a\delta^d_b u_2(\beta) + \delta_{ab}\delta^{dc}v_2(\beta). \end{cases} \tag{4.64}$$

However for massless particles backward scattering is unacceptable and we therefore set $u_2(\beta) = v_2(\beta) = 0$, [57].

The particle-particle S-matrix element is given by:

$$\begin{aligned} {}^{out}< R_c(\beta_1')R_d(\beta_2')|R_a(\beta_1)R_b(\beta_2)>^{in} &= \delta(p_1' - p_1)\delta(p_2' - p_2)S^{cd}_{ab}(\beta) \\ &-\delta(p_1' - p_2)\delta(p_2' - p_1)S^{dc}_{ab}(\beta), \end{aligned} \tag{4.65}$$

with

$$S^{cd}_{ab}(\beta) = \sigma_T(\beta)\delta^c_a\delta^d_b + \sigma_R(\beta)\delta^d_a\delta^c_d. \tag{4.66}$$

$\sigma_T(\beta)$ and $\sigma_R(\beta)$ are the transition and reflection amplitudes, respectively. It is also convenient to introduce the 2-particle amplitudes in the isovector and isoscalar channels:

$$\begin{cases} S_V(\beta) = \sigma_T(\beta) + \sigma_R(\beta), \\ S_0(\beta) = \sigma_T(\beta) - \sigma_R(\beta). \end{cases} \tag{4.67}$$

[4] The minus sign in this expression arises because the particles are fermions. We shall come to this point again later.

Using the requirements of factorizability, unitarity and crossing symmetry, the following minimal solution was suggested in ref.[57]:

$$\begin{cases} u_1(\beta) = -\sigma_T(\beta) - \sigma_R(\beta), \\ v_1(\beta) = \sigma_R(\beta), \\ \sigma_T(\beta) = \frac{i}{\pi}\beta\sigma_R(\beta), \\ \sigma_R(\beta) = -\frac{i\pi}{\beta - i\pi} S_V(\beta), \end{cases} \tag{4.68}$$

where

$$S_V(\beta) = \frac{\Gamma(\frac{1}{2}+\frac{\beta}{2i\pi})\Gamma(-\frac{\beta}{2i\pi})}{\Gamma(\frac{1}{2}-\frac{\beta}{2i\pi})\Gamma(\frac{\beta}{2i\pi})} = \\ = exp\left\{-\int_0^{+\infty} dk \frac{e^{-\pi k/2}}{2cosh(\pi k/2)}\frac{sin(\beta k)}{k}\right\}. \tag{4.69}$$

Of course we get exactly the same expression for the L-L scattering. The non-trivial right-left scattering is defined by the commutation relations:

$$R_a(\beta)L_{\bar{a}}(\beta') = U_{a\bar{a}}^{b\bar{b}}(\beta - \beta')L_{\bar{b}}(\beta')R_b(\beta). \tag{4.70}$$

As we discussed before, this scattering breaks the scale invariance thus spoiling the $\hat{SU}(2)\times \hat{SU}(2)$ current algebra symmetry (4.9). However action (4.1) is invariant under the global $SU(2)_L \times SU(2)_R$ isotopic symmetry (4.4) at all distances. The only form of $U_{a\bar{a}}^{b\bar{b}}(\beta)$ preserving this symmetry is:

$$U_{a\bar{a}}^{b\bar{b}}(\beta) = U_{RL}(\beta)\delta_a^b\delta_{\bar{a}}^{\bar{b}}. \tag{4.71}$$

The factorization constraint is trivially met for this choice. For massless particles there is a combined unitarity-crossing restriction [83],

$$U_{RL}(\beta + i\pi)U_{RL}(\beta) = 1. \tag{4.72}$$

The simplest non-trivial solution proposed in ref.[57] is:

$$U_{RL}(\beta) = \frac{1}{U_{LR}(-\beta)} = tanh\left(\frac{\beta}{2} - \frac{i\pi}{4}\right). \tag{4.73}$$

It is worth noting that both amplitudes (4.66) and (4.73) have no poles on the physical sheet. Also, we see from the soft scattering ($\beta \to -\infty$) in eq.(4.73) that the fields behave as fermions. And since $S_{aa}^{aa}(0) = 1$, we conclude that we have a selection rule preventing any two particles of the same type to be in exactly the same quantum state (cf. section 3.5).

In ref.[57] it was shown that the TBA equations based on this background scattering lead to the correct central charge in the IR limit.

# Chapter 5

# Boundary Conformal Field Theory

*In this chapter we extend the previous methods to conformal systems with boundaries and determine what novel features arise in this new situation.*

In the presence of boundaries it is natural to address the following issues:

1) What happens with the correlation functions? Do they still satisfy partial differential equations as in the bulk? And in particular, is conformal invariance by itself sufficient to pin down the 2- and 3-point functions as before?

2) Does the theory still have the same set of primary operators and if so do they have the same set of scaling dimensions?

3) Is there some way of classifying the possible boundary conditions that give a consistent conformal theory?

## 5.1 Correlation functions

Our prototype geometry will be that of the semi-infinite complex plane. The boundary will be taken to lie along the real axis. Intuitively, we see that since only the reparametrizations that preserve the boundary are allowed, the number of constraints that we are able to impose on the Green's functions near the boundary are necessarily less than in the bulk. In fact, there will be half of the symmetries available as in the bulk case. We also note that the decay of correlations of boundary operators along the boundary, will be dictated by just one number, which is called the *surface scaling dimension*, [58]. For instance, if $|x_1 - x_2|$ is the distance between the locations of two operators on the boundary (that correspond to the order parameter), then the correlation will behave as $|x_1 - x_2|^{-2\tilde{\Delta}}$, where $\tilde{\Delta}$ is the surface scaling dimension. This is in contrast with the correlations between surface and bulk quantities which are completely determined by conformal invariance.

Consider the infinitesimal conformal transformation (2.11). To preserve the geometry, only transformations for which $\zeta(z)$ is real analytic (i.e. $\overline{\zeta(z)} = \zeta(\bar{z})$) are allowed, [58]. The Ward identity (2.30) sill remains valid, with the contour $C$ now being restricted to the upper half-plane for the analytic part and the contour $\bar{C}$ to the lower half-plane for the antianalytic part. However we can no longer decouple the two sectors. The way to proceed is to analytically continue the definition of the stress tensor into the lower half-plane, according to:

$$T(z) = \bar{T}(z), \qquad \text{for } Imz \leq 0. \tag{5.1}$$

The boundary condition (5.1) at $Imz = 0$, corresponds to $T_{xy} = 0$ in Cartesian coordinates. This has a precise physical meaning, namely that there is no flux of energy across the surface.

More generally, if we have a system enjoying a certain symmetry generated by the chiral currents $(W, \bar{W})$ such that $\partial_{\bar{z}} W = \partial_z \bar{W} = 0$, then the boundary condition is, [60]:

$$W = \bar{W}, \qquad \text{for } Imz = 0. \tag{5.2}$$

This generalization plays a rôle whenever we consider extensions of the Virasoro algebra such as the current algebra symmetry and superconformal invariance.

The conformal Ward identity (2.30) can thus be rewritten in the form, [58]:

$$\begin{aligned} &\tfrac{1}{2i\pi} \oint_C dw \zeta(w) < T(w) \Pi_{i=1}^N \phi(z_i, z_i') > - \tfrac{1}{2i\pi} \oint_{\bar{C}} dw \zeta(w) < T(w) \Pi_{i=1}^N \phi(z_i, z_i') > = \\ &\quad = \sum_{i=1}^N \left[ \zeta(z_i) \partial_{z_i} + \zeta(z_i') \partial_{z_i'} + h \partial_{z_i} \zeta(z_i) + \bar{h} \partial_{z_i'} \zeta(z_i') \right] < \Pi_{i=1}^N \phi(z_i, z_i') >, \end{aligned} \tag{5.3}$$

where we relabelled $\bar{z}_i = z_i'$ and where $C$ and $\bar{C}$ are the contours in 5.1 enclosing $z_i$ and $z_i'$, respectively.

Because of the boundary condition (5.1) the two straight portions of $C$ and $\bar{C}$ exactly cancel and the two integrals on the l.h.s. of eq.(5.3) collapse into one single integral around a large contour enclosing all the $z_i$ and $z_i'$. Using Cauchy's theorem, we get:

$$\begin{aligned} < T(z) \Pi_{i=1}^N \phi(z_i, z_i') > = \sum_{i=1}^N \left\{ \tfrac{h}{(z-z_i)^2} + \tfrac{1}{z-z_i} \partial_{z_i} + \tfrac{\bar{h}}{(z-z_i')^2} + \tfrac{1}{z-z_i'} \partial_{z_i'} \right\} \times \\ \times < \Pi_{i=1}^N \phi(z_i, z_i') > . \end{aligned} \tag{5.4}$$

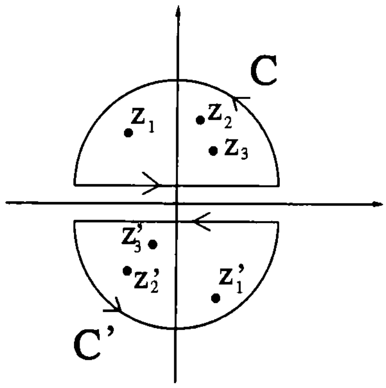


Figure 5.1: contours

This means that the correlation function $<\phi(z_1,\bar{z}_1)\cdots\phi(z_N,\bar{z}_N)>$ in the semi-infinite geometry, regarded as a function of $(z_1,\cdots,z_N,\bar{z}_1,\cdots,\bar{z}_N)$ satisfies the same differential equation as does the bulk correlation function $<\phi(z_1,\bar{z}_1)\cdots\phi(z_{2N},\bar{z}_{2N})>$ regarded as a function of $(z_1,\cdots,z_{2N})$ only. In particular the 2-point function satisfies the same differential equations as the bulk 4-point function. The only difference lies in the distinct boundary conditions that they obey. We conclude that conformal invariance alone is not sufficient to fully determine any of the correlators in the surface geometry. We need additional information (e.g. having a degenerate theory, finite number of primary operators, etc.) to obtain sufficient constraints.

Let us now consider an example of a model with an extended conformal symmetry, namely the WZW model. This new situation might appear awkward at the first sight, from the very definition of the WZW action (4.1). This is because analyzing boundaries in this context would imply considering the boundary of a boundary, which is evidently an empty set. Nevertheless, we can be cavalier about it by ignoring the classical action (4.1) altogether and going directly to the quantum theory. We can then define the theory in an axiomatic fashion by introducing the generators of the current algebra $(J,\bar{J})$ and defining the stress tensor via the Sugawara construction (4.23). We assume as our boundary condition, [63]:

$$J(z)=\bar{J}(z), \qquad \text{for } Im z\leq 0, \tag{5.5}$$

in agreement with (5.2). In practical terms this implies not only the conservation of the current algebra symmetry, but equally, as we shall see, of conformal invariance. Given the variation of a primary field $\phi$ (4.21) under an infinitesimal current algebra transformation

(4.14), then (5.5) translates into the following boundary condition:

$$\omega^a(z)t^a\phi(z) = -\phi(z)\omega^a(z)\bar{t^a}, \qquad Imz \leq 0, \tag{5.6}$$

where we used $\overline{\omega^a(z)} = \omega^a(\bar{z})$. Following the same procedure as before, we get the following Ward identity:

$$\begin{aligned} \tfrac{1}{2i\pi}\oint_C dz\omega^a(z) < J^a(z)\Pi_{i=1}^N\phi(z_i,z_i') >= \textstyle\sum_{i=1}^N \Big[\omega^a(z_i)t^a < \Pi_{j=1}^N\phi(z_j,z_j') > \\ - < \Pi_{j=1}\phi(z_j,z_j') > \omega^a(z_i')\bar{t^a}\Big], \end{aligned} \tag{5.7}$$

where the contour $C$ encloses all the points $z_i$ and $z_i'$. Using eq.(5.6), we finally get:

$$< J^a(z)\Pi_{i=1}^N\phi(z_i,z_i') >= \sum_{i=1}^N\left[\frac{t^a}{z-z_i} + \frac{t^a}{z-z_i'}\right] < \Pi_{j=1}^N\phi(z_j,z_j') > . \tag{5.8}$$

Again, the conclusion is the same as before. There is a correspondence:

$$\text{2-point function on the surface} \leftrightarrow \text{4-point function in the bulk}, \tag{5.9}$$

in the same sense as before. Because the choice of boundary conditions (5.2) leads to one surviving conformal algebra (or extended conformal algebra) the rest of the analysis concerning the representations and existence of null vectors still goes through. It remains to find out what operators are allowed for a consistent theory and with what surface scaling dimensions.

## 5.2 Boundary conformal field theory on a strip. The genuine Casimir effect

Let us now go back to our approach on the strip, [59]. We consider a strip of width $R$ with for instance *free* (the order parameter is unconstrained at the boundary) or *fixed* (the order parameter takes a fixed value on the surface) boundary conditions at the ends. This is called the "*L-channel*" because the time-direction was chosen to be the $L$-axis, [86]. This time, since there is only one set of Virasoro modes, one gets:

$$T_{strip}(w) = \left(\frac{R}{\pi z}\right)^2 T_{plane}(z) + \frac{c}{24z^2}, \tag{5.10}$$

under the conformal transformation (2.58). Consequently, if we call $H_F$ the Hamiltonian on the strip with free or fixed boundary conditions, we get:

$$L_0 \to \frac{R}{\pi}H_F + \frac{c}{24}. \tag{5.11}$$

Again there is a one-to-one correspondence between the eigenvalues of $H_F$ and the surface scaling dimensions $\tilde{\Delta}_n$ (eigenvalues of $L_0$). The free energy this time takes the finite size scaling form, [15], [17]:

$$F \sim fR + f^x - \frac{\pi c}{24R}, \tag{5.12}$$

where $f^x/2$ is the surface free energy. The anomaly term is called the *genuine Casimir effect.* This is because we have a finite geometry with a real boundary, which is closer to the spirit of the Casimir effect, [3].

We are considering a situation where we have the same boundary condition on both sides of the strip. However, nothing prevents us from having different boundary conditions, say $\alpha$ and $\beta$, on either side of the strip and more general than free or fixed. The existence of the surviving Virasoro algebra depends only on having conformally invariant boundary conditions, $T_{\tau\sigma} = 0$ at $\sigma = 0, \pi$, [60]. The corresponding Hamiltonian is denoted $H_{\alpha\beta}$ and of course (because conformal invariance is conserved) its eigenvalues will fall into irreducible representations of the Virasoro algebra. Let us denote by $n^i_{\alpha\beta}$ the number of times the representation $i$ appears in its spectrum. If we go back to the geometry on the upper half-plane, we realize that there is a discontinuity in the boundary condition at the origin. $T(z)|0>$ is therefore not regular at the origin and in particular the new vacuum is no longer annihilated by $L_{-1}$. Cardy [60] interpreted this discontinuity as a consequence of the insertion of a boundary operator $\phi_{\alpha\beta}(0)$ acting on the true vacuum. By definition the new "vacuum" is a state with the lowest eigenvalue of $L_0$ in the theory. We conclude that the operator $\phi_{\alpha\beta}$ must be the primary operator corresponding to the representation $i$ for which $n^i_{\alpha\beta}$ is nonvanishing and the eigenvalue of $L_0$ is the lowest. These considerations have the important consequence that there will be a biunivocal correspondence between the set of conformal blocks and the set of conformally invariant boundary conditions, [60]. To find out what the operator content is, we consider again the geometry of the strip with periodic boundary conditions in the 'time' direction. Because the resulting topology is that of an annulus, the system is no longer invariant under the full modular group. However, the direction of time remains arbitrary and therefore eq.(2.69) still holds. The partition function in the "*L-channel*" is given by:

$$Z_{\alpha\beta}(q) = Tre^{-RH_{\alpha\beta}} = \sum_i n^i_{\alpha\beta}\chi_i(q) = \sum_{i,j} n^i_{\alpha\beta} S^j_i \chi_j(\tilde{q}), \tag{5.13}$$

where $\chi_i(q)$ is as before the character of the representation $i$, $n^i_{\alpha\beta}$ is its degeneracy in the spectrum of $H_{\alpha\beta}$ and we used eq.(2.77) in the last step. We can equally well define the Hamiltonian $H^{(P)}$ in the "*R-channel*" (*R*-axis is the time-direction):

$$H^{(P)} = -\frac{\pi}{2L}\left[L_0^{(P)} + \bar{L}_0^{(P)} - \frac{c}{12}\right]. \tag{5.14}$$

We then get:

$$Z_{\alpha\beta}(q) = <\alpha|e^{-RH^{(P)}}|\beta>, \tag{5.15}$$

where $|\alpha>$ and $|\beta>$ are the boundary states corresponding to the boundary conditions $\alpha$ and $\beta$, respectively. On the strip the Fourier components of the generators $(W, \bar{W})$ satisfy (cf.(5.2)):

$$\left(W_n - (-1)^s \bar{W}_{-n}\right)|\alpha>= 0, \tag{5.16}$$

together with a similar equation for $|\beta>$. $s$ is the spin of $W$. We also assume that the Fourier modes satisfy the hermiticity condition $W_n^\dagger = W_{-n}$. For $W = T$, we have from the above equation:

$$(L_n - \bar{L}_{-n})|\alpha>= 0. \tag{5.17}$$

Although $|\alpha>$ is a boundary state, in this picture it belongs to the Fock space of the periodic system (spanned by both $\{L_n\}$ and $\{\bar{L}_m\}$). The solution of eqs.(5.17) was found by Ishibashi, [63]. $|\alpha>$ will be some linear combination of states of the form:

$$|j>\equiv \sum_n |j;n>\otimes\overline{|j;n>}, \tag{5.18}$$

where $j$ labels the irreducible representations of the Virasoro algebra and $\{|j;n>\}$, $\left\{\overline{|j;m>}\right\}$ are orthonormal basis for the representations $j$ of the holomorphic and antiholomorphic Virasoro algebras. And this yields the following linear combination:

$$|\alpha>\equiv \sum_{j\in J} <j0|\alpha>|j>, \tag{5.19}$$

where $J$ is the set of permissible representations of the Virasoro algebra and:

$$|j0>\equiv |j;0>\otimes\overline{|j;0>}. \tag{5.20}$$

From (5.15) and (5.19), we then have:

$$Z_{\alpha\beta} = \sum_{j\in J} <\alpha|j0><j0|\beta><j|e^{-RH^{(P)}}|j>. \tag{5.21}$$

Assuming that our theory is diagonal in the sense of eq.(4.39) then we can show that:

$$<j|e^{-RH^{(P)}}|j>=<j|(\tilde{q}^{1/2})^{L_0^{(P)}+\bar{L}_0^{(P)}-c/12}|j>= \chi_j(\tilde{q}). \tag{5.22}$$

Equating (5.13) with (5.21) and assuming that we are dealing with a rational theory with a finite number of linearly independent characters, we obtain *Cardy's equations*, [60]:

$$\sum_i n^i_{\alpha\beta} S^j_i =< \alpha|j0 >< j0|\beta > . \tag{5.23}$$

There always exists a boundary state $|\tilde{0}>$ satisfying $n^i_{\tilde{0}\tilde{0}} = \delta^i_0$. Substituting in eq.(5.23), we obtain $S^j_0 = |<\tilde{0}|j0>|^2$. It can be shown [60] that $S^j_0$ is always a positive real number. Consequently:

$$<\tilde{0}|j0>= (S^j_0)^{\frac{1}{2}}. \tag{5.24}$$

Similarly there is a state $|\tilde{l}>$ such that $n^i_{\tilde{0}\tilde{l}} = \delta^i_l$. From (5.23), (5.24) and (5.19), we get:

$$|\tilde{l}>= \sum_j \frac{S^j_l}{(S^j_0)^{\frac{1}{2}}} |j> . \tag{5.25}$$

Let us assume that we know $n^i_{\alpha\alpha}$ for a boundary state $|\alpha>$, then we can construct more general solutions to Cardy's equations by fusing a new conformal tower, say $l$, with the previous one yielding a new boundary state $|\beta>$, according to, [60]:

$$n^i_{\alpha\beta} = \sum_k N^i_{kl} n^k_{\alpha\alpha}, \tag{5.26}$$

$$< j0|\beta >=< j0|\alpha > \frac{S^j_l}{S^j_0}, \tag{5.27}$$

where we are implicitly assuming that the fusion rule coefficients $N^j_{il}$ are a solution of Verlinde's formula (2.78). To show that eqs.(5.26) and (5.27) constitute a solution to Cardy's equations, let us first consider the r.h.s. of eq.(5.23):

$$< \alpha|j0 >< j0|\beta >=< \alpha|j0 >< j0|\alpha > \frac{S^j_l}{S^j_0}, \tag{5.28}$$

where we used eq.(5.27). From eq.(5.26) and Verlinde's formula, the l.h.s. becomes:

$$\sum_i n^i_{\alpha\beta} S^j_i = \sum_{i,k} N^i_{kl} n^k_{\alpha\alpha} S^j_i = \sum_k n^k_{\alpha\alpha} \frac{S^j_k S^j_l}{S^j_0} = \frac{S^j_l}{S^j_0} < \alpha|j0 >< j0|\alpha >, \tag{5.29}$$

which is indeed equal to (5.28). Let us consider as an example the Ising model. From the modular matrix (2.72) for minimal theories and (5.25), we obtain:

$$\begin{cases} |\tilde{0}> & = \frac{1}{\sqrt{2}}|0> + \frac{1}{\sqrt{2}}|\epsilon> + \frac{1}{\sqrt[4]{2}}|\sigma>, \\ |\tilde{\frac{1}{2}}> & = \frac{1}{\sqrt{2}}|0> + \frac{1}{\sqrt{2}}|\epsilon> - \frac{1}{\sqrt[4]{2}}|\sigma>, \\ |\tilde{\frac{1}{16}}> & = |0> - |\epsilon> . \end{cases} \tag{5.30}$$

Note that $|\tilde{0}>$ and $|\tilde{\frac{1}{2}}>$ differ only in the sign of the coefficient of the $\mathcal{Z}_2$-odd state $|\sigma>$. It is then natural to identify $|+>$ (all spins up on the boundary) and $|->$ (all spins down on the boundary) with $|\tilde{0}>$ and $|\tilde{\frac{1}{2}}>$, respectively. Under duality, the energy-like state $|\epsilon>$ changes sign and the free boundary state $|f>$ goes into an equal superposition of the $|+>$ and $|->$ states. Thus we may identity $|f>$ with $|\tilde{\frac{1}{16}}>$. With this identification, we get from eq. (5.23): $n^{+}_{++} = n^{+}_{--} = n^{+}_{ff} = n^{-}_{ff} = 1$ and all the other $n^{i}_{\alpha\alpha} = 0$. Substituting in (5.26) we obtain the following operator content in the different sectors:

$$\begin{aligned} (++),\quad (--): \quad & h = 0 \\ (ff): \quad & h = 0, \tfrac{1}{2} \\ (+-): \quad & h = \tfrac{1}{2} \\ (+f),\quad (-f): \quad & h = \tfrac{1}{16} \end{aligned} \tag{5.31}$$

in agreement with [59], [62]. We see from this analysis that although we have an explicit formula (5.25) for the boundary states it is actually quite difficult to interpret what the boundary conditions correspond to at the level of the microscopic degrees of freedom in the statistical system. We usually need some additional information about the symmetries of the statistical model (e.g. duality, $\mathcal{Z}_2$ symmetry, etc). Another important point is that the scaling dimensions in (5.31) do not, in general, correspond to the same operators as in the bulk. For instance, in the sector $(ff)$ the operator corresponding to $h = 1/2$ was shown to be odd under the $\mathcal{Z}_2$ symmetry, [59]. It is therefore interpreted as the magnetization and not as the energy density.

Let us now move on to the WZW model. According to (5.16) the boundary condition is:

$$(J^a_n + \bar{J}^a_{-n})|\alpha >= 0. \tag{5.32}$$

Using the analogy with the oscillator modes $(\alpha^\mu_n, \tilde{\alpha}^\mu_n)$ (4.25) (or taking the group $G$ to be abelian, e.g. $G = U(1)$), we see that this would correspond to a Neumann boundary condition in the flat case, [67]. From the Kac-Moody algebra (4.19) it is straightforward to show that (5.32) form first class constraints as do (5.17). If the stress tensor is related to the Kac-Moody generators by the Sugawara form (4.24), then (5.32) actually imply (5.17) as we claimed before:

$$L_n|\alpha > \; = \tfrac{1}{c_v+k}\textstyle\sum_m : J^a_{n-m} J^a_m : |\alpha >= \tfrac{1}{c_v+k}\sum_m : \bar{J}^a_{-m}\bar{J}^a_{m-n} : |\alpha >= \bar{L}_{-n}|\alpha > . \tag{5.33}$$

For a $k = 1$ Kac-Moody symmetry, we get from (4.43) and (5.25) the following boundary states:

$$\left\{ \begin{array}{l} |\tilde{0}> = \frac{1}{\sqrt[4]{2}} \left( |0> + |\frac{1}{2}> \right), \\ \\ |\tilde{\frac{1}{2}}> = \frac{1}{\sqrt[4]{2}} \left( |0> - |\frac{1}{2}> \right). \end{array} \right. \tag{5.34}$$

Also from (4.44), (5.23) and (5.26), we have the following operator content:

$$\begin{array}{c} (\tilde{0}, \tilde{0}), \quad \left(\tilde{\frac{1}{2}}, \tilde{\frac{1}{2}}\right): \; h = 0 \\ \\ \left(\tilde{0}, \tilde{\frac{1}{2}}\right), \quad \left(\tilde{\frac{1}{2}}, \tilde{0}\right): \; h = \frac{1}{4} \end{array} \tag{5.35}$$

We are of course using the fact that as we discussed in section (4.1) the $SU(2)_k$ WZW model is a diagonal theory.

# Chapter 6

# Boundary Integrable Models

*In this chapter we consider two-dimensional systems constrained to the half-line $x \in (-\infty, 0]$ with a boundary located at the origin. The boundary is therefore the vertical axis as opposed to the situation in the previous chapter. Our intention in doing so is to have our conventions as close as possible to the literature on the subject, [70], [72], [71].*

We start by assuming a boundary conformal field theory (CFT) with conformally invariant boundary conditions (CBC) and action $\mathcal{S}_{CFT+CBC}$, [68]. Suppose also that the relevant scalar primary field $\Phi(x, y)$ provides an integrable perturbation in the bulk. We can also have a boundary perturbation induced by the relevant boundary operator $\Phi_B(y)$. For this boundary perturbation to be compatible with CBC in the conformal limit, the field $\Phi_B(y)$ has to be one of the degenerate primary fields of the conformal theory such that the fusion rule coefficients are non-zero, [60]. Altogether we have:

$$\mathcal{S} = \mathcal{S}_{CFT+CBC} + \lambda \int_{-\infty}^{+\infty} dy \int_{-\infty}^{0} dx \Phi(x, y) + \lambda_B \int_{-\infty}^{0} dy \Phi_B(y). \tag{6.1}$$

Now suppose that the conservation laws (3.9), (3.19) and (3.20) with the corresponding conserved charges hold in the bulk for any spin $s \in S$, where S is some infinite set of positive integers. Ghoshal and Zamolodchikov [68] argued that the boundary theory (6.1) is integrable, provided we can find an infinite number of functionals $\theta_s(y)$ ($s \in S_B$)[1] of the boundary fields such that:

$$[T_{s+1} + \bar{\Theta}_{s-1} - \bar{T}_{s+1} - \Theta_{s-1}] \,|_{x=0} = \frac{d}{dy} \theta_s(y). \tag{6.2}$$

Indeed, it is straightforward to show that with $\theta_s$ satisfying eq.(6.2), the following quantity

[1] $S_B$ is an infinite subset of $S$

is a nontrivial IM:

$$H_B^{(s)} = \int_{-\infty}^{0} dx\, [T_{s+1}(x,y) + \Theta_{s-1}(x,y) + \bar{T}_{s+1}(x,y) + \bar{\Theta}_{s-1}(x,y)] + \theta_s. \tag{6.3}$$

Classically, eq.(6.2) amounts to solving a differential equation for $\theta_s(y)$. We will see an explicit example of this calculation in chapter 9 for the super-Liouville theory, [109]. In ref.[70] it was shown that for Toda theories based on the affine algebras $a_n^{(1)}$ and $d_n^{(1)}$ the boundary term $\mathcal{L}_{boundary} = -\delta(x)\mathcal{B}(\phi)$ preserving the classical integrability is of the form:

$$\mathcal{B} = \frac{m}{\beta^2}\sum_{i=0}^{r} A_i e^{\frac{\beta}{2}\alpha_i\cdot\phi}, \tag{6.4}$$

where $m$ is a mass scale, $\beta$ a coupling constant and $\alpha_i$ $(i = 0,\cdots,r)$ the set of simple roots of the underlying algebra and $\alpha_0 = -\sum_{i=1}^{r} n_i\alpha_i$ for some set of integers $\{n_i\}$. The real numbers $A_i$ either all vanish (corresponding to Neumann boundary condition) or satisfy:

$$|A_i| = 2\sqrt{n_i}. \tag{6.5}$$

In the quantum theory we have to perform a dimensional analysis similar to the one in chapter 3 for each value of $s$, to see whether the l.h.s. of eq.(6.4) yields a total derivative. As an example consider the case $s = 1$. First suppose that $\lambda = 0$, [68]. We then have in analogy with (3.6):

$$\begin{aligned} &< [T(y+ix) - \bar{T}(y-ix)]\, \phi_1(x_1,y_1)\cdots > = < [T(y+ix) - \bar{T}(y-ix)]\, \phi_1(x_1,y_1)\cdots >_0 \\ &\quad -\lambda_B \textstyle\int_{-\infty}^{+\infty} dy' < [T(y+ix) - \bar{T}(y-ix)]\, \Phi_B(y')\phi_1(x_1,y_1)\cdots >_0 + \mathcal{O}(\lambda_B^2). \end{aligned} \tag{6.6}$$

In the limit $x \to 0$ the first term on the r.h.s. vanishes and the second term is controlled by the OPE:

$$\begin{aligned} &[T(y+ix) - \bar{T}(y-ix)]\, \Phi_B(y') = \\ &= \left\{\tfrac{h_B}{(y-y'+ix)^2} - \tfrac{h_B}{(y-y'-ix)^2} + \tfrac{1}{y-y'+ix}\tfrac{\partial}{\partial y'} - \tfrac{1}{y-y'-ix}\tfrac{\partial}{\partial y'}\right\}\Phi_B(y') + \cdots \\ &\to \left\{h_B\delta'(y-y') + \delta(y-y')\tfrac{\partial}{\partial y'}\right\}\Phi_B(y'), \end{aligned} \tag{6.7}$$

where $h_B$ is the surface scaling dimension of $\Phi_B$. Substituting in (6.6) and integrating over $y'$, we get to first order in $\lambda_B$:

$$T_{xy}|_{x=0} = -i(T - \bar{T})|_{x=0} = \frac{d}{dy}\theta_1(y), \tag{6.8}$$

with

$$\theta_1(y) = -i\lambda_B(1 - h_B)\Phi_B(y). \tag{6.9}$$

Dimensional analysis shows that in most cases equations (6.6) and (6.7) remain valid even if we turn on the bulk perturbation $\lambda \neq 0$. In any case, even if there are any additional terms in $\lambda^n \lambda_B^m$ they will be finite in number as before (cf. section 3.1).

As always we have the freedom to choose the direction of Euclidean time arbitrarily. We can quantize our system on the half-line $x \in (-\infty, 0]$ with Hamiltonian $H_B (= H_B^{(1)})$ and evolving in Euclidean time $y \in (-\infty, +\infty)$. Alternatively, we can consider our system to lie on the whole line $y \in (-\infty, +\infty)$ and evolving in Euclidean time between the remote past $x = -\infty$ and the boundary state $|B>$ at the instant $x = 0$. In this latter picture, we see that $|B>$ is a state in the Fock space of the bulk theory, satisfying the equations (cf.(6.2)), [68]:

$$(P_s - \bar{P}_s)|B>= 0, \qquad s \in S_B. \tag{6.10}$$

On the half-line we can again define the asymptotic incoming states $|A_{a_1}(\theta_1) \cdots A_{a_N}(\theta_N)>_B^{in}$ moving towards the boundary, i.e. all the rapidities are positive. In the infinite future $t \to \infty$ we have a superposition of outgoing states $|A_{b_1}(\theta'_1) \cdots A_{b_M}(\theta'_M)>_B^{out}$ (all the rapidities are negative). Again these states are chosen to be simultaneous eigenstates of the IM $H_B^{(s)}$ (6.3). The same analysis as before reveals that the conservation of these charges implies pure elasticity of the reflections, in the sense that the mass spectrum is conserved ($N = M$) and the set of rapidities $\{\theta'_1, \theta'_2, \cdots, \theta'_N\}$ can differ only by a permutation from $\{-\theta_1, -\theta_2, \cdots, -\theta_N\}$. The Fock space is defined by multiple action of the creation operators $A_a(\theta)$ on the ground state $|0>_B$ of $H_B$: $A_{a_1}(\theta_1) \cdots A_{a_N}(\theta_N)|0>_B$ with $\theta_1 > \cdots > \theta_N$. The vacuum state is created by an operator $B$ representing an infinitely heavy impenetrable particle sitting at $x = 0$:

$$|0>_B = B|0> . \tag{6.11}$$

The scattering factorizes into products of one-particle reflection amplitudes $R_a^b(\theta)$ defined by:

$$A_a(\theta)B = R_a^b(\theta) A_b(-\theta)B. \tag{6.12}$$

If we apply the above algebra twice, we obtain the *boundary unitarity condition*:

$$R_a^c(\theta) R_c^b(-\theta) = \delta_a^b. \tag{6.13}$$

The requirement of factorizability is encoded in the following *boundary Yang-Baxter rela-*

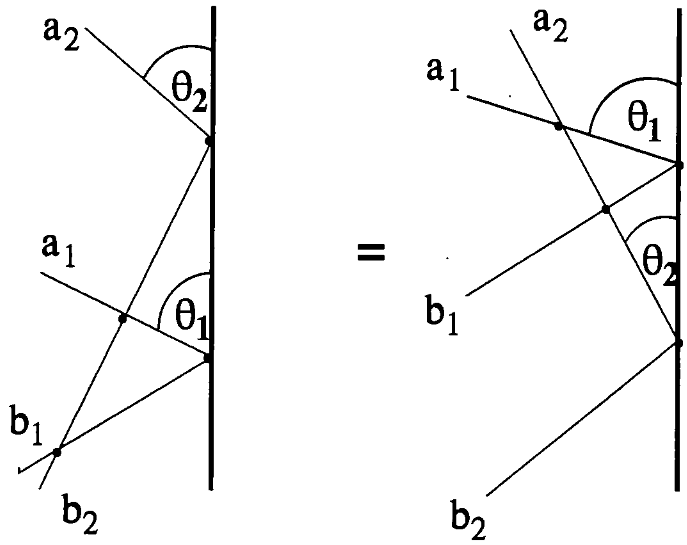


Figure 6.1: Boundary Yang-Baxter equation

*tion*:

$$\begin{aligned} &R^{c_2}_{a_2}(\theta_2) S^{c_1 d_2}_{a_1 c_2}(\theta_1+\theta_2) R^{d_1}_{c_1}(\theta_1) S^{b_2 b_1}_{d_2 d_1}(\theta_1-\theta_2) = \\ &= S^{c_1 c_2}_{a_1 a_2}(\theta_1-\theta_2) R^{d_1}_{c_1}(\theta_1) S^{d_2 b_1}_{c_2 d_1}(\theta_1+\theta_2) R^{b_2}_{d_2}(\theta_2), \end{aligned} \tag{6.14}$$

which is depicted in fig.6.1.

To obtain the analog of the crossing symmetry condition let us first construct the boundary states $|B>$ satisfying (6.8). In this picture $|B>$ belongs to the Fock space of the Hamiltonian $H(=H_1)$ defined on the whole line. The eigenvalue of $P_s - \bar{P}_s$ acting on a state with $N$ particles of rapidities $\{\theta_i\}$ $(i = 1, \cdots, N)$ is:

$$2\sum_{i=1}^{N} \chi^{(s)}_{a_i} sinh(s\theta_i). \tag{6.15}$$

Consequently $|B>$ is made up of pairs $A_a(\theta)A_b(-\theta)$ of particles of equal mass and opposite rapidities. Ghoshal and Zamolodchikov [68] showed that $|B>$ is of the form:

$$|B>= g exp\left[\int_0^{+\infty} d\theta K^{ab}(\theta) A_a(-\theta) A_b(\theta)\right] |0>, \tag{6.16}$$

where $g$ is a normalization and[2]:

$$K^{ab}(\theta) = R^{b}_{\bar{a}}\left(\frac{i\pi}{2} - \theta\right). \tag{6.17}$$

[2] $\bar{a}$ is the anti-particle of $a$.

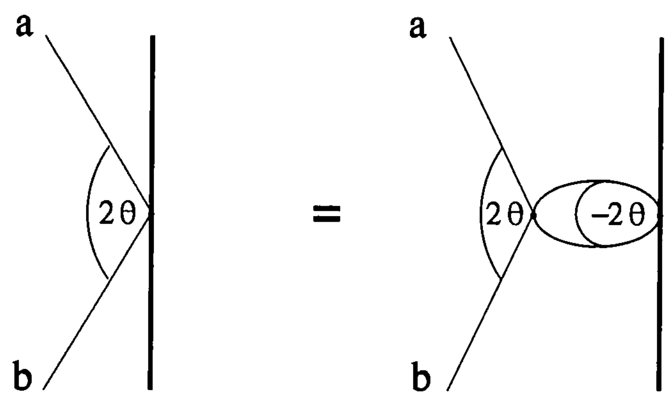


Figure 6.2: Boundary cross-unitarity

Notice that we could equally write (6.16) in terms of out-states by continuing the definition of $K^{ab}(\theta)$ for negative rapidities:

$$|B>= gexp\left[\int_0^{+\infty} d\theta K^{ab}(-\theta)A_a(\theta)A_b(-\theta)\right]|0> . \qquad (6.18)$$

By equating the two expressions we obtain the *boundary cross-unitarity condition*:

$$K^{ab}(\theta) = S^{ab}_{a'b'}(2\theta)K^{b'a'}(-\theta). \qquad (6.19)$$

This equation is represented diagrammatically in fig.6.2.

There are also boundary bootstrap conditions for the reflections of bound-state particles. Furthermore, we can also consider boundary bound states of the boundary particles, [69]. For all the applications in this thesis these situations will not occur, and we shall not discuss them any further.

As before the boundary reflection matrix is defined up to CDD factor.

# Chapter 7

# The Kondo Effect

## 7.1 The Kondo Model

*In this chapter we give a concise overview of the conformal approach to the Kondo problem. All the results presented are based on refs.[96], [98], [97] and [94].*

We described in the previous chapters the general approach to boundary quantum problems. There are many applications in quantum systems with impurities, the most celebrated one being the Kondo problem. The Kondo effect consists of the resistivity of metals $\rho(T)$ increasing as $T \to 0$, contrary to the standard behaviour of $\rho(T)$ decreasing either to zero (phonons or electron-electron interactions), or $\rho(T) \to$ *constant* (non-magnetic impurities). This anomalous pattern is caused by the existence of magnetic impurities.

Kondo [95] proposed the following asymptotically free theory which predicts the correct behaviour for the resistivity:

$$H = \sum_{\vec{k}\alpha} \psi^{\dagger\alpha}_{\vec{k}} \psi_{\vec{k}\alpha} \epsilon(k) + \lambda \vec{S} \cdot \sum_{\vec{k}\vec{k}'} \psi^{\dagger}_{\vec{k}} \frac{\vec{\sigma}}{2} \psi_{\vec{k}'}, \tag{7.1}$$

where $\psi_{\vec{k}\alpha}$ is the annihilation operator for a conduction electron with momentum $\vec{k}$ and spin $\alpha = \pm$. $\vec{S}$ represents the spin of the magnetic impurity, with

$$\left[S^a, S^b\right] = i\epsilon^{abc} S^c. \tag{7.2}$$

We assume that the representation of the above $SU(2)$ algebra is $(2s+1)$-dimensional, i.e. $\vec{S}^2 = s(s+1)$. In eq.(7.1), $\epsilon(k)$ denotes the kinetic energy for the excitations above the Fermi sea ($\epsilon_F$). The analysis of this model can be drastically simplified if we assume a spherically symmetric $\epsilon(k)$,

$$\epsilon(k) = \frac{k^2}{2m} - \epsilon_F \approx v_F(k - k_F), \tag{7.3}$$

a $\delta$-function Kondo interaction and look at s-wave scattering only. We can thus restrict ourselves to the radial coordinate ($r \geq 0$) and this becomes a $(1+1)$-dimensional problem, with the fermions being constrained to the half-line and the impurity sitting at the boundary. The resulting Hamiltonian is:

$$\begin{aligned} H_0 &= \tfrac{1}{2\pi}\int_0^{+\infty} dx \left(i\psi_L^\dagger \tfrac{d}{dx}\psi_L - i\psi_R^\dagger \tfrac{d}{dx}\psi_R\right), \\ H_{int} &= \tfrac{\lambda}{2}\psi_L^\dagger(0)\vec{\sigma}\psi_L(0)\cdot\vec{S}, \end{aligned} \tag{7.4}$$

where we have set $v_F = 1$ and $r = x$. $H_{int}$ is the interaction Hamiltonian and $\psi_L, \psi_R$ are left- and right-movers:

$$\psi_L(x,\tau) = \psi_L(\tau + ix), \qquad \psi_R(x,\tau) = \psi_R(\tau - ix), \tag{7.5}$$

where $\tau$ is the imaginary time. The quantization of (7.4) leads to the following propagator:

$$< \psi_L^\alpha(x)\psi_L^{\dagger\beta}(y) >= \frac{-i\delta^{\alpha\beta}}{x-y}, \tag{7.6}$$

with a similar expression for the propagator involving right-movers only. Notice from $H_0$ in eq.(7.4) that the theory is conformally invariant in the bulk with two sectors of non-interacting left-movers and right-movers. Since $\psi_L(0,\tau) = \pm\psi_R(0,\tau)$, the two are independent and the L-L interaction is identical to the R-R interaction, we may consider $\psi_R$ to be the continuation of $\psi_L$ to the negative $r$-axis:

$$\psi_R(x,\tau) \equiv \psi_L(-x,\tau), \tag{7.7}$$

thus obtaining a chiral (left-movers only) $(1+1)$-dimensional theory with:

$$H_0 = \frac{i}{2\pi}\int_{-\infty}^{+\infty} dx \psi_L^{\dagger\alpha}\frac{d}{dx}\psi_{L\alpha}, \tag{7.8}$$

and $H_{int}$ given by (7.4). This Hamiltonian is manifestly $SU(2)$ invariant. The RG flow interpolates between an unstable (high-temperature) UV fixed point where the impurity is decoupled ($\lambda = 0$) and a strongly coupled (low temperature) IR one ($\lambda = 2/3$) where the spin of the impurity is "*screened*". The precise meaning of the latter statement will become apparent when we describe the conformal approach to this problem. As usual the crossover between the two fixed points introduces a scale $T_K$ called the *Kondo temperature.*

## 7.2 Non-abelian bosonization and conformal field theory

The conformal field theory approach to the Kondo problem consists of the method of non-abelian bosonization. In two dimensions it is possible to construct fermionic fields from bosonic ones. Although this yields highly complicated expressions in terms of composite fields, one is often capable of constructing currents associated with the particular symmetries of the system, which have usually simple expressions in terms of the bosonic fields. This is of course advantageous, because bosonic fields are easier to deal with than fermionic ones. The WZW currents of chapter 4, constitute an example of bosonization when the symmetries of the system are non-abélian.

The idea is to try to write the Hamiltonian density in terms of currents. This, as we shall see will allow us to decouple the charge and spin degrees of freedom. Let us define the charge and spin currents:

$$J =: \psi^{\alpha\dagger}\psi_\alpha :, \qquad \vec{J} =: \psi^{\alpha\dagger}\frac{\vec{\sigma}_\alpha^\beta}{2}\psi_\beta : . \tag{7.9}$$

As an illustration we compute the following OPE using eq.(7.6):

$$\begin{aligned} J(x)J(y) =&: \psi^{\alpha\dagger}(x)\psi_\alpha(x)\psi^{\beta\dagger}(y)\psi_\beta(y) : + < \psi^{\alpha\dagger}(x)\psi_\beta(y) >: \psi_\alpha(x)\psi^{\beta\dagger}(y) : + \\ &+ < \psi_\alpha(x)\psi^{\beta\dagger}(y) >: \psi^{\alpha\dagger}(x)\psi_\beta(y) : + < \psi^{\alpha\dagger}(x)\psi_\beta(y) >< \psi_\alpha(x)\psi^{\beta\dagger}(y) >= \\ &= -\tfrac{2}{(x-y)^2} + : \psi^{\alpha\dagger}(y)\psi_\alpha(y)\psi^{\beta\dagger}(y)\psi_\beta(y) : + 2i\psi^{\alpha\dagger}(y)\tfrac{d}{dy}\psi_\alpha(y) + \mathcal{O}(x-y). \end{aligned} \tag{7.10}$$

The normal ordered product is:

$$\begin{aligned} : (J(y))^2 : \; &= \lim_{x\to y}\left[J(x)J(y) - \text{singular terms}\right] \\ &=: \psi^{\alpha\dagger}\psi_\alpha\psi^{\beta\dagger}\psi_\beta : + 2i : \psi^{\alpha\dagger}\tfrac{d}{dx}\psi_\alpha : . \end{aligned} \tag{7.11}$$

Similarly, we can show that:

$$: \vec{J}^2 := -\frac{3}{4} : \psi^{\alpha\dagger}\psi_\alpha\psi^{\beta\dagger}\psi_\beta : + \frac{3i}{2} : \psi^{\alpha\dagger}\frac{d}{dx}\psi_\alpha :, \tag{7.12}$$

where we used the identity:

$$\vec{\sigma}_\alpha^\beta \cdot \vec{\sigma}_\gamma^\delta = 2\delta_\gamma^\beta\delta_\alpha^\delta - \delta_\beta^\alpha\delta_\delta^\gamma. \tag{7.13}$$

Consequently, the bulk free Hamiltonian density can be written up to a c-number as:

$$\mathcal{H} = \frac{1}{8\pi}J^2 + \frac{1}{6\pi}\vec{J}^2. \tag{7.14}$$

From (7.10), we see that the charge current satisfies the algebra:

$$[J(x), J(y)] = 4\pi i\delta'(x-y). \tag{7.15}$$

Similarly, one can show that the spin currents satisfy a $SU(2)$ Kac-Moody algebra at level $k = 1$:

$$\left[J^a(x), J^b(y)\right] = 2\pi i\epsilon^{abc}J^c(x)\delta(x-y) + i\pi\delta^{ab}\frac{d}{dx}\delta(x-y). \tag{7.16}$$

Also, the two algebras are independent, i.e. $\left[J, \vec{J}\right] = 0$. The charge and spin degrees of freedom are therefore completely decoupled. We can consider our system to lie in a finite box of length $L$ with periodic (or anti-periodic) boundary conditions at $x = -\frac{L}{2}$ and $x = \frac{L}{2}$. We can thus introduce the Fourier modes of the spin currents according to:

$$\vec{J}_n \equiv \frac{1}{2\pi} \int_{-L/2}^{L/2} dx e^{-2in\frac{\pi}{L}x} \vec{J}(x). \tag{7.17}$$

In terms of these eq.(7.16) reads:

$$\left[J_n^a, J_m^b\right] = i\epsilon^{abc} J_{n+m}^c + \frac{1}{2} n \delta^{ab} \delta_{n+m,0}. \tag{7.18}$$

The spin part of the free Hamiltonian $H_0$ is:

$$H_s^0 = \frac{2\pi}{3L} \sum_{-\infty}^{+\infty} \vec{J}_{-n} \cdot \vec{J}_n. \tag{7.19}$$

Similarly, we can write the interaction Hamiltonian as:

$$H_{int} = \lambda \vec{J}(0) \cdot \vec{S} = \frac{2\pi\lambda}{L} \sum_{n=-\infty}^{+\infty} \vec{J}_n \cdot \vec{S}. \tag{7.20}$$

Obviously, $\left[S^a, J_n^b\right] = 0$. Since the Kondo interaction involves only the spin degrees of freedom, we can consider the following Hamiltonian:

$$H_K = H_s^0 + H_{int} = \frac{2\pi}{L} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3} \vec{J}_{-n} \cdot \vec{J}_n + \lambda \vec{J}_n \cdot \vec{S}\right). \tag{7.21}$$

The spectrum of this Hamiltonian at the weak coupling fixed point ($\lambda = 0$) corresponds to the Kac-Moody conformal towers at level $k = 1$, as can be seen from:

$$H_K(\lambda = 0) = \frac{2\pi}{3L} \sum_{n=-\infty}^{+\infty} : \vec{J}_{-n} \cdot \vec{J}_n : . \tag{7.22}$$

At the IR fixed point ($\lambda = 2/3$), we have:

$$H_K(\lambda = 2/3) = \frac{2\pi}{3L} \sum_{n=-\infty}^{+\infty} \left(\vec{J}_{-n} \cdot \vec{J}_n + 2\vec{J}_n \cdot \vec{S}\right). \tag{7.23}$$

The remarkable feature of this Hamiltonian is that we can complete the squares, yielding:

$$H_K(\lambda = 2/3) = \frac{2\pi}{3L} \sum_{n=-\infty}^{+\infty} \left[(\vec{J}_{-n} + \vec{S}) \cdot (\vec{J}_n + \vec{S}) - s(s+1)\right], \tag{7.24}$$

where we used $\vec{S}^2 = s(s+1)$. Notice also that:

$$\left[J_n^a + S^a, J_m^b + S^b\right] = i\epsilon^{abc}(J_{n+m}^c + S^c) + \frac{1}{2} n \delta^{ab} \delta_{n+m,0}. \tag{7.25}$$

This means that $H_K$ is quadratic in the new currents, $\vec{\mathcal{J}}_n \equiv \vec{J}_n + \vec{S}$, which obey the same Kac-Moody algebra as the old ones, $\vec{J}_n$. The explicit dependence of the spin $\vec{S}$ of the impurity has disappeared. This is what is meant by "*screening*".

## 7.3 Multi-channel Kondo problem and boundary conditions

Suppose now that we consider different "*channels*" of electrons-e.g. different d-shell orbitals. This would require the introduction of $k$ species of electrons falling into multiplets of a $SU(k)$-"*flavour*" symmetry. The Hamiltonian (7.1) would be modified to:

$$H = \sum_{\vec{k}\alpha}\sum_{i=1}^{k} \epsilon(k)\psi_{\vec{k}}^{\alpha i\dagger}\psi_{\vec{k}\alpha i} + \lambda\vec{S}\cdot\sum_{\vec{k}\vec{k}'\alpha\beta}\sum_{i=1}^{k}\psi_{\vec{k}}^{\alpha i\dagger}\vec{\sigma}_{\alpha}^{\beta}\psi_{\vec{k}'\beta i}. \tag{7.26}$$

This is known as the *multi-channel Kondo problem.* Again we follow the same procedure of mapping the previous model to a $(1+1)$ QFT and introduce a form of bosonization that separates the spin, charge and flavour degrees of freedom. This representation is also known as *conformal embedding.* The $SU(k)$ group has $k^2-1$ generators. They can be represented by the traceless hermitean matrices $T^A$ $(A = 1,\cdots,k^2-1)$ normalized so that

$$Tr(T^AT^B) = \frac{1}{2}\delta^{AB}, \tag{7.27}$$

and obeying the completeness relation:

$$\sum_A (T^A)_a^b(T^A)_c^d = \frac{1}{2}\left[\delta_c^b\delta_a^d - \frac{1}{k}\delta_a^b\delta_c^d\right]. \tag{7.28}$$

The structure constants $f^{ABC}$ of the $su(k)$ algebra are defined by the set of commutation relations:

$$\left[T^A, T^B\right] = if^{AB}{}_C T^C, \tag{7.29}$$

and the quadratic Casimir associated with the adjoint representation is:

$$c_v(SU(k)) = \frac{1}{k^2-1}f^{AC}{}_D f_{AC}{}^D = k. \tag{7.30}$$

The definition of the charge and spin currents (7.9) has to be slightly altered to incorporate the additional flavour degrees of freedom:

$$J \equiv: \psi^{\alpha i\dagger}\psi_{\alpha i}:, \qquad \vec{J} \equiv: \psi^{\alpha i\dagger}\frac{\vec{\sigma}_{\alpha}^{\beta}}{2}\psi_{\beta i}:. \tag{7.31}$$

Their commutation relations are not significantly modified, except that now we have $k$ copies of fermions. The charge current which corresponded to a level 4 abelian Kac-Moody algebra now has level $4k$. Similarly, the spin currents now satisfy a level $k$ $SU(2)$ Kac-Moody algebra.

Finally we define the flavour current according to:

$$J^A \equiv: \psi^{\alpha i\dagger}(T^A)_i^j\psi_{\alpha j}:. \tag{7.32}$$

The currents $J^A$ obey the $SU(k)$ current algebra with central charge 2:

$$\left[J_n^A, J_m^B\right] = if^{AB}{}_C J_{n+m}^C + n\delta^{AB}\delta_{n+m,0}, \tag{7.33}$$

and the Hamiltonian density can be written in terms of these as:

$$\mathcal{H} = \frac{1}{8\pi k}J^2 + \frac{1}{2\pi(k+2)}\vec{J}^2 + \frac{1}{2\pi(k+2)}J^A J^A. \tag{7.34}$$

Since the three types of currents are mutually commutative, we conclude that we have managed to decouple again the three types of degrees of freedom: $SU(2)_k \otimes SU(k)_2 \otimes U(1)$. Alternatively, using the Sugawara construction (4.23), we can re-express (7.34) in terms of the stress tensors with the different current algebras:

$$\mathcal{H} = \frac{1}{2\pi}(T_{charge} + T_{spin} + T_{flavour}), \tag{7.35}$$

with central charges:

$$c_{charge} = 1, \qquad c_{spin} = \frac{3k}{k+2}, \qquad c_{flavour} = \frac{2(k^2-1)}{2+k}. \tag{7.36}$$

The total value of the central charge is thus:

$$c_{tot} = c_{charge} + c_{spin} + c_{flavour} = 2k. \tag{7.37}$$

As before, since the impurity couples only to the spin degrees of freedom, we consider the reduced Kondo Hamiltonian:

$$H_K = \int dx \left[\frac{1}{2\pi(k+2)}\vec{J}^2 + \lambda\delta(x)\vec{J}\cdot\vec{S}\right]. \tag{7.38}$$

In terms of the modes (7.17) this reads:

$$H_K = \frac{2\pi}{L}\sum_{n=-\infty}^{+\infty}\left[\frac{1}{k+2}\vec{J}_n\cdot\vec{J}_{-n} + \lambda\vec{J}_n\cdot\vec{S}\right]. \tag{7.39}$$

Again, we can complete the squares for the critical value,

$$\lambda_c = \frac{2}{k+2}, \tag{7.40}$$

yielding:

$$H_K = \frac{2\pi}{L(k+2)}\sum_{n=-\infty}^{+\infty}\left[(\vec{J}_{-n}+\vec{S})\cdot(\vec{J}_n+\vec{S}) - s(s+1)\right]. \tag{7.41}$$

And the Kac-Moody algebra remains unchanged:

$$\left[J_n^a + S^a, J_m^b + S^b\right] = i\epsilon^{abc}(J_{n+m}^c + S^c) + \frac{1}{2}kn\delta^{ab}\delta_{n+m,0}. \tag{7.42}$$

The new currents at the infrared fixed point, $\vec{\mathcal{J}}$, are related to the old ones, $\vec{J}$, by:

$$\vec{\mathcal{J}}_n = \vec{J}_n + \vec{S}. \tag{7.43}$$

If $\vec{J}$ and $\vec{\mathcal{J}}$ were just ordinary spin operators, then the new spectrum would be given by the ordinary angular momentum addition rules. In particular, if $s$ is half-integer, states of integer total spin are mapped into states of half-integer spin, and vice-versa. Furthermore, we recall the fusion rule coefficients for a level $k$ $SU(2)$ WZW model (cf.(4.45)):

$$N_{jp}^{l} = \begin{cases} 1, & \text{if } |j-p| \le l \le min\{j+p, k-j-p\}, \\ 0, & \text{otherwise,} \end{cases} \tag{7.44}$$

where of course $j, p = 0, \frac{1}{2}, \cdots, \frac{k}{2}$. The striking resemblance between these coefficients and those for the addition of angular momentum, together with the fact that they also encode the set of permissible conformally invariant boundary conditions at the conformal point suggest a way of determining the boundary conditions for an arbitrary number of channels $k$ and impurity spin $s$. It consists of fusions with the spin-$s$ representation. However, this approach requires some caution, as the spin-$s$ representation is only allowed in the $SU(2)$ WZW model for $s \le k/2$. In the *underscreened case*, $s > k/2$, we assume fusion with the maximal possible spin, namely $k/2$. The same happens in the *exactly screened case* $s = k/2$. In both cases, from (7.44), we have the following fusions:

$$j \otimes \frac{k}{2} = \frac{k}{2} - j. \tag{7.45}$$

Each conformal tower $j$ is mapped into a unique conformal tower $\left(\frac{k}{2} - j\right)$. In this situation, one electron from each species binds to the impurity, effectively reducing its spin to

$$q \equiv s - \frac{k}{2}. \tag{7.46}$$

In particular in the exactly screened case, we have:

$$q = 0. \tag{7.47}$$

## 7.4 Impurity entropy

As we have seen in chapter 5 the partition function on the strip for a set of boundary conditions $(\alpha, \beta)$ is (cf.(5.21), (5.22)):

$$Z_{\alpha\beta} = \sum_j < \alpha|j >< j|\beta > \chi_j(q). \tag{7.48}$$

In the limit $R \to \infty$ this is controlled by the ground state, $|0>$, i.e. the state with lowest eigenvalue of $L_0$:

$$Z_{\alpha\beta} \sim e^{\frac{\pi Rc}{12L}} < \alpha|0 >< 0|\beta > . \tag{7.49}$$

The free energy is:

$$F_{\alpha\beta} = -\frac{\pi c T^2 R}{12} - Tln < \alpha|0 >< 0|\beta >, \tag{7.50}$$

where we used $L = 1/T$. The first term gives the specific heat:

$$C = \frac{\pi c T R}{6}, \tag{7.51}$$

and the second gives the impurity entropy:

$$S_{imp} = S_\alpha + S_\beta = ln < \alpha|0 > +ln < 0|\beta > . \tag{7.52}$$

Each boundary $i = \alpha, \beta$ therefore contributes with the entropy $S_i = lng_i$, where $g_i$ is called the "*ground state degeneracy*" associated with the boundary condition $i$. But we know what this is. It comes from the fusion with the spin-$s$ (or $k/2$) operator (cf.(5.27)):

$$g = \frac{S_s^0}{S_0^0}. \tag{7.53}$$

Using the expression (4.42) for the modular matrix S, we get:

$$g(s,k) = \frac{sin\left[\frac{\pi(2s+1)}{k+2}\right]}{sin\left[\frac{\pi}{k+2}\right]}. \tag{7.54}$$

In the exact screened case, we replace $s$ by $k/2$, yielding:

$$g(k/2,k) = 1. \tag{7.55}$$

In the underscreened case, we must multiply the previous result by the number of *non-screened degrees of freedom.* Since the impurity has effective spin $q = s - k/2$ it falls into a $(2q+1)$-dimensional multiplet of $SU(2)$. Consequently:

$$g(q,k) = 2q + 1. \tag{7.56}$$

For the overscreened case, the ground state degeneracy is noninteger. For instance, for $k = 2$ and $s = 1/2$, we have:

$$g(1/2,2) = \sqrt{2}. \tag{7.57}$$

## 7.5 Kinks and background scattering

As we have seen in the previous section, we managed to decouple the charge, spin and flavour degrees of freedom using the technique of non-abelian bosonization. Furthermore, since the impurity couples only to the spin degrees of freedom, we can discard the remaining and consider only the $SU(2)_k$ spin current algebra. As we discussed in chapter 4 the corresponding WZW model has $k+1$ degenerate vacua. Consequently, the spectrum of particles consists of the stable massless sectors of right- and left-movers, containing non-trivial internal symmetries of $SU(2)$ isotopic spin and a kink structure associated with the degeneracy of the colored vacua. In this description in terms of "*quasi-particles*" we are studying directly the excitations above the Fermi sea. Also from the conformal invariance in the bulk, we were able to extend our system to the whole line by considering only say left-movers with the impurity sitting at the origin. If we assume the model to be integrable, then the S-matrix $S_{BL}$ for the scattering between the impurity and a left-mover of rapidity $\beta$ will be determined by the usual requirements of unitarity, crossing symmetry and factorizability. Let us consider first the exactly screened case. The effective spin of the boundary impurity is $q=0$. Thus, it is represented by an $SU(2)$ singlet with no kink structure. Under these circumstances, the particle cannot exchange isotopic spin or kink degrees of freedom with the impurity and $S_{BL}$ satisfies precisely the same equations as does $U_{RL}$ in chapter 4:

$$S_{BL}(\beta) = U_{RL}(\beta) = tanh\left(\frac{\beta}{2} - \frac{i\pi}{4}\right). \tag{7.58}$$

Now consider the particular underscreened case when the boundary particle is a $SU(2)$ doublet, i.e. $k=2s-1$ or $q=1/2$. The particle is now allowed to exchange isotopic degrees of freedom with the impurity in a spin preserving fashion. The constraints are therefore the same as for the bulk L-L scattering:

$$(S_{BL})_{ab}^{cd}(\beta) = S_{ab}^{cd}(\beta) = \sigma_T(\beta)\delta_a^c\delta_b^d + \sigma_R(\beta)\delta_a^d\delta_b^c, \tag{7.59}$$

where $\sigma_T(\beta)$ and $\sigma_R(\beta)$ are given by (4.68), (4.69).

In the overscreened case the impurity's spin is completely screened and therefore it is a $SU(2)$ singlet as in the exactly screened case. However there are still $p=k-2s$ leftover electrons. Since the spin degrees of freedom of the impurity are saturated, if it is to have a non-trivial structure it has to couple to the flavour symmetry (i.e. kink structure) of these leftover electrons. The boundary conditions in this case are obtained by generalizing the previous procedure to a sort of kink version of the fusion which preserves the integrability of the model, [77], [78]. The *boundary "incidence" matrix* $I_p$ encodes the kink structure of the $p$ kinks. Its rows and columns correspond to the vacua. If the vacua $a$ and $b$ are connected by a kink, then the entry $(I_p)^a_b$ is 1, otherwise it is zero. The incidence matrix for the bulk kinks is $I_1$ and $I_0$ is defined to be the $k \times k$ identity matrix. The analog of the angular-momentum multiplication is [98]:

$$I_1 I_p = I_{p-1} + I_{p+1}. \tag{7.60}$$

As an illustration, consider the simplest nontrivial overscreened case, $k = 3$ and $s = 1/2$. The bulk WZW model has $k + 1 = 4$ degenerate vacua, with the kinks interpolating between adjacent vacua. On the other hand $p = k - 2s = 2$. From (7.60):

$$I_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad I_2 = I_1 I_1 - I_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{7.61}$$

In this case the boundary spectrum consists of the kink doublets (1,3), (3,1), (2,4), (4,2), (2,2) and (3,3).

We shall not consider any further the overscreened case. The scattering matrices are those for *restricted solid on solid* (RSOS) models, which can be found in ref.[79].

# Chapter 8

# The principal chiral model on the half-line

## 8.1 Boundary conditions and reflection amplitudes

*In this chapter we consider the PCM (at level $k = 1$) on the half-line.* The determination of the boundary conditions compatible with integrability and the corresponding reflection amplitudes will involve some amount of guesswork. We shall use as guideline some knowledge coming from the symmetries of the problem, the limiting IR conformal field theory and a related problem (Kondo). The difference between this and the Kondo problem lies in the fact that in the former the scale invariance is broken in the bulk by the mass scale associated with a very unstable $O(4)$-isovector resonance[1], whereas in the latter the scale invariance is broken at the boundary. We therefore assume that the boundary conditions are conformally invariant at the IR fixed point and that the RG flow is controlled by the bulk perturbation. We would thus have a system of the form (6.1) with $\lambda_B = 0$. In chapter 5 it was shown that at the IR point there are two boundary conditions compatible with conformal invariance, corresponding to the identity operator with isospin $l = 0$ and the fundamental field $g$ of the WZW action with isospin $l = 1/2$. These boundary conditions could also be regarded as the IR limit of the Kondo model when $k = 1$. The bulk spectrum of the two problems is the same with identical L-L and R-R scattering. The difference lies in whether the R-L scattering is trivial (Kondo) or not (PCM). The overscreened case is not allowed as this would require $s = 0$ and the impurity would completely decouple from the system. In the underscreened case, we could have in principle $s \geq 1$, but we shall only consider the case $s = 1$. When we derive the boundary consistency equations

[1] We assume that the boundary introduces no additional mass scale.

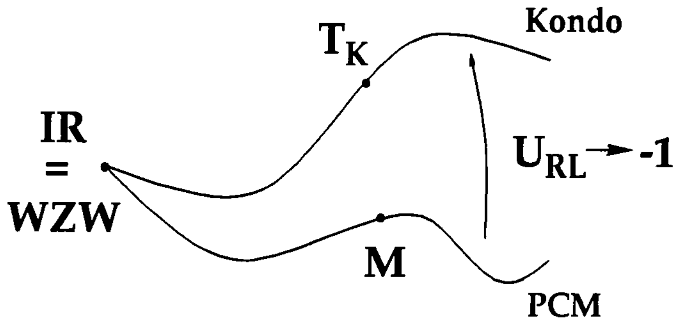


Figure 8.1: RG trajectories

(e.g.unitarity), in the limit when the bulk theory becomes scale invariant ($U_{RL} \to -1$), the scattering amplitudes of the Kondo problem should solve these equations. This will be an important consistency check. This program is formally depicted in the diagram of fig.8.1.

In the exactly screened case ($k = 1, s = 1/2$), the boundary particle is a $SU(2)$ singlet and the scattering matrix is given by eq.(7.58). By analogy, we start by assuming that the boundary impurity has no internal structure. We call this "*fixed*" boundary condition. The reflection matrix $R_a^{\bar{b}}$ is defined by (cf.(6.12)):

$$R_a(\beta)B = R_a^{\bar{b}}(\beta)L_{\bar{b}}(-\beta)B. \tag{8.1}$$

Since the total isospin has to be conserved and the boundary particle is a $SU(2)$ singlet, we conclude that the reflection matrix has to be diagonal:

$$R_a^{\bar{b}}(\beta) = \delta_a^{\bar{b}} R_{RL}(\beta). \tag{8.2}$$

This amplitude automatically satisfies the boundary Yang-Baxter equation (6.14) irrespective of $U_{RL}$. Let us now consider the boundary crossing symmetry condition:

$$K^{\bar{a}b}(\beta) = U_{\bar{c}d}^{\bar{a}b}(2\beta)K^{d\bar{c}}(-\beta), \tag{8.3}$$

where $K^{a\bar{b}}(\beta) = R_{\underline{a}}^{\bar{b}}(i\pi/2 - \beta)$. From eqs.(7.58) and (8.2), we get:

$$R_{LR}(-\beta) = -\frac{R_{RL}(i\pi + \beta)}{U_{RL}(2\beta)}. \tag{8.4}$$

Notice that $R_{LR}$ would correspond to a left-moving particle being reflected into a right-moving one. This does not seem to make much sense given that our system is defined on the half-line $(-\infty, 0]$. However, as we shall see, it will prove fruitful to ignore this and see it only as a formal tool to derive consistency conditions for the reflection factors.

Next consider the boundary unitarity condition:

$$R_a^{\bar{c}}(\beta)R_{\bar{c}}^{b}(-\beta) = \delta_a^b. \tag{8.5}$$

Using (8.2), we get:

$$R_{RL}(\beta)R_{LR}(-\beta) = 1. \tag{8.6}$$

We can thus express $R_{LR}$ in terms of $R_{RL}$. Eq.(8.4) then reads:

$$R_{RL}(\beta)R_{RL}(i\pi+\beta) = -U_{RL}(2\beta). \tag{8.7}$$

Notice that we cannot take $R_{RL} = R_{LR}$ as can readily be verified if we substitute $\beta = -i\pi/2$ in eqs.(8.4) and (8.6). As we discussed before, if we take $U_{RL} \to -1$, then the exactly screened amplitude (7.58) of the Kondo problem is a solution of eq.(8.7). Let us now consider (8.7) with non-trivial R-L scattering. This has the minimal solution:

$$R_{RL}(\beta) = exp\left(-\frac{i\pi}{4}\right)\left\{\frac{sinh\left(\frac{\beta}{2}-\frac{i\pi}{8}\right)}{sinh\left(\frac{\beta}{2}+\frac{i\pi}{8}\right)}\right\}. \tag{8.8}$$

This amplitude has no poles on the physical sheet. The only pole lies on the second sheet at $\beta = -i\pi/4$ and is, of course, associated with the mass scale of the bulk theory.

We now move on to the underscreened case, where the boundary has an effective spin $q = 1/2$. This will be denoted "*free*" boundary condition. We then have:

$$R_a(\beta)B_b = \sum_{\bar{c},d=\pm} R_{ab}^{\bar{c}d}(\beta)L_{\bar{c}}(-\beta)B_d, \tag{8.9}$$

where $B_b$ creates a boundary state with isotopic spin $b = \pm$:

$$|B>_d = B_d|0> . \tag{8.10}$$

The boundary Yang-Baxter equation has to be slightly modified to incorporate this additional structure, [68]:

$$\begin{aligned} &R_{bc}^{\bar{b}'c'}(\beta_2)U_{a\bar{b}'}^{a'\bar{b}''}(\beta_1+\beta_2)R_{a'c'}^{\bar{a}'d}(\beta_1)S_{\bar{b}''\bar{a}'}^{\bar{b}\bar{a}}(\beta_1-\beta_2) = \\ &= S_{ab}^{a'b'}(\beta_1-\beta_2)R_{a'c}^{\bar{a}'c'}(\beta_1)U_{b'\bar{a}'}^{b''\bar{a}}(\beta_1+\beta_2)R_{b''c'}^{\bar{b}d}(\beta_2). \end{aligned} \tag{8.11}$$

Substituting (4.71), we get:

$$R_{bc}^{\bar{b}'c'}(\beta_2)R_{ac'}^{\bar{a}'d}(\beta_1)S_{\bar{b}'\bar{a}'}^{\bar{b}\bar{a}}(\beta_1-\beta_2) = S_{ab}^{a'b'}(\beta_1-\beta_2)R_{a'c}^{\bar{a}c'}(\beta_1)R_{b'c'}^{\bar{b}d}(\beta_2). \tag{8.12}$$

We see that the bulk R-L scattering decouples as before. It will only play a rôle in the boundary crossing-symmetry condition. This, of course, is a consequence of the R-L scattering being diagonal. Substituting (4.66), this yields:

$$\begin{aligned}&\left[R^{\bar{a}d}_{ac'}(\beta_1)R^{\bar{b}c'}_{bc}(\beta_2) - R^{\bar{a}c'}_{ac}(\beta_1)R^{\bar{b}d}_{bc'}(\beta_2)\right]\sigma_T(\beta_1-\beta_2) = \\ &= \left[R^{\bar{a}c'}_{bc}(\beta_1)R^{\bar{b}d}_{ac'}(\beta_2) - R^{\bar{b}d}_{ac'}(\beta_1)R^{\bar{a}c'}_{bc}(\beta_2)\right]\sigma_R(\beta_1-\beta_2).\end{aligned} \tag{8.13}$$

Since the total isospin has to be conserved, we assume the following $SU(2)$ symmetric combination:

$$R^{\bar{c}d}_{ab}(\beta) = \delta^{\bar{c}}_a\delta^d_b f_{RL}(\beta) + \delta^d_a\delta^{\bar{c}}_b g_{RL}(\beta). \tag{8.14}$$

We then get using (4.68):

$$\frac{i}{\pi}(\beta_1-\beta_2)g_{RL}(\beta_1)g_{RL}(\beta_2) = f_{RL}(\beta_1)g_{RL}(\beta_2) - g_{RL}(\beta_1)f_{RL}(\beta_2), \tag{8.15}$$

which implies:

$$f_{RL}(\beta) = \frac{i}{\pi}\beta g_{RL}(\beta). \tag{8.16}$$

The boundary unitarity condition,

$$R^{\bar{a}'b'}_{ab}(\beta)R^{cd}_{\bar{a}'b'}(-\beta) = \delta^c_a\delta^d_b, \tag{8.17}$$

reads:

$$\begin{aligned}&f_{RL}(\beta)f_{LR}(-\beta) + g_{RL}(\beta)g_{LR}(-\beta) = 1, \\ &f_{RL}(\beta)g_{LR}(-\beta) + g_{RL}(\beta)f_{LR}(-\beta) = 0.\end{aligned} \tag{8.18}$$

Finally we consider the boundary crossing-unitarity condition. We assume the following generalization:

$$R^{bd}_{\bar{a}c}(\beta) = R^{\bar{a}d}_{\underline{b}c}(i\pi-\beta)U_{RL}(i\pi-2\beta). \tag{8.19}$$

Following Berg et al.[56], we define the crossing symmetric matrix:

$$G^{ad}_{bc}(\beta) \equiv R^{\bar{a}d}_{\underline{b}c}(\beta) = \delta^a_b\delta^d_c u_{RL}(\beta) + \delta_{bc}\delta^{ad}v_{RL}(\beta). \tag{8.20}$$

As before the matrix $H^{ad}_{bc}(\beta) \equiv R^{\bar{d}\underline{a}}_{\underline{b}c}(\beta)$ vanishes because it is associated with the exchange of momenta, which is not possible since the boundary particle has to stay at rest after the interaction. In terms of $u$ and $v$, (8.19) reads:

$$f_{LR}(\beta) = -U_{RL}(2\beta)u_{RL}(i\pi-\beta), \qquad g_{LR}(\beta) = -U_{RL}(2\beta)v_{RL}(i\pi-\beta). \tag{8.21}$$

The unitarity conditions,

$$G^{a'd'}_{bc}(\beta)G^{ad}_{a'd'}(-\beta) = \delta^a_b\delta^d_c, \tag{8.22}$$

yield the equations of $u, v$:

$$\begin{aligned} &u_{RL}(\beta)v_{LR}(-\beta) + v_{RL}(\beta)u_{LR}(-\beta) + 2v_{RL}(\beta)v_{LR}(-\beta) = 0, \\ &u_{RL}(\beta)u_{LR}(-\beta) = 1. \end{aligned} \tag{8.23}$$

Notice that if we choose,

$$f_{RL}(\beta) = -u_{RL}(\beta) - v_{RL}(\beta), \qquad g_{RL}(\beta) = v_{RL}(\beta), \tag{8.24}$$

then eq.(8.15) is automatically satisfied. The boundary Yang-Baxter equation for antipaticles imposes that

$$u_{RL}(\beta) = \frac{i}{\pi}\beta v_{RL}(\beta), \tag{8.25}$$

which is perfectly compatible with (8.16) for the choice (8.24). Solving this whole system is tantamount to finding $g_{RL}, g_{LR}$ such that:

$$\begin{cases} g_{RL}(\beta)g_{LR}(-\beta) = \frac{\pi^2}{\pi^2+\beta^2}, \\ g_{LR}(\beta) = -U_{RL}(2\beta)g_{RL}(i\pi - \beta). \end{cases} \tag{8.26}$$

Suppose that the R-L scattering becomes trivial, $U_{RL}(\beta) \to -1$. In that case it is perfectly consistent to take $g_{RL} = g_{LR} \equiv g$:

$$g(\beta)g(-\beta) = \frac{\pi^2}{\pi^2 + \beta^2}, \qquad g(i\pi - \beta) = g(\beta). \tag{8.27}$$

This system is solved by $g(\beta) = \sigma_R(\beta)$ in agreement with Fendley, [98]. The system (8.26), with $U_{RL}(\beta)$ given by (4.73), is not consistent for $g_{RL} = g_{LR}$. Again this is checked immediately for $\beta = -i\pi/2$. However, we can take $g_{LR}(\beta) = \gamma g_{RL}(\beta)$, where $\gamma$ is some constant. Consistency of the system (8.26) requires $\gamma = -U_{RL}(i\pi) = -i$. We then have:

$$\begin{cases} g_{RL}(\beta)g_{RL}(-\beta) = \frac{i\pi^2}{\pi^2+\beta^2}, \\ ig_{RL}(\beta) = U_{RL}(2\beta)g_{RL}(i\pi - \beta). \end{cases} \tag{8.28}$$

This system has the minimal solution:

$$g_{RL}(\beta) = iR_{RL}(\beta)\sigma_R(\beta), \tag{8.29}$$

where $R_{RL}(\beta)$ is given by (8.8).

## 8.2 Boundary TBA

In this section, we compute the boundary ground state energy and the boundary degeneracy for the system defined on the topology of the annulus with periodic boundary conditions in the $L$-direction and "*fixed*" boundary conditions on both sides of the annulus in the $R$-direction. This is accomplished by the technique of boundary TBA, [86]. Because the theory in the bulk is nondiagonal the computations involved are quite cumbersome. The best we can hope for is the situation we have just described. The "*free*" boundary condition entails a nondiagonal reflection matrix, thus rendering the equations even more complicated.

### 8.2.1 TBA in the $R$-channel

We start by assuming our system to lie in a periodic interval of length $L$, and evolving between the two boundary states $|B>$ (associated with (8.1)) during a lapse of time $R$. The boundary state is defined by (cf.(6.16)):

$$|B>= gexp\left\{\int_{-\infty}^{+\infty} d\beta K^{\bar{a}b}(\beta)L_{\bar{a}}^{\dagger}(-\beta)R_b^{\dagger}(\beta)\right\}|0> . \tag{8.30}$$

In this section we use the symbol $\dagger$ to distinguish between creation and annihilation operators. They satisfy the following non-commutative algebra:

$$R_a(\beta_1)R_b^{\dagger}(\beta_2) = S_{ab}^{a'b'}(\beta_1-\beta_2)R_{b'}^{\dagger}(\beta_2)R_{a'}(\beta_1) + \delta_{ab}\delta(\beta_1-\beta_2). \tag{8.31}$$

In eq.(8.30) $g$ is a normalization, which we consider equal to 1 unless otherwise stated. Notice also that we are performing the integration in the interval $-\infty < \beta < +\infty$. This is because the particles are massless and therefore e.g. $R_a^{\dagger}(\beta)$ will always be an incoming particle moving towards the right boundary regardless of what the sign of $\beta$ is, [86].

The partition function in the $R$-channel is, [86]:

$$Z =< B|exp(-RH)|B >= \sum_{\alpha} \frac{< B|\alpha >< \alpha|B >}{< \alpha|\alpha >} exp(-RE_\alpha). \tag{8.32}$$

The sum is *a priori* over any state in the Hilbert space. $H$ is the Hamiltonian for the periodic system and $E_\alpha$ the energy of the state $|\alpha>$. Since the theory is integrable, the number of particles and momenta are conserved. Consequently, the only states $|\alpha>$ that contribute to the sum are of the form:

$$|2N >= L_{\bar{c}_N}^{\dagger}(-\beta_N)R_{d_N}^{\dagger}(\beta_N)\cdots L_{\bar{c}_1}^{\dagger}(-\beta_1)R_{d_1}^{\dagger}(\beta_1)|0> . \tag{8.33}$$

We then claim that:

$$\frac{< B|2N >< 2N|B >}{< 2N|2N >} = \frac{2^N}{(N!)^2}\Pi_{i=1}^N \bar{K}(\beta_i)K(\beta_i). \tag{8.34}$$

As an illustration let us consider the first few terms. For $g = 1$ and assuming that the vacuum state is properly normalized ($<0|0>= 1$), we have immediately:

$$\frac{<B|0><0|B>}{<0|0>} = 1. \tag{8.35}$$

The first non trivial term is:

$$<B|2>= \int_{-\infty}^{+\infty} d\beta_1' \bar{K}^{\bar{a}_1 b_1}(\beta_1') <0|R_{b_1}(\beta_1')L_{\bar{a}_1}(-\beta_1')L^{\dagger}_{\bar{c}_1}(-\beta_1)R^{\dagger}_{d_1}(\beta_1)|0> . \tag{8.36}$$

From eq.(8.31) and the fact that $L_{\bar{a}}(-\beta)|0>= 0$, we get:

$$<B|2><2|B>= \delta^2(0)\delta^{d_1}_{\underline{\bar{c}_1}}\delta^{d_1}_{\underline{\bar{c}_1}}\bar{K}(\beta_1)K(\beta_1). \tag{8.37}$$

In the above expression there is no summation over the indices yet. Similarly:

$$<2|2>=<0|R_{d_1}(\beta_1)L_{\bar{c}_1}(-\beta_1)L^{\dagger}_{\bar{c}_1}(-\beta_1)R^{\dagger}_{d_1}(\beta_1)|0>= \delta^2(0). \tag{8.38}$$

Finally, we can sum over the internal isotopic degrees of freedom. This is because for the states $|\alpha>= |2N>$ in the partition function (8.32), the exponential term $exp(-RE_\alpha)$ is the same for all the states $|2N>$ with the same number $N$ of pairs of particles, i.e. it depends upon the rapidity but not on the isospin. We then get:

$$\frac{<B|2><2|B>}{<2|2>} = 2\bar{K}(\beta_1)K(\beta_1). \tag{8.39}$$

Next consider:

$$\begin{aligned}<B|4>= &\tfrac{1}{2!}\textstyle\int_{-\infty}^{+\infty} d\beta_1' \int_{-\infty}^{+\infty} d\beta_2' \bar{K}^{\bar{a}_1 b_1}(\beta_1')\bar{K}^{\bar{a}_2 b_2}(\beta_2')\times \\ &\times I_4(\beta_1',-\beta_1',\beta_2',-\beta_2',-\beta_2,\beta_2,-\beta_1,\beta_1)_{b_1\bar{a}_1 b_2\bar{a}_2\bar{c}_2 d_2\bar{c}_1 d_1},\end{aligned} \tag{8.40}$$

where

$$\begin{aligned}&I_4(\beta_1',-\beta_1',\beta_2',-\beta_2',-\beta_2,\beta_2,-\beta_1,\beta_1)_{b_1\bar{a}_1 b_2\bar{a}_2\bar{c}_2 d_2\bar{c}_1 d_1} = \\ &=<0|R_{b_1}(\beta_1')L_{\bar{a}_1}(-\beta_1')R_{b_2}(\beta_2')L_{\bar{a}_2}(-\beta_2')L^{\dagger}_{\bar{c}_2}(-\beta_2)R^{\dagger}_{d_2}(\beta_2)L^{\dagger}_{\bar{c}_1}(-\beta_1)R^{\dagger}_{d_1}(\beta_1)|0>= \\ &= \tfrac{U_{RL}(\beta_1+\beta_2)}{U_{RL}(\beta_1'+\beta_2')} I_4^{(R)}(\beta_1',\beta_2';\beta_2,\beta_1)_{b_1 b_2,d_2 d_1} \times I_4^{(L)}(-\beta_1',-\beta_2';-\beta_2,-\beta_1)_{\bar{a}_1\bar{a}_2,\bar{c}_2\bar{c}_1}.\end{aligned} \tag{8.41}$$

In the last equality the commutation relations (4.70) were used. $I_4^{(R)}$ and $I_4^{(L)}$ are the amplitudes:

$$\begin{cases} I_4^{(L)}(-\beta_1',-\beta_2';-\beta_2,-\beta_1)_{\bar{a}_1\bar{a}_2,\bar{c}_2\bar{c}_1} & =<0|L_{\bar{a}_1}(-\beta_1')L_{\bar{a}_2}(-\beta_2')L^{\dagger}_{\bar{c}_2}(-\beta_2)L^{\dagger}_{\bar{c}_1}(-\beta_1)|0>, \\ I_4^{(R)}(\beta_1',\beta_2';\beta_2,\beta_1)_{b_1 b_2,d_2 d_1} & =<0|R_{b_1}(\beta_1')R_{b_2}(\beta_2')R^{\dagger}_{d_2}(\beta_2)R^{\dagger}_{d_1}(\beta_1)|0> . \end{cases} \tag{8.42}$$

From the commutation relations, we get:

$$I_4^{(R)} = \delta_{b_1 d_1}\delta_{b_2 d_2}\delta(\beta_1 - \beta_1')\delta(\beta_2 - \beta_2') + \delta(\beta_1 - \beta_2')\delta(\beta_2 - \beta_1')S^{d_1 b_1}_{b_2 d_2}(\beta_1 - \beta_2). \tag{8.43}$$

Substituting in eq.(8.41), we obtain[2]:

$$\begin{aligned} I_4 = {} & \delta^2(0)\left[\delta_{b_1 d_1}\delta_{b_2 d_2}\delta_{\bar{a}_1\bar{c}_1}\delta_{\bar{a}_2\bar{c}_2}\delta(\beta_1 - \beta_1')\delta(\beta_2 - \beta_2') + \right. \\ & \left. +\delta(\beta_1 - \beta_2')\delta(\beta_2 - \beta_1')S^{d_1 b_1}_{b_2 d_2}(\beta_1 - \beta_2)S^{\bar{c}_1\bar{a}_1}_{\bar{a}_2\bar{c}_2}(\beta_2 - \beta_1)\right] + \\ & +\delta(0)\left[\delta_{b_1 d_1}\delta_{b_2 d_2}\delta(\beta_1 - \beta_2)\delta(\beta_1 - \beta_1')\delta(\beta_2 - \beta_2')S^{\bar{c}_1\bar{a}_1}_{\bar{a}_2\bar{c}_2}(0) + \right. \\ & \left. +\delta_{\bar{a}_1\bar{c}_1}\delta_{\bar{a}_2\bar{c}_2}\delta(\beta_1 - \beta_2)\delta(\beta_1 - \beta_2')\delta(\beta_2 - \beta_1')S^{d_1 b_1}_{b_2 d_2}(0)\right]. \end{aligned} \tag{8.44}$$

Consequently:

$$\begin{aligned} < B|4 > = {} & \tfrac{1}{2!}\left\{\delta^2(0)\left[\bar{K}^{\bar{c}_1 d_1}(\beta_1)\bar{K}^{\bar{c}_2 d_2}(\beta_2) + \bar{K}^{\bar{a}_1 b_1}(\beta_2)\bar{K}^{\bar{a}_2 b_2}(\beta_1)\times\right.\right. \\ & \left. \times S^{d_1 b_1}_{b_2 d_2}(\beta_1 - \beta_2)S^{\bar{c}_1\bar{a}_1}_{\bar{a}_2\bar{c}_2}(\beta_2 - \beta_1)\right] + \delta(0)\left[\bar{K}^{\bar{a}_1 d_1}(\beta_1)\bar{K}^{\bar{a}_2 d_2}(\beta_2)\delta(\beta_1 - \beta_2)S^{\bar{c}_1\bar{a}_1}_{\bar{a}_2\bar{c}_2}(0) + \right. \\ & \left.\left. +\bar{K}^{\bar{c}_1 b_1}(\beta_2)\bar{K}^{\bar{c}_2 b_2}(\beta_1)\delta(\beta_1 - \beta_2)S^{d_1 b_1}_{b_2 d_2}(0)\right]\right\}. \end{aligned} \tag{8.45}$$

Now consider the term:

$$\bar{K}^{\bar{a}_1 b_1}(\beta_2)\bar{K}^{\bar{a}_2 b_2}(\beta_1)S^{d_1 b_1}_{b_2 d_2}(\beta_1-\beta_2)S^{\bar{c}_1\bar{a}_1}_{\bar{a}_2\bar{c}_2}(\beta_2-\beta_1) = \bar{K}(\beta_1)\bar{K}(\beta_2)S^{d_1\bar{a}_1}_{\bar{a}_2 d_2}(\beta_1-\beta_2)S^{\bar{c}_1\bar{a}_1}_{\bar{a}_2\bar{c}_2}(\beta_2-\beta_1).$$

This term vanishes because it involves the backward scattering matrix B of eq.(4.64). Next consider:

$$\bar{K}^{\bar{a}_1 d_1}(\beta_1)\bar{K}^{\bar{a}_2 d_2}(\beta_2)\delta(\beta_1 - \beta_2)S^{\bar{c}_1\bar{a}_1}_{\bar{a}_2\bar{c}_2}(0) = \delta_{d_1 d_2}\delta_{\bar{c}_1\bar{c}_2}\bar{K}(\beta_1)\bar{K}(\beta_2)\delta(\beta_1 - \beta_2),$$

where we used $S^{\bar{c}_1\bar{a}_1}_{\bar{a}_2\bar{c}_2}(0) = \delta^{\bar{a}_1}_{\bar{a}_2}\delta^{\bar{c}_1}_{\bar{c}_2}$. Again, this term has to vanish. To understand why this is so, let us go back to the state $|2N>$ in eq.(8.33). The delta function $\delta(\beta_1 - \beta_2)$ and the Kronecker symbols $\delta_{d_1 d_2}\delta_{\bar{c}_1\bar{c}_2}$ mean that we have a state of the form:

$$L^\dagger_{\bar{c}_1}(-\beta_1)R^\dagger_{d_1}(\beta_1)L^\dagger_{\bar{c}_1}(-\beta_1)R^\dagger_{d_1}(\beta_1)|0>.$$

However this is not allowed by the selection rules (cf. sections 3.5 and 4.3).

Finally consider the term:

$$\bar{K}^{\bar{c}_1 b_1}(\beta_2)\bar{K}^{\bar{c}_2 b_2}(\beta_1)\delta(\beta_1 - \beta_2)S^{d_1 b_1}_{b_2 d_2}(0) = \bar{K}(\beta_1)\bar{K}(\beta_2)\delta(\beta_1 - \beta_2)\delta_{\bar{c}_1\bar{c}_2}\delta_{d_1 d_2}.$$

[2]Note that the contribution of $U_{RL}$ in (8.41) exactly cancels due to the $\delta$-functions.

Again this is not allowed because of the selection rules. Altogether, we have:

$$< B|4 >< 4|B >= \frac{\delta^4(0)}{(2!)^2}\bar{K}(\beta_1)K(\beta_1)\bar{K}(\beta_2)K(\beta_2)\delta^{d_1}_{\bar{c}_1}\delta^{d_1}_{\bar{c}_1}\delta^{d_2}_{\bar{c}_2}\delta^{d_2}_{\bar{c}_2}. \qquad (8.46)$$

On the other hand:

$$< 4|4 >= I_4^{(R)}(\beta_1,\beta_2;\beta_2,\beta_1)_{d_1d_2,d_2d_1} \times I_4^{(L)}(-\beta_1,-\beta_2;-\beta_2,-\beta_1)_{\bar{c}_1\bar{c}_2,\bar{c}_2\bar{c}_1}.$$

Due to the selection rules this is equal to $\delta^4(0)$. After summing over the internal degrees of freedom, we get:

$$\frac{< B|4 >< 4|B >}{< 4|4 >} = \frac{2^2}{(2!)^2}\bar{K}(\beta_1)K(\beta_1)\bar{K}(\beta_2)K(\beta_2). \qquad (8.47)$$

And this is equally in agreement with eq.(8.34). We assume that formula (8.34) is exact for all N. It is worth noting that although we started out with a nondiagonal theory, we ended up with a remarkably simple formula. Drastic simplifications were achieved due to the selection rules that prevent any two particles of the same type to be in exactly the same quantum state. But also the fact that the theory is massless prevents the backward scattering of particles and antiparticles.

The $\delta(0)$ quantities appearing in the computation of the internal products are not well defined. However, they are unavoidable if one wants to compute the partition function in the R-channel (see e.g.[86]). One must keep in mind that they require some regularization that will of course be controlled by the length $L$ of the interval. Another interesting feature about eq.(8.34) is the fact that it is independent of the R-L scattering $U_{RL}$.

Let us now go back to eq.(8.32). In each state $|\alpha>$ there are exactly the same number of left-movers and right-movers. For each Cooper-pair $i = 1, 2, \cdots, N$, we have an energy term, $(M/2)(e^{\beta_i} + e^{-(-\beta_i)}) = Mexp(\beta_i)$. Substituting in (8.32), we get:

$$Z = \sum_{\alpha} \frac{< B|\alpha >< \alpha|B >}{< \alpha|\alpha >} exp\left\{-RM\sum_{i(\alpha)} e^{\beta_i}\right\}, \qquad (8.48)$$

where the sum in the exponential is over all pairs in the state $|\alpha >= |2N >$. Let us now consider eq.(8.34). We shall drop the normalization $2^N/(N!)^2$ (it can be absorbed into the measure of integration in eq.(8.49) below). In the thermodynamic limit, when the number of particles tends to infinity, we can introduce the density $\rho_0(\beta)$ of pairs of right- and left-movers with opposite rapidities. Eq.(8.48) reads, [86], [87]:

$$Z \propto \int \mathcal{D}\rho_0(\beta) exp\left\{L\int_{-\infty}^{+\infty} d\beta \left[log\left(\bar{K}(\beta)K(\beta)\right) - RMe^{\beta}\right]\rho_0(\beta) + LS[\rho_0]\right\}. \qquad (8.49)$$

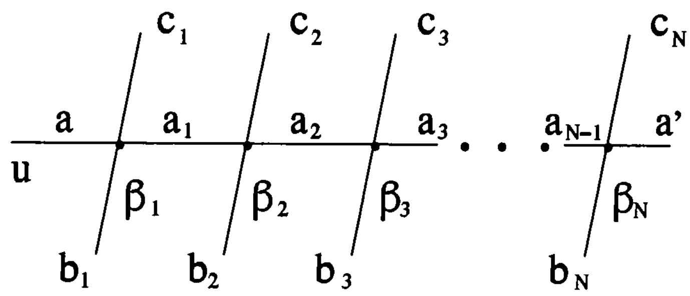


Figure 8.2: Transfer matrix

$S[\rho_0]$ is the entropy associated with the configuration $\rho_0$. We see that the contribution from the boundary can be interpreted as a rapidity dependent chemical potential, [86].

The system is constrained by the quantization condition, that arises from placing our system in a finite periodic box of length $L$. We get equations of the form, [81]:

$$\begin{aligned} &e^{ip_1L}R_{a_1}(\beta_1)L_{\bar{d}_1}(-\beta_1)R_{c_2}(\beta_2)L_{\bar{d}_2}(-\beta_2)\cdots R_{c_N}(\beta_N)L_{\bar{d}_N}(-\beta_N)= \\ &= L_{\bar{d}_1}(-\beta_1)R_{c_2}(\beta_2)L_{\bar{d}_2}(-\beta_2)\cdots R_{c_N}(\beta_N)L_{\bar{d}_N}(-\beta_N)R_{a_1}(\beta_1), \end{aligned} \tag{8.50}$$

where $p_1 = \frac{M}{2}e^{\beta_1}$ is the momentum of $R_{a_1}(\beta_1)$ and we dropped the symbol † for simplicity. By commuting $R_{a_1}(\beta_1)$ on the l.h.s. of eq.(8.50) with all the other operators, we get:

$$\begin{aligned} &e^{ip_1L}\left[\Pi_{l=1}^N U_{RL}(\beta_1+\beta_l)\right] S_{a_1c_2}^{a_2b_2}(\beta_1-\beta_2)S_{a_2c_3}^{a_3b_3}(\beta_1-\beta_3)\cdots S_{a_{N-1}c_N}^{a_Nb_N}(\beta_1-\beta_N)= \\ &= \delta_{c_2}^{b_2}\delta_{c_3}^{b_3}\cdots\delta_{c_N}^{b_N}\delta_{a_1}^{a_N}. \end{aligned} \tag{8.51}$$

We define the $(2^N \times 2^N)$ "*colour transfer matrix*" for $N$ right-moving particles with the following matrix elements, [57]:

$$T_a^{a'}(u|\beta_1,\cdots,\beta_N)_{c_1\cdots c_N}^{b_1\cdots b_N} = S_{ac_1}^{a_1b_1}(u-\beta_1)S_{a_1c_2}^{a_2b_2}(u-\beta_2)\cdots S_{a_{N-1}c_N}^{a'b_N}(u-\beta_N), \tag{8.52}$$

where $u$ is called the spectral parameter. This object [57] is represented in fig.8.2.

Each node represents an interaction with S-matrix given by eq.(4.66). Multiplying eq.(8.51) by $S_{a_Nc_1}^{a_1b_1}(0) = \delta_{a_N}^{b_1}\delta_{c_1}^{a_1}$ and summing over repeated indices, we get:

$$e^{ip_1L}\left[\Pi_{l=1}^N U_{RL}(\beta_1+\beta_l)\right] T(\beta_1|\beta_1,\cdots,\beta_N) = 1. \tag{8.53}$$

where $T(u) \equiv \sum_a T_a^a(u)$ is the trace of the transfer matrix. Since we chose the first particle randomly and the system is periodic, eq.(8.53) can be generalized to the following set of Yang equations [75], [57]:

$$e^{ip_kL}\left[\Pi_{l=1}^N U_{RL}(\beta_k+\beta_l)\right] T(\beta_k|\beta_1,\cdots,\beta_N) = 1, \qquad k=1,2,\cdots,N. \tag{8.54}$$

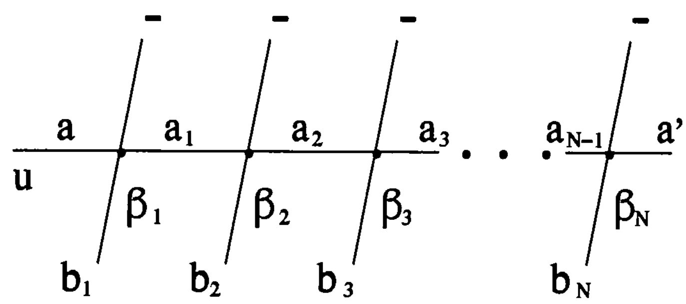


Figure 8.3: Bare vacuum

As usual for the higher Bethe Ansatz (i.e. nondiagonal scattering), we try to diagonalize the transfer matrix using the method of "*quantum inverse scattering*", [57]. The commutation relations for the transfer matrix can be obtained by successive application of the Yang-Baxter relation. They read, [57]:

$$T_a^{a''}(u)T_b^{b''}(v)S_{a''b''}^{a'b'}(u-v) = S_{ab}^{a''b''}(u-v)T_{b''}^{b'}(v)T_{a''}^{a'}(u). \tag{8.55}$$

Writing these out explicitly, we have:

$$\begin{aligned}
&[T(u),T(v)] = [T_+^+(u),T_+^+(v)] = [T_-^-(u),T_-^-(v)] = [T_+^-(u),T_+^-(v)] = [T_-^+(u),T_-^+(v)] = 0,\\
&(u-v)T_+^+(u)T_+^-(v) = i\pi T_+^-(u)T_+^+(v) + (u-v-i\pi)T_+^-(v)T_+^+(u),\\
&(u-v)T_-^-(v)T_+^-(u) = i\pi T_+^-(v)T_-^-(u) + (u-v-i\pi)T_+^-(u)T_-^-(v).
\end{aligned} \tag{8.56}$$

From these commutation relations we see that it makes sense to find the simultaneous eigenstates of $T(u)$ for different values of $u$. We start by defining the 'bare vacuum' state $|0>$, which corresponds to a state where all the $N$ frame particles have spin "*down*", [57]. It is an eigenstate of $T(u)$. For example, $T_+^+(u)$ acting on $|0>$ would correspond to the situation in fig.8.3.

The first node yields a contribution $S_{+c_1}^{a_1-}(u-\beta_1) = \delta_+^{a_1}\delta_{c_1}^-\sigma_T(u-\beta_1)$. Consequently, this will vanish unless $a_1 = +$ and $c_1 = -$. Proceeding with this reasoning, we conclude that all the $a_i$ are "up" and all the $c_i$ are "down". Notice also that this is compatible with having $+$ at the end of the frame and that each node $i$ gives a contribution $\sigma_T(u-\beta_i)$. Therefore:

$$T_+^+(u)|0> = \left[\Pi_{l=1}^N \sigma_T(u-\beta_l)\right]|0> . \tag{8.57}$$

Similarly, we can show that $|0>$ is also an eigenstate of $T_-^-(u)$. Altogether we have:

$$T(u)|0> = \left[\Pi_{l=1}^N \sigma_T(u-\beta_l) + \Pi_{l=1}^N S_V(u-\beta_l)\right]|0> . \tag{8.58}$$

The space of states is constituted by the set of fictitious particle states ("*magnons*") of the form, [57]:

$$|\lambda_1, \cdots, \lambda_M >= T^-_+(\lambda_1) \cdots T^-_+(\lambda_M)|0> . \tag{8.59}$$

From eq.(8.56) we see that the order of the rapidities $\lambda_1, \cdots, \lambda_M$ is immaterial. In general it is not an eigenstate of $T(u)$. In fact it can be shown that, [57]:

$$\begin{aligned} T(u)|\lambda_1, \cdots, \lambda_M >= & \left[\Pi_{j=1}^M \tfrac{\lambda_j - u + i\pi}{\lambda_j - u} \Pi_{k=1}^N \sigma_T(u - \beta_k) + \right. \\ & \left. + \Pi_{j=1}^M \tfrac{u - \lambda_j + i\pi}{u - \lambda_j} \Pi_{k=1}^N S_V(u - \beta_k)\right] |\lambda_1, \cdots, \lambda_M > + \\ & + \textstyle\sum_{j=1}^M \tfrac{i\pi}{\lambda_j - u} \left[ -\Pi_{i \neq j}^M \tfrac{\lambda_i - \lambda_j + i\pi}{\lambda_i - \lambda_j} \Pi_{k=1}^N \sigma_T(\lambda_j - \beta_k) + \right. \\ & \left. + \Pi_{i\neq j}^M \tfrac{\lambda_j - \lambda_i + i\pi}{\lambda_j - \lambda_i} \Pi_{k=1}^N S_V(\lambda_j - \beta_k)\right] |\lambda_1, \cdots \not\lambda_j, \cdots, \lambda_M, u >, \end{aligned} \tag{8.60}$$

where $\not\lambda_j$ means that this rapidity is omitted. This state will be an eigenstate of the trace of the transfer matrix, provided the shifted rapidities $y_j = \lambda_j - i\pi/2$ $(j = 1, \cdots, M)$ satisfy the constraints, [57]:

$$\Pi_{i\neq j}^M \frac{y_j - y_i + i\pi}{y_j - y_i - i\pi} \Pi_{k=1}^M \frac{y_j - \beta_k - i\pi/2}{y_j - \beta_k + i\pi/2} = 1, \qquad j = 1, \cdots, M, \tag{8.61}$$

in which case:

$$\begin{aligned} T(u)|\lambda_1, \cdots, \lambda_M >= & \left\{\Pi_{j=1}^M \tfrac{u - \lambda_j - i\pi}{u - \lambda_j} \Pi_{k=1}^N \sigma_T(u - \beta_k) + \right. \\ & \left. + \Pi_{j=1}^M \tfrac{u - \lambda_j + i\pi}{u - \lambda_j} \Pi_{k=1}^N S_V(u - \beta_k)\right\} |\lambda_1, \cdots, \lambda_M > . \end{aligned} \tag{8.62}$$

Equations (8.61) can be interpreted as the periodicity conditions for the Bethe wave function of $M$ magnons subject to the diagonal factorized scattering with magnon-magnon scattering amplitude:

$$S_{11}(y) = \frac{y + i\pi}{y - i\pi}. \tag{8.63}$$

The scattering between the magnons and the frame particles is described by the amplitude:

$$S_1(y - \beta) = \frac{y - \beta - i\pi/2}{y - \beta + i\pi/2}. \tag{8.64}$$

The system (8.61) has been analyzed in the context of the Heisenberg spin chain, [74], [76]. In the thermodynamic limit ($N \to \infty$), its solutions are either the isolated real roots $y_0$, corresponding to the magnon $M_1(y_0)$ proper rapidity $y_0$, or the strings of arbitrary number n of roots:

$$y_\nu = y_0 + \frac{i\pi}{2}\nu, \qquad \nu = -n+1, -n+3, \cdots, n-1,$$

which can be interpreted as the n-magnon bound state $M_n(y_0)$ ($n = 1, 2, \ldots, \infty$) of real rapidity $y_0$. The amplitudes ($S_{nm}$) for the $M_n - M_m$ ($m \geq n$) bound state scattering and for the scattering between the n-magnons $M_n$ and the frame particles ($S_n$) can be derived from the bootstrap fusions (3.52). They are given by, [57]:

$$\begin{aligned} S_n(y-\beta) &= \tfrac{y-\beta-in\pi/2}{y-\beta+in\pi/2}, \\ S_{mn}(y) &= \tfrac{y+i\pi(m+n)/2}{y-i\pi(m+n)/2} \times \left[\tfrac{y+i\pi(m+n-2)/2}{y-i\pi(m+n-2)/2} \cdots \tfrac{y+i\pi(m-n-2)/2}{y-i\pi(m-n-2)/2}\right]^2 \times \tfrac{y+i\pi(m-n)/2}{y-i\pi(m-n)/2}. \end{aligned} \tag{8.65}$$

In the thermodynamic limit $N, M \to \infty$, we introduce the densities $\rho_n(y)$ and the densities of states $\Lambda_n(y)$ of n-magnon bound states ($n = 1, 2, \cdots, \infty$). Eq.(8.61) then reads, [57]:

$$2\pi\Lambda_n = \phi_n * \rho_0 + \sum_{m=1}^{\infty} \phi_{mn} * \rho_m, \tag{8.66}$$

where the kernels $\phi_n$, $\phi_{mn}$ are defined as:

$$\begin{cases} \phi_n(y) &= -i\frac{\partial}{\partial y} log S_n(y), \\ \phi_{mn}(y) &= -i\frac{\partial}{\partial y} log S_{mn}(y). \end{cases} \tag{8.67}$$

The rapidity convolution is given by:

$$(\phi * \rho)(y) = \int_{-\infty}^{+\infty} dy' \phi(y-y')\rho(y'). \tag{8.68}$$

We can rewrite Eqs.(8.66) in a more tractable form by noticing that if we work with Fourier transforms, convolutions are replaced by ordinary products. We define the Fourier transform of a quantity $A(y)$ by:

$$A(k) = \int_{-\infty}^{+\infty} dy A(y) e^{iky}, \tag{8.69}$$

and its inverse by:

$$A(y) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} A(k) e^{-iky}. \tag{8.70}$$

Then we can show that the Fourier transforms of the kernels (8.67) satisfy the following identities, [57]:

$$\begin{aligned} &\tfrac{1}{2\pi}\phi_n(k) = exp\left(-\tfrac{\pi n}{2}|k|\right), \\ &\textstyle\sum_{p=1}^{\infty} \left(\delta_{m,p} - \tfrac{1}{2\pi}\phi_{mp}(k)\right) \times \left(\delta_{p,n} - \tfrac{1}{2cosh(\pi k/2)} l_{pn}\right) = \delta_{m,n}. \end{aligned} \tag{8.71}$$

$l_{mn}$ ($m, n = 1, 2, \cdots, \infty$) is called the incidence matrix, [57]. Its elements vanish unless the nodes $m$ and $n$ are connected (i.e. adjacent) in the diagram of fig.8.4.

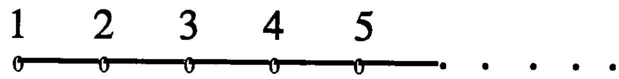

Figure 8.4: Incidence matrix

From eqs.(8.65) and (8.67), we have:

$$\phi_n(y) = -i\frac{\partial}{\partial\beta} log S_n(y) = \frac{n\pi}{\left(y + i\frac{n\pi}{2}\right)\left(y - i\frac{n\pi}{2}\right)}. \tag{8.72}$$

Consequently the integrand in (8.69) has two poles situated at $y = \pm in\pi/2$. If $k > 0$, the integral diverges unless $Imy > 0$. We therefore consider the contour $C_+$ in fig.8.5, which encloses only the singularity $y = in\pi/2$. Using Cauchy's theorem, we have:

$$\phi_n(k) = 2\pi i Res(y = in\pi/2) = 2\pi e^{-\pi nk/2}. \tag{8.73}$$

For $k < 0$, the integral converges for $Imy < 0$ and we perform the integral over $C_-$:

$$\phi_n(k) = -2\pi i Res(y = -in\pi/2) = 2\pi e^{\pi nk/2}. \tag{8.74}$$

The additional minus sign comes from the opposite orientation of the contour $C_-$. Altogether we recover the first equation in (8.71). The second equation can be obtained in a similar fashion. If we Fourier transform eq.(8.66), we get:

$$2\pi\Lambda_n(k) = \phi_n(k)\rho_0(k) + \sum_{m=1}^{\infty} \phi_{mn}(k)\rho_m(k). \tag{8.75}$$

Using this equation, we get for $n \geq 2$:

$$\begin{aligned}
&\Lambda_{n+1}(k) + \Lambda_{n-1}(k) = \tfrac{1}{\pi}cosh(\pi k/2)\phi_n(k)\rho_0(k) + \tfrac{1}{2\pi}\textstyle\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} l_{pn}\rho_m(k)\phi_{mp}(k) = \\
&= \tfrac{1}{\pi}cosh(\pi k/2)\phi_n(k)\rho_0(k) + \textstyle\sum_{m=1}^{\infty} l_{mn}\rho_m(k) + \tfrac{1}{\pi}cosh(\pi k/2)\textstyle\sum_{m=1}^{\infty}\phi_{mn}(k)\rho_m(k) = \\
&= \rho_{n+1}(k) + \rho_{n-1}(k) + \tfrac{1}{\pi}cosh(\pi k/2)\left[\phi_n(k)\rho_0(k) + \textstyle\sum_{m=1}^{\infty}\phi_{mn}(k)\rho_m(k)\right] = \\
&= 2cosh(\pi k/2)\Lambda_n(k) + \rho_{n+1}(k) + \rho_{n-1}(k).
\end{aligned} \tag{8.76}$$

Consequently, if we define the densities of holes, $\tilde{\Lambda}_n = \Lambda_n - \rho_n$ $(n = 0, 1, \cdots, \infty)$, we get, [57]:

$$\Lambda_n(k) = \frac{1}{2cosh(\pi k/2)}\left(\tilde{\Lambda}_{n+1}(k) + \tilde{\Lambda}_{n-1}(k)\right). \tag{8.77}$$

Equivalently:

$$\Lambda_n = \frac{1}{2\pi}\varphi * \left(\tilde{\Lambda}_{n+1} + \tilde{\Lambda}_{n-1}\right), \qquad n \geq 2, \tag{8.78}$$

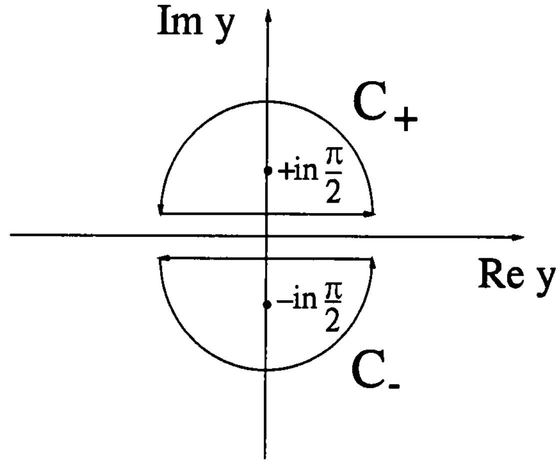


Figure 8.5: Contours

where $\varphi(y)$ is the unified kernel, [57]:

$$\varphi(y) \equiv \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\pi}{cosh(\pi k/2)} e^{-iky} = \frac{1}{coshy}. \tag{8.79}$$

Following the same procedure for $n = 1$, we get:

$$\Lambda_1 = \frac{1}{2\pi} \varphi * \left(\tilde{\Lambda}_2 + \rho_0\right). \tag{8.80}$$

Let us consider again eq.(8.77). If we multiply by $2\pi e^{-\pi n|k|/2}$ and sum over $n \geq 2$ we get from (8.71):

$$\sum_{n=2}^{\infty} \Lambda_n(k)\phi_n(k) = \frac{1}{2cosh(\pi k/2)} \left\{ e^{\pi|k|/2} \sum_{n=3} \tilde{\Lambda}_n(k)\phi_n(k) + e^{-\pi|k|/2} \sum_{n=1}^{\infty} \tilde{\Lambda}_n(k)\phi_n(k) \right\}. \tag{8.81}$$

This yields:

$$\Lambda_1(k)\phi_1(k) = \frac{e^{\pi|k|/2}}{2cosh(\pi k/2)} \left(\tilde{\Lambda}_1(k)\phi_1(k) + \tilde{\Lambda}_2(k)\phi_2(k)\right) + \sum_{n=1}^{\infty} \phi_n(k)\rho_n(k). \tag{8.82}$$

Replacing $\tilde{\Lambda}_2$ by eq.(8.80), we get:

$$\begin{aligned} \Lambda_1(k)\phi_1(k) = & \tfrac{e^{\pi|k|/2}}{2cosh(\pi k/2)} \phi_1(k)\tilde{\Lambda}_1(k) + e^{\pi|k|/2}\phi_2(k)\Lambda_1(k) + \\ & + \textstyle\sum_{n=1}^{\infty} \phi_n(k)\rho_n(k) - \tfrac{e^{\pi|k|/2}}{2cosh(\pi k/2)}\phi_2(k)\rho_0(k). \end{aligned} \tag{8.83}$$

Notice that $e^{\pi|k|/2}\phi_1(k) = 2\pi$ and $e^{\pi|k|/2}\phi_2(k) = \phi_1(k)$, and so:

$$\sum_{n=1}^{\infty} \phi_n(k)\rho_n(k) = -\frac{\pi}{cosh(\pi k/2)}\tilde{\Lambda}_1(k) + \frac{\pi}{cosh(\pi k/2)} e^{-\pi|k|/2}\rho_0(k). \tag{8.84}$$

After Fourier transforming, we obtain the following useful identity, [57]:

$$\sum_{n=1} \phi_n * \rho_n = \phi * \rho_0 - \varphi * \tilde{\Lambda}_1, \tag{8.85}$$

where

$$\phi(\beta) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\pi}{cosh(\pi k/2)} e^{-\pi|k|/2} e^{-i\beta k} = -i\frac{\partial}{\partial\beta} log S_V(\beta). \tag{8.86}$$

In the last step we used eq.(4.69) to show that $\phi(\beta)$ is the kernel associated with the isovector amplitude $S_V(\beta)$, [57].

Let us now go back to the set of Yang equations (8.54). We consider as eigenvector of the transfer matrix the $M$ magnon state (8.59) with rapidities $y_1, \cdots, y_M$ subject to the constraints (8.61). Taking into account the fact that $\sigma_T(0) = 0$, we get:

$$e^{ip_k L} \Pi_{j=1}^{M} \frac{\beta_k - y_j + i\pi/2}{\beta_k - y_j - i\pi/2} \Pi_{l=1}^{N} \left[ S_V(\beta_k - \beta_l) U_{RL}(\beta_k + \beta_l) \right] = 1. \tag{8.87}$$

In the thermodynamic limit $N, M \to \infty$, this yields:

$$2\pi\Lambda_0 = \phi_S * \rho_0 - \sum_{n=1}^{\infty} \phi_n * \rho_n + \frac{M}{2} e^{\beta}, \tag{8.88}$$

where

$$\phi_S(\beta - \beta') = -i\frac{\partial}{\partial\beta} \left[ log S_V(\beta - \beta') + log U_{RL}(\beta + \beta') \right] = \phi(\beta - \beta') + \varphi(\beta + \beta'). \tag{8.89}$$

Using the identity (8.85), we get:

$$2\pi\Lambda_0 = \varphi_S * \rho_0 + \varphi * \tilde{\Lambda}_1 + \frac{M}{2} e^{\beta}, \tag{8.90}$$

where $\varphi_S = \varphi(\beta + \beta')$.

To compute the ground state energy, we perform a saddle point evaluation of the partition function (8.49), or equivalently, we minimize the free energy, [73],

$$\mathcal{F} = \frac{L}{2} \int_{-\infty}^{+\infty} d\beta \left\{ \left[ RMe^{\beta} - log(\bar{K}(\beta) K(\beta)) \right] \rho_0(\beta) - S\left[ \rho_n, \Lambda_n \right] \right\}, \tag{8.91}$$

with respect to the macroscopic quantities $\rho_n(\beta)$, $\Lambda_n(\beta)$ taking into account the constraints (8.78), (8.80) and (8.90). The magnons being fictitious particles do not contribute to the energy of the system. However, they account for the isospin degrees of freedom and must therefore contribute to the total entropy, [57]. We have mentioned above that all particles in the system- fictitious or not -obey certain selection rules. The sets of left- and right-movers are represented by anti-commuting operators. Because, the corresponding S-matrix element is symmetric for particles with the same rapidity and isospin ($S_{aa}^{aa}(0) = 1$) we concluded that they obey an exclusion principle preventing any two of them to be simultaneously in the exact same quantum state. The magnons, on the other hand, are bosons, but $(S_n(0) = -1)$, $(n = 0, 1, \cdots)$. Consequently, they equally obey Fermi statistics, [57].

Consider a system with $N$ types of fermionic particles. Suppose that there are $N_a^{(r)}$ particles of type $a(= 1, \cdots, N)$ and $N_a$ states available. The entropy for such a system is known to be of the form, [80]:

$$\delta s \sim ln \left\{ \Pi_{a=1}^{N} \frac{N_a!}{(N_a - N_a^{(r)})! N_a^{(r)}!} \right\}. \tag{8.92}$$

In the thermodynamic limit $N_a^{(r)}, L \to \infty$, (with $N_a/L$ fixed) we define the density of particles of type $a$ as $\rho_a^{(r)}(\beta) L \Delta\beta = N_a^{(r)}$ and the density of states as $\rho_a(\beta) L \Delta\beta = N_a$. Using Stirling's formula, we get, [73]:

$$\delta s \sim L\Delta\beta \sum_{a=1}^{N} \left[ \rho_a ln\rho_a - (\rho_a - \rho_a^{(r)}) ln(\rho_a - \rho_a^{(r)}) - \rho_a^{(r)} ln\rho_a^{(r)} \right]. \tag{8.93}$$

To take into account the constraints (8.77), (8.80) and (8.90), we introduce an infinite number of Lagrange's multipliers (one for each constraint), $\mu_n(\beta)$ $(n = 0, 1, \cdots, \infty)$, and rewrite (8.91) as:

$$\begin{gathered} \mathcal{F} = \tfrac{L}{2} \int_{-\infty}^{+\infty} d\beta \Big\{ RMe^{\beta}\rho_0(\beta) - log\left[\bar{K}(\beta)K(\beta)\right]\rho_0(\beta) \\ - \textstyle\sum_{n=0}^{\infty} \left[ \Lambda_n log\Lambda_n - \rho_n log\rho_n - \tilde{\Lambda}_n log\tilde{\Lambda}_n \right] + \mu_0(\beta) \left[ \Lambda_0 - \tfrac{1}{2\pi}\varphi_S * \rho_0 - \tfrac{1}{2\pi}\varphi * \tilde{\Lambda}_1 - \tfrac{M}{4\pi} e^{\beta} \right] + \\ + \mu_1(\beta) \left[ \Lambda_1 - \tfrac{1}{2\pi}\varphi * (\tilde{\Lambda}_2 + \rho_0) \right] + \textstyle\sum_{n=2}^{\infty} \mu_n(\beta) \left[ \Lambda_n - \tfrac{1}{2\pi}\varphi * (\tilde{\Lambda}_{n+1} + \tilde{\Lambda}_{n-1}) \right] \Big\}. \end{gathered} \tag{8.94}$$

It is convenient to define the pseudo-energies, [57]:

$$\begin{cases} \frac{\rho_0}{\Lambda_0} = \frac{e^{-\epsilon_0}}{1+e^{-\epsilon_0}}, \\ \frac{\tilde{\Lambda}_n}{\Lambda_n} = \frac{e^{-\epsilon_n}}{1+e^{-\epsilon_n}}, \qquad n = 1, 2, \cdots, \infty, \end{cases} \tag{8.95}$$

and the functions:

$$L_n(\beta) \equiv log(1 + e^{-\epsilon_n(\beta)}), \qquad n = 0, 1, \cdots, \infty. \tag{8.96}$$

Minimizing the functional (8.94) with respect to $\rho_n$, $\Lambda_n$ and $\mu_n$ leads to the set of equations:

$$\begin{cases} -\nu_0 + \epsilon_0 + \frac{1}{2\pi}\varphi_S * L_0 + \frac{1}{2\pi}\varphi * L_1 = 0, \\ -\nu_n + \epsilon_n + \frac{1}{2\pi}\varphi * \sum_{m=0}^{\infty} l_{mn} L_m = 0, \qquad n = 1, 2 \cdots, \infty, \\ \mu_n(\beta) = L_n(\beta), \qquad n = 0, 1, \cdots, \infty, \end{cases} \tag{8.97}$$

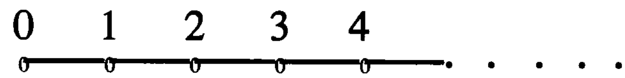


Figure 8.6: incidence matrix

where $l_{mn}$ is the incidence matrix [57] obtained by adding an extra node to the previous one in eq.(8.71) (fig.8.6).

The boundary energy terms are given by:

$$\begin{cases} \nu_0(\beta) = RMe^{\beta} - log(\bar{K}(\beta)K(\beta)), \\ \\ \nu_n(\beta) = 0, \qquad n = 1, 2, \cdots, \infty. \end{cases} \tag{8.98}$$

The minimum value of the free energy is then:

$$\mathcal{F}|_{min} = -\frac{ML}{8\pi}\int_{-\infty}^{+\infty} d\beta e^{\beta} L_0(\beta). \tag{8.99}$$

We would like to compute the previous integral in the IR limit $R \to \infty$. We start by rewriting (8.99) in the form, [81]:

$$\mathcal{F}|_{min} = -\frac{L}{8\pi}\sum_{n=0}^{\infty} m_n \int_{-\infty}^{+\infty} d\beta e^{\beta}\tilde{L}_n(\beta), \tag{8.100}$$

where $m_0 = M$, $m_n = 0$ $(n = 1, 2, \cdots, \infty)$ and $\tilde{L}_n(\beta) \equiv log(1 + \lambda_n(\beta)e^{-\epsilon_n(\beta)})$. The boundary terms are $\lambda_n(\beta) \equiv 1 - \delta_{n,0}\,(1 - \bar{K}(\beta)K(\beta))$ and the pseudo-energies satisfy the equations:

$$\begin{cases} -m_0Re^{\beta} + \epsilon_0 + \frac{1}{2\pi}\varphi_S * \tilde{L}_0 + \frac{1}{2\pi}\varphi * \tilde{L}_1 = 0, \\ \\ -m_nRe^{\beta} + \epsilon_n + \frac{1}{2\pi}\varphi * \sum_{m=0}^{\infty} l_{mn}\tilde{L}_m = 0. \end{cases} \tag{8.101}$$

In the limit $R \to \infty$, we see from (8.101) that $\epsilon_0(+\infty) = \infty$. However the same does not hold for the remaining pseudo-energies. Rather they tend to a limiting constant value in the region $\beta << RM$. These constant values are determined by the set of equations:

$$\epsilon_n(+\infty) + \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} d\beta\varphi(\beta)\right)\sum_{m=0}^{\infty} l_{mn} log(1 + e^{-\epsilon_m(+\infty)}) = 0, \tag{8.102}$$

where we used the fact that for $K(\beta)$ given by (8.8), $\lambda_n(\beta) = 1$ for *all* $n = 0, 1, \cdots, \infty$. Now:

$$\int_{-\infty}^{+\infty} d\beta\varphi(\beta) = \int_{-\infty}^{+\infty} \frac{d\beta}{cosh\beta} = 2\int_{0}^{\infty} \frac{dt}{1+t^2} = 2\left[argtg(t)\right]_0^{\infty} = \pi. \tag{8.103}$$

Defining $y_n \equiv e^{-\epsilon_n(+\infty)}$, we get the set of equations:

$$y_n^2 = \Pi_{m=1}^{\infty}(1+y_n)^{\tilde{l}_{mn}} = (1+y_{n+1})(1+y_{n-1}), \qquad n = 1,2,\cdots,\infty, \tag{8.104}$$

where $\tilde{l}_{mn}$ is obtained from $l_{mn}$ by omitting the zero-th node. To solve this system we truncate it for some positive integer $p$ and eventually take the limit $p \to \infty$. The general solution is, [85]:

$$1 + y_n = \frac{sin^2\left[\frac{\pi(n+a)}{p+b}\right]}{sin^2\left[\frac{\pi}{p+b}\right]}, \tag{8.105}$$

where $a$ and $b$ are arbitrary constants. They are fixed by imposing the "*initial values*": $y_0 = y_{p+1} = 0$. We then get:

$$1 + y_n = \frac{sin^2\left[\frac{\pi(n+1)}{p+3}\right]}{sin^2\left[\frac{\pi}{p+3}\right]}, \qquad n = 0,1,\cdots,p. \tag{8.106}$$

On the other hand, when $\beta \to -\infty$, we get the following set of equations, in the region $-RM << \beta$:

$$\epsilon_0(-\infty) + \frac{1}{2}log\left(1 + e^{-\epsilon_0(+\infty)}\right) + log\left(1 + e^{-\epsilon_1(-\infty)}\right) = 0, \tag{8.107}$$

and

$$\epsilon_n(-\infty) + \frac{1}{2}\sum_{m=0}^{\infty} l_{mn}log\left(1 + e^{-\epsilon_m(-\infty)}\right) = 0, \qquad n = 1,2,\cdots,\infty. \tag{8.108}$$

Using $\epsilon_0(+\infty) = \infty$, we can rewrite the two previous expressions in the compact form:

$$\epsilon_n(-\infty) + \frac{1}{2}\sum_{m=0}^{\infty} l_{mn}log\left(1 + e^{-\epsilon_m(-\infty)}\right) = 0, \qquad n = 0,1,\cdots,\infty. \tag{8.109}$$

Defining $x_n \equiv e^{-\epsilon_n(-\infty)}$, we get:

$$x_n^2 = \Pi_{m=0}^{\infty}(1+x_n)^{l_{mn}}, \qquad n = 0,1,\cdots,\infty. \tag{8.110}$$

The solution is of the form (8.105). However, this time, the initial values are $x_{-1} = x_{p+1} = 0$, and we get:

$$1 + x_n = \frac{sin^2\left[\frac{\pi(n+2)}{p+4}\right]}{sin^2\left[\frac{\pi}{p+4}\right]}, \qquad n = 0,1,\cdots,\infty. \tag{8.111}$$

Let us now go back to eq.(8.100). We can replace $m_n e^{\beta}$ by the derivatives of (8.101) with respect to $\beta$, [73]:

$$\begin{aligned}\mathcal{F}|_{min} = & -\tfrac{L}{8\pi R}\int_{-\infty}^{+\infty} d\beta L_0(\beta)\left[\epsilon_0'(\beta) + \tfrac{1}{2\pi}\left(\varphi_S' * L_0\right)(\beta) + \tfrac{1}{2\pi}\left(\varphi' * L_1\right)(\beta)\right] \\ & -\tfrac{L}{8\pi R}\textstyle\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty} d\beta L_n(\beta)\left[\epsilon_n'(\beta) + \tfrac{1}{2\pi}\left(\varphi' * \textstyle\sum_{m=0}^{\infty} l_{mn}L_m\right)(\beta)\right] \qquad (8.112) \\ \equiv & \textstyle\sum_{n=0}^{\infty} I_1^{(n)} + I_2^{(0)} + \sum_{n=1}^{\infty} I_2^{(n)},\end{aligned}$$

where

$$I_1^{(n)} \equiv -\frac{L}{8\pi R}\int_{-\infty}^{+\infty} d\beta L_n(\beta)\epsilon_n'(\beta) = -\frac{L}{8\pi R}\int_{\epsilon_n(-\infty)}^{\epsilon_n(+\infty)} d\epsilon log(1+e^{-\epsilon}). \tag{8.113}$$

In the last step we performed the substitution $\beta \to \epsilon = \epsilon_n(\beta)$. This is allowed, because the pseudo-energies are monotonic functions of $\beta$. The next term is:

$$\sum_{n=1}^{\infty} I_2^{(n)} = -\frac{L}{16\pi^2 R}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} l_{mn}\int_{-\infty}^{+\infty} d\beta \int_{-\infty}^{+\infty} d\beta' L_n(\beta)\frac{\partial}{\partial\beta}\varphi(\beta-\beta')L_m(\beta'). \tag{8.114}$$

Using $\frac{\partial}{\partial\beta}\varphi(\beta-\beta') = -\frac{\partial}{\partial\beta'}\varphi(\beta-\beta')$ and the symmetry of both the kernel $\varphi(\beta)$ and the incidence matrix $l_{mn}$, we obtain after an integration by parts:

$$\sum_{n=1}^{\infty} I_2^{(n)} = \frac{L}{8\pi R}\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty} d\beta\frac{e^{-\epsilon_n(\beta)}}{1+e^{-\epsilon_n(\beta)}}\epsilon_n'(\beta)\times\left[\frac{1}{2\pi}\sum_{m=1}^{\infty} l_{mn}(\varphi * L_m)(\beta)\right]. \tag{8.115}$$

Similarly,

$$\begin{aligned} I_2^{(0)} = \tfrac{L}{8\pi R}\textstyle\int_{-\infty}^{+\infty} d\beta \Big[&\tfrac{e^{-\epsilon_0(\beta)}}{1+e^{-\epsilon_0(\beta)}}\epsilon_0'(\beta)\tfrac{1}{2\pi}(\varphi_S * L_0)(\beta)+ \\ &+\tfrac{e^{-\epsilon_1(\beta)}}{1+e^{-\epsilon_1(\beta)}}\epsilon_1'(\beta)\tfrac{1}{2\pi}(\varphi * L_1)(\beta)\Big]. \end{aligned} \tag{8.116}$$

Adding up (8.115) and (8.116), we get:

$$\begin{aligned} I_2^{(0)} + \textstyle\sum_{n=1}^{\infty} I_2^{(n)} &= \tfrac{L}{8\pi R}\textstyle\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty} d\beta\tfrac{e^{-\epsilon_n(\beta)}}{1+e^{-\epsilon_n(\beta)}}\epsilon_n'(\beta)\times\left[\left(\tfrac{1}{2\pi}\varphi * \sum_{m=0}^{\infty} l_{mn}L_m\right)(\beta)\right] + \\ &+\tfrac{L}{8\pi R}\textstyle\int_{-\infty}^{+\infty} d\beta\tfrac{e^{-\epsilon_0(\beta)}}{1+e^{-\epsilon_0(\beta)}}\epsilon_0'(\beta)\times\left[\tfrac{1}{2\pi}(\varphi_S * L_0)(\beta) + \tfrac{1}{2\pi}(\varphi * L_1)(\beta)\right]. \end{aligned} \tag{8.117}$$

From (8.101), we see that this is equal to:

$$-\frac{L}{8\pi R}\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty} d\beta\frac{e^{-\epsilon_n(\beta)}}{1+e^{-\epsilon_n(\beta)}}\epsilon_n'(\beta)\epsilon_n(\beta) = -\frac{L}{8\pi R}\sum_{n=0}^{\infty}\int_{\epsilon_n(-\infty)}^{\epsilon_n(+\infty)} d\epsilon\frac{e^{-\epsilon}\epsilon}{1+e^{-\epsilon}}. \tag{8.118}$$

Substituting (8.113) and (8.118) in (8.112), we get:

$$\mathcal{F}|_{min} = -\frac{L}{8\pi R}\sum_{n=0}^{+\infty}\int_{\epsilon_n(-\infty)}^{\epsilon_n(+\infty)} d\epsilon\left[log(1+e^{-\epsilon}) + \frac{e^{-\epsilon}\epsilon}{1+e^{-\epsilon}}\right]. \tag{8.119}$$

We now perform the substitution,

$$\epsilon \to t = \frac{1}{1+e^{\epsilon}}, \tag{8.120}$$

yielding:

$$\mathcal{F}|_{min} = \frac{L}{4\pi R}\sum_{n=0}^{\infty}\left\{\mathcal{L}\left[\frac{y_n}{1+y_n}\right] - \mathcal{L}\left[\frac{x_n}{1+x_n}\right]\right\}, \tag{8.121}$$

where $\mathcal{L}(x)$ is Rodger's dilogarithm function, [73]:

$$\mathcal{L}(x) = -\frac{1}{2}\int_0^x dt\left[\frac{logt}{1-t} + \frac{log(1-t)}{t}\right]. \tag{8.122}$$

This function satisfies the property, [82]:

$$\mathcal{L}(1-x) = \mathcal{L}(1) - \mathcal{L}(x), \tag{8.123}$$

and the following sum rules, [85]:

$$\sum_{n=1}^{N} \mathcal{L}\left\{ \frac{sin^2\left[\frac{\pi}{N+3}\right]}{sin^2\left[\frac{\pi(n+1)}{N+3}\right]} \right\} = \frac{2N}{N+3}\mathcal{L}(1). \tag{8.124}$$

Consider the first term in eq.(8.121):

$$\begin{aligned} &\tfrac{L}{4\pi R} \lim_{p\to\infty} \textstyle\sum_{n=0}^{p} \mathcal{L}\left\{ 1 - \frac{sin^2\left[\frac{\pi}{p+3}\right]}{sin^2\left[\frac{\pi(n+1)}{p+3}\right]} \right\} = \\ &= \tfrac{L}{4\pi R} \lim_{p\to\infty} \textstyle\sum_{n=0}^{p} \left\{ \mathcal{L}(1) - \mathcal{L}\left[ \frac{sin^2\left(\frac{\pi}{p+3}\right)}{sin^2\left(\frac{\pi(n+1)}{p+3}\right)} \right] \right\} = \\ &= \tfrac{L\pi}{24R} \lim_{p\to\infty} \left[ \tfrac{p^2+3p+6}{p+3} \right], \end{aligned} \tag{8.125}$$

where we used the fact that $\mathcal{L}(1) = \frac{\pi^2}{6}$. Similarly, the second term in eq.(8.121) is:

$$\begin{aligned} &-\tfrac{L}{4\pi R} \lim_{p\to\infty} \textstyle\sum_{n=0}^{p} \mathcal{L}\left\{ 1 - \frac{sin^2\left[\frac{\pi}{p+4}\right]}{sin^2\left[\frac{\pi(n+2)}{p+4}\right]} \right\} = \\ &= -\tfrac{L\pi}{24R} \lim_{p\to\infty} \left[ \tfrac{p^2+3p+2}{p+4} \right]. \end{aligned} \tag{8.126}$$

Adding up the contributions (8.124) and (8.126) to (8.121), we get:

$$\mathcal{F}|_{min} = \frac{L\pi}{24R} \lim_{p\to\infty} \left[ \frac{p^2+7p+18}{p^2+7p+12} \right] = \frac{L\pi}{24R}. \tag{8.127}$$

Consequently:

$$E(R) = -\frac{\pi c}{24R}, \tag{8.128}$$

where $c = 1$. Comparing with (5.11), we conclude that $L_0^{min} = 0$ in complete agreement with (5.35).

### 8.2.2 TBA in the L-channel

Let us now perform the TBA in the L-channel. In this section we shall assume $g \neq 1$ in eq.(8.30). We consider a system of $N$ right-movers with rapidities $\beta_1, \cdots, \beta_N$ and $M$ left-movers with rapidities $\beta'_1, \cdots, \beta'_M$ in an interval of length $R$ with fixed boundary conditions on both ends. Since the reflection matrix is diagonal on both sides, we can assume that the Bethe wave function vanishes at the ends of the interval. We can thus impose a standing wave quantization condition, [88]. Mathematically this condition can be expressed in the form[3]:

$$\begin{aligned}&exp(2iRp_k)R_{RL}^2(\beta_k)\tfrac{1}{U_{RL}(2\beta_k)}\Pi_{l=1}^N U_{RL}(\beta_k+\beta_l)\Pi_{j=1}^M U_{RL}(\beta_k-\beta_j')\times\\ &\times\textstyle\sum_{a,b}T_a^b(\beta_k|\{\beta\}_{1\cdots N})\times\bar{T}_{\bar{b}}^{\bar{a}}(-\beta_k|\{\beta'\}_{1\cdots M})=1,\qquad k=1,\cdots,N.\end{aligned}\tag{8.129}$$

In the above equation it is understood that $a=\bar{a}$ and $b=\bar{b}$. The reflection amplitude $R_{RL}$ is given by eq.(8.8). Following the same procedure as before, we obtain the TBA system:

$$\begin{aligned}&-\nu_0+\epsilon_0+\tfrac{1}{2\pi}\varphi_S*L_0+\tfrac{1}{2\pi}\varphi*L_1=0,\\ &-\nu_n+\epsilon_n+\tfrac{1}{2\pi}\varphi*\textstyle\sum_{m=0}^{\infty}l_{mn}L_m=0,\qquad n=1,2,\cdots,\infty,\end{aligned}\tag{8.130}$$

where this time $\rho_0(\beta)$ is the combined density of right- and left-movers with rapidities in the intervals $(\beta,\beta+\Delta\beta)$ and $(-\beta,-\beta-\Delta\beta)$, respectively. The energy terms are:

$$\begin{aligned}&\nu_0(\beta)=LMe^{\beta},\\ &\nu_n(\beta)=0,\qquad n=1,2,\cdots,\infty.\end{aligned}\tag{8.131}$$

The saddle point evaluation of the partition function yields:

$$logZ\sim\frac{1}{4\pi}\int_{-\infty}^{+\infty}d\beta\left\{RMe^{\beta}+\Theta(\beta)\right\}L_0(\beta),\tag{8.132}$$

where:

$$\Theta(\beta)\equiv\frac{1}{i}\frac{\partial}{\partial\beta}log(R_{RL}(\beta)R_{RL}(\beta))-\frac{1}{i}\frac{\partial}{\partial\beta}logU_{RL}(2\beta).\tag{8.133}$$

The $\Theta$ term in eq.(8.132) is independent of $R$ may be regarded as a boundary free energy, [88]. In the large-$R$ limit the next to leading order of the partition function (8.32) is:

$$Z\approx<B|0><0|B>exp(-RE_0(R)),\tag{8.134}$$

where $E_0$ is the ground-state energy of the periodic Hamiltonian. Equating (8.132) and (8.134), we get:

$$log<B|0><0|B>=\frac{1}{4\pi}\int_{-\infty}^{+\infty}d\beta\Theta(\beta)L_0(\beta)+const.\tag{8.135}$$

The boundary free energy is thus defined up to an additive constant. This is because ([88], [86]) we have discarded corrections to the Stirling formula in computing the entropy (eqs.(8.92), (8.93)) and loop corrections in the saddle point computation. Also the correspondence between the entropy of the field theory and the one evaluated using the particle description (e.g. kinks) might involve some constant.

[3]The term $1/U_{RL}(2\beta_k)$ arises because the particle does not interact with itself.

In the IR limit when $L \to \infty$, we see from (8.130) that $\epsilon_0 \to \infty$ and so the first term on the r.h.s. of eq.(8.135) vanishes. This is compatible with $g = +1$, which is the expected result for the exactly screened case (cf. eq.(7.55)), provided we take *constant*$= 0$. On the contrary, when $L \to 0$, $\epsilon_0$ becomes constant. $\Theta$ can be shown to be equal to:

$$\Theta(\beta) = \frac{2\sqrt{2}}{2cosh\beta - \sqrt{2}} - \frac{2}{cosh(2\beta)}. \tag{8.136}$$

Consequently:

$$\int_{-\infty}^{+\infty} d\beta \Theta(\beta) = 2\pi. \tag{8.137}$$

We then have:

$$log\,(g_{UV}) = \frac{1}{4} log(1 + e^{-\epsilon_0}), \tag{8.138}$$

where $e^{-\epsilon_0}$ is equal to $x_0$ in eq.(8.111). And do:

$$g_{UV} = \lim_{p\to\infty} log \left[ \frac{sin^2\left(\frac{2\pi}{p+4}\right)}{sin^2\left(\frac{\pi}{p+4}\right)} \right]^{1/4} = \sqrt{2}. \tag{8.139}$$

We conclude that the internal product $g =< 0|B >$ varies along the RG flow. This is partly unexpected because the RG flow is controlled by a bulk perturbation. However, both in the IR and UV limits the conformal states can be represented as combinations of massless particle states. The boundary states corresponding to conformal states in these limits have different scalar products with the conformal ground state. This supports the conjecture of ref.[97] which states that the boundary entropy is a characteristic (like the central charge) of universality classes. Models interpolating between distinct universality classes are therefore expected to have distinct boundary entropies at the extremes of the RG trajectory.

# Chapter 9

# The super-Liouville equation

*In this chapter I will discuss different aspects of the N=1 super-Liouville (SL) theory. I will derive a recursive formula for an infinite number of conservation laws using Lax-pair techniques. After some algebraic manipulations these are shown to be the super-Korteweg-de Vries (sKdV) hierarchy of Hamiltonians, thus establishing a connection with other integrable theories. I will investigate the boundary interactions which are classically compatible with the superconformal symmetry. The Poisson brackets are then established on the light-cone and used to prove the involutive nature of the integrals of motion (IM). They are also used to quantize the theory and consequently determine how the classical IM are modified into their quantum counterparts.*

## 9.1 Integrable supersymmetric theories

In the past few years, there has been renewed interest in the problem of incorporating fermions in integrable two-dimensional quantum field theory [100], [101], [102], [117]. Introducing fermions seems to be a natural extension to the more standard purely bosonic theories. It is well known, for instance, that there are striking similarities between the two-dimensional O(N) sigma model and four-dimensional pure Yang-Mills theories, notably asymptotic freedom. However, if the purpose is to mimic the full QCD, it is natural to include fermions. Shankar and Witten [103] have considered a supersymmetric version of this model and determined the exact factorizable S-matrix using the usual bootstrap program and some knowledge coming from the large N expansion.

Supersymmetry is an indispensable ingredient in the context of string theory applications [6]. But also in statistical mechanics some lattice models (e.g tricritical Ising model in two dimensions, [105]) realize superconformal symmetry.

The bootstrap program for finding exact S-matrices for integrable theories with N=1 Supersymmetry is thoroughly described in ref.[106] and the *thermodynamic Bethe ansatz* (TBA) developed in ref.[107]. It is argued that although the scattering is nondiagonal, the S-matrices satisfy a technical condition called the "*free fermion condition*" that renders the diagonalization of the transfer matrix feasible.

Another type of integrable theories with N=1 Supersymmetry, called *super-KdV like equations*, has been studied by Kuperschmidt, [110], [111]. These equations have profound implications in the study of integrable perturbed superconformal theories and their conserved charges, [28].

There have also been some attempts at supersymmetrizing Toda field theories, [100], [119]. It turns out that with the exceptions of the Liouville and sinh-Gordon theories this is not a simple matter. It is rather striking that Supersymmetry, which improves dramatically the quantum properties of four-dimensional theories, [2] seems to be compatible with integrability in two dimensions only under very restrictive circumstances. However, if one focuses attention on the integrability of the models rather than Supersymmetry, it is possible to construct a new class of Toda models with fermions where the underlying algebra is a Lie superalgebra, [104], [118]. Some exact S-matrices have been proposed for this class of theories, [115], [116].

Recently, Inami et al. [114] considered the supersymmetric extension of the sine-Gordon theory on the half-line and found that the requirements of integrability and Supersymmetry fully determine the boundary potential up to an overall sign. Moriconi and Schoutens [108] subsequently conjectured the exact reflection amplitudes for the breather multiplets of this model.

Here, I will apply similar considerations to the N=1 super-Liouville theory (SLT), [109]. Besides its applications in statistical mechanics [112], the SLT arises in Polyakov's approach to the superstring, [124], [112]. It is the simplest example of a Toda theory based on a contragradient Lie superalgebra. This superalgebra is labeled B(0,1) in the classification of Kac [104] and it possesses three bosonic generators and two fermionic ones. A realization of B(0,1) is provided by $Osp(1|2;C)$. The theory based on this finite superalgebra is conformally invariant. Furthermore, the SL equation also happens to be supersymmetric and therefore superconformal, [112].

## 9.2 The super-Liouville theory

Let me first establish my notation. Consider two-dimensional superspace, with units such that $\hbar = c = 1$, and the superspace coordinate,

$$Z = (x^\mu, \theta_A) = (x^0, x^1; \theta_1, \theta_2),$$

where $x^\mu$ is the coordinate on 2-dimensional Minkowski space and $\theta_A$ are Grassmann variables, which are the components of a Majorana spinor. We also introduce the scalar superfield $\Phi$ with components,

$$\Phi(x, \theta, \bar{\theta}) = \varphi(x) + \bar{\theta}\chi(x) + \frac{1}{2}\bar{\theta}\theta F(x),$$

where $\chi$ is a Majorana spinor $\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, $\varphi(x)$, $F(x)$ are scalar fields and where we used the fact that $\bar{\chi}\theta = \bar{\theta}\chi$. We also consider the two-dimensional Dirac matrices in the Majorana representation (1.14). The super-Liouville theory is then defined by the superspace action,

$$S = \int d^2x d^2\theta \left\{ \frac{1}{2}\bar{D}\Phi D\Phi - \frac{2}{\beta}e^{\beta\Phi} \right\}, \tag{9.1}$$

where $\beta$ is a dimensionless coupling constant and the superderivatives are:

$$D = \frac{\partial}{\partial\bar{\theta}} - i\theta\gamma^\mu\partial_\mu, \qquad \bar{D} = -\frac{\partial}{\partial\theta} + i\bar{\theta}\gamma^\mu\partial_\mu.$$

In this form the action (9.1) is manifestly supersymmetric. Integrating the $\theta$-coordinates, we get:

$$S = \frac{1}{2}\int d^2x \left\{ (\partial\varphi)^2 + i\bar{\chi}\gamma^\mu\partial_\mu\chi + F^2 - 2Fe^{\beta\varphi} + \beta\bar{\chi}\chi e^{\beta\varphi} \right\}.$$

As usual, the auxiliary field is nondynamical, and we can use its equation of motion, $F = e^{\beta\varphi}$, to eliminate it from the action and we get:

$$S = \frac{1}{2}\int d^2x \left\{ (\partial\varphi)^2 + i\bar{\chi}\gamma^\mu\partial_\mu\chi - e^{2\beta\varphi} + \beta\bar{\chi}\chi e^{\beta\varphi} \right\}. \tag{9.2}$$

From this action we can derive the classical field equations,

$$\begin{cases} \partial^2\varphi + \beta e^{2\beta\varphi} - \frac{1}{2}\beta^2\bar{\chi}\chi e^{\beta\varphi} = 0, \\ (i\gamma^\mu\partial_\mu + \beta e^{\beta\varphi})\chi = 0, \end{cases} \tag{9.3}$$

and the canonical stress-energy tensor,

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + \frac{i}{4}\left[\bar{\chi}\gamma_\mu\partial_\nu\chi - \partial_\nu\bar{\chi}\gamma_\mu\chi\right] - g_{\mu\nu}\mathcal{L}, \tag{9.4}$$

associated with the invariance under rigid space-time displacements. The action is also invariant under the conformal transformation,

$$\begin{cases} x^\pm & \rightarrow \quad f^\pm(x^\pm) = x^\pm + \epsilon^\pm(x^\pm), \\ \varphi(x) & \rightarrow \quad \tilde{\varphi}(x) = \varphi(f) + \frac{1}{2\beta}ln(\partial_+ f^+ \partial_- f^-), \\ \chi(x) & \rightarrow \quad \chi(x) + \frac{1}{2}\epsilon^\mu\partial_\mu\chi(x), \end{cases} \tag{9.5}$$

where $x^\pm = x^0 \pm x^1$ are the light-cone variables. For the stress tensor to be the generator of these transformations, it has to be traceless, and therefore a conformal improvement is in order. It can be shown [112], [124] that

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{\beta}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\varphi \tag{9.6}$$

satisfies this requirement. The action (9.1) is also invariant under the supersymmetric transformation,

$$\delta_\eta\Phi = [\bar{\eta}Q, \Phi] = \bar{\eta}\chi + \bar{\theta}\left(\frac{1}{2}\eta F - i\gamma^\mu\eta\partial_\mu\varphi\right) - \frac{i}{2}\bar{\theta}\theta\bar{\eta}\gamma^\mu\partial_\mu\chi, \tag{9.7}$$

where Q is the supersymmetry generator,

$$Q_A = \frac{\partial}{\partial\bar{\theta}^A} + i(\gamma^\mu\theta)_A\partial_\mu,$$

and $\eta$ is some constant infinitesimal fermionic parameter. The associated Noether current is:

$$J^\mu = \gamma^\nu\gamma^\mu\chi\partial_\nu\varphi - \frac{1}{\beta}\gamma^\mu\gamma^\nu\partial_\nu\chi. \tag{9.8}$$

For this current to generate a superconformal symmetry, it has to satisfy a condition analogous to the tracelessness of the stress tensor, namely:

$$\gamma_\mu J^\mu = 0. \tag{9.9}$$

As we will see, this condition will render the theory fully "*chiral*", with the sectors of '*right-movers*' and '*left-movers*' completely decoupled. The "*conformally improved*" supercurrent satisfying (9.9) is found to be:

$$J^\mu = \gamma^\nu\gamma^\mu\chi\partial_\nu\varphi - \frac{1}{\beta}(2\partial^\mu - \gamma^\mu\gamma^\nu\partial_\nu)\chi. \tag{9.10}$$

## 9.3 Lax-pair and conservation laws

Let us consider the components of the superderivatives,

$$D_1 = -i\frac{\partial}{\partial\theta_2} - 2\theta_2\partial_-; \qquad D_2 = i\frac{\partial}{\partial\theta_1} + 2\theta_1\partial_+.$$

They have the properties,

$$D_1^2 = 2i\partial_-; \qquad D_2^2 = 2i\partial_+.$$

In terms of these the SL equation is written:

$$D_1 D_2 \Phi = -ie^{2\Phi}, \tag{9.11}$$

which yields component wise the following set of equations:

$$\begin{cases} \partial_+\partial_-\varphi & = -\frac{\beta}{4}e^{2\beta\varphi} - i\frac{\beta^2}{4}\chi_1\chi_2 e^{\beta\varphi}, \\ \partial_+\chi_1 & = \frac{\beta}{2}e^{\beta\varphi}\chi_2, \\ \partial_-\chi_2 & = -\frac{\beta}{2}e^{\beta\varphi}\chi_1, \\ F & = e^{\beta\varphi}. \end{cases} \tag{9.12}$$

Equation (9.11) can be cast in the following linear system:

$$\begin{cases} D_1\zeta = A_1(\lambda)\zeta, \\ D_2\zeta = A_2(\lambda)\zeta, \end{cases} \tag{9.13}$$

where $\zeta$ is a column vector, whose components are the bosonic superfields $V_1$, $V_2$ and the fermionic superfield $V_3$; $\lambda$ is an arbitrary parameter, and $A_1$, $A_2$ are the graded matrices:

$$A_1(\lambda) = -\sqrt{\frac{2}{\lambda}}\begin{pmatrix} 0 & 0 & e^{\beta\Phi} \\ 0 & 0 & ie^{\beta\Phi} \\ e^{\beta\Phi} & ie^{\beta\Phi} & 0 \end{pmatrix}; \qquad A_2(\lambda) = \begin{pmatrix} \lambda\theta_1 & -i\beta D_2\Phi & 0 \\ i\beta D_2\Phi & \lambda\theta_1 & \beta\sqrt{\frac{\lambda}{2}} \\ 0 & -\beta\sqrt{\frac{\lambda}{2}} & \lambda\theta_1 \end{pmatrix}. \tag{9.14}$$

I will now use a method developed in refs. [121], [122] in the context of the super-sine-Gordon theory to extract the IM.

We define two new scalar superfields $U$, $Z$ and a fermionic superfield $Y$ as:

$$U = lnV_1 - \frac{\lambda}{2}x^+; \qquad Z = \frac{V_2}{V_1}; \qquad Y = \frac{V_3}{V_1}.$$

We then have:

$$\begin{aligned} &D_1U = -\sqrt{\tfrac{2}{\lambda}}e^{\beta\Phi}Y, \qquad D_2U = -i\beta D_2\Phi\cdot Z, \\ &\beta\sqrt{\tfrac{\lambda}{2}}Y = -i\beta D_2\Phi + D_2Z - i\beta D_2\Phi\cdot Z\cdot Z, \\ &\beta\sqrt{\tfrac{\lambda}{2}}Z = -D_2Y + i\beta D_2\Phi\cdot Z\cdot Y. \end{aligned} \tag{9.15}$$

Using the fact that the square of a fermionic superfield vanishes, we obtain the following differential equation for Y:

$$\beta\sqrt{\frac{\lambda}{2}}Y = -i\beta D_2\Phi - \frac{2i}{\beta}\sqrt{\frac{2}{\lambda}}\partial_+ Y + \frac{4}{\beta\lambda}\partial_+\Phi \cdot D_2 Y \cdot Y. \tag{9.16}$$

We now assume an expansion of $Y$ in powers of $\lambda^{-1}$:

$$Y = \frac{1}{i\sqrt{2\lambda}}\sum_{n=0}^{\infty}\frac{Y^{(n+1/2)}}{(2i\lambda)^n}. \tag{9.17}$$

Substituting this expansion in eq.(9.16) and equating powers of $\lambda^{-1}$, we obtain the following recursive formula:

$$\begin{aligned} &Y^{(1/2)} = 2D_2\Phi,\\ &Y^{(n+1/2)} = \tfrac{8}{\beta^2}\partial_+ Y^{(n-1/2)} + \tfrac{16i}{\beta^2}\partial_+\Phi \cdot \textstyle\sum_{l=1}^{n-1} D_2 Y^{(l-1/2)} \cdot Y^{(n-l-1/2)},\\ &n = 1,2,3,\cdots \end{aligned} \tag{9.18}$$

The integrability condition,

$$D_1 D_2 U = -D_2 D_1 U,$$

can be interpreted as an infinite number of supersymmetric covariant conservation laws:

$$D_1 J_2^{(n+1/2)} = D_2 J_1^{(n+1/2)}, \qquad n = 1,2,3,\cdots, \tag{9.19}$$

where

$$\begin{cases} J_1^{(n+1/2)} = e^{\beta\Phi} \cdot Y^{(n-1/2)},\\ J_2^{(n+1/2)} = iD_2\Phi \cdot D_2 Y^{(n-1/2)}. \end{cases} \tag{9.20}$$

Notice that eq.(9.19) is invariant under the '*gauge transformation*':

$$\begin{cases} J_1^{(n+1/2)} \to J_1^{(n+1/2)} + D_1 V^{(n+1/2)},\\ J_2^{(n+1/2)} \to J_2^{(n+1/2)} - D_2 V^{(n+1/2)}. \end{cases} \tag{9.21}$$

It is straightforward to show that up to one of these transformations one has

$$\begin{cases} J_A^{(5/2)} = \partial_+ J_A^{(3/2)},\\ \\ J_A^{(9/2)} = \partial_+ J_A^{(7/2)}, \end{cases} \qquad (A = 1,2) \tag{9.22}$$

where

$$\begin{aligned} V^{(3/2)} = &-2i\left(1-\tfrac{8}{\beta^2}\right)\chi_2\partial_+\chi_2 - 4\left(1-\tfrac{4}{\beta^2}\right)(\partial_+\varphi)^2 +\\ &+\theta_1\left[4i\left(1-\tfrac{8}{\beta^2}\right)\partial_+^2\varphi\chi_2 + 4i\partial_+\varphi\partial_+\chi_2\right] +\\ &+\theta_2\left[-2i\beta\partial_+\varphi e^{\beta\varphi}\chi_2 - 2i\left(1-\tfrac{8}{\beta^2}\right)e^{\beta\varphi}\partial_+\chi_2\right] +\\ &+\theta_1\theta_2\left[4i\left(1-\tfrac{8}{\beta^2}\right)\partial_+^2\varphi e^{\beta\varphi} + 4i\beta(\partial_+\varphi)^2 e^{\beta\varphi} - \tfrac{16}{\beta}e^{\beta\varphi}\chi_2\partial_+\chi_2\right], \end{aligned} \tag{9.23}$$

and:

$$V^{(7/2)} = \tfrac{128}{\beta^4}\Big\{i\left(1-\tfrac{8}{\beta^2}\right)\chi_2\partial_+^3\chi_2 + \tfrac{8i}{\beta^2}\partial_+\chi_2\partial_+^2\chi_2 + 2\left(1-\tfrac{8}{\beta^2}\right)\partial_+\varphi\partial_+^3\varphi + \tfrac{8}{\beta^2}(\partial_+^2\varphi)^2$$

$$-2\beta^2\left(1-\tfrac{10}{\beta^2}\right)(\partial_+\varphi)^4 - i\beta^2\left(3-\tfrac{40}{\beta^2}\right)(\partial_+\varphi)^2\chi_2\partial_+\chi_2 +$$

$$+\theta_1\Big[-2i\left(1-\tfrac{8}{\beta^2}\right)\partial_+^4\varphi\chi_2 - 2i\partial_+^3\varphi\partial_+\chi_2 +$$

$$+2i\beta^2\left(3-\tfrac{40}{\beta^2}\right)(\partial_+\varphi)^2\partial_+^2\varphi\chi_2 + 2i\beta^2(\partial_+\varphi)^3\partial_+\chi_2\Big] +$$

$$+\theta_2\Big[i\beta\partial_+\varphi e^{\beta\varphi}\partial_+^2\chi_2 + i\left(1-\tfrac{8}{\beta^2}\right)e^{\beta\varphi}\partial_+^3\chi_2 - 2i\beta^2\left(1-\tfrac{12}{\beta^2}\right)\partial_+\varphi\partial_+^2\varphi e^{\beta\varphi}\chi_2$$

$$-i\beta^3(\partial_+\varphi)^3 e^{\beta\varphi}\chi_2 - i\beta^2\left(1-\tfrac{16}{\beta^2}\right)(\partial_+\varphi)^2 e^{\beta\varphi}\partial_+\chi_2\Big] +$$

$$+\theta_1\theta_2\Big[-2i\left(1-\tfrac{8}{\beta^2}\right)\partial_+^4\varphi e^{\beta\varphi} - 2i\beta\partial_+\varphi\partial_+^3\varphi e^{\beta\varphi} + 2i\beta^2\left(3-\tfrac{40}{\beta^2}\right)(\partial_+\varphi)^2\partial_+^2\varphi e^{\beta\varphi} +$$

$$+2i\beta^3(\partial_+\varphi)^4 e^{\beta\varphi} - 2\beta^2\left(1-\tfrac{12}{\beta^2}\right)\partial_+^2\varphi e^{\beta\varphi}\chi_2\partial_+\chi_2 - 2\beta^3\left(1+\tfrac{8}{\beta^2}\right)(\partial_+\varphi)^2 e^{\beta\varphi}\chi_2\partial_+\chi_2 +$$

$$+3\beta^2\left(\tfrac{8}{\beta^2}-1\right)\partial_+\varphi e^{\beta\varphi}\chi_2\partial_+^2\chi_2 - \beta e^{\beta\varphi}\partial_+\chi_2\partial_+^2\chi - \beta\left(1-\tfrac{8}{\beta^2}\right)e^{\beta\varphi}\chi_2\partial_+^3\chi_2\Big]\Big\}. \tag{9.24}$$

This means that the charges associated with $J_A^{(5/2)}$, $J_A^{(9/2)}$ are trivial and there will thus be a gap in the sequence (9.19). This results in all even spin charges being omitted, as will become apparent in what follows.

We will henceforth work in Euclidean space,

$$\begin{cases} x = x_1 \\ y = ix_0 \end{cases} \qquad \begin{cases} z = -x + iy \\ \bar{z} = -x - iy \end{cases}$$

unless otherwise stated and redefine the fields,

$$\varphi = \phi/2, \qquad \chi_1 = \alpha\bar{\psi}, \qquad \chi_2 = \alpha\psi,$$

for future convenience. The parameter $\alpha$ is such that $\alpha^2 = i/2$. The equations of motion then become:

$$\begin{cases} \partial_z\partial_{\bar{z}}\phi &= \frac{\beta}{2}e^{\beta\phi} - \frac{\beta^2}{4}e^{\beta\phi/2}\bar{\psi}\psi, \\ \partial_{\bar{z}}\psi &= -\frac{\beta}{2}e^{\beta\phi/2}\bar{\psi}, \\ \partial_z\bar{\psi} &= -\frac{\beta}{2}e^{\beta\phi/2}\psi, \\ F &= e^{\beta\phi/2}. \end{cases} \tag{9.25}$$

The energy-momentum tensor has the two independent components:

$$\begin{cases} T \equiv \Theta_{zz} = \frac{1}{2}(\partial_z\phi)^2 + \frac{1}{2}\psi\partial_z\psi - \frac{1}{\beta}\partial_z^2\phi, \\ \bar{T} \equiv \Theta_{\bar{z}\bar{z}} = \frac{1}{2}(\partial_{\bar{z}}\phi)^2 - \frac{1}{2}\bar{\psi}\partial_{\bar{z}}\bar{\psi} - \frac{1}{\beta}\partial_{\bar{z}}^2\phi, \\ \Theta_{z\bar{z}} = \Theta_{\bar{z}z} = 0. \end{cases} \tag{9.26}$$

And the components of the supercurrent are:

$$\begin{cases} J \equiv J^- = \frac{1}{2}\partial_z\phi\psi - \frac{1}{\beta}\partial_z\psi, \\ \bar{J} \equiv J^+ = \frac{1}{2}\partial_{\bar{z}}\phi\bar{\psi} - \frac{1}{\beta}\partial_{\bar{z}}\bar{\psi}. \end{cases} \tag{9.27}$$

As was mentioned earlier, the conformal improvements of the energy-momentum tensor and the supercurrent lead to a '*chiral*' theory, i.e. a decoupling of two independent sectors of right- and left-movers. This is displayed manifestly in eqs.(9.26) and (9.27).

From the conservation law (9.19) a pattern will emerge for all $n$. The $\theta_2$ and $\theta_1\theta_2$ components lead to the conservation laws, whereas the remaining components are just identities.

For $n = 1$ the $\theta_2$ component of eq.(9.19) is:

$$\partial_{\bar{z}}(\partial_z\phi\psi) = \frac{\beta}{2}e^{\beta\phi}\psi - \frac{\beta}{2}\partial_z\phi e^{\beta\phi/2}\bar{\psi}.$$

However, the right-hand-side of this equation can be integrated using the equations of motion (9.25):

$$\partial_{\bar{z}}(\partial_z\phi\psi) = \frac{2}{\beta}\partial_z\partial_{\bar{z}}\psi.$$

And we get the conservation law:

$$\partial_{\bar{z}}U_{3/2} = 0, \tag{9.28}$$

where

$$U_{3/2} = J = \frac{1}{2}\partial_z\phi\psi - \frac{1}{\beta}\partial_z\psi.$$

The $\theta_1\theta_2$ component of eq.(9.19) can be written as:

$$\partial_{\bar{z}}T_{s+1} = \partial_z\Theta_{s-1}, \qquad s = 1, 3, 5, \cdots \tag{9.29}$$

Here are the first elements of this sequence:

$$
\begin{aligned}
T_2 = &\ \tfrac{1}{2}(\partial_z\phi)^2 + \tfrac{1}{2}\psi\partial_z\psi, \\
T_4 = &\ \tfrac{1}{4}(\partial_z\phi)^4 + \tfrac{3}{4}(\partial_z\phi)^2\psi\partial_z\psi + \tfrac{1}{\beta^2}(\partial_z^2\phi)^2 + \tfrac{1}{\beta^2}\partial_z\psi\partial_z^2\psi, \\
T_6 = &\ \tfrac{1}{8}(\partial_z\phi)^6 - \tfrac{1}{\beta^4}\partial_z^2\phi\partial_z^4\phi - \tfrac{11}{4\beta^2}(\partial_z\phi)^2(\partial_z^2\phi)^2 - \tfrac{7}{4\beta^2}(\partial_z\phi)^3\partial_z^3\phi - \tfrac{1}{2\beta^2}(\partial_z\phi)^2\partial_z\psi\partial_z^2\psi \\
&-\tfrac{4}{\beta^2}\partial_z\phi\partial_z^3\phi\psi\partial_z\psi - \tfrac{5}{\beta^2}\partial_z\phi\partial_z^2\phi\psi\partial_z^2\psi - \tfrac{7}{4\beta^2}(\partial_z\phi)^2\psi\partial_z^3\psi + \tfrac{5}{8}(\partial_z\phi)^4\psi\partial_z\psi \\
&-\tfrac{1}{\beta^4}\partial_z\psi\partial_z^4\psi - \tfrac{11}{4\beta^2}(\partial_z^2\phi)^2\psi\partial_z\psi, \\
\Theta_0 = &\ \tfrac{1}{2}e^{\beta\phi} - \tfrac{\beta}{4}e^{\beta\phi/2}\bar{\psi}\psi, \\
\Theta_2 = &\ \tfrac{1}{2}(\partial_z\phi)^2e^{\beta\phi} + \tfrac{1}{4}e^{\beta\phi}\psi\partial_z\psi - \tfrac{1}{4}\partial_z\phi e^{\beta\phi/2}\bar{\psi}\partial_z\psi - \tfrac{\beta}{8}(\partial_z\phi)^2e^{\beta\phi/2}\bar{\psi}\psi, \\
\Theta_4 = &\ -\tfrac{1}{2\beta^2}\partial_z\phi\partial_z^3\phi e^{\beta\phi} - \tfrac{17}{8\beta}(\partial_z\phi)^2\partial_z^2\phi e^{\beta\phi} - \tfrac{1}{2}(\partial_z\phi)^4e^{\beta\phi} + \tfrac{1}{8\beta}\partial_z\phi\partial_z^3\phi e^{\beta\phi/2}\bar{\psi}\psi + \\
&+\tfrac{11}{8\beta}\partial_z\phi\partial_z^2\phi e^{\beta\phi/2}\bar{\psi}\partial_z\psi - \tfrac{3}{16}(\partial_z\phi)^2\partial_z^2\phi e^{\beta\phi/2}\bar{\psi}\psi - \tfrac{21}{16}(\partial_z\phi)^2e^{\beta\phi}\psi\partial_z\psi + \\
&+\tfrac{3}{16}(\partial_z\phi)^3e^{\beta\phi/2}\bar{\psi}\partial_z\psi + \tfrac{3}{4\beta}(\partial_z\phi)^2e^{\beta\phi/2}\bar{\psi}\partial_z^2\psi - \tfrac{11}{8\beta}\partial_z\phi e^{\beta\phi}\psi\partial_z^2\psi - \tfrac{1}{4\beta^2}e^{\beta\phi}\psi\partial_z^3\psi + \\
&+\tfrac{1}{4\beta^2}\partial_z\phi e^{\beta\phi/2}\bar{\psi}\partial_z^3\psi - \tfrac{11}{8\beta}\partial_z^2\phi e^{\beta\phi}\psi\partial_z\psi - \tfrac{\beta}{16}(\partial_z\phi)^4e^{\beta\phi/2}\bar{\psi}\psi.
\end{aligned}
$$

These coincide with the results of ref. [123], which were obtained by using Bäcklund transformations. Note also that the system (9.25) is invariant under $z \leftrightarrow \bar{z}$ and $\psi \to i\bar{\psi}$, $\bar{\psi} \to i\psi$, and this yields a corresponding set of conserved quantities:

$$
\partial_z\bar{T}_{s+1} = \partial_{\bar{z}}\bar{\Theta}_{s-1}, \qquad s = 1,3,5,\cdots \tag{9.30}
$$

Given the fact that the theory is conformally invariant, the form of the conservation laws (9.29) may appear awkward, as it seems to prevent the decoupling of the left- and right-moving sectors. The desired form of the chiral conservation laws is:

$$
\partial_{\bar{z}}U_{s+1} = 0, \qquad s = 1,3,5,\cdots
$$

I will now show that all the functions $\Theta_{s-1}$ ($s = 1,3,5,\cdots$) can be integrated using the equations of motion, yielding:

$$
\Theta_{s-1} = \partial_{\bar{z}}A_{s-2}, \qquad s = 1,3,5,\cdots
$$

This will act as a conformal improvement for all the higher spin conservation laws. We start with $\Theta_0$ which is immediately identified with:

$$
\Theta_0 = \partial_{\bar{z}}A_1 = \frac{1}{\beta}\partial_z\partial_{\bar{z}}\phi.
$$

The first chiral bosonic conserved density is just the conformally improved stress tensor (cf.(9.26)),

$$U_2 \equiv T = T_2 - \partial_z A_1 = \frac{1}{2}(\partial_z\phi)^2 + \frac{1}{2}\psi\partial_z\psi - \frac{1}{\beta}\partial_z^2\phi. \tag{9.31}$$

The next element in the sequence is:

$$\Theta_2 = \partial_{\bar{z}} A_3 = \partial_{\bar{z}}\left[\frac{1}{3\beta}(\partial_z\phi)^3 + \frac{1}{2\beta}\partial_z\phi\psi\partial_z\psi\right].$$

And the corresponding chiral conservation law reads in terms of $T$, $J$ and their derivatives:

$$U_4 = T^2 + J\partial_z J. \tag{9.32}$$

Finally,

$$\begin{aligned}\Theta_4 = \;& \partial_{\bar{z}} A_5 = \partial_{\bar{z}}\Big[\tfrac{3}{20\beta}(\partial_z\phi)^5 + \tfrac{1}{\beta^3}\partial_z\phi(\partial_z^2\phi)^2 - \tfrac{7}{4\beta^2}(\partial_z\phi)^3\partial_z^2\phi - \tfrac{1}{\beta^4}\partial_z^2\phi\partial_z^3\phi \\ & -\tfrac{1}{2\beta^3}\partial_z\phi\psi\partial_z^3\psi + \tfrac{1}{2\beta^3}\partial_z^3\phi\psi\partial_z\psi + \tfrac{1}{2\beta^3}\partial_z^2\phi\psi\partial_z^2\psi - \tfrac{13}{4\beta^2}\partial_z\phi\partial_z^2\phi\psi\partial_z\psi + \\ & +\tfrac{1}{2\beta}(\partial_z\phi)^3\psi\partial_z\psi - \tfrac{3}{2\beta^2}(\partial_z\phi)^2\psi\partial_z^2\psi\Big].\end{aligned}$$

And in terms of $T$, $J$ and their derivatives $U_6 = T_6 - \partial_z A_5$ reads:

$$U_6 = T^3 + \frac{1}{4\beta^2}(\partial_z T)^2 + 2TJ\partial_z J - \frac{1}{4\beta^2}J\partial_z^3 J. \tag{9.33}$$

With hindsight, we saw the supercurrent $J$ emerge in eq.(9.28) from the $\theta_2$ component of the first conservation law. It is the superpartner of the stress-energy tensor (9.31) which corresponds to the $\theta_1\theta_2$ component. Similarly we might expect to see the superpartners of the higher spin conserved quantities (9.31), (9.33) arising from the $\theta_2$ component of the appropriate superspace conservation law. However I have checked explicitly that the conserved densities $U_{7/2}$ and $U_{11/2}$ are total derivative terms, e.g.

$$U_{7/2} = \partial_z^2 J. \tag{9.34}$$

This means that the corresponding charges are trivial. Although there is no proof known to me, it seems plausible to assume that this will also be true for the higher ($s > 5$) spin densities. Note that $U_2$, $U_{3/2}$, $U_4$ and $U_6$ constitute a supersymmetric extension of the KdV hierarchy (3.25) to which it reduces in the zero fermion limit.

## 9.4 The super-Liouville equation on the half-line

Let us now assume a boundary located at $x = 0$. The action on the half-line $x \in (-\infty, 0]$ is the sum of two contributions,

$$S = S_0 + S_{\mathcal{B}} \equiv \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \left\{ \theta(-x)\mathcal{L}_0 + \frac{1}{4}\delta(x)\mathcal{B}(\phi, \psi, \bar{\psi}) \right\},$$

where $\mathcal{L}_0$ is the bulk lagrangian density and the boundary potential $\mathcal{B}$ is assumed to be independent of the field derivatives. $\theta$ is the Heaviside step function. Our purpose is to investigate under what circumstances will the boundary potential $\mathcal{B}$ lead to a supersymmetry preserving action in the presence of a boundary.

Minimizing the action leads to the SL field equations for $x \leq 0$. Furthermore, we get the boundary conditions at $x = 0$:

$$\frac{\partial \mathcal{L}_0}{\partial(\partial_x \phi)} = -\frac{1}{4}\frac{\partial \mathcal{B}}{\partial \phi}, \qquad \frac{\partial \mathcal{L}_0}{\partial(\partial_x \psi)} = -\frac{1}{4}\frac{\partial \mathcal{B}}{\partial \psi}, \qquad \frac{\partial \mathcal{L}_0}{\partial(\partial_x \bar{\psi})} = -\frac{1}{4}\frac{\partial \mathcal{B}}{\partial \bar{\psi}}. \tag{9.35}$$

In this section, we set $\beta = 2$ for simplicity. In Euclidean space the Lagrangian density $\mathcal{L}_0$ is written (cf(9.2)):

$$\mathcal{L}_0 = \frac{1}{2}\left[\partial_z \phi \partial_{\bar{z}} \phi - \bar{\psi}\partial_z \bar{\psi} + \psi \partial_{\bar{z}} \psi + e^{2\phi} - 2e^{\phi}\bar{\psi}\psi\right]. \tag{9.36}$$

Equations (9.35) read:

$$\partial_x \phi + \frac{\partial \mathcal{B}}{\partial \phi} = 0, \qquad \psi - \frac{\partial \mathcal{B}}{\partial \psi} = 0, \qquad \bar{\psi} + \frac{\partial \mathcal{B}}{\partial \bar{\psi}} = 0. \tag{9.37}$$

Besides being conformally invariant, the action (9.36) is also invariant under the supersymmetry transformations (cf.(9.7)),

$$\begin{cases} \delta_S \phi = \eta\psi + \bar{\eta}\bar{\psi}, \\ \delta_S \psi = -(\eta \partial_z \phi + \bar{\eta} e^{\phi}), \\ \delta_S \bar{\psi} = \bar{\eta}\partial_{\bar{z}}\phi + \eta e^{\phi}, \end{cases} \tag{9.38}$$

where $\eta$ and $\bar{\eta}$ are infinitesimal fermionic parameters. It is straightforward to show that the variation of the action (9.36) under these transformations amounts to a total derivative:

$$\begin{aligned} \delta_S \mathcal{L}_0 = \; & \tfrac{1}{2}\Big\{\partial_z \left[\delta_S \phi \partial_{\bar{z}} \phi - \bar{\psi}\delta_S \bar{\psi} - 2\partial_{\bar{z}}\phi \bar{\eta}\bar{\psi} - 2e^{\phi}\eta\bar{\psi}\right] + \\ & + \partial_{\bar{z}}\left[\delta_S \phi \partial_z \phi + \psi \delta_S \psi - 2\partial_z \phi \eta \psi - 2e^{\phi}\bar{\eta}\psi\right]\Big\}. \end{aligned} \tag{9.39}$$

In the presence of a boundary, we have [1]:

$$\delta_S S_0 = \frac{1}{4}\int_{-\infty}^{+\infty} dy \left\{ \frac{1}{2}(\eta\psi + \bar{\eta}\bar{\psi})\phi_x + (\bar{\psi}\eta + \psi\bar{\eta})e^{\phi} + \frac{i}{2}(\eta\psi - \bar{\eta}\bar{\psi})\phi_y \right\} |_{x=0}. \tag{9.40}$$

[1] The subscripts $x$, $y$ stand for $\partial/\partial x$, $\partial/\partial y$, respectively.

This expression can be compensated for by adding a boundary term. On dimensional grounds, we consider a boundary potential of the form:

$$\mathcal{B}_S = c_S e^{\phi} + M_S \bar{\psi}\psi.$$

The boundary equations of motion arising from this term are (cf(9.37)):

$$\phi_x = -c_S e^{\phi}, \qquad \bar{\psi} + M_S \psi = 0, \qquad (M_S^2 = 1).$$

Under a supersymmetry transformation, we have:

$$\tfrac{1}{4}\delta_S \int_{-\infty}^{+\infty} dy \mathcal{B}_S = \tfrac{1}{4}\int_{-\infty}^{+\infty} dy \left\{\tfrac{1}{2}M_S(\bar{\eta}\psi + \eta\bar{\psi})\phi_x + e^{\phi}(M_S + c_S)(\eta\psi + \bar{\eta}\bar{\psi}) + \right.$$
$$\left. + \tfrac{i}{2}M_S(\bar{\eta}\psi + \bar{\psi}\eta)\phi_y\right\}|_{x=0}.$$

It is only possible to keep half of the supersymmetries. We therefore choose $\bar{\eta} = \pm\eta$. The sum of the two contributions is thus:

$$\delta_S S_0 + \delta_S S_{\mathcal{B}_S} = \frac{1}{4}\int_{-\infty}^{+\infty} dy \left\{\frac{1}{2}(1 \pm M_S)\eta(\psi \pm \bar{\psi})\phi_x + \right.$$
$$\left. +(c_S + M_S \mp 1)\eta(\psi \pm \bar{\psi})e^{\phi} + \frac{i}{2}(1 \pm M_S)\eta(\psi \mp \bar{\psi})\phi_y\right\}|_{x=0}. \tag{9.41}$$

The integrand in the above expression vanishes, if:

$$\psi \mp \bar{\psi} = 0, \qquad c_S = \pm 2.$$

Therefore the boundary potential,

$$\mathcal{B}_S = \pm(2e^{\phi} - \bar{\psi}\psi), \tag{9.42}$$

restores supersymmetry.

Let us now consider additional terms in the boundary potential, of the form $\epsilon_S\psi + \bar{\epsilon}_S\bar{\psi}$. Under a supersymmetry transformation such that $\bar{\eta} = \pm\eta$:

$$\delta_S(\epsilon_S\psi + \bar{\epsilon_S}\bar{\psi}) = -\frac{1}{2}(\epsilon_s \mp \bar{\epsilon}_S)\eta\phi_x \mp (\epsilon_S \mp \bar{\epsilon}_S)\eta e^{\phi} + \frac{i}{2}(\epsilon_s \pm \bar{\epsilon}_S)\eta\phi_y.$$

This will be a total y-derivative if:

$$\begin{array}{ll} (1) & \epsilon_S \mp \bar{\epsilon}_S = 0, \\ (2) & \epsilon_S \pm \bar{\epsilon}_S = 0, \quad \phi_x = \mp 2e^{\phi}. \end{array}$$

The latter implies $\epsilon_S = 0$. In the former case, we have the following boundary potential:

$$\mathcal{B}_S = \pm(2e^{\phi} - \bar{\psi}\psi + \epsilon_S(\bar{\psi} \pm \psi)). \tag{9.43}$$

This yields the boundary conditions (cf(9.37)):

$$\phi_x = \mp 2e^{\phi}, \qquad \psi \mp \bar{\psi} + \epsilon_S = 0. \tag{9.44}$$

Alternatively, we can equally well impose the boundary conditions by hand [114]. We require the integrand in eq.(9.40) to be a total y-derivative for $\bar{\eta} = \pm\eta$. It turns out that there is a single boundary condition satisfying this requirement that is left invariant under a supersymmetry transformation:

$$\phi_y = 0, \qquad \psi \pm \bar{\psi} = 0. \tag{9.45}$$

Let us now discuss the integrability of the theory. According to Cardy [60], [113], invariance of the boundary conditions under a symmetry generated by some set of conserved currents $(W, \bar{W})$ requires $W = \bar{W}$ on the boundary. If we take $W$ to be the stress tensor, we get from eq.(9.26):

$$-i\phi_x\phi_y + \psi\psi_x - i\psi\psi_y + i\phi_{xy} = i\phi_x\phi_y - \bar{\psi}\bar{\psi}_x - i\bar{\psi}\bar{\psi}_y - i\phi_{xy}.$$

The bosonic part, $-i\phi_x\phi_y + i\phi_{xy}$, vanishes for

$$\phi_x = ce^{\phi}, \tag{9.46}$$

where c is some constant. Using the equations of motion (9.25) to eliminate the x-derivatives, we get, $\psi\psi_y = \bar{\psi}\bar{\psi}_y$. There are two solutions to this equation:

$$\begin{array}{ll} (1) & \bar{\psi} = \pm\psi, \\ (2) & \psi_y = \bar{\psi}_y = 0. \end{array} \tag{9.47}$$

These conditions together with eq.(9.46) preserve the conformal invariance of the theory. We can still impose similar constraints on the supercurrents $(J, \bar{J})$ and this should fix c. The boundary condition is $\eta J = \bar{J}\bar{\eta}$. Remember that we want to keep half of the supersymmetries, by setting $\bar{\eta} = \pm\eta$. Accordingly, we impose the boundary condition $\bar{J} = \mp J$ and get:

$$(\phi_x - i\phi_y \pm 2e^{\phi})\psi + 2i\psi_y = \mp(\phi_x + i\phi_y \pm 2e^{\phi})\bar{\psi} \pm 2i\bar{\psi}_y. \tag{9.48}$$

This equation is solved by (9.44) provided $\epsilon_S = 0$. We might expect (9.44) and (9.45) to solve this equation without any additional constraints, since $(J, \bar{J})$ generate supersymmetry transformations. However, $J = \mp\bar{J}$ is more restrictive than simple supersymmetry conservation due to the orthogonality condition (9.9).

If we want superconformal symmetry to be preserved, we have to use as Ansatz the conditions (9.46), (9.47) obtained above for the conservation of the conformal invariance. We then get:

$$\begin{array}{lllll} (1) & \bar{\psi} = \pm\psi, & \phi_x = \mp 2e^{\phi}, & & \\ (2) & \psi_y = \bar{\psi}_y = 0, & \phi_x = \mp 2e^{\phi}, & \phi_y = 0. \end{array} \tag{9.49}$$

Both conditions are left invariant under supersymmetry transformations. We see that these conditions are compatible with supersymmetry conservation (cf (9.44), (9.45)), although far more constrained than the latter. It is worth remarking that they are unambiguously determined, in the sense that there are no unfixed parameters up to a sign. A similar situation occurs in the super-sine-Gordon theory [114] and appears to be a consequence of supersymmetry. Indeed, as we shall see in the next section, our analysis of the non-supersymmetric $B^{(1)}(0,1)$ theory reveals that, in contrast to the two models above, the boundary potential depends on free parameters [109].

It is also interesting to investigate, whether any of the IM $U_{s+1}$, $\bar{U}_{s+1}$ (s=1,3,5,...) survive on the half-line for the set of boundary conditions (9.49). One could then perturb the theory on the half-line with some relevant primary operator and thus obtain an integrable massive theory (e.g. sinh-Gordon) with boundary interactions. The IM for this theory could be seen as deformations of the surviving conformal ones. Also if one wants to describe the boundary conformal field theory in terms of massless particles, [89], [90], [91], then these IM are indispensable ingredients, as the massless particles are their simultaneous eigenvectors.

It is easy to verify that the conditions (9.49) preserve the following combinations of the IM:

$$I_s = \int_{-\infty}^{0} dx (U_{s+1} + \bar{U}_{s+1}), \qquad (s = 1, 3, 5, \cdots). \tag{9.50}$$

We just have to prove that $U_{s+1} = \bar{U}_{s+1}$ at $x = 0$ as a consequence of the stress tensor and the supercurrent satisfying $T = \bar{T}$ and $\bar{J} = \mp J$. All polynomials $T^n$ $(n > 1)$ automatically satisfy $T^n = \bar{T}^n$. The first non trivial term is $(\partial_z T)^2$. From the conservation of the stress tensor, we have $T_x = -iT_y$ and $\bar{T}_x = i\bar{T}_y$. This implies that at $x = 0$,

$$(\partial_z T)^2 = -T_y^2 = -\bar{T}_y^2 = (\partial_{\bar{z}} \bar{T})^2.$$

Similarly, from $J_x = -iJ_y$, $\bar{J}_x = i\bar{J}_y$, we have:

$$J\partial_z J = -iJJ_y = -i(\mp\bar{J})(\mp\bar{J}_y) = -i\bar{J}\bar{J}_y = -\bar{J}\partial_{\bar{z}}\bar{J}.$$

Altogether, this means that $U_4 = \bar{U}_4$. Next, we consider the term $J\partial_z^3 J$. We use the following identities,

$$\begin{cases} J_{xyy} = iJ_{xxy} = -J_{xxx} = -iJ_{yyy}, \\ \bar{J}_{xyy} = -i\bar{J}_{xxy} = -\bar{J}_{xxx} = i\bar{J}_{yyy}, \end{cases}$$

to show that

$$J\partial_z^3 J = \frac{1}{8}J(J_{xxx} - 3iJ_{xxy} - 3J_{xyy} + iJ_{yyy}) = iJJ_{yyy} = i\bar{J}\bar{J}_{yyy} = -\bar{J}\partial_{\bar{z}}^3\bar{J}.$$

Again, we have $U_6 = \bar{U}_6$. In summary, we found that the conditions preserving the superconformal invariance in the surface configuration, also ensure the conservation of half of the IM. This is not surprising, since the IM, being composite fields of the stress tensor, the supercurrent and their derivatives, are deeply connected with the superconformal symmetry of the theory.

## 9.5 The $B^{(1)}(0,1)$ theory

In this section, I shall briefly compute the boundary potential for the $B^{(1)}(0,1)$ theory, [109]. This theory is in fact not a very interesting one. The distortions on its mass spectrum arising from quantum corrections render this theory incompatible with an exact, factorizable S-matrix [116]. However we shall ignore this fact, since at the classical level it suits our purposes of checking whether there are any unfixed parameters in the boundary potential. The action in superspace leads to the following equation:

$$D_1 D_2 \Phi = ie^{2\Phi} - \frac{1}{2}\theta_1\theta_2 e^{-4\Phi}. \tag{9.51}$$

I chose a normalization such that the coupling constant is absorbed into the definition of the fields.

The second term on the right-hand side spoils invariance under supersymmetry transformations. This is a common feature of Toda theories based on contragradient Lie superalgebras, [100], [118].

Alternatively, eq.(9.51) can be seen as the compatibility condition for a linear system similar to (9.13), where this time the graded matrices take the form:

$$A_1(\lambda) = \begin{pmatrix} -2D_1\Phi & -i\lambda\sqrt{2} & 0 \\ 0 & 0 & i\lambda\sqrt{2} \\ -\lambda\theta_2 & 0 & 2D_1\Phi \end{pmatrix}, \qquad A_2(\lambda) = \frac{1}{\lambda}\begin{pmatrix} 0 & 0 & \theta_1 e^{-4\Phi} \\ \sqrt{2}e^{2\Phi} & 0 & 0 \\ 0 & \sqrt{2}e^{2\Phi} & 0 \end{pmatrix}.$$

Expressing eq.(9.51) in components, we get in Euclidean space:

$$\begin{cases} F = -ie^{\phi}, \qquad \partial_z \bar{\psi} = -e^{\phi}\psi, \qquad \partial_{\bar{z}}\psi = -e^{\phi}\bar{\psi}, \\ \partial_z \partial_{\bar{z}}\phi = e^{2\phi} - e^{\phi}\bar{\psi}\psi - \frac{1}{4}e^{-2\phi}. \end{cases} \tag{9.52}$$

The bosonic limit of this theory is the $a_1^{(1)}$ bosonic Toda theory. It was conjectured [118] that the gaps in the sequence of conservation laws be periodic with period equal to 2. Specifically, there will be an infinite set of conserved densities, $\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}$, with s=1,3,5,...

Considering the most general Ansatz, I obtained the following elements:

$$\begin{cases} T_2 = & (\partial_z\phi)^2 - \partial_z\psi\psi, \\ T_4 = & (\partial_z^2\phi)^2 + (\partial_z\phi)^4 + 3(\partial_z\phi)^2\psi\partial_z\psi + \partial_z\psi\partial_z^2\psi + 3\partial_z\phi\partial_z^2\psi\psi, \\ & \\ \Theta_0 = & e^{2\phi} - e^{\phi}\bar{\psi}\psi + 1/4e^{-2\phi}, \\ \Theta_2 = & 2(\partial_z\phi)^2 e^{2\phi} + 4(\partial_z\phi)^2 e^{\phi}\psi\bar{\psi} + 1/2(\partial_z\phi)^2 e^{-2\phi} + \\ & +2e^{2\phi}\partial_z\psi\psi + 3/2e^{-2\phi}\psi\partial_z\psi + 2\partial_z\phi e^{\phi}\bar{\psi}\partial_z\psi. \end{cases} \tag{9.53}$$

Again, the action on the half-line is defined according to:

$$S = S_0 + S_{\mathcal{B}} \equiv \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \left\{ \theta(-x)\mathcal{L}_0 + \delta(x)\mathcal{B}(\phi, \psi, \bar{\psi}) \right\},$$

leading to the boundary conditions:

$$\phi_x = -\frac{\partial \mathcal{B}}{\partial \phi}, \qquad \psi = \frac{\partial \mathcal{B}}{\partial \psi}, \qquad \bar{\psi} = -\frac{\partial \mathcal{B}}{\partial \bar{\psi}}. \tag{9.54}$$

In addition, we have:

$$\frac{\partial^2 \mathcal{B}}{\partial\phi\partial\psi} = \frac{\partial^2 \mathcal{B}}{\partial\phi\partial\bar{\psi}} = \frac{\partial^2 \mathcal{B}}{\partial\psi\partial\bar{\psi}} = 0.$$

Consequently:

$$\phi_{xy} = -\frac{\partial^2 \mathcal{B}}{\partial \phi^2}\phi_y. \tag{9.55}$$

Suppose that the boundary potential $\mathcal{B}$ can be chosen in such a way that at $x = 0$, there is a $\theta_3$ such that eq.(6.2) holds.

From the equations of motion we have:

$$\begin{cases} \psi_{xy} = -2e^{\phi}\bar{\psi}_y - 2\phi_y e^{\phi}\bar{\psi} - i\psi_{yy}, \\ \psi_{xx} = 4e^{2\phi}\psi - 2\phi_x e^{\phi}\bar{\psi} + 2i\phi e^{\phi}\bar{\psi} - \psi_{yy}, \\ \bar{\psi}_{xy} = -2e^{\phi}\psi_y - 2\phi_y e^{\phi}\psi + i\bar{\psi}_{yy}, \\ \bar{\psi}_{xx} = 4e^{2\phi}\bar{\psi} - 2\phi_x e^{\phi}\psi - 2i\phi e^{\phi}\psi - \bar{\psi}_{yy}, \end{cases}$$

$$\phi_{xx} = 4e^{2\phi} - \phi_{yy} - 4e^{\phi}\bar{\psi}\psi - e^{-2\phi}.$$

Using these expressions and eq.(9.55), we get:

$$T_4 - \bar{T}_{-4} + \bar{\Theta}_{-2} - \Theta_2 = \mathcal{W}_b + \mathcal{W}_R,$$

where $\mathcal{W}_b$ is a purely bosonic contribution,

$$\mathcal{W}_b = -\frac{i}{4}\frac{\partial^2\mathcal{B}}{\partial\phi^2}\phi_{yy}\phi_y - \frac{i}{8}\frac{\partial\mathcal{B}}{\partial\phi}\phi_y^3 + \frac{i}{2}\left\{\frac{1}{4}\left(\frac{\partial\mathcal{B}}{\partial\phi}\right)^3 + \frac{\partial^2\mathcal{B}}{\partial\phi^2}\left(e^{2\phi} - \frac{1}{4}e^{-2\phi}\right) - \frac{\partial\mathcal{B}}{\partial\phi}\left(e^{2\phi} + \frac{1}{4}e^{-2\phi}\right)\right\}\phi_y.$$

We look for solutions of the form $\mathcal{B}(\phi,\psi,\bar{\psi}) = \mathcal{B}_b(\phi) + \mathcal{B}_f(\psi,\bar{\psi})$. It is straightforward to show that for $\mathcal{B}_b(\phi) = ae^{\phi} + be^{-\phi}$, where $a$ and $b$ are arbitrary constants, $\mathcal{W}_b$ will automatically be a total y-derivative. The remaining contribution $\mathcal{W}_R$ is given by:

$$\mathcal{W}_R = -\tfrac{i}{2}\bar{\psi}\psi\phi_y\left(\tfrac{\partial^2\mathcal{B}_b}{\partial\phi^2} + \tfrac{\partial\mathcal{B}}{\partial\phi}\right)e^{\phi} - \tfrac{3i}{8}\left[\left(\tfrac{\partial\mathcal{B}_b}{\partial\phi}\right)^2 - \phi_y^2 - 4e^{2\phi} - e^{-2\phi}\right](\bar{\psi}_y\bar{\psi} - \psi_y\psi) +$$

$$+\tfrac{3}{4}\tfrac{\partial\mathcal{B}_b}{\partial\phi}\phi_y(\bar{\psi}\psi_y + \psi\bar{\psi}_y) + \tfrac{i}{2}\tfrac{\partial\mathcal{B}_b}{\partial\phi}e^{\phi}(\psi_y\bar{\psi} - \bar{\psi}_y\psi) + 2\phi_y e^{\phi}(\psi_y\bar{\psi} + \bar{\psi}_y\psi) +$$

$$+i(\bar{\psi}_{yy}\bar{\psi}_y - \psi_{yy}\psi_y) + e^{\phi}(\bar{\psi}\psi_{yy} - \bar{\psi}_{yy}\psi) + \tfrac{3}{4}\tfrac{\partial\mathcal{B}_b}{\partial\phi}(\psi_{yy}\psi + \bar{\psi}_{yy}\bar{\psi}) + \tfrac{3i}{4}\phi_y(\psi_{yy}\psi - \bar{\psi}_{yy}\bar{\psi}).$$

Because $\psi$, $\bar{\psi}$ are Grassmann variables, $\mathcal{B}_f$ takes the form $\mathcal{B}_f(\psi,\bar{\psi}) = M\bar{\psi}\psi + \epsilon\psi + \bar{\epsilon}\bar{\psi}$, where $M$, $\epsilon$, $\bar{\epsilon}$ are constant parameters, $M$ being bosonic and the remaining fermionic. From eq.(9.54), we have the following possibilities at $x = 0$:

$$(1)\quad \psi = -\tfrac{\epsilon + M\bar{\epsilon}}{1-M^2}, \qquad \bar{\psi} = \tfrac{\bar{\epsilon} + M\epsilon}{1-M^2}, \quad (M \neq \pm 1),$$

$$(2)\quad \bar{\psi} = \mp(\psi + \epsilon), \quad \bar{\epsilon} = \mp\epsilon, \qquad (M = \pm 1).$$

In the former case, $\mathcal{W}_R$ is automatically a total y-derivative irrespective of the values of $a$, $b$, $\epsilon$, $\bar{\epsilon}$ and $M(\neq \pm 1)$. In the latter case, we get $\epsilon = \bar{\epsilon} = 0$ and $a = \mp 2$, corresponding to $M = \pm 1$. In summary, there will be a spin $s = 3$ conserved charge in the following cases:

(1) $\mathcal{B}(\phi,\psi,\bar{\psi}) = ae^{\phi} + be^{-\phi} + M\bar{\psi}\psi + \epsilon\psi + \bar{\epsilon}\bar{\psi}$,

$\phi_x = -ae^{\phi} + be^{-\phi}, \qquad \psi = -\tfrac{\epsilon + M\bar{\epsilon}}{1-M^2}, \qquad \bar{\psi} = \tfrac{\bar{\epsilon} + M\epsilon}{1-M^2},$

$a, b, \epsilon, \bar{\epsilon}$ and $M(\neq \pm 1)$ are arbitrary.

(2) $\mathcal{B}(\phi,\psi,\bar{\psi}) = \mp 2e^{\phi} + be^{-\phi} \pm \bar{\psi}\psi$,

$\phi_x = \pm 2e^{\phi} + be^{-\phi}, \qquad \psi \pm \bar{\psi} = 0,$

$b$ is arbitrary.

So, as we can see the integrability is not sufficient to fix all the parameters in the boundary potential.

## 9.6 Quantization of the super-Liouville theory

Let us now return to the action in Minkowski space:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\beta\phi} + i(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+) + i\beta\psi_+\psi_-e^{\beta\phi}. \tag{9.56}$$

I redefined $\chi_1 \equiv \psi_-$ and $\chi_2 \equiv \psi_+$. This theory has been quantized on the light cone in ref. [112]. The light cone has two branches $\mathcal{C}_\pm$, corresponding to $x^\pm = 0$, respectively. The Dirac quantization for constrained systems leads to the following Dirac brackets on the branch $\mathcal{C}_-$:

$$\begin{aligned} [\phi(x), 2\partial_+\phi(y)]_{D.B.} &= \delta(x-y), \\ \{\psi_+(x), \psi_+(y)\}_{D.B.} &= i\delta(x-y). \end{aligned} \tag{9.57}$$

On this branch, $\psi_-$ is expressed as a function of $\phi$ and $\psi_+$ via its equation of motion. In Minkowski space the stress-tensor and the supercurrent are given by:

$$\begin{cases} \Theta_{++} &= \frac{1}{2}(\partial_+\phi)^2 - \frac{i}{4}\psi_+\partial_+\psi_+ - \frac{1}{2\beta}\partial_+^2\phi, \\ J_+ &= \frac{1}{2}\partial_+\phi\psi_+ - \frac{1}{2\beta}\partial_+\psi_+. \end{cases} \tag{9.58}$$

Using this expressions and eq.(9.57), we obtain the following Dirac brackets:

$$\begin{aligned} \{J_+(x), J_+(y)\}_{D.B.} &= -\tfrac{i}{4\beta^2}\delta''(x-y) + \tfrac{i}{2}\delta(x-y)\Theta_{++}(y), \\ [J_+(x), \Theta_{++}(y)]_{D.B.} &= \tfrac{3}{4}\delta'(x-y)J_+(y) - \tfrac{1}{4}\delta(x-y)\partial_+J_+(y), \\ [\Theta_{++}(x), \Theta_{++}(y)]_{D.B.} &= -\tfrac{1}{8\beta^2}\delta'''(x-y) + \tfrac{1}{2}\delta'(x-y)\left[\Theta_{++}(x) + \Theta_{++}(y)\right]. \end{aligned} \tag{9.59}$$

The IM are (cf. (9.31)-(9.33)):

$$\begin{cases} H_{1/2} &= \int dx U_{3/2}(x) &= \int dx J_+(x), \\ H_1 &= \int dx U_2(x) &= \int dx \Theta_{++}(x), \\ H_3 &= \int dx U_4(x) &= \int dx \left[\Theta_{++}^2(x) - \frac{i}{2}J_+(x)\partial_+J_+(x)\right], \\ H_5 &= \int dx U_6(x) &= \int dx \left[\Theta_{++}^3(x) + \frac{1}{16\beta^2}(\partial_+\Theta_{++}(x))^2\right. \\ & & \left. -i\Theta_{++}(x)J_+(x)\partial_+J_+(x) + \frac{i}{32\beta^2}J_+(x)\partial_+^3J_+(x)\right]. \end{cases} \tag{9.60}$$

It is also possible to compute the Dirac brackets for the charge densities. For example:

$$\begin{aligned}
\left[U_{3/2}(x), U_4(y)\right]_{D.B.} = & -\tfrac{1}{8\beta^2}\delta'''(x-y)J_+(y) - \tfrac{1}{8\beta^2}\delta''(x-y)\partial_+ J_+(y)+, \\
& +\tfrac{7}{4}\delta'(x-y)\Theta_{++}(y)J_+(y) - \tfrac{1}{4}\delta(x-y)\partial_+\left(\Theta_{++}(y)J_+(y)\right) \\
[U_2(x), U_4(y)]_{D.B.} = & -\tfrac{1}{4\beta^2}\delta'''(x-y)\Theta_{++}(y) + \tfrac{3i}{8}\delta''(x-y)J_+(x)J_+(y)+ \\
& +\delta'(x-y)\left[\Theta_{++}(x)\Theta_{++}(y) + \Theta_{++}^2(y) - \tfrac{3i}{8}J_+(x)\partial_+ J_+(y)\right. \\
& \left.-\tfrac{i}{8}J_+(y)\partial_+ J_+(y)\right] + \tfrac{i}{8}\delta(x-y)\partial_+\left[J_+(y)\partial_+ J_+(y)\right].
\end{aligned} \tag{9.61}$$

We can see that because of non-linear terms like $\delta'(x-y)\Theta_{++}(y)J_+(y)$, these quantities do not form a Lie algebra. We then get, after a few integrations by parts:

$$\begin{aligned}
[H_{1/2}, H_3]_{D.B.} &= -2\int dx\partial_+\left[\theta_{++}(x)J_+(x)\right] = 0, \\
[H_1, H_3]_{D.B.} &= \int dx\partial_+\left[\tfrac{1}{4\beta^2}\partial_+^2\Theta_{++}(x) + \tfrac{1}{2}\Theta_{++}^2(x) - \tfrac{3i}{8}J_+(x)\partial_+ J_+(x)\right] = 0.
\end{aligned}$$

We see that these integrals of motion are classically in involution. They are known to be hierarchy of Hamiltonians of the super KdV equation, [110], [111].

Let us now proceed to the quantum theory. Following the usual quantization prescription,

$$[\alpha, \beta]_{D.B.} \rightarrow \frac{i}{\hbar}[\alpha, \beta]_{commutator},$$

we get:

$$\begin{aligned}
[\phi(x), 2\partial_+\phi(y)] = &\quad i\hbar\delta(x-y), \\
\{\psi_+(x), \psi_+(y)\} = &\quad \hbar\delta(x-y).
\end{aligned} \tag{9.62}$$

If $x \neq y$, we have:

$$\begin{aligned}
< \phi(x^+)\phi(y^+) >= &\quad \hbar\Delta(x^+ - y^+), \\
< \psi_+(x^+)\psi_+(y^+) >= &\quad 2i\hbar\Delta_+(x^+ - y^+),
\end{aligned} \tag{9.63}$$

where

$$\Delta_+(x^+ - y^+) = \partial_+\Delta(x^+ - y^+) = -\frac{1}{4\pi(x^+ - y^+)}. \tag{9.64}$$

The quantum counterparts of the stress tensor and the supercurrent were found to be:

$$\left\{\begin{aligned}
\Theta_{++} &= -\tfrac{1}{2} : \partial_+\phi\partial_+\phi : + \tfrac{1}{2}\left(\tfrac{1}{\beta} + \tfrac{\hbar\beta}{4\pi}\right)\partial_+^2\phi - \tfrac{i}{4} : \psi_+\partial_+\psi_+ :, \\
J_+ &= \tfrac{1}{2}\partial_+\phi\psi_+ - \tfrac{1}{2}\left(\tfrac{1}{\beta} + \tfrac{\hbar\beta}{4\pi}\right)\partial_+\psi_+
\end{aligned}\right. \tag{9.65}$$

in ref.[112]. Besides being conserved and properly regularized, they also satisfy the requirements of tracelessness and orthogonality (cf.(9.9)). If we choose units such that $\hbar = 4\pi$, we have the standard operator product expansions (OPE) for a superconformal theory:

$$\begin{cases} \Theta_{++}(x^+)\Theta_{++}(y^+) & = \frac{3\hat{c}/4}{(x^+-y^+)^4} + \frac{2}{(x^+-y^+)^2}\Theta_{++}(y^+) + \frac{1}{x^+-y^+}\partial_+\Theta_{++}(y^+) + \cdots \\ \Theta_{++}(x^+)J_+(y^+) & = \frac{3/2}{(x^+-y^+)^2}J_+(y^+) + \frac{1}{x^+-y^+}\partial_+J_+(y^+) + \cdots \\ J_+(x^+)J_+(y^+) & = \frac{i\hat{c}/2}{(x^+-y^+)^3} + \frac{i}{x^+-y^+}\Theta_{++}(y^+) + \cdots \end{cases} \tag{9.66}$$

where $\hat{c}$ is related to the central charge $c$ according to $\hat{c} = 2c/3$. These equations can be checked using the propagators

$$\begin{aligned} < \phi(x^+)\phi(y^+) > & = -log(x^+ - y^+), \\ < \psi_+(x^+)\psi_+(y^+) > & = -\frac{2i}{x^+-y^+}, \end{aligned} \tag{9.67}$$

which can be obtained from (9.63), (9.64) by setting $\hbar = 4\pi$. As an illustration, we consider the OPE $J_+(x^+)J_+(y^+)$. This OPE can be computed using Wick's theorem and Taylor expanding when $x^+ \to y^+$.

$$\begin{aligned} & J_+(x^+)J_+(y^+) = \tfrac{1}{4} : \partial_+\phi(x^+)\partial_+\phi(y^+) :: \psi_+(x^+)\psi_+(y^+) : + \\ & +\tfrac{1}{4} < \partial_+\phi(x^+)\partial_+\phi(y^+) >: \psi_+(x^+)\psi_+(y^+) : +\tfrac{1}{4} < \psi_+(x^+)\psi_+(y^+) >: \partial\phi(x^+)\partial_+\phi(y^+) : + \\ & +\tfrac{1}{4} < \partial_+\phi(x^+)\partial_+\phi(y^+) >< \psi_+(x^+)\psi_+(y^+) > -\tfrac{1}{4}\left(\beta + \tfrac{1}{\beta}\right)\partial_+\phi(x^+) : \psi_+(x^+)\partial_+\psi_+(y^+) : \\ & -\tfrac{1}{4}\left(\beta + \tfrac{1}{\beta}\right) < \psi_+(x^+)\partial\psi_+(y^+) > \partial_+\phi(x^+) - \tfrac{1}{4}\left(\beta + \tfrac{1}{\beta}\right)\partial_+\phi(y^+) : \partial_+\psi_+(x^+)\psi_+(y^+) : \\ & -\tfrac{1}{4}\left(\beta + \tfrac{1}{\beta}\right)\partial_+\phi(y^+) < \partial_+\psi_+(x^+)\psi_+(y^+) > +\tfrac{1}{4}\left(\beta + \tfrac{1}{\beta}\right)^2 : \partial_+\psi_+(x^+)\partial_+\psi_+(y^+) : + \\ & +\tfrac{1}{4}\left(\beta + \tfrac{1}{\beta}\right)^2 < \partial_+\psi_+(x^+)\partial_+\psi_+(y^+) >= \\ & = \frac{\frac{i}{2}\left[1+2\left(\beta+\frac{1}{\beta}\right)^2\right]}{(x^+-y^+)^3} + \frac{i}{x^+-y^+}\Theta_{++}(y^+) - \tfrac{i}{2} : \partial_+\phi(y^+)\partial_+^2\phi(y^+) : +\tfrac{1}{8} : \psi_+(y^+)\partial_+^2\psi_+(y^+) : + \\ & +\tfrac{i}{4}\left(\beta + \tfrac{1}{\beta}\right)\partial_+^3\phi(y^+) + \mathcal{O}(x^+ - y^+). \end{aligned}$$

From (9.66) we conclude that:

$$\hat{c} = 1 + 2\left(\beta + \frac{1}{\beta}\right)^2. \tag{9.68}$$

There are two remarkable features about eq.(9.68). First, we notice that there is a strong-weak coupling duality, in the sense that the stress tensor, the supercurrent and the central charge are invariant under $\beta \to 1/\beta$. Secondly, we conclude that the super-Liouville theory describes a superconformal theory in the continuum region $\hat{c} > 1$. Moreover, it makes sense

[112] to analytically continue $\beta \to i\tilde{\beta}$ (or alternatively set $\hbar$ less than zero). For the choice $\tilde{\beta}^2 = (m+2)/m$, we recover the central charge of the superconformal unitary series (cf. eq.(4.58)):

$$\hat{c} = 1 - \frac{8}{m(m+2)}, \qquad m = 3, 4, 5, , \cdots \tag{9.69}$$

We have now assembled all the necessary ingredients to compute the quantum versions of the super KdV hierarchy of IM. We assume as Ansatz:

$$\begin{cases} U_{3/2} &= J_+, \\ U_2 &= \Theta_{++}, \\ U_4 &=: \Theta_{++}^2 : + \alpha : J_+ \partial_+ J_+ :, \\ U_6 &=: \Theta_{++}^3 : + \gamma : (\partial_+ \Theta_{++})^2 : + \lambda : \Theta_{++} J_+ \partial_+ J_+ : + \sigma : J_+ \partial_+^3 J_+ : . \end{cases} \tag{9.70}$$

These quantities are obviously conserved. We just have to compute the coefficients $\alpha$, $\gamma$, $\lambda$ and $\sigma$ which will generically have quantum corrections involving the central charge. They are fixed by the constraint that the charges corresponding to (9.70) should be in involution. It turns out that supersymmetry is restrictive enough a constraint to pin down all these coefficients.

One strategy consists in imposing that the first order pole in the OPEs of the densities (9.70) be a total derivative. To illustrate this consider:

$$[H_1, H_{1/2}] = \oint dx^+ \oint dy^+ [\Theta_{++}(x^+), J_+(y^+)].$$

Using the usual techniques to transform commutators into OPEs (cf. section 2.2), we get:

$$\begin{aligned} [H_1, H_{1/2}] &\propto \oint dx^+ \oint dy^+ \Theta_{++}(x^+) J_+(y^+) \\ &\propto \oint dx^+ \oint dy^+ \left\{ \tfrac{3/2}{(x^+ - y^+)^2} J_+(y^+) + \tfrac{1}{x^+ - y^+} \partial_+ J_+ + \cdots \right\}. \end{aligned}$$

Using Cauchy's theorem the first term reads:

$$\oint dx^+ \frac{3}{2} \partial_+ J_+(x^+) = 0.$$

Poles of order greater than one therefore give zero contribution, since they lead to total derivative terms.

The second term yields:

$$\oint dx^+ \partial_+ J_+(x^+).$$

This also vanishes because, in this case, the first order pole happened to be a total derivative. The recipe for computing $\alpha$, $\gamma$, $\lambda$ and $\sigma$ consists in cooking up these coefficients in order to produce total derivative terms in the first order poles for the OPEs of the densities (9.70).

Applying this to $[H_{1/2}, H_s]$ $(s = 3, 5)$, we get:

$$\alpha = \frac{i}{2}, \qquad \gamma = \frac{25}{2}\hat{c}, \qquad \lambda = i, \qquad \sigma = -\frac{i}{8}(3 + 50\hat{c}). \tag{9.71}$$

Although very involved it is straightforward to check that for this choice, all the IM are mutually in involution.

# Chapter 10

# Conclusions and outlook

Let us restate our results. The WZW theory for the $\widehat{SU}(2)_1$ Kac-Moody algebra was defined in an axiomatic way by introducing the current algebra symmetry and constructing the conformal symmetry according to Sugawara's procedure. Cardy's approach shows that if we impose the conservation of these two symmetries in the presence of a boundary there will be two permissible boundary conditions which we denoted as free and fixed. There are two RG trajectories that terminate at this theory in the IR limit. One of them - the Kondo theory - allows us to interpret the two boundary conditions as the exactly screened and underscreened situations. The factorized scattering is well known for this theory. The second trajectory represents the principal chiral model with scale invariant boundary conditions. The symmetries of the model, the IR limiting WZW theory and comparison with the Kondo model allowed us to construct the reflection amplitudes for the PCM. Subsequently we derived the boundary TBA equations in the $R$- and $L$-channels for fixed boundary conditions. The former lead to the correct prediction for the boundary ground-state energy in the IR limit whereas the latter yield distinct boundary entropies in the IR and UV limits in agreement with the conjecture of ref.[97].

Of course there is much room for further developments. We have to consider the TBA in the $R$-channel involving free boundary conditions. In ref.[84] some remarks were made regarding the universal structure of TBA systems. Similarly, it seems plausible to assume that the same might be conjectured for boundary TBA. If the reflection amplitudes are scalar functions, $K(\beta)$, then the contribution of the boundary to the ground-state energy is (in all known cases) of the form $\sim log(\bar{K}(\beta)K(\beta))$. How this generalises to nondiagonal reflection matrices could be speculated by exploring the conformal approach (in the IR or UV limits) in the lines of ref.[97]. In the $L$-channel nondiagonal reflection also poses many problems. The standing wave condition can no longer be imposed. Some progress has been made by LeClair et al.[86] for the sine-Gordon model by generalizing the Destri-de Vega theory in the presence of reflecting boundaries. However this approach requires a lattice regularization and it is not clear to me that this is possible for the PCM. An alternative solution might consist of considering as boundary condition the zero flux of energy across the boundary. At the level of the Bethe wave function this should correspond to imposing a Neumann condition. I also expect to obtain some results in the near future [99] concerning the $SU(2)$ sigma model with $\theta$ term, [57]. In this model the spectrum is the same as in the PCM. However the R-R and L-L scattering are trivial and the R-L scattering is nondiagonal. This should provide some more insight for the program described above.

Finally, we considered the super-Liouville model. This theory provides another extension of the Virasoro algebra. We constructed the super KdV hierarchy of Hamiltonians by considering the integrability of this theory. We derived the boundary conditions that preserve both the integrability and the supersymmetry at the classical level. Usually a detailed knowledge of the integrability properties of a theory provide guidelines for approaching the theory quantum mechanically. After quantizing the theory on the light-cone [112], we constructed the quantum versions of the integrals of motion. Using the techniques described in chapter 3, we expect [126] to show that these yield the conservation laws of the super-sinh-Gordon theory. As a by-product we will be able to infer the spins of integrable perturbed super minimal conformal models, [125], by fine-tuning the coupling constant. We also hope to construct a supersymmetric version of the quantum Bousinessq equation and of the $W_3$ algebra using this super Coulomb gas description.

# Bibliography


[1] M.Kaku, *Introduction to Superstrings*, 1988 Springer Verlag.

[2] D.Bailin, A.Love, *Supersymmetric Gauge Field Theory and String Theory*, 1994 Institute of Physics Publishing.

[3] C.Itzykson, J.-B.Zuber, *Quantum Field Theory*, 1980 McGraw-Hill.

[4] C.Itzykson, J.-M.Drouffe, *Statistical Field Theory*, 1989 Cambridge University Press.

[5] D.J.Amit, *Field Theory, the Renormalization Group and Critical Phenomena*, 1984 World Scientific.

[6] M.B.Green, J.H.Schwarz, E.Witten, *Superstring Theory*, 1987 Cambridge University Press.

[7] J.Polchinski, *What is String Theory?*, NSF-ITP-94-97.

[8] V.E.Korepin, N.M.Bogoliubov, A.G.Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, 1993 Cambridge University Press.

[9] T.D.Schultz, D.C.Mattis, E.H.Lieb, *Two-Dimensional Ising Model as a Soluble Problem of Many Fermions*, Rev. Mod. Phys. 36 (1964) 856.

[10] J.B.Kogut, *An Introduction to Lattice Gauge Theory and Spin Systems*, Rev. Mod. Phys. 51 (1979) 659.

[11] L.Onsager, Phys. Rev. 65 (1944) 117.

[12] A.A.Belavin, A.M.Polyakov, A.B.Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, Nucl. Phys. B241 (1984) 333.

[13] P.Ginsparg, *Applied Conformal Field Theory, Les Houches Lectures*, 1988 North Holland.

[14] Y.Grandati, *Elements d'Introduction à l'Invariance Conforme*, Ann. Phys. F. 17 (1992) 159.

[15] H.W.J.Blöte, J.L.Cardy, M.P.Nightingale, *Conformal Invariance, the Central Charge, and Universal Finite-Size Amplitudes at Criticality*, Phys. Rev. Lett. 56 (1986) 742.

[16] D.Friedan, Z.Qiu, S.Shenker, *Conformal Invariance, Unitarity, and Critical Exponents in Two Dimensions*, Phys. Rev. Lett. 52 (1984) 1575.

[17] I.Affleck, *Universal Term in the Free Energy at a Critical Point and the Conformal Anomaly*, Phys. Rev. Lett. 56 (1986) 746.

[18] Vi.S.Dotsenko, *Lectures on Conformal Field Theory*, Adv. Stud. P. Math. 16 (1988) 123.

[19] Vi.S.Dotsenko, V.A.Fateev, *Conformal Algebra and Multipoint Correlation Functions in 2D Statistical Models*, Nucl. Phys. B240 (1984) 312.

[20] J.L.Cardy, *Operator Content of Two-Dimensional Conformally Invariant Theories*, Nucl. Phys. B270 (1986) 186.

[21] A.Rocha-Caridi, *Vertex Operators in Mathematics and Physics*, 1984 Springer Verlag.

[22] I.Affleck, *Monopole-Catalysed Baryon Decay: A Boundary Conformal Field Theory Approach*, Nucl. Phys. B417 (1994) 374.

[23] A.J.Leggett, S.Chakravarty, A.T.Dorsey, M.P.A.Fisher, A.Garg, N.Zweger, *Dynamics of the Dissipative Two-State System*, Rev. Mod. Phys. 59 (1987) 1.

[24] K.Moon, H.Yi, C.L.Kane, S.M.Girvin, M.P.A.Fisher, *Resonant Tunneling Between Quantum Hall Edge States*, Phys. Rev. Lett. 71 (1983) 4381.

[25] G.Mussardo, *Off-Critical Statistical Models: Factorized Scattering Theories and Bootstrap Program*, Phys. Rep. 218 (1992) 215.

[26] S.Coleman, J.Mandula, *All Possible Symmetries of the S Matrix*, Phys. Rev. 159 (1967) 1251.

[27] F.A.Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, 1992 World Scientific, Singapore.

[28] B.A.Kuperschmidt, P.Mathieu, *Quantum Korteweg-de Vries Like Equations and Perturbed Conformal Field Theories*, Phys. Lett. B227 (1989) 245.

[29] A.B.Zamolodchikov, *Integrable Field Theory from Conformal Field Theory*, Adv. Stud. P. Math. 19 (1989) 641.

[30] A.B.Zamolodchikov, *Higher-Order Integrals of Motion in Two-Dimensional Field Theory with a Broken Conformal Symmetry*, JETP Lett. 46 (1987) 160.

[31] A.B.Zamolodchikov, *Integrals of Motion in Scaling 3-State Potts Model Field Theory*, Int. J. Mod. Phys. A3 (1988) 743.

[32] A.B.Zamolodchikov, *"Irreversibility" of the Flux of the Renormalization Group in a 2D Field Theory*, JETP Lett. 43 (1986) 730.

[33] A.W.W.Ludwig, J.L.Cardy, *Perturbative Evaluation of the Conformal Anomaly at New Critical Points with Applications to Random Systems*, Nucl. Phys. B285 (1987) 687.

[34] P.Fendley, H.Saleur, Al.B.Zamolodchikov, *Massless Flows I: the Sine-Gordon and $O(n)$ Models*, Int. J. Mod. Phys. A8 (1993) 5717.

[35] P.Fendley, H.Saleur, Al.B.Zamolodchikov, *Massless Flows II: the Exact S-Matrix Approach*, Int. J. Mod. Phys. A8 (1993) 5751.

[36] A.B.Zamolodchikov, Al.B.Zamolodchikov, *Factorized S Matrices in Two Dimensions as the Exact Solution of Certain Relativistic Quantum Field Models*, Ann. Phys. 120 (1979) 253.

[37] H.W.Braden, E.Corrigan, P.E.Dorey, R.Sasaky, *Affine Toda Field Theory and Exact S-Matrices*, Nucl. Phys. B338 (1990) 689.

[38] A.B.Zamolodchikov, *Higher-Order Integrals of Motion in Two-Dimensional Models with a Broken Conformal Symmetry*, JETP Lett. 46 (1987) 160.

[39] K.Schoutens, A.Sevrin, P.v.Nieuwenhuizen, *Towards a Theory of $W_3$ Gravity*, Proceedings of the Argonne Workshop on Quantum Groups, May 1990, World Scientific.

[40] E.Witten, *Non-Abelian Bosonization in 2 Dimensions*, Commun. Math. Phys. 92 (1984) 455.

[41] S.P.Novikov, Usp. Mat. Nauk. 37 (1982) 3.

[42] D.Gepner, E.Witten, *String Theory on Group Manifolds*, Nucl. Phys. B278 (1986) 493.

[43] A.Cappelli, C.Itzykson, J.-B.Zuber, *Modular Invariant Partition Functions in Two Dimensions*, Nucl. Phys. B280 (1987) 445.

[44] V.G.Knizhnik, A.B.Zamolodchikov, *Current Algebra and Wess-Zumino Model in Two Dimensions*, Nucl. Phys. B247 (1984) 83.

[45] A.B.Zamolodchikov, V.A.Fateev, *Operator Algebra and Correlation Functions in the Two-Dimensional* $SU(2) \times SU(2)$ *Chiral Wess-Zumino Model*, Yad. Fiz. 43 (1986) 1031.

[46] P.Mejean, F.A.Smirnov, *Form Factors for Principal Chiral Field Model with Wess-Zumino-Novikov-Witten Term*, Int. J. Mod. Phys. A12 (1997) 3383.

[47] Y.Y.Goldschmidt, E. Witten, *Conservation Laws in Some Two-Dimensional Models*, Phys. Lett. B91 (1980) 392.

[48] M.C.B.Abdalla, *Integrability of Chiral Nonlinear Sigma Models with a Wess-Zumino Term*, Phys. Lett. B152 (1985) 215.

[49] A.M.Polyakov, P.B.Wiegmann, *Goldstone Fields in Two Dimensions with Multivalued Actions*, Phys. Lett. B141 (1984) 223.

[50] A.M.Polyakov, *Hidden Symmetry of the Two-Dimensional Chiral Fields*, Phys. Lett. B72 (1977) 224.

[51] A.M.Polyakov, P.B.Wiegmann, *Theory of Goldstone Bosons in 2 Dimensions*, Phys. Lett. B131 (1983) 121.

[52] A.N.Kirillov, F.A.Smirnov, *A Representation of the Current Algebra Connected with the* $SU(2)$*-Invariant Thirring Model*, Phys. Lett. B198 (1987) 506.

[53] P.Goddard, A.Kent, D.Olive, *Virasoro Algebras and Coset Space Models*, Phys. Lett. B152 (1984) 88.

[54] P.Goddard, D.Olive, *Kac-Moody Algebras, Conformal Symmetry, and Critical Exponents*, Nucl. Phys. B257 (1985) 226.

[55] P.Goddard, A.Kent, D.Olive, *Unitary Representations of the Virasoro Algebra and Super-Virasoro Algebras*, Commun. Math. Phys. 103 (1986) 105.

[56] B.Berg, M.Karowski, P.Weisz, V.Kurak, *Factorized U(n) Symmetric S Matrices in Two Dimensions*, Nucl. Phys. B134 (1978) 125.

[57] A.B.Zamolodchikov, Al.B.Zamolodchikov, *Massless Factorized Scattering and Sigma Models with Topological Terms*, Nucl. Phys. B379 (1992) 602.

[58] J.L.Cardy, *Conformal Invariance and Surface Critical Behavior*, Nucl. Phys. B240 (1984) 514.

[59] J.L.Cardy, *Effect of Boundary Conditions on the Operator Content of Two-Dimensional Conformally Invariant Theories*, Nucl. Phys. B275 (1986) 200.

[60] J.L.Cardy, *Boundary Conditions, Fusion Rules and the Verlinde Formula*, Nucl. Phys. 324 (1989) 581.

[61] J.L.Cardy, D.Lewellen, *Bulk and Boundary Operators in Conformal Field Theory*, Phys. Lett. B259 (1991) 274.

[62] M.Bauer, H.Saleur, *On Some Relations Between Local Height Probabilities and Conformal Invariance*, Nucl. Phys. B320 (1989) 591.

[63] N.Ishibashi, *The Boundary and Crosscap States in Conformal Field Theories*, Mod. Phys. Lett. A4 (1989) 251.

[64] N.Ishibashi, T.Onogi, *Conformal Field Theories on Surfaces with Boundaries and Crosscaps*, Mod. Phys. Lett. A4 (1989) 161.

[65] E.Verlinde, *Fusion Rules and Modular Transformations in 2D Conformal Field Theory*, Nucl. Phys. B300 (1988) 360.

[66] N.Ishibashi, T.Onogi, *Open String Model Building*, Nucl. Phys. B318 (1989) 239.

[67] A.Yegulalp, *New Boundary Conformal Field Theories Indexed by the Simply-Laced Lie Algebras*, Nucl. Phys. B450 (1995) 641.

[68] S.Ghoshal, A.B.Zamolodchikov, *Boundary S Matrix and Boundary State in Two-Dimensional Integrable Quantum Field Theory*, Int. J. Mod. Phys. A9 (1994) 3841.

[69] S.Ghoshal, *Bound State Boundary S Matrix of the Sine-Gordon Model*, Int. J. Mod. Phys. A9 (1994) 4801.

[70] P.Bowcock, E.Corrigan, P.E.Dorey, R.H.Rietdijk, *Classically Integrable Boundary Conditions for Affine Toda Field Theories*, Nucl. Phys. B445 (1995) 469.

[71] J.D.Kim, *Boundary Reflection Matrix in Perturbative Quantum Field Theory*, Phys. Lett. B353 (1995) 213.

[72] E.Corrigan, *Reflections*, DTP-96/1.

[73] T.R.Klassen, E.Melzer, *Purely Elastic Scattering Theories and Their Ultraviolet Limits*, Nucl. Phys. B338 (1990) 485.

[74] C.N.Yang, C.P.Yang, *One-Dimensional Chain of Anisotropic Spin-Spin Interactions. I. Proof of Bethe's Hypothesis for Ground State in a Finite System*, Phys. Rev. 150 (1966) 321.

[75] C.N.Yang, *Some Exact Results for the Many-Body Problem in One Dimension with Repulsive Delta-Function Interaction*, Phys. Rev. Lett. 19 (1967) 1312.

[76] M.Takahashi, *One-Dimensional Heisenberg Model at Finite Temperature*, Prog. Theor. Phys. 46 (1971) 401.

[77] P.Kulish, N.Reshetikhin, E.Sklyanin, *Yang-Baxter Equation and Representation Theory:I.*, Lett. Math. Phys. 5 (1981) 393.

[78] E.Date, M.Jimbo, T.Miwa, M.Okado, *Fusion of the Eight Vertex SOS Model*, Lett. Math. Phys. 12 (1986) 209.

[79] G.Andrews, R.Baxter, J.Forrester, *Eight Vertex SOS Model and Generalized Rogers-Ramanujan Type Identities*, J. Stat. Phys. 35 (1984) 193.

[80] Al.B.Zamolodchikov, *Thermodynamic Bethe Ansatz in Reletivistic Models: Scaling 3-State Potts and Lee-Yang Models*, Nucl. Phys. B342 (1990) 695.

[81] Al.B.Zamolodchikov, *Thermodynamic Bethe Ansatz for RSOS Scattering Theories*, Nucl. Phys. B358 (1991) 497.

[82] H.Itoyama, P.Moxhay, *Thermodynamic Bethe Ansatz, Sine-Gordon Theory, and Minimal Models*, Proceedings of the Argonne Workshop on Quantum Groups, May 1990, World Scientific.

[83] Al.B.Zamolodchikov, *From Tricritical to Critical Ising by Thermodynamic Bethe Ansatz*, Nucl. Phys. B358 (1991) 524.

[84] Al.B.Zamolodchikov, *On the Thermodynamic Bethe Ansatz Equations for Reflectionless ADE Scattering Theories*, Phys. Lett. B 253 (1991) 391.

[85] Al.B.Zamolodchikov, *TBA Equations for Integrable Perturbed* $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$ *Coset Models*, Nucl. Phys. B366 (1991) 122.

[86] A.LeClair, G.Mussardo, H.Saleur, S.Skorik, *Boundary Energy and Boundary States in Integrable Quantum Field Theories*, Nucl. Phys. B453 (1995) 581.

[87] P.Dorey, A.Pocklington, R.Tateo, G.Watts, *TBA and TCSA with Boundaries and Excited States*, DTP-97-67.

[88] P.Fendley, H.Saleur, *Deriving Boundary S Matrices*, Nucl. Phys. B428 (1994) 681.

[89] V.V.Bazhanov, S.L.Lukyanov, A.B.Zamolodchikov, *Integrable Structure of Conformal Field Theory, Quantum KdV Theory and Thermodynamic Bethe Ansatz*, Commun. Math. Phys. 177 (1996) 381.

[90] V.V.Bazhanov, S.L.Lukyanov, A.B.Zamolodchikov, *Integrable Structure of Conformal Field Theory II. Q-Operator and DDV Equation*, Commun. Math. Phys. 190 (1997) 247.

[91] D.Fioravanti, F.Ravanini, M.Stanishkov, *Generalized KdV and Quantum Inverse Scattering Description of Conformal Minimal Models*, Phys. Lett. B367 (1996) 113.

[92] G.Feverati, E.Quattrini, F.Ravanini, *Infrared Behaviour of Massless Integrable Flows Entering the Minimal Models from* $\phi_{31}$, Phys. Lett. B374 (1996) 64.

[93] P.Fendley, H.Saleur, *Massless Integrable Quantum Field Theories and Massless Scattering in 1+1 Dimensions - Lectures Presented at Strings 93, May 1993 and at the Trieste Summer School on High Energy Physics and Cosmology, July 1993*, USC-93-022.

[94] N.Andrei, K.Furuya, J.H.Lowenstein, *Solution of the Kondo Problem*, Rev. Mod. Phys. 55 (1983) 331.

[95] J.Kondo, *Resistance Minimum in Dilute Manetic Alloys*, Prog. Theor. Phys. 32 (1964) 37.

[96] I.Affleck, *Lectures Given at the XXXVth Cracow School of Theoretical Physics, Zakopane, Poland, June 1995*, Acta Phys. Polon. B26 (1995) 1869.

[97] I.Affleck, A.W.W.Ludwig, *Universal Noninteger "Ground-State Degeneracy" in Critical Quantum Systems*, Phys. Rev. Lett. 67 (1991) 161.

[98] P.Fendley, *Kinks in the Kondo Problem*, Phys. Rev. Lett. 71 (1993) 2485.

[99] J.Prata, *Boundary States and Boundary Entropy for Sigma Models with Topological Terms*, in preparation.

[100] J.M.Evans, J.O.Madsen, *Integrability vs. Supersymmetry*, Phys. Lett. B389 (1996) 665.

[101] G.Papadopoulos, *Supersymmetric Toda Field Theories*, Phys. Lett. B365 (1996) 98.

[102] D.Sorokin, F.Toppan, *An* $n = (1|1)$ *Super-Toda Model Based on* $Osp(1|4)$, Lett. Math. Phys. 42 (1997) 139.

[103] R.Shankar, E.Witten, *S Matrix of the Supersymmetric Nonlinear* $\sigma$ *Model*, Phys. Rev. D17 (1978) 2134.

[104] V.G.Kac, Adv. Math. 30 (1978) 85.

[105] D.Friedan, Z.Qiu, S.Shenker, *Superconformal Invariance in Two Dimensions and the Tricritical Ising Model*, Phys. Lett. B151 (1985) 37.

[106] T.J.Hollowood, E.Mavrikis, *The N=1 Supersymmetric Bootstrap and Lie Algebras*, Nucl. Phys. B484 (1997) 631.

[107] M.Moriconi, K.Schoutens, *Thermodynamic Bethe Ansatz for N=1 Supersymmetric Theories*, Nucl. Phys. B464 (1996) 472.

[108] M.Moriconi, K.Schoutens, *Reflection Matrices for Integrable N=1 Supersymmetric Theories*, Nucl. Phys. B487 (1997) 756.

[109] J.Prata, *The Super-Liouville Theory on the Half-Line*, Phys. Lett. B405 (1997) 271.

[110] B.A.Kuperschmidt, *A Super Korteweg-de Vries Equation: an Integrable System*, Phys. Lett. A102 (1984) 213.

[111] B.A.Kuperschmidt, *Super Korteweg-de Vries Equations Associated to Super Extensions of the Virasoro Algebra*, Phys. Lett. A109 (1985) 417.

[112] H.C.Liao, P.Mansfield, *Light-Cone Quantization of the Super-Liouville Theory*, Nucl. Phys. B344 (1990) 696.

[113] J.Schulze, *Coulomb Gas on the Half Plane*, Nucl. Phys. 489 (1997) 580.

[114] T.Inami, S.Odake, Y.-Z.Zhang, *Supersymmetric Extension of the Sine-Gordon Theory With Integrable Boundary Interactions*, Phys. Lett. B359 (1995) 118.

[115] G.W.Delius, M.T.Grisaru, S.Penati, D.Zanon, *The Exact S Matrices of Affine Toda Theories Based on Lie Superalgebras*, Phys. Lett. B256 (1991) 164.

[116] M.T.Grisaru, S.Penati, D.Zanon, *Mass Corrections in Affine Toda Theories Based on Lie Superalgebras*, Phys. Lett. B252 (1991) 357.

[117] H.C.Liao, P.Mansfield, *Solution of N=2 Superconformal Liouville Theory*, Phys. Lett. B255 (1991) 230.

[118] M.A.Olshanetsky, *Supersymmetric Two-Dimensional Toda Lattice*, Commun. Math. Phys. 88 (1983) 63.

[119] J.Evans, T.Hollowood, *Integrable N=2 Supersymmetric Field Theories*, Phys. Lett. B293 (1992) 100.

[120] J.-L.Gervais, *Infinite Family of Polynomial Functions of the Virasoro Generators With Vanishing Poisson Brackets*, Phys. Lett. B160 (1985) 277.

[121] S.Ferrara, L.Girardello, S.Sciuto, *An Infinite Set of Conservation Laws of the Supersymmetric Sine-Gordon Theory*, Phys. Lett. B76 (1978) 303.

[122] L.Girardello, S.Sciuto, *Inverse Scattering-Like Problem for Supersymmetric Models*, Phys. Lett. B77 (1978) 267.

[123] M.Chaichian, P.P.Kulish, *On the Method of Inverse Scattering Problem and Bäcklund Transformations for Supersymmetric Equations*, Phys. Lett. 78 (1978) 413.

[124] E.D'Hoker, *Classical and Quantal Supersymmetric Liouville Theory*, Phys. Rev. D28 (1983).

[125] P.Mathieu, G.Watts, *Probing Integrable Perturbations of Conformal Theories Using Singular Vectors II: N=1 Superconformal Theories*, KCL-MTH-97-41.

[126] J.Prata, *Super-KdV Like Equations, Super-Coulomb Gas and Perturbed Superconformal Theories*, in preparation.