Density Perturbations in Hybrid Inflation

by

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Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Master of Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2009

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Abstract

Inflation is a substantial modification to the big bang theory, and supernatural inflation is a hybrid inflation motivated from supersymmetry. In this thesis we carry out a one dimensional numerical simulation to verify the untested analytic approximation of Radall et al. The results show a good agreement for a wide range of parameters. We also propose a new method for calculating density perturbations in hybrid inflation, which shows an excellent agreement with the simulation in one dimension.

Thesis Supervisor: Alan H. Guth
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Acknowledgments

I first wish to thank my advisor Alan Guth for his invaluable help on both physics and non-physics matters. It has always been a pleasure and honor to work with him. I wish to thank Hong Liu and Xingang Chen for their knowledge and advices. I wish to thank my collaborator Carlos Santana for his patience and enthusiasm. I am honored to be a fellow of the Vietnam Education Foundation. And last but not least, I wish to thank my parents Nguyen Trung Thuan and Doan Thi Phong for their endless support and encouragement, my companion Dang Thi La Thu for her understanding and sharing, my undergraduate research advisor Nguyen Quoc Khanh for leading me through the very first steps of scientific research, and all friends and teachers that I have been lucky enough to have.

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Chapter 1

Hybrid Inflation

Inflation is the epoch when the energy density of the universe is dominated by the potential energy of a scalar field. During this epoch, the scalar field is either trapped in a false vacuum or slowly rolling down a hill, the energy density of the universe is approximately constant, and the scale factor is exponentially growing with time.

Inflation was first introduced to solve the problem of magnetic monopoles predicted by Grand Unified Theories[3]. By adding an epoch of exponential expansion, any unwanted relic produced before inflation will be diluted away. It was shown later that inflation also provides an explanation for the flatness and the horizon problems of the Standard Model of Cosmology. Recent measurements confirmed the prediction of inflation for scale-invariant density perturbations and a flat universe.

However, inflation does face the naturalness problem: most inflation potentials have small parameters, either to have the correct order of magnitude for the density perturbations or to have enough e-foldings. One way to get over this difficulty is to construct an inflation potential with small parameters that already existed in particle physics so that we do not need to introduce new ones. This is the motivation for Supernatural Inflation[7], a hybrid inflation model with a potential motivated from supersymmetry. As with other hybrid inflation models, supernatural inflation does not have a classical solution for the evolution of the scalar field, to which quantum fluctuations can be treated as small perturbations. The authors of Supernatural inflation suggested using root mean square of the scalar field as the classical value.

We will validate the approximation by comparing with a Monte Carlo simulation. Kristin Burgess in her Ph.D. thesis [2] has reached the conclusion that the analytic approximation and the simulation are in good agreement provided that we rescale both the amplitude and the wave number of the analytic calculation for density perturbation by two factors of order 1. The aim of this thesis is to go one step further by investigating the dependence of these factors on the model parameters.

1.1 Classical evolution

We will consider a scalar field in a fixed background de-Sitter space, with the scale factor

$$a(t) = e^{Ht} \tag{1.1}$$

where H is the Hubble constant during inflation. The scalar field ϕ is described by the Lagrangian density

$$\mathcal{L}_\phi = e^{Ht}\left[\frac{1}{2}|\dot{\phi}|^2 - \frac{1}{2}e^{-2Ht}|\nabla\phi|^2 - \frac{1}{2}m_\phi^2(t)|\phi|^2\right] \tag{1.2}$$

In a typical single field inflation the mass term would be time-independent. In a hybrid inflation, however, the mass term is controlled by the second scalar field which acts as the switch to end inflation. We will consider the mass term of the form:

$$m_\phi^2(t) = -m_0^2\left[1 - \left(\frac{\psi(t)}{\psi_c}\right)^r\right] \tag{1.3}$$

We will set $r = 4$ in the simulation. The second scalar field is described by another Lagrangian density but with fixed mass

$$\mathcal{L}_\psi = e^{Ht}\left[\frac{1}{2}\dot{\psi}^2 - \frac{1}{2}e^{-2Ht}(\nabla\psi)^2 - \frac{1}{2}m_\psi^2\psi^2\right] \tag{1.4}$$

Note that the Lagrangian densities are defined up to additive constants. These constants will be defined to have sufficient values for negligible variation of H during inflation.

The equations of motion are

$$\ddot{\phi} + 3H\dot{\phi} - e^{-2Ht}\nabla^2\phi = -m_\phi^2(t)\phi \tag{1.5}$$

$$\ddot{\psi} + H\dot{\psi} - e^{-2Ht}\nabla^2\psi = -m_\psi^2\psi \tag{1.6}$$

Using the slow-roll approximation for the ψ field, we get the evolution as:

$$\psi(t) = \psi_c e^{-(m_\psi^2/H)t} \tag{1.7}$$

The integration constant was chosen so that $\psi = \psi_c$ and $m_\phi^2(t) = 0$ at $t = 0$. The mass term for ϕ becomes

$$m_\phi^2 = -m_0^2\left[1 - e^{-r\mu_\psi^2 N}\right] \tag{1.8}$$

where we have defined

$$N \equiv Ht$$

$$\mu_\psi \equiv m_\psi/H \tag{1.9}$$

With the mass term (1.8), the scalar field ϕ has two separate stages of evolution

- Oscillation $N < 0$, $m_\phi^2 > 0$: The ϕ field is trapped at its local minimum. Classically, there will be no fluctuation, and ϕ will stay at the same value even when the potential flipped. Quantum mechanically, ϕ will oscillate around the minimum, and those quantum fluctuations will provide the initial deviations for rolling down in the tachyon stage.

- Tachyon $N > 0$, $m_\phi^2 < 0$: potential flipped, ϕ has negative effective squared mass and will roll down the potential hill until reaching a new minimum.

In our toy model, which is constructed as a free field theory, the ϕ field will roll down forever. For ending inflation, instead of reaching a true minimum, the ϕ field will reach an ending value ϕ_{end}.

1.2 Expansion in Modes

We will approximate space by a lattice so that the problem can be solved numerically To make the calculations as efficient as possible, we work in one space dimension. Although we are really interested in three dimensions, the one-dimensional model will allow us to test the analytic approximation that will be described in Section 2.3. Assume, therefore, that the space consists of a box of fixed coordinate length b, with periodic boundary conditions with Q independent points. In the computer program, the number of points Q will be chosen as a power of 2 ($Q = 2^{17}$) so that we can use the Fast Fourier Transform algorithm to speed up the calculation.

$$x = \frac{b}{Q} l \tag{1.10}$$

where $l = 0, 1, ..., (Q-1)$. And the wave number is

$$k = \frac{2\pi}{b} n \tag{1.11}$$

with n running from $-Q/2$ to $(Q/2)-1$. Because of the periodic boundary condition, only one value of k on the boundary is needed.

The mode expansion of the scalar field is

$$\begin{aligned} \phi(x,t) &= \frac{1}{\sqrt{2\pi}} \left[\frac{2\pi}{b}\right]^{1/2} \sum_{n=-Q/2}^{(Q/2)-1} \left[c(k)e^{ikx}u(k,t) + d^\dagger(k)e^{-ikx}u^*(k,t)\right] \\ &= \frac{1}{\sqrt{b}} \sum_{n=-Q/2}^{(Q/2)-1} e^{ikx} \left[c(k)u(k,t) + d^\dagger(-k)u^*(-k,t)\right] \end{aligned} \tag{1.12}$$

where $c(k)$ and $d^\dagger(k)$ are creation and annihilation operators. The canonical commutation relation of the scalar field is

$$[\phi(x,t)\pi(x',t)] = i\delta_{x,x'} \tag{1.13}$$

where

$$\pi(x) = \frac{\partial L}{\partial \dot{\phi}(x)} \tag{1.14}$$

Using the normalization convention of $u(k,t)$ as in [4] and [2], we can get the commutation relations for the creation and annihilation operators as

$$[c(k)c^\dagger(k')] = [d(k)d^\dagger(k')] \;=\; \delta_{k,k'} \tag{1.15}$$

$$[(k), d^\dagger(k')] = [c(k)c(k')] \;=\; 0 \tag{1.16}$$

Substituting the mode expansion back into the equation of motion, we get the equation for the modes

$$\ddot{u} + H\dot{u} + e^{-2Ht}\vec{k}^2 u = -m_\phi^2(t)u \tag{1.17}$$

Define new function and variable as

$$\tilde{k} \;\equiv\; \frac{|\vec{k}|}{H} \tag{1.18}$$

$$u(k,t) \;\equiv\; \frac{1}{\sqrt{2\tilde{k}H}}R(k,t)e^{i\theta(k,t)} \tag{1.19}$$

then the equation of motion becomes

$$\ddot{R} \;=\; -\dot{R} + R[\mu_\phi^2(1 - e^{-\tilde{\mu}_\psi^2 N}) - \tilde{k}^2 e^{-2N}] + \frac{\tilde{k}^2 e^{-2N}}{R^3} \tag{1.20}$$

$$\dot{\theta} \;=\; -\frac{\tilde{k}e^{-N}}{R^2} \tag{1.21}$$

At early time, the $\vec{k}$ term in equation (1.17) dominates over the $m_\phi(t)$ term and the solution will be [4]

$$u(\vec{k},t) = \frac{1}{2}\sqrt{\frac{\pi}{H}}e^{-N/2}Z(z) \tag{1.22}$$

where

$$z \equiv \frac{k}{H}e^{-N} \tag{1.23}$$

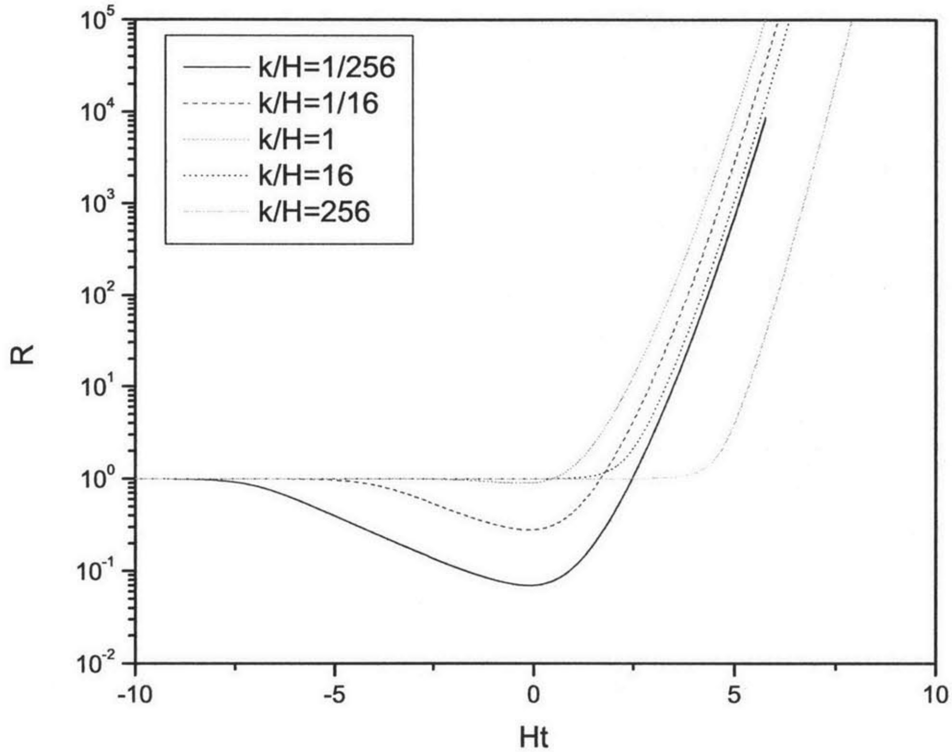

Figure 1-1: Mode function for k/H =1/256, 1/16, 1, 16, 256

and $Z(z)$ satisfies

$$z^2\frac{d^2Z}{dz^2} + z\frac{dZ}{dz} + \left(z^2 - \frac{1}{4}\right) z = 0 \tag{1.24}$$

At early time, the mode functions have the form

$$u(\vec{k}, t) \sim \frac{1}{\sqrt{2H\tilde{k}}} e^{ike^{-N}} \tag{1.25}$$

or in term of R and θ

$$R \rightarrow 1 \tag{1.26}$$

$$\theta \rightarrow \tilde{k}e^{-N} \tag{1.27}$$

These equations will be used as the initial conditions for the differential equations (1.20) and (1.21). For numerical calculation, given the values of $R(k,t)$ and $\dot{R}(k,t)$, we can always use the Runge Kutta method to evaluate the values of $R(k, t+dt)$ and $\dot{R}(k, t+dt)$. With the step size $dt = 5 \times 10^{-4}$ in units with $H \equiv 1$, the error in $\Delta R/R$ is less than 10^{-8} for the entire rage of t. Some results from these integrations are shown in Fig. 1-1. We can see that all the mode functions increase exponentially at the same rate at late time, which is consistent with the asymptotic form of the mode equations when $N \rightarrow \infty$

$$\ddot{R} \rightarrow -\dot{R} + R[\mu_\phi^2(1 - e^{-\tilde{\mu}_\psi^2 N})] \tag{1.28}$$

$$\dot{\theta} \rightarrow 0 \tag{1.29}$$

.

Chapter 2

Time Delay Method

Time delay is a method of calculating the primordial density perturbation[5],[6],[8]. At the end of inflation, the inflation field rolls down a hill in its potential energy diagram toward its true vacuum value. Because of the quantum fluctuations, some regions may reach the end of inflation earlier than others, and therefore have less inflation.

This method has the advantage of being simple and intuitive, but it also has some limitations:

- Valid only if $H \sim$const during inflation.
- Valid only for single field inflation. However, we will make an attempt to use it for hybrid inflation.
- Requires an assumption of instantaneous ending of inflation. This approximation can be justified as follows: The characteristic time scale is not the Hubble scale size, but the time for light to travel a wave length of the fluctuation modes. But the modes that we are interested in, observable in the CMB, have already exited the horizon many e-foldings before the end of inflation. So, even if the transition of inflation takes a couple of Hubble times, it is still 20 order of magnitude smaller than the characteristic time scale.

2.1 Time Delay Definition

We will work in the de-Sitter background

$$ds^2 = -dt^2 + a(t)^2 dx^2 \tag{2.1}$$

where

$$a(t) = e^{Ht} \tag{2.2}$$

The equation of motion for the scalar field in a de-Sitter background is

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi}\phi + \frac{1}{a^2}\nabla^2\phi \tag{2.3}$$

where

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} \tag{2.4}$$

The equation of motion (2.3) has a homogeneous classical solution $\phi_0(t)$. The true scalar field is of course a quantum field described by a feld operator, but we assume that at suffciently late times the field is accurately modeled by a classical scalar field which has small perturbations about the homogeneous classical solution. Thus we write

$$\phi(x,t) = \phi_0(t) + \delta\phi(x,t) \tag{2.5}$$

Substituting the above expansion back into (2.3), we have the equation for the perturbation part

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} = -\frac{\partial^2 V}{\partial \phi_0^2}\delta\phi + \frac{1}{a^2}\nabla^2\delta\phi \tag{2.6}$$

We want to understand the behavior of $\delta\phi$ at late time during inflation when the Hubble parameter H is constant. As the scale factor $a(t)$ grows exponentially, the term proportional to $1/a^2$ can be neglected and the equation for $\delta\phi(x,t)$ becomes

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} = -\frac{\partial^2 V}{\partial \phi_0^2}\delta\phi \tag{2.7}$$

If we introduce a new function $\chi(t) \equiv \dot{\phi}_0$, then taking a derivative with respect to time on both sides of the equation for ϕ_0 gives the equation for χ.

$$\ddot{\chi} + 3H\dot{\chi} = -\frac{\partial^2 V}{\partial \phi_0^2}\chi \tag{2.8}$$

where we ignore for the moment the explicit time-dependence that $V(\phi)$ has for the toy model of hybrid inflation that we are studying. With this proviso, equation (2.8) is identical to the equation for $\delta\phi$. This second order differential equation has two independent solutions, and the most general solution is a linear combination of these two solutions with two free parameters. However, we can show that one of the solutions is damped at late time. So, at late time $\delta\phi(x,t)$ is proportional to $\dot{\phi}_0$, and we call the parameter of proportionality $\delta\tau(x)$. Thus,

$$\delta\phi(x,t) \rightarrow -\delta\tau(x)\phi_0(x) \tag{2.9}$$

The scalar field can now be rewritten as

$$\begin{aligned} \phi(x,t) &= \phi_0(t) + \delta(x,t) \\ &= \phi_0(t) - \delta\tau(x)\dot{\phi}_0(t) \\ &= \phi_0(t - \delta\tau(x)) \end{aligned} \tag{2.10}$$

So we can consider the perturbed scalar field at different spatial points as the homogeneous classical solution, but with a spatially-dependent time delay in its evolution.

The above argument, however, is invalidated in our toy model by the explicit time dependence of $V(\phi)$, which leads to an additional term in Eq. (2.8). Nonetheless, the time-dependent mass of Eq. (1.8) approaches the constant $-m_0$ at very late times, $\mu_\psi^2 N \gg 1$, so the time-delay description will still be valid at these very late times.

2.2 Proof That Time Delay Is Fixed

From the definition of time delay (2.9), we will need to show that $\frac{d}{dt}\left(\frac{\delta\phi(x,t)}{\dot{\phi}_0}\right)$ approaches zero at late time

$$\frac{d}{dt}\left(\frac{\delta\phi}{\dot{\phi}_0}\right) = \frac{\delta\dot{\phi}\dot{\phi}_0 - \ddot{\phi}_0\delta\phi}{\dot{\phi}_0^2} \tag{2.11}$$

During inflation the scalar field will roll faster and faster, so one will only need to show that the numerator approaches zero at late time.

Define the Wronskian as the numerator

$$W = \delta\dot{\phi}\dot{\phi}_0 - \ddot{\phi}_0\delta\phi \tag{2.12}$$

It follows that

$$\dot{W} = -3HW \tag{2.13}$$

Solution for the Wronskian is then

$$W \sim e^{-3Ht} \tag{2.14}$$

So the Wronskian falls off exponentially at late time, or in other words $\frac{\delta\phi(x,t)}{\dot{\phi}_0}$ is time-independent at late time.

2.3 Randall-Soljacic-Guth Approximation

The root mean square of the field can be evaluated as

$$\phi_{\text{rms}}(t) = \sqrt{\langle 0|\phi(x,t)\phi^*(x,t)|0\rangle} \tag{2.15}$$

where $|0\rangle$ is the Bunch-Davies vacuum [1]

$$c(k)|0\rangle = d(k)|0\rangle = 0 \tag{2.16}$$

Using the mode expansion (1.12) from previous section, we get

$$\begin{aligned}
\phi_{\rm rms}(t) &= \frac{1}{b}\sum_k\sum_{k'} e^{ikx}e^{-ik'x}\langle 0|\left(c_k u_k + d^\dagger_{-k}u^*_{-k}\right)\left(c^\dagger_{k'}u^*_{k'} + d_{-k'}u_{-k'}\right)|0\rangle \\
&= \frac{1}{b}\sum_k\sum_{k'} e^{i(k-k')x}|u_k|^2\delta_{kk'} \\
&= \frac{1}{b}\sum_k \frac{R(k,t)^2}{2|k|} \qquad (2.17)
\end{aligned}$$

The scalar field ϕ was originally trapped at the local minimum. When the potential flipped over, it would roll down the hill with an initial velocity determined by quantum fluctuations. Without the quantum fluctuations, this scalar field would stay for arbitrarily long time at the point of unstable equilibrium. But once it starts to roll down, its evolution can be treated as classsical, but with inhomogeneities determined by the quantum fluctuations at earlier times. We want to consider this evolution as a perturbation about a classical homogeneous solution. Following the approximation in Randal et. al. [7], we will approximate the homogeneous solution by the root mean square of the quantum field

$$\phi_{\rm classical}(t) = \phi_{\rm rms}(t) \qquad (2.18)$$

The two point correlation function of the scalar field is

$$\begin{aligned}
\Delta\phi(k,t) &= \sqrt{\frac{2\pi k}{b}\langle 0|\phi^*(k,t)\phi(k,t)|0\rangle} \\
&= \frac{R(k,t)}{2\sqrt{\pi}} \qquad (2.19)
\end{aligned}$$

The time-delay field is approximately

$$\Delta\tau(k) \approx \frac{\Delta\phi(k,t)}{\dot{\phi}_{\rm rms}(t)} \qquad (2.20)$$

and evaluated at the end of inflation. The time derivative of the rms (2.17) is

$$\dot{\phi}(k,t) = \frac{1}{\sqrt{2b}}\left(\sum_k \frac{R(k,t)^2}{|k|}\right)^{-1/2}\left(\sum_{k'} \frac{R(k',t)\dot{R}(k',t)}{|k'|}\right) \qquad (2.21)$$

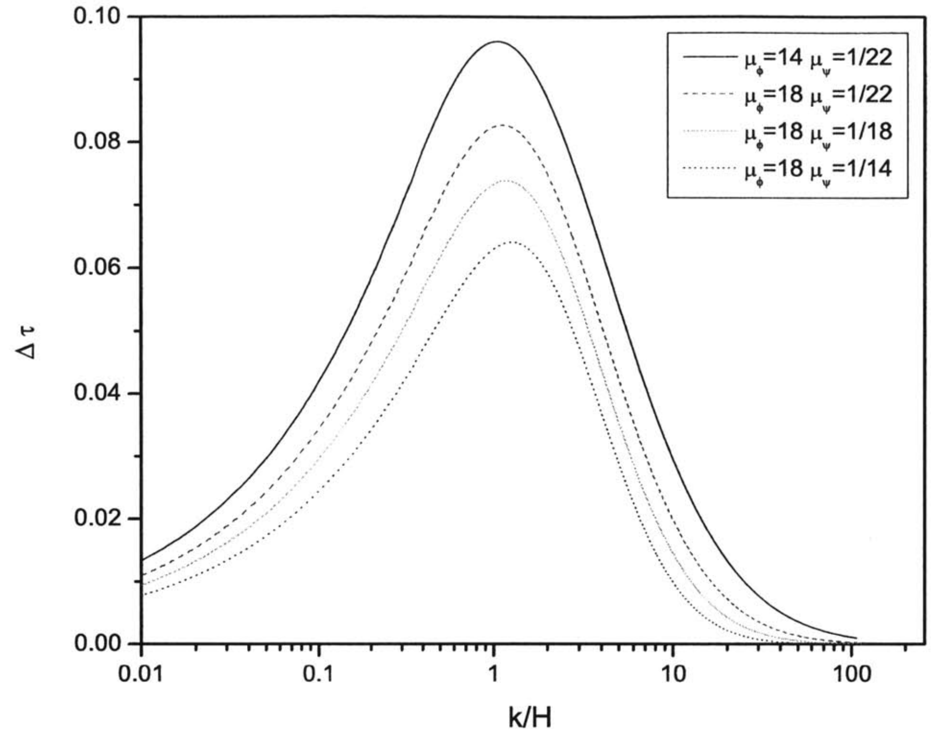

Figure 2-1: RSG approximation for different sets of physical parameters.

Using the mode functions in the previous chapter, we can calculate the mode functions at arbitrary value of k and t. Some examples of the analytic evaluation for the time delay field are shown in Fig. 2-1.

Chapter 3

Numerical simulation

3.1 Field evolution

The scalar field is a quantum field, and its evolution is a quantum process. We will use random numbers to simulate the creation and annihilation operators in (1.12)

$$\begin{aligned} c(k) &= a_1 + ia_2 \\ d^\dagger(k) &= a_3 + ia_4 \end{aligned} \tag{3.1}$$

where a_1, a_2, a_3, and a_4 are random numbers with normal distribution of standard deviation $1/\sqrt{2}$. Each Monte Carlo realization of the vacuum corresponds to a particular set of random numbers. The vacuum expectation will be approximated by taking the average over a large run with many different sets of random numbers. We can see that the numerical process actually simulates the effect of quantum operators.

$$\langle 0|c^\dagger(k)c(k)|0\rangle = \langle (a_1 - ia_2)(a_1 + ia_2)\rangle = \langle a_1^2 + a_2^2\rangle = 2\langle a^2\rangle = 1 \tag{3.2}$$

$$\langle 0|c^\dagger(k)c(k' \neq k)|0\rangle = \langle (a_1 - ia_2)(a_1' + ia_2')\rangle = 0 \tag{3.3}$$

We are considering a toy model in which the scalar field ϕ is a free field with quadratic potential. Therefore, each mode can evolve independently of the others. Once we have initialized a random value for the field, we can simply follow the evolution of that numerical value until the end of inflation.

Having the mode functions at arbitrary time and wave number, we can evaluate the scalar field at arbitrary time and space using the mode expansion. Consider the first component of (1.12)

$$\begin{aligned} \sum_{m=-Q/2}^{Q/2-1} c(k)e^{ikx}u(k,t) &= \sum_{m=-Q/2}^{-1} c(k)e^{ikx}u(k,t) + \sum_{m=0}^{Q/2-1} c(k)e^{ikx}u(k,t) \\ &= \sum_{m=Q/2}^{Q-1} c(k)e^{ikx}u(k-Q,t) + \sum_{m=0}^{Q/2-1} c(k)e^{ikx}u(k,t) \end{aligned} \tag{3.4}$$

where we have used

$$e^{ikx} = e^{i(k-Q)x} \tag{3.5}$$

and

$$k = \frac{2\pi}{b} \tag{3.6}$$

In the computer program, we used the value of b such that the maximum value $k_{\max} = Q\frac{2\pi}{b} = 512$ in the units where $H = 1$. The numerical values of $k_{\max}$ was chosen well beyon the position of the peak of the spectrum and should not have effect on the numerical results.

Similarly for the second term

$$\begin{aligned} \sum_{m=-Q/2}^{Q/2-1} d^{\dagger}(k)e^{-ikx}u^{*}(k,t) &= \sum_{m=-Q/2}^{Q/2-1} d^{\dagger}(k)e^{ikx}u^{*}(-k,t) \\ &= \sum_{m=Q/2}^{Q-1} d^{\dagger}(k)e^{ikx}u^{*}(k-Q,t) + \sum_{m=0}^{Q/2-1} d^{\dagger}(k)e^{ikx}u^{*}(k,t) \end{aligned} \tag{3.7}$$

where we have used $u(k,t) = u(-k,t)$. The mode expansion becomes

$$\phi(x,t) = \frac{2\pi}{b}\left(\sum_{m=0}^{Q/2-1} e^{ikx} f(k,t) + \sum_{m=Q/2}^{Q-1} e^{ikx} f(k-Q,t)\right) \tag{3.8}$$

where

$$f(k,t) = \frac{b}{2\pi}\frac{1}{\sqrt{2kb}} R(k,t)\left[c(k)e^{i\theta(k,t)} + d^{\dagger}(k)e^{-i\theta(k,t)}\right] \tag{3.9}$$

If we define

$$\phi(k,t) = \begin{cases} f(k,t) & m = 0 \ldots Q/2-1 \\ f(k-Q,t) & m = Q/2 \ldots Q-1 \end{cases} \tag{3.10}$$

then

$$\phi(x,t) = \frac{2\pi}{b}\sum_{m=0}^{Q-1} e^{ikx}\phi(k,t) \tag{3.11}$$

This equation will be used in the computer program to calculate the ramdom realizations of the scalar field in x space.

3.2 The ending time

The condition for ending inflation is

$$\phi(x, t_{\text{end}}(x)) = \phi_{\text{end}} \tag{3.12}$$

At each value of x, we solve the above equation and get $t_{\text{end}}(x)$, the inflation-ending time for each value of x. The strength of fluctuations in the time delay $t_{\text{end}}(x)$ at a given wave number k can be measured by

$$\Delta\tau(k) = \left(\frac{2\pi k}{b}\langle t^{*}_{\text{end}}(k) t_{\text{end}}(k)\rangle\right)^{1/2} \tag{3.13}$$

where $t_{\text{end}}(k)$ is the Fourier transform of $t_{\text{end}}(x)$

$$t_{\text{end}}(k) = \frac{1}{2\pi}\frac{b}{Q}\sum_{x} e^{-ikx} t_{\text{end}}(x) \tag{3.14}$$

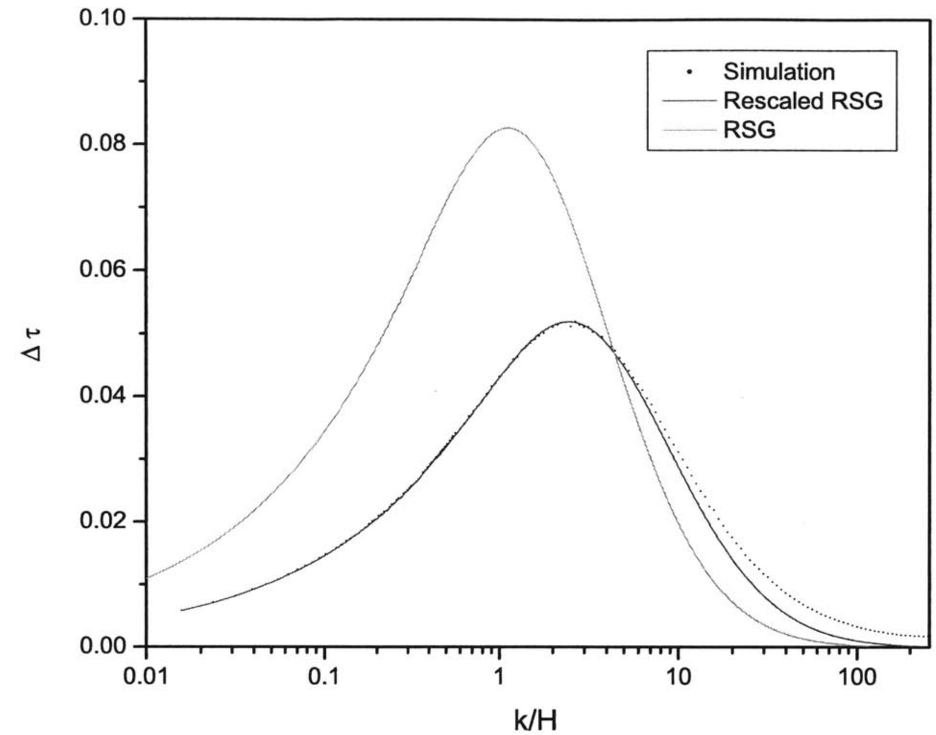

Figure 3-1: Compare the RSG approximation with the Monte Carlo simulation. The rescaling parameters are $k_{\text{rescaled}} = 2.2k$, $\Delta\tau_{\text{rescaled}} = 0.63\Delta\tau$.

The bracket in (3.13) is the vacuum expectation, or in the simulation it is average over many different sets of random numbers.

3.3 Monte Carlo Simulation

As we have discussed in the previous chapters, the quantum fluctuations set the initial values at the top of the hill for the scalar field to roll down. Without the quantum fluctuations, the field would stay at the unstable equilibrium for an arbitrary long time. If we look at an arbitrary time-slice when the field is rolling down, the fluctuations are at the same order of magnitude as the field itself.

In our simulations, for each set of physical parameters, we run the simulation for 10000 times, which is sufficient to have the accuracy better than 10^{-2} when averaging

μ_ψ	μ_ϕ	A	B	$\delta\tau_{\max}$	t_{end}
1/22	18	0.6280	2.203	0.052	12.79
1/33	27	0.6280	2.203	0.052	12.71
1/55	45	0.6280	2.203	0.051	12.68
1/110	90	0.6280	2.203	0.051	12.66

Table 3.1: Change the physical parameters μ_ϕ and μ_ψ with fixed product $\mu_\psi\mu_\phi$

for the vacuum expectation. As we can see in Fig. 3-1, the RSG approximation is at the same order of magnitude as the Monte Carlo simulation. Burgess has pointed out in her thesis that the RSG approximation would be in very good agreement with the simulation if we rescale $\Delta\tau$ and shift k, each by a factor of order 1. To find these rescaling factors, we first define a rescaled-RSG time delay field as a function with two parameters:

$$\Delta\tau_{\text{rescaled}}(k) = A\Delta\tau_{\text{RSG}}(Bk) \tag{3.15}$$

The total distance between this rescaled function and the simulation value can be calculated as

$$D(A,B) = \sum_k [\Delta\tau_{\text{simulation}}(k) - \Delta\tau_{\text{rescaled}}(k)]^2 \tag{3.16}$$

Then we will find the values of A and B such that $D(A,B)$ is minimized.

To understand the behavior of the rescaling, we ran the simulation for a wide range of physical parameters m_ϕ and m_ψ. It turned out that the amplitude rescaling A was almost constant, stayed at 0.63 in the entire range of examined parameters, while the wave number shifting factor B was more sensitive to the physical parameters.

We also noticed that not only the rescaling factors, but also the peak of the time delay field and the time of ending inflation would be constant if we vary both two physical parameters μ_ψ and μ_ϕ in such a way that their product is constant as in Table 3.1. This can be explained as the mass dependence of the equation of motion for the mode functions

$$\mu_\phi^2(1 - e^{-\mu_\psi^2 N}) \approx \mu_\phi^2\left(-\mu_\psi^2 N + \frac{1}{2}(\mu_\psi^2 N)^2 + \cdots\right) \tag{3.17}$$

k	Burgess' simulation	Current Simulation
0.0156	0.0059	0.0058
0.0234	0.007	0.0071
0.0312	0.0083	0.0082
0.0935	0.0143	0.0143
0.125	0.0165	0.0164
0.382	0.0282	0.0283
0.502	0.0324	0.0321
0.666	0.0364	0.0365
0.88	0.0409	0.0412
1.52	0.0488	0.049
2.01	0.0512	0.0512
2.66	0.0517	0.0517
3.51	0.0499	0.0501
4.63	0.0466	0.0462
6.10	0.0416	0.0414
8.05	0.0358	0.0356
10.6	0.0298	0.0302
14.0	0.024	0.0239
18.5	0.0189	0.0187
24.4	0.0146	0.0144
32.2	0.0111	0.0109

Table 3.2: Compare our simulation results with Burgess' for $\mu_\phi = 18$ and $\mu_\psi = 1/22$.

$\mu_\psi \mu_\phi$	A	B	$\delta\tau_{\max}$	t_{end}
0.450	0.629	2.821	0.0716	19.6
0.600	0.628	2.448	0.0617	15.9
0.818	0.628	2.203	0.0520	12.8
1.00	0.627	2.09	0.0460	11.1
1.28	0.626	1.982	0.0401	9.35
1.80	0.623	1.876	0.0331	7.45
3.00	0.620	1.770	0.0234	5.19
4.00	0.618	1.727	0.0192	4.24
5.00	0.617	1.704	0.0165	3.63
8.00	0.615	1.667	0.0120	2.63
11.0	0.614	1.649	0.00967	2.11
15.0	0.613	1.635	0.00784	1.71
20.0	0.613	1.628	0.00646	1.41
30.0	0.613	1.622	0.00492	1.07
50.0	0.613	1.614	0.00350	0.76

Table 3.3: Change the physical parameters with fixed ratio μ_ϕ / μ_ψ.

is a function of $\mu_\phi \mu_\psi$ when $N \approx 0$. So the density perturbations are determined by the initial fluctuations of the field ϕ at the time when the potential flips sign.

3.4 The Direct Integration Method

The Monte Carlo simulation can accurately describe the evolution of the scalar fied and evaluate the time delay. However, the amount of numerical work makes it extremely difficult to expand to a realistic 3-dimensional system. In this section we will introduce a new method of evaluating the time delay field with excellent agreement with the Monte Carlo simulation in 1-dimensional system.

As we have discussed at the end of chapter 2, the equations for the asymptotic behavior of the mode functions were (1.28) and (1.29). So we can write

$$u(k, t \to \infty) \sim e^{\lambda t} u(k) \tag{3.18}$$

where λ can be defined as

$$\lambda \equiv \frac{\dot{\phi}_{\text{rms}}(t)}{\phi_{\text{rms}}(t)} \tag{3.19}$$

Then we have the approximate expression for the scalar field at late time is

$$|\phi(x,t)|^2 = |\phi(x,t_0)|^2 e^{2\lambda H \delta t} \tag{3.20}$$

where $t = t_0 + \delta t$ and t_0 satisfies

$$\phi^2_{\text{rms}}(t_0) = \phi^2_{\text{end}} \tag{3.21}$$

Solving for the time of ending inflation, we get

$$\delta t(x) = -\frac{1}{2\lambda} \log \left(\frac{|\phi(x,t_0)|^2}{\phi^2_{\text{rms}}(t_0)} \right) \tag{3.22}$$

If we rescale the scalar field by its root mean square $\tilde{\phi}(x,t) = \frac{\phi(x,t)}{\phi_{\rm rms}(t)}$ then

$$\delta t = -\frac{1}{2\lambda} \log |\tilde{\phi}(x,t_0)|^2 \tag{3.23}$$

We will evaluate the time-delay field in x space

$$\langle \delta t(x) \delta t(0) \rangle = \frac{1}{4\lambda^2} \langle \log |\tilde{\phi}(x,t_0)|^2 \log |\tilde{\phi}(0,t_0)|^2 \rangle \tag{3.24}$$

where the bracket $\langle\rangle$ denotes the vacuum expectation. Rewrite the complex scalar field as

$$\begin{aligned} \tilde{\phi}(x,t) &= X_1 + iX_2 & (3.25) \\ \tilde{\phi}(0,t) &= X_3 + iX_4 & (3.26) \end{aligned}$$

where X_i's are real. The vacuum expectation can be written as the integration over the jointly Gaussian distribution of four random variables,

$$\langle F[X] \rangle = \int dX \frac{1}{(2\pi)^2 \sqrt{\det \Sigma}} \exp\{-\frac{1}{2} X^T \Sigma^{-1} X\} F[X] \tag{3.27}$$

where

$$\begin{aligned} dX &= dX_1 dX_2 dX_3 dX_4 & (3.28) \\ \Sigma_{ij} &= \langle X_i X_j \rangle & (3.29) \end{aligned}$$

The mode expansion of the scalar field $\phi(x,t)$ can be written as

$$\begin{aligned} X_1 &= \frac{1}{2} \left[\tilde{\phi}(x) + \tilde{\phi}^*(x) \right] & (3.30) \\ X_2 &= \frac{1}{2i} \left[\tilde{\phi}(x) - \tilde{\phi}^*(x) \right] & (3.31) \end{aligned}$$

We can get the components of the matrix Σ as

$$\Sigma_{11} = \Sigma_{22} = \frac{1}{2}\langle\tilde{\phi}(x)\tilde{\phi}^*(x)\rangle = \frac{1}{2} \tag{3.32}$$

$$\Sigma_{12} = \Sigma_{21} = \frac{1}{4i}\langle\tilde{\phi}(x)\tilde{\phi}(x) + \tilde{\phi}^*(x)\tilde{\phi}^*(x)\rangle = 0 \tag{3.33}$$

In the end, we will get the variance matrix as

$$\Sigma = \begin{bmatrix} \frac{1}{2} & 0 & \Delta & 0 \\ 0 & \frac{1}{2} & 0 & \Delta \\ \Delta & 0 & \frac{1}{2} & 0 \\ 0 & \Delta & 0 & \frac{1}{2} \end{bmatrix} \tag{3.34}$$

where

$$\Delta = \langle X_1 X_3\rangle = \langle X_2 X_4\rangle = \frac{1}{2b}\sum_k |\tilde{u}(k, t_0)|^2 e^{ikx} \tag{3.35}$$

and

$$\tilde{u}(k,t) = \frac{u(k,t)}{\phi_{\text{rms}}(t)} \tag{3.36}$$

Equation (3.24) becomes

$$\langle \delta t(x)\delta(0)\rangle = \frac{1}{4\lambda^2}\int \frac{dX_1 dX_2 dX_3 dX_4}{(2\pi)^2[\frac{1}{4} - \Delta^2]} \log(X_1^2 + X_2^2)\log(X_3^2 + X_4^2)$$
$$\exp\{-\frac{1}{4[\frac{1}{4} - \Delta^2]}[X_1^2 + X_2^2 + X_3^2 + X_4^2 - 4(X_1X_3 + X_2X_4)\Delta]\} \tag{3.37}$$

Changing the variables to polar coordinates

$$X_1 = r_1 \cos\theta_1 \tag{3.38}$$

$$X_2 = r_1 \sin\theta_1 \tag{3.39}$$

$$X_3 = r_2 \cos\theta_2 \tag{3.40}$$

$$X_4 = r_2 \sin\theta_2 \tag{3.41}$$

then

$$\langle \delta t(x)\delta(0)\rangle = \frac{2\pi}{\lambda^2}\int_0^{2\pi} d\theta \int_0^{\infty} r_1 dr_1 \int_0^{\infty} r_2 dr_2 \frac{1}{(2\pi)^2[\frac{1}{4}-\Delta^2]}$$
$$\exp\{-\frac{1}{4[\frac{1}{4}-\Delta^2]}[r_1^2+r_2^2-4\Delta r_1 r_2\cos\theta]\}\log(r_1)\log(r_2) \tag{3.42}$$

where $\theta = \theta_1 - \theta_2$. Changing variables again

$$r_1 = r\cos\phi \tag{3.43}$$
$$r_2 = r\sin\phi \tag{3.44}$$

equation (3.37) becomes

$$\langle \delta t(x)\delta(0)\rangle = \frac{1}{(2\pi)[\frac{1}{4}-\Delta^2]\lambda^2}\int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi\cos\phi\sin\phi \int_0^{\infty} r^3 dr\log(r\cos\phi)\log(r\sin\phi)$$
$$\exp\{-\frac{1}{4[\frac{1}{4}-\Delta^2]}[r^2(1-4\Delta\cos\phi\sin\phi\cos\theta)]\} \tag{3.45}$$

The integration over r can be done analytically

$$\int_0^{\infty} r^3 dr\log(ar)\log(br)\exp(-cr^2)$$
$$= \frac{1}{8c^2}\left[(\gamma-2)\gamma+\frac{\pi^2}{6}-2\log ab(\gamma-1+\log c)+4\log a\log b+\log c(2\gamma-2+\log c)\right] \tag{3.46}$$

where

$$a \equiv \cos\phi \tag{3.47}$$
$$b \equiv \sin\phi \tag{3.48}$$
$$c \equiv \frac{1}{4[\frac{1}{4}-\Delta^2]}(1-4\Delta\cos\phi\sin\phi\cos\theta) \tag{3.49}$$
$$\gamma \equiv 0.577215664901532 \tag{3.50}$$

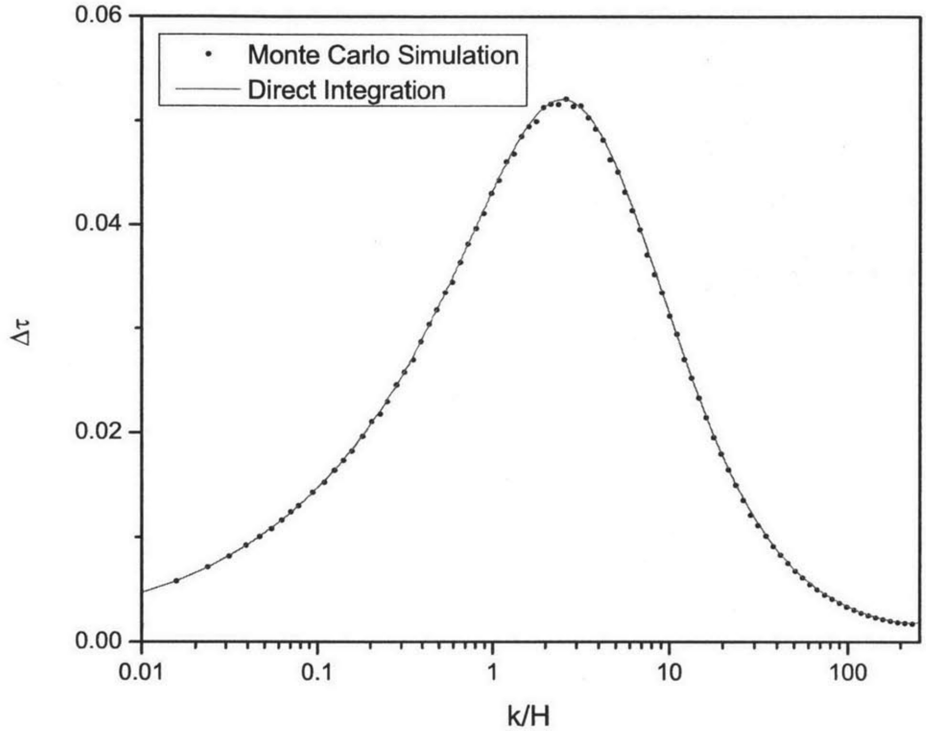

Figure 3-2: Compare the direct integration method with the Monte Carlo simulation. The parameters are $\mu_\phi = 18$ and $\mu_\psi = 1/22$.

Consider the special case

$$\langle \delta t(0)\delta t(0)\rangle = \frac{1}{4\lambda^2}\int \frac{dX_1 dX_2}{2\pi(\frac{1}{2})} \exp\{-(X_1^2+X_2^2)\}\log^2(X_1^2+X_2^2) \tag{3.51}$$

Changing variables lead to

$$\begin{aligned}\langle \delta t(0)\delta t(0)\rangle &= \frac{1}{4\lambda^2}\frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^\infty r dr e^{-r^2}[\log r^2]^2 \\ &= \frac{1}{4\lambda^2}8\int_0^\infty r dr e^{-r^2}[\log r]^2 \\ &= \frac{1}{4\lambda^2}\left(\gamma^2+\frac{\pi^2}{6}\right) \end{aligned} \tag{3.52}$$

The integration (3.45) will give us the time delay field in x-space. The corresponding k-space time delay field was plotted in Fig. 3-2 along with the Monte Carlo simulation for reference. The difference between the two results is about 1%, which can be think of as the limit of the convergence of the Monte Carlo simulation. It would be possible to reduce the difference by increasing the number of runs in the Monte Carlo simulation.

3.5 Variation of the time delay field

From the previous sections, we have seen that the both the time delay field and the mean ending time depend only on $\mu_\phi\mu_\psi$ but not μ_ϕ/μ_ψ. Since we have investigated a wide range of physical parameters, it would be possible the see the pattern of the dependence. Define a new parameter as

$$\zeta \equiv \mu_\phi\mu_\psi \tag{3.53}$$

The dependence of the ending time $t_{\rm end}$ and the time delay peak $\delta\tau_{\rm max}$ on ζ can be fitted as

$$\delta\tau_{\rm max}(\zeta) = a\zeta^b \tag{3.54}$$

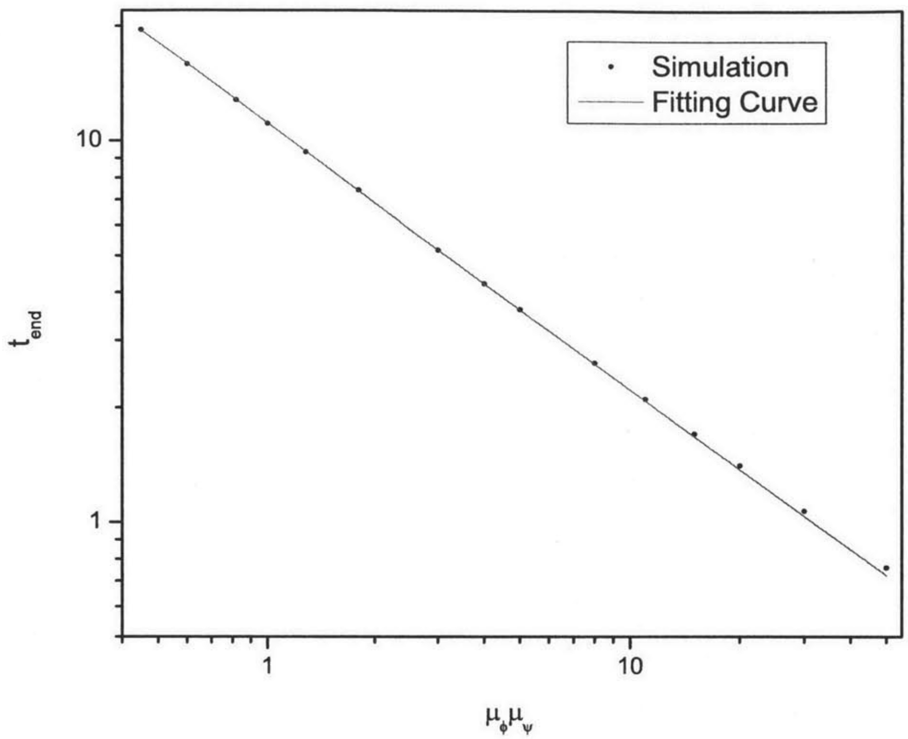

Figure 3-3: The mean ending time.

where

$$a \approx 0.045 \quad (3.55)$$

$$b \approx -0.60 \quad (3.56)$$

and

$$t_{\text{end}}(\zeta) = c\zeta^d \quad (3.57)$$

where

$$c \approx 11.2 \quad (3.58)$$

$$d \approx -0.70 \quad (3.59)$$

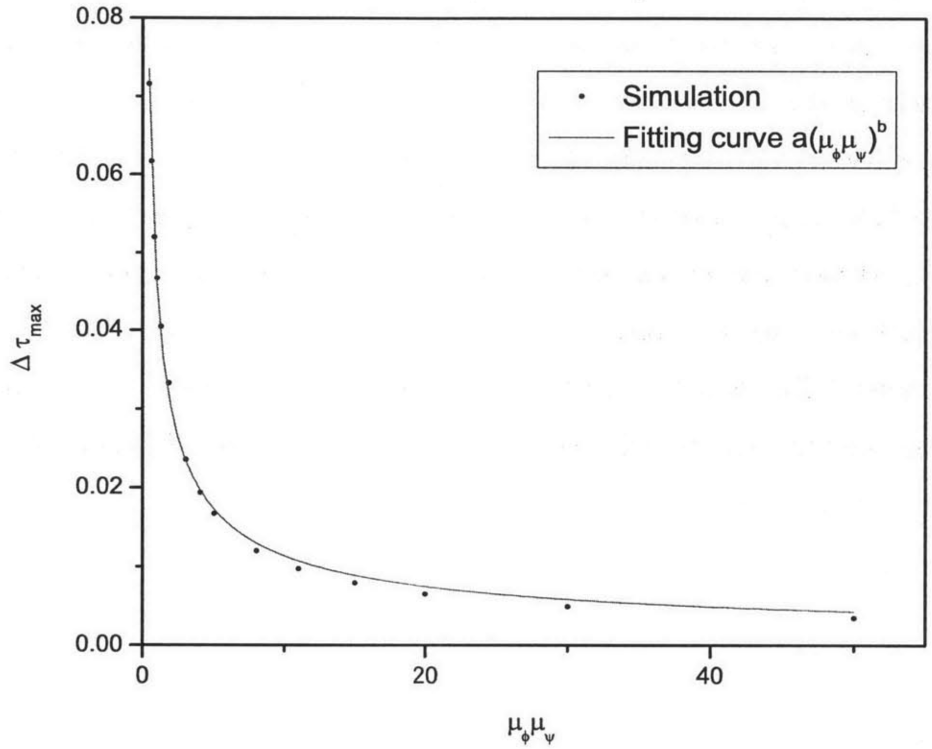

Figure 3-4: The peak of the time delay fields.

3.6 Conclusions

The time delay field calculated from the simple approximation proposed by Randall et. al. shows a very good agreement with that of the Monte Carlo simulation up to a rescaling in the amplitude and a shifting in the wave number both by factors of order 1. This is the same conlcusion as Burgess but with wider range of mass parameters μ_ϕ and μ_ψ. We also noticed that the time delay field only depends on the product but not the ratio of the two mass parameters. By varying this product from 0.45 to 50, we can have the peak of the time delay field changing from 0.072 to 0.004.

The Monte Carlo simulation is a powerful tool to investigate the evolution of the fields in 1-dimensional systems, but very difficult to expand to 3-dimensional due to an enormous computational requirement. By integrating over the probability distribution of the scalar field, the direct integration method can accurately produce the result of the Monte Carlo simulation in the asymtotic regime when all mode functions increase with time at the same rate independent of k. The direct integration method is very suitable for 3-dimensional expansion since the probability distribution integration has the same form, with the only parameter dependence arising from the two point function $\Delta(x)$.

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