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Equivalent permeability tensor in fractured media : an algebraic approach.

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Abstract. *This work is part of an extensive investigation on the equivalent permeability of heterogeneous and fractured media. We focus here on the problem of Darcian/Poiseuille flow in an irregular network of fracture segments (in 2D) or conduits (in 2D or 3D). An exact algebraic relation between the mean flux vector(Q) and the mean hydraulic gradient (J) is developed through a mathematical analyzis of the network flow problem, based on concepts from graph theory, leading to a discrete definition of DIV and GRAD operators. The resulting equivalent permeability is a 2nd rank tensor Kij, not necessarily symmetric and not necessarily positive-definite. Its properties are analyzed for given types of boundary conditions and averaging procedures.*

1 Introduction

A previous survey of permeability upscaling or homogenization techniques [1] indicates that there is a wide variety of approaches for defining and calculating the equivalent permeability or conductivity of a heterogeneous porous medium. Consequently, the resulting equivalent conductivities obtained by these different methods are not necessarily the same, and in particular, they can have different mathematical properties (tensorial or not, symmetric or not, definite positive or not).

In addition, different cases can be distinguished regarding the spatial structure of the heterogeneous media: (1) continuous porous media (including the case of random media with spatially correlated permeability); (2) discontinuous and/or boolean porous media (multilayered media, composite media with low or high permeability inclusions); (3) discontinuous fractured porous media. In the latter case (3), the discontinuities are due to thin fractures or fissures embedded in the porous matrix. It is assumed that the fractures have very large length/aperture aspect ratio ($\ell/a >> 1$), and that the embedded fractures are more "conductive" than the embedding porous matrix ($K_F/K_M > 1$). [2]

A fracture network can be viewed as a special type of discontinuous heterogeneous medium (3), in the limit case of infinite fracture/matrix permeability contrast. We are interested in the equivalent flow behavior of irregular networks of conductive features, such as fracture segments (in 2D), conduits (in 2D or 3D), and networks of planar fractures (in 3D). However, in this work, we focus on the case of 2D networks (fracture segments), using an algebraic method that can be extended directly to the case of 3D networks (conduits).

We develop an exact algebraic relation between the mean flux vector(**Q**) and the mean hydraulic gradient (**J**) through a an algebraic and graph-theoretic analyzis of the network flow problem, based on algebraic formulations of the DIV, GRAD and LAPLACIAN operators. [3] With the proposed definitions, the resulting equivalent permeability is indeed a 2nd rank tensor Kij, not necessarily symmetric positive-definite. Its properties are analyzed for given types of boundary conditions and "criteria" (averaging operators).

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[1] Renard P. and G. de Marsily (1997). "Calculating equivalent permeability: A review". Advances in Water Resources, 20(5-6):253278, 1997.

[2] Note: at least two different models can be used for the flow inside the fractures: either (a) Darcy-type flow (e.g. if the fracture is filled with a coarse porous medium), or else (b) Poiseuille-Couette type flow (e.g. parallel plate flow model). A fracture/matrix permeability contrast can be defined in either case (a) or (b), albeit in a different way.

[3] See for instance the enlightning paper by Strang (1988): A framework for equilibrium equations. SIAM Review 30: 283-297. Or else Strang's 2007 book: Computational science and engineering. Wellesley-Cambridge Press, MA, USA, 2007.

2 Basic definitions and assumptions

In the case of fracture networks assuming Poseuille flow, the flux is assumed to follow the cubic law. In two dimensions, the fracture network is represented by a graph of segments and nodes. The flux Q_jk $[L^2T^{-1}]$ along the segment jk is modeled by:

$$Q_{jk} = \frac{ga_{jk}^3}{12\mu}\frac{h_j - h_k}{l_{jk}} \tag{1}$$

where a_{jk} is the aperture of segment jk, μ the dynamic viscosity of water, h_j and h_k the nodal heads at the end points of segment jk, and l_{jk} is the distance (oriented with respect to the system of coordinate) between the two nodes. The conservation equation is written for each node j:

$$\sum_k Q_{jk} = 0 \tag{2}$$

where k is a node index running over the end points of all segments jk connected to node j. After upscaling, it is assumed (or proved rigorously in the present case) that the ”macroscale” flow can be described by an equivalent flow model analogous to Darcy’s law:

$$\nabla \cdot \boldsymbol{V} = 0, \qquad \boldsymbol{V} = -\boldsymbol{K}\nabla H, \tag{3}$$

with $\boldsymbol{K}$ $[LT^{-1}]$ the upscaled hydraulic conductivity tensor, H $[L]$ the hydraulic head within the upscaled block, and $\boldsymbol{V}$ the specific discharge or flux density $[LT^{-1}]$ [4]. The macroscale H and $\boldsymbol{K}$ and $\boldsymbol{V}$ are continuous functions of space, while the corresponding microscale quantities could be discontinuous in the original heterogeneous or fractured domain. Note that the macroscale H and $\boldsymbol{V}$ are *not* defined as simple ”averages” of the local variables (except in very special cases). Defining the equivalent hydraulic conductivity requires (i)an equivalence criterion and (usually) specific definitions of mean flux and mean gradient; and (ii) a set of boundary conditions to be imposed both on the heterogeneous domain and the equivalent domain (medium). The Boundary Conditions (*BC’s*) considered in this work on network flow are essentially *Head Gradient Immersion (Linearly Varying Head) BC’s*: a multilinear function $H(\boldsymbol{x})$ is prescribed on every boundary point $\boldsymbol{x}$, such that the $\boldsymbol{J} = -\nabla H$ is a given macroscale hydraulic gradient vector.

Different equivalence criteria (averaging procedures) are then considered to define and compute the equivalent conductivity of the sample: *Net surface*

[4] Specific discharge can be defined differently, e.g., depending on space dimensions.

flux (NSF); *Vectorial surface flux (VSF)*; *Volume average flux (VAF)*; *Volume average gradient (VAG)*. Using these different criteria and BC's yields different results for the equivalent K_{ij}. We then investigate how the properties of K_{ij} depend on the method: tensorial behavior, symmetry, positiveness, and other criteria, e.g., consistency of equivalent conductivity *vs.* resistivity tensors (Fadili A., R. Ababou 2004: Dual homogenisation of immiscible steady two-phase flow in random porous media. Water Resour. Res., Vol.40, W01513, doi:10.1029/2003WR002465).

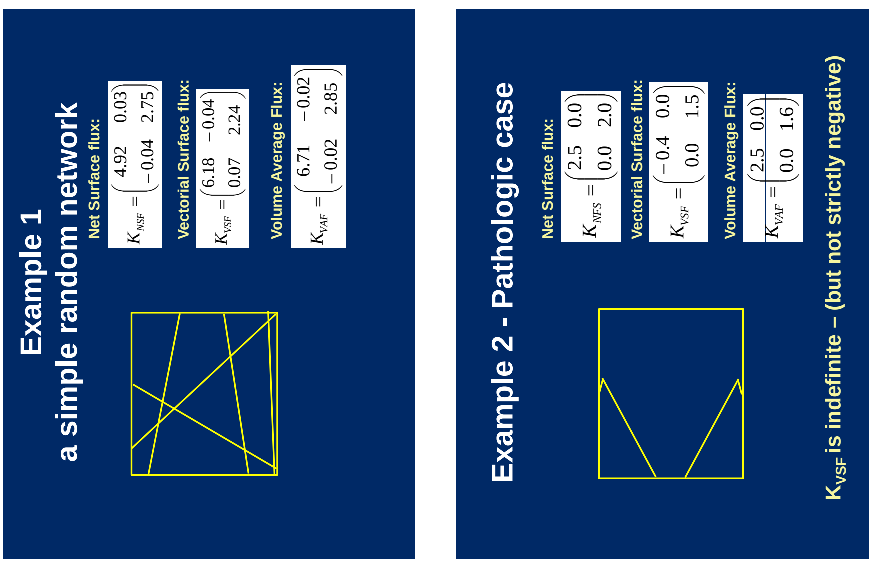
3 Algebraic analyzis of network flow and Kij via graph algebra

Operators $Div = \nabla^T$, $Grad = \nabla$, $DivGrad = \Delta = \nabla^T \nabla$, are defined on the network using node-arc, arc-node, node-node *adjacency matrices*.

4 A few examples, results and conclusions on Kij for networks

See **Figures** illustrating the Kij tensor obtained on simple small networks.

The general results obtained for irregular networks indicate that the VAF procedure, combined with Gradient Immersion BC's, always yields a symmetric permeability tensor **K**. Non-symmetric **K** can occur with other averaging procedures (criteria): NSF and VSF. Furthermore, it was found that non strictly positive (e.g. indefinite **K** tensor) can occur only with the VSF procedure.



(a) (b)

Figure 1: Examples of results on the equivalent Kij tensor for small fracture networks. **(a)** A small random fracture network. **(b)** A pathological "HerringBone" type network.