To cite this version:
Lavalle, Gianluca and Laurent, Claire and Charru, François and Vila, Jean-Paul *Study of the dynamics of thin liquid films sheared by gas flows*. (2013) In: 8th International Conference on Multiphase Flow - ICMF 2013, 26 May 2013 - 31 May 2013 (Jeju, Korea, Republic Of)

Study of the dynamics of thin liquid films sheared by gas flows

Gianluca Lavalle[1], Claire Laurent[2], François Charru[3], Jean Paul Vila[4]

[1]*Department of Chemical Engineering, Jeju National University, Jeju, Korea*
[2]*Affiliation #2, Place #2*
[3]*Affiliation #3, Place #3*

Keywords: Saint-Venant equations, thin film, gas flow, stability analysis

Abstract

We investigate the long-wave instability of a thin liquid film sheared by a concurrent gas flow. The film flows down an inclined plane and is driven by a constant shear stress and a pressure gradient. The height of the film is small compared to the wavelength, which justifies a long-wave approach. We adopt this procedure to approximate the Saint-Venant (or shallow water) equations. A linear stability analysis is thus developed and the limit between stable and unstable conditions is analyzed in terms of a critical Reynolds number. It comes out that the constant shear stress, applied at the free surface of the film, tends to stabilize the liquid layer to long waves. This is true when the mass flow of the film is kept unaltered. On the other side, the role of the pressure gradient is the same as the longitudinal component of gravity, which is known to be a destabilizing factor for the liquid film.

Introduction

Instabilities of thin liquid films have been studied for many years, since the pioneering works of Kapitzas. When a film flows down an inclined plane a wide range of wave evolution can appear at the free surface. The development of long waves manifests itself with the inception of linear structures and the transition to non-linear waves, such as solitons and periodic waves. When a gas flows above the film, its dynamics involves more phenomena, and a strong influence between the two phases rises at the interface.
Both natural phenomena and industrial applications are related with the film dynamics. Particularly, in the domains of aeronautics and space, there exists many typical configurations, such as: water ingestion in aircraft engines, primary atomization in fuel combustion, and de-icing of aircraft systems. In order to model the latter, for example, it is necessary considering that the flow around a wing profile manifests strong pressure gradient at the leading edge and relevant shear stresses all along the profile. Therefore, a deep comprehension of the physics of long-wave instabilities of thin liquid layers is still required.
When the height of waves at the interface remains smaller than their wavelength, an asymptotic approach based on long wave approximation results to be more suitable to model the film dynamics, due to its low-dimensional construction. This procedure was firstly proposed by Benney [Benney]. Furthermore, the use of Saint-Venant equations, integral form of the full Navier-Stokes, has certain benefits, especially for low Reynolds numbers [Tseluiko and Kalliadasis].
Shkadov firstly proposed a model of Saint-Venant equations where all the variables are enslaved to the film thickness and average velocity [Shkadov]. The limitation of his model is the lack of consistency, because the disagreement of the dispersion relation with the Orr-Sommerfeld theory. Later, Ruyer-Quil and Manneville have improved Shkadov's model, expanding the velocity field by using polynomial functions [Ruyer-Quil and Manneville 1/2]. In this way, they were able to ensure the consistency of their Saint-Venant equations, that is, the linearized model provides the right dispersion relation and the correct critical Reynolds number Re_{cr}, above which the liquid film is unstable.
Subsequently, Boutounet et al. have proposed a frame to obtain several models at given order accurate [Boutounet et al.].
However, in order to develop such first-order models, it is necessary to calculate the whole velocity field, that could be very tricky. Luchini and Charru have thus presented a way to avoid the calculation of the first-order velocity field [Luchini and Charru 1/2]. Their development is based on the use of the energy equation rather than the momentum equation, in order to get the proper reductions to prevent the hard calculation of the velocity correction.
All these already cited Saint-Venant models are consistent and expressed in a non-conservative form.
In this paper we present a consistent system of Saint-Venant equations, combined with a Benney-like asymptotic development, which satisfies two more properties, namely it is expressed in a conservative form and the concerning numerical flux does not depend on the possible bottom velocity. The former is relevant in numerical developments, i.e., a conservative numerical flux provides benefits for finite volume method. The latter, instead, is significant when computing industrial calculations involving rotating components, such as turbines and fans. In these configurations, the bottom velocity, represented by the blade speed, reaches very high values; if it is contained into the numerical flux, besides in the source terms, this velocity can be greatly high compared to the other terms of the expression.

Secondly, a linear stability analysis is developed for a thin liquid film falling on an inclined plane, with a constant shearing applied at the free surface.
The instability of such a liquid layer has been developed by Miles [Miles 1960] and Benjamin [Benjamin 1959], then improved by Smith and Davis [Smith and Davis 1982). Later, Smith gave a critical condition for the instability of such a film [Smith 1990]. In his configuration the film thickness is supposed to be fixed.
Hereby, we provide a critical Reynolds number depending on a prescribed shear stress, when the mass flow of the film is kept unchanged. Our result is different from the cited one, in a way that the effect of the shear stress is to stabilize the film to large wavelength disturbances.
Furthermore, the effect of the pressure gradient in the stability condition of the liquid layer has been analyzed. Its role is the same as the longitudinal component of gravity, thus destabilizes the film interface.

Nomenclature

g Gravity field (ms^{-1})
p Pressure (Nm^{-2})

Greek letters
β Inclination angle of the plane (rad)
μ Dynamic viscosity ($kg\ m^{-1}\ s^{-1}$)

Subscripts
0 Equilibrium state
c Critical number
Imaginary part

The governing equations

For a 2D incompressible flow, such as a thin liquid film flowing down an inclined plane and sheared by a concurrent gas flow (Figure 1), Navier-Stokes equations reduce to

$$\begin{cases} \tilde u_x + \tilde v_y = 0 \\ \tilde u_t + \tilde u \tilde u_x + \tilde v \tilde u_y = -\dfrac{1}{\rho}\tilde p_x + g\sin\beta + \nu\nabla^2 \tilde u \\ \tilde v_t + \tilde u \tilde v_x + \tilde v \tilde v_y = -\dfrac{1}{\rho}\tilde p_y - g\cos\beta + \nu\nabla^2 \tilde v \end{cases} \quad (1)$$

where u and v are the streamwise (x) and cross-stream (y) velocity components, p the pressure, g the gravity, β the angle of the inclined plane and $\nu = \mu\ \rho$ the kinematic viscosity. These equations must be completed with boundary conditions at the wall and at the free surface of the film, where $y = 0$ and $y = h$ respectively. The no-slip condition reads

$$\tilde u|_0 = 0\ , \qquad \tilde v|_0 = 0 \quad (2)$$

At the free surface, boundary conditions state the continuity of y-velocity and the balance of tangential and normal stress components. The first of such conditions sets the free surface a material line, and reads

$$\tilde h_t + \tilde u|_h\, \tilde h_x = \tilde v|_h\ . \quad \textbf{(3)}$$

With reference to Figure 1, the two other conditions yield

$$\tilde{\boldsymbol n}\tilde{\boldsymbol T}^{(\tilde n)} = -\tilde p_e(\tilde x, \tilde t) \quad \textbf{(4)}$$

$$\tilde{\boldsymbol t}\cdot\tilde{\boldsymbol T}^{(\tilde n)} = \tilde\tau(\tilde x, \tilde t) \quad \textbf{(5)}$$

where $\boldsymbol T^{(n)} = \Sigma \cdot \boldsymbol n$ is the normal $\boldsymbol n$ stress vector, with Σ stress tensor, p_e the gas pressure and τ shear stress of the gas at the free surface.
By the development of conditions (4), (5) one gets for normal and tangential stresses respectively

$$\frac{2\mu}{1+\tilde h_x^2}\left(\tilde h_x(\tilde u_y|_h + \tilde v_x|_h) - \tilde u_x|_h\, \tilde h_x^2 - \tilde v_y|_h\right) + \tilde p|_h - \tilde p_e = 0 \quad \textbf{(6)}$$

$$\frac{1}{1+\tilde h_x^2}\left(2\mu\tilde h_x(\tilde v_y|_h - \tilde u_x|_h) + \mu(1-\tilde h_x^2)(\tilde u_y|_h + \tilde v_x|_h)\right) - \tilde\tau(x,t) = 0 \quad \textbf{(7)}$$

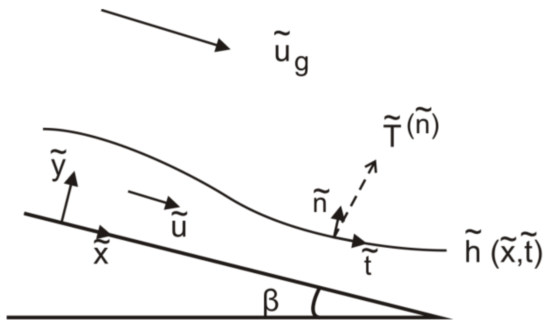

Figure 1: Inclined plane and boundary conditions.

Hence, the equations (1) with boundary conditions (2), (3) and (6), (7) describe the whole film dynamics. Let now consider the following dimensionless parameters,

$$x = \frac{\tilde x}{\tilde L} \qquad y = \frac{\tilde y}{\tilde h}$$
$$u = \frac{\tilde u}{\tilde U} \qquad v = \frac{\tilde v}{\tilde U\tilde h / \tilde L}$$
$$t = \frac{\tilde t}{\tilde L / \tilde U} \qquad \tau = \frac{\tilde \tau}{\tilde U \mu / \tilde h} \quad \textbf{(8)}$$
$$p = \frac{\tilde p}{\rho g \tilde L} \qquad p_e = \frac{\tilde p_e}{\rho g \tilde L}\ .$$

The system (1) thus becomes

$$\begin{cases} u_x + v_y = 0 \\ u_t + uu_x + vu_y = -\dfrac{p_x}{\varepsilon Fr\cos\beta} + \dfrac{1}{\varepsilon Fr}\tan\beta \\ \qquad + \dfrac{1}{Re}\left(\varepsilon u_{xx} + \dfrac{1}{\varepsilon}u_{yy}\right) \\ v_t + uv_x + vv_y = -\dfrac{1}{\varepsilon^2 Fr}\left(\dfrac{p_y}{\varepsilon\cos\beta} + 1\right) \\ \qquad + \dfrac{1}{Re}\left(\varepsilon v_{xx} + \dfrac{1}{\varepsilon}v_{yy}\right) \end{cases} \quad \textbf{(9)}$$

with appropriate boundary conditions at the wall and the free surface, namely

$$u|_0 = 0 \qquad v|_0 = 0 \quad \textbf{(10)}$$

and

$$\begin{cases} h_t + u|_h\, h_x = v|_h \\ 2\left(h_x(u_y|_h + \varepsilon^2 v_x|_h) - v_y|_h - \varepsilon^2 h_x^2 u_x|_h\right) \\ \qquad + \dfrac{Re}{\varepsilon^2 Fr\cos\beta}(p|_h - p_e)(1+\varepsilon^2 h_x^2) = 0 \\ \dfrac{1}{1+\varepsilon^2 h_x^2}\Big(2\varepsilon h_x(v_y|_h - u_x|_h) \\ \qquad + (1-\varepsilon^2 h_x^2)\left(\dfrac{1}{\varepsilon}u_y|_h + \varepsilon v_x|_h\right)\Big) - \dfrac{1}{\varepsilon}\tau = 0 \end{cases} \quad \textbf{(11)}$$

Two dimensionless numbers appear in the equations above, the Reynolds and Froude numbers, defined as

$$Re = \frac{\tilde{U}\tilde{h}}{\nu} \qquad Fr = \frac{\tilde{U}^2}{g\tilde{h}\cos\beta} \quad \textbf{(12)}$$

On the other hand, ε defines the ratio $h\ L$.
By integrating over the film thickness and then by imposing that $\varepsilon \ll 1$, which corresponds with considering a long-wave assumption, one can obtain the Saint-Venant (or shallow water) equations, namely

$$\begin{cases} h_t + (Uh)_x = 0 \\ (Uh)_t + \left(\int_0^h u(y)^2\,dy\right)_x + \dfrac{h}{Fr}h_x = \dfrac{h}{\varepsilon Fr}\tan\beta \\ \qquad - p_{ex}\dfrac{h}{\varepsilon Fr\cos\beta} + \dfrac{1}{\varepsilon Re}(u_y|_h - u_y|_0) \end{cases} \quad \textbf{(13)}$$

where U is the dimensionless average film velocity. Note that the y-momentum equation has been already replaced into the equations (13). These are expressed in terms of the unknowns h and $q = hU,$ film thickness and flow rate respectively, except for the momentum transport and wall shear stress terms. In order to enslave all the terms of the equation to h and q only, two closure models are required. This consists in assuming a proper shape for the velocity profile.

The base state

A trivial solution of system (9-11), in line with the wavyless solution of Nusselt, is based on the assumption that the film thickness is steady and constant, in a way that both the derivatives with respect to x and t are zero. By developing the calculations, the equilibrium velocity and pressure profiles read

$$u_0 = \frac{Re}{Fr\cos\beta}\left(\sin\beta - p_{ex}\right)\frac{y}{2}(2h-y) + \tau y \quad \textbf{(14)}$$

$$\mathrm{p}_0 = p_e \quad \textbf{(15)}$$

The average velocity, instead, is given by

$$U_0 = \frac{Re}{Fr\cos\beta}\left(\sin\beta - p_{ex}\right)\frac{h^2}{3} + \tau\frac{h}{2} \quad \textbf{(16)}$$

Thin liquid film modelling

Following Benney's development [Benney], let expand the velocity and pressure fields as

$$\begin{aligned} u &= u^{(0)} + \varepsilon u^{(1)} + \dots \\ v &= v^{(0)} + \varepsilon v^{(1)} + \dots \\ p &= p^{(0)} + \varepsilon p^{(1)} + \dots \end{aligned} \quad \textbf{(17)}$$

where the zeroth-order development coincides with the base state solution. According to the dimensionless pressure (8), the basic state consists in a constant pressure inside the liquid layer. The classic hydrostatic behavior results just shifted to the next order. The first-order variables can be found by substitution and then development of (17) into (9-11). Once the quantities (17) are known until the first order, the closure models for the momentum transport and wall shear stress terms can be obtained.
The former is enslaved to the zero order velocity only and is given by

$$\int_0^h (u(y))^2\,dy \approx \int_0^h u^{(0)^2}\,dy = hU^2 + \mathcal{F}(h) \quad \textbf{(18)}$$

The latter, instead, requires also the first order velocity development (included into the flow rate q), such as

$$\tau_w = 3\frac{q^{(0)}}{h^2} - \frac{\tau}{2} + \varepsilon\tau_w^{(1)} = 3\frac{q}{h^2} - \frac{\tau}{2} + \tau_w^{(c)} \quad \textbf{(19)}$$

Let now define the quantity

$$\Lambda = \frac{Re}{Fr\cos\beta}\left(\sin\beta - p_{ex}\right) \quad \textbf{(20)}$$

which takes into account both the gravity and pressure gradient effects. From the expressions (18) and (19), the unknowns $\mathcal{F}(h)$, $\tau^{(1)}$ are thus calculated, and the system (13) can be finally closed. Particularly, $\mathcal{F}(h)$ turns to be

$$\mathcal{F} = \frac{1}{45}\Lambda^2 h^5 + \frac{1}{12}\tau^2 h^3 + \frac{1}{12}\Lambda\tau h^4 \quad \textbf{(21)}$$

whereas the first order correction of the wall shear stress leads to

$$\tau_w^{(c)} = -\frac{Re}{15}\Lambda(\Lambda h + \tau)h^3 h_x \quad \textbf{(22)}$$

Apart from the shear stress and the pressure gradient, that were not considered by Ruyer-Quil and Manneville [Ruyer-Quil and Manneville 1/2] and by Luchini and Charru [Luchini and Charru 1/2] in their analysis, this correction term corresponds to their. The consistency of the model above is thus ensured. Note that the use of Benney's development already guarantees the consistency of the Saint-Venant equations. Finally, by replacing (21), (22) into (13), the Saint-Venant equations read

$$\begin{cases} h_t + \left(Uh\right)_x = 0 \\ (Uh)_t + \Big(hU^2 + \frac{2}{225}\Lambda^2 h^5 + \frac{1}{15}\Lambda\tau h^4 + \frac{1}{12}\tau^2 h^3 \\ \quad + \frac{1}{2}\frac{h^2}{Fr}\Big)_x = \frac{\Lambda}{\varepsilon Re}h + \frac{1}{\varepsilon Re}\Big(\frac{3}{2}\tau - 3\frac{U}{h}\Big) \end{cases} \quad \textbf{(23)}$$

Unlike the previously cited works, this system of consistent shallow water equations is expressed in conservative form. This turns to have certain advantages when using a finite volume method. Furthermore, if we introduce a bottom velocity in the analysis, the numerical flux of equations (23) remains unaltered. The bottom velocity will appear as a source term only. When solving industrial calculations with rotating components, typical of an aerospace environment, this properties results to be greatly relevant for numerical stability.

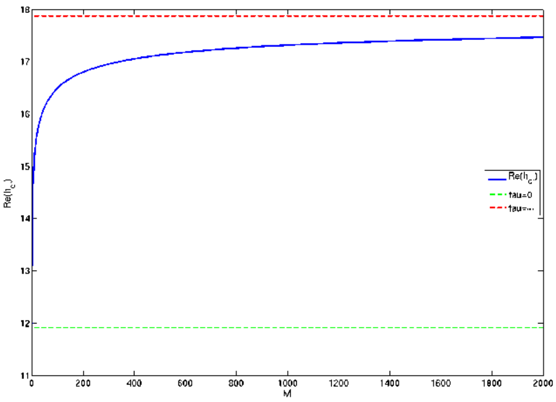

Figure 2: Critical Reynolds number versus $M = \tau^3(3\mu^2 g^2 (\tan\beta)^2\rho)^{-1}$. $\beta = 4°$.

Linear stability analysis

Let analyze the linear stability associated to the system of shallow water equations defined previously. Let thus calculate the dispersion relation and then the conditions setting a stable or unstable liquid layer.

We admit plane wave solutions, such that both film thickness and flow rate can be written in the form

$$h = h_0 + \hat{h}\exp[i(kx - \omega t)]\,, \qquad q = q_0 + \hat{q}\exp[i(kx - \omega t)] \quad \textbf{(24)}$$

where the subscript 0 refers to the base equilibrium solution defined before, and k and ω are the wavenumbers and the angular velocity respectively. We admit real wavenumbers k and complex ω, so that the angular velocity $\omega = kc$, where c is the complex wave velocity, can be written as

$$\omega = \omega_r + i\omega_i = kc_r + ikc_i \quad \textbf{(25)}$$

The sign of the amplification or damping rate kc_i defines the conditions for a stable or unstable film. When kc_i is negative the solution is stable to large wavelengths, whereas $kc_i = 0$ represents the transition and thus defines the neutral-stability curve. Inversely, when kc_i is positive the film is unstable to large wavelength disturbances.

Let replace the wavy solutions (24) into the quasi-linear form of the SW equations (23). The mass conservation equation yields

$$\hat{h}\omega = k\hat{q} \quad \textbf{(26)}$$

On the other hand, the momentum equation produces two contributions: one is the non-exponential part, which gives the equilibrium flow rate, and the other is the exponential term. The former reads

$$q_0 = \frac{h_0^3}{3}\Lambda + \frac{1}{2}\tau h_0^2 \quad \textbf{(27)}$$

The second contribution, instead, is given by

$$\begin{aligned} &-\hat{q}i\omega + 2\frac{q_0}{h_0}\hat{q}ik - \frac{q_0^2}{h_0^2}\hat{h}ik + \frac{2}{45}\Lambda^2 h_0^4 ik\hat{h} \\ &+ \frac{4}{15}\tau\Lambda h_0^3 ik\hat{h} + \frac{1}{4}\tau^2 h_0^2 ik\hat{h} + \frac{1}{Fr}h_0 ik\hat{h} \\ &= \frac{\Lambda}{\varepsilon Re}\hat{h} - \frac{3}{\varepsilon Re h_0^2}\hat{q} + \frac{6}{\varepsilon Re}\frac{q_0}{h_0^3}\hat{h} \end{aligned} \quad \textbf{(28)}$$

The system of equations (26) and (28) allows to find the dispersion relation, which governs the relationship between k and ω, and subsequently the neutral-stability curve. Let replace the equation (26) into (28). With the help of (27), the dispersion relation reads

$$\begin{aligned} &-i\omega^2 \\ &+\Big(\frac{3}{\varepsilon Re h_0^2} + \frac{2}{3}\Lambda h_0^2 ik + h_0\tau ik\Big)\omega \\ &-\frac{3}{\varepsilon Re}\Big(\Lambda + \frac{\tau}{h_0}\Big)k \\ &+\Big(-\frac{1}{15}\Lambda^2 h_0^4 - \frac{1}{15}h_0^3\tau\Lambda + \frac{h_0}{Fr}\Big)ik^2 = 0 \end{aligned} \quad \textbf{(29)}$$

Note that the assumptions of h_0 and q_0 equal to unity permits to semplify the calculations, without loss of generality. By an asymptotic expansion of ω in powers of the wavenumber k (thus, in the limit $k \ll 1$, which corresponds to large wavelengths), the solution of (29) can be written as (we express here the wave velocity c)

$$c = \Lambda + \tau + \frac{\varepsilon Re}{3}i\Big(-\frac{2}{5}\Lambda^2 - \frac{2}{5}\tau\Lambda + \frac{1}{Fr}\Big)k + O(k^2) \quad \textbf{(30)}$$

Note that the wave velocity is complex. As stated before, the real part represents the propagation velocity, and the imaginary part the growth rate.

Different cases have been studied. When only gravity acts on the liquid layer, such as a falling film along an inclined plane, the wave velocity reduces to

$$c = \frac{Re}{Fr}\tan\beta + \frac{\varepsilon Re}{3} i\Big(-\frac{2}{5}\frac{Re^2}{Fr^2}(\tan\beta)^2 + \frac{1}{Fr}\Big)k + O(k^2) \tag{31}$$

and the result coincides with the analysis of Benjamin [Benjamin 1957] and Yih [Yih 1963]. In this configuration $c_r = 3$ (since $\Lambda = 3$ from the equilibrium relationship (27)), and the waves move three times faster than the film. On the other hand, $kc_i = 0$ imposes the stability condition, which can be expressed in terms of either a critical Reynolds number or a critical Froude number. In details,

$$Re_{cr} = \frac{5}{6}\cot\beta\ , \qquad Fr_{cr} = \frac{5}{18} \tag{32}$$

If $Re < Re_{cr}$, or $Fr < Fr_{cr}$, the film is stable to large wavelength disturbances. It is manifest that a horizontal film is always stable, whereas a vertical film is always unstable.

When a constant shear stress supports the film, such as the case of a falling film sheared by a concurrent gas flow, the wave velocity changes both in the real and imaginary part. Now we obtain $c_r = 3 - 1\ \ 2\,\tau$. Hence, the propagation velocity of the waves, related to the film velocity, decreases in respect to the falling film configuration, and this happens because the shear stress increases the film velocity more than the wave speed. The stability condition now reads

$$Re_{cr} = \frac{5/6\cot\beta}{1-\tau/6}\ , \qquad Fr_{cr} = \frac{5/18}{(1-\tau/2)(1-\tau/6)} \tag{33}$$

Therefore, the value of Re_{cr}, as for Fr_{cr}, increases when a constant shearing is applied in the direction of the film flow. Its effect is thus to stabilize the film to large wavelength disturbances.

Let rewrite now the expression of τ using the equilibrium velocity. It yields

$$\tau = \frac{\tilde\tau}{\mu\tilde U_0/\tilde h_0} = \frac{\tilde\tau}{\frac{\mu}{\tilde h_0}\Big(\rho g\tan\beta\,\frac{\tilde h_0^2}{3\mu} + \frac{\tilde h_0\tilde\tau}{2\mu}\Big)} = \frac{1}{\rho g\tan\beta\,\frac{\tilde h_0}{3\tilde\tau} + \frac{1}{2}} \tag{34}$$

For any τ, the value of τ is thus confined in the range $\ 0{,}2\ $. This makes the critical Reynolds number lying in the range $\ 5\ \ 6\,\mathbf{cot}\,\beta, 5\ \ 4\,\mathbf{cot}\,\beta\ $, as sketched in Figure 2. Therefore, although Re_{cr} increases, it is limited up to $5\ \ 4\,\mathbf{cot}\,\beta$.

The expression of Re_{cr} as function of the dimensionless shear stress τ depends on the definition of the Reynolds number. In our analysis it depends on the average velocity of the liquid layer. Apart from the reference velocity, the results of Smith [Smith 1990] differs from this one in the definition of the Reynolds number. In his work, it does not depend on the shear stress. Hence, his results shows a destabilizing effect of τ, but when the film thickness is kept unchanged in the experience.

Another way to quantify the effect of the shear stress over the free surface of the film is to redefine the dimensionless setting of τ and introduce a dimensionless film thickness h, as follows

$$h = \frac{\tilde h_0}{\tilde\tau/(\rho g\tan\beta)} \tag{35}$$

By expressing both the Reynolds number and its critical value in terms of h, a stability condition for the thickness of the film can be found, which reads

$$h_{cr}^2(h_{cr}+1) = \frac{5}{6}\cot\beta\,\frac{3\mu^2 g^2(\tan\beta)^2\rho}{\tilde\tau^3} = \frac{1}{M}\frac{5}{6}\cot\beta \tag{36}$$

When including also the pressure gradient, it turns useful to express Λ in terms of fictitious angle θ and Froude number $Fr^{\star}$, namely

$$\Lambda = \frac{Re}{Fr^{\star}}\tan\theta \tag{37}$$

This idea is supported by the fact that the pressure gradient modifies only the longitudinal component of gravity force, keeping unchanged the normal component instead. As sketched in Figure 3, this involves a modification of the angle of the inclined plane and the gravity norm.

Therefore, keeping Re fixed, the wave velocity c reads

$$c = \frac{Re}{Fr^{\star}}\tan\theta + \tau + \frac{\varepsilon Re}{3} i\Big(-\frac{2}{5}\frac{Re^2}{Fr^{\star 2}}(\tan\theta)^2 - \frac{2}{5}\tau\frac{Re}{Fr^{\star}}\tan\beta + \frac{1}{Fr^{\star}}\Big)k + O(k^2) \tag{38}$$

In this way, we can easily compare this case to the configuration of falling films. The analysis of kc_i provides the relationship

$$Re_{cr}^{(\theta)} = \frac{5/6\cot\theta}{1-\tau/6} \tag{39}$$

If the pressure gradient is negative (gas flowing concurrent to the film), the fictitious angle θ results being greater than β. Hence,

$$Re_{cr}^{(\theta)} < Re_{cr}^{(\beta)} \tag{40}$$

One can thus summarize that a negative p_{e_x} tends to destabilize the film, because the tangential component of gravity increases.

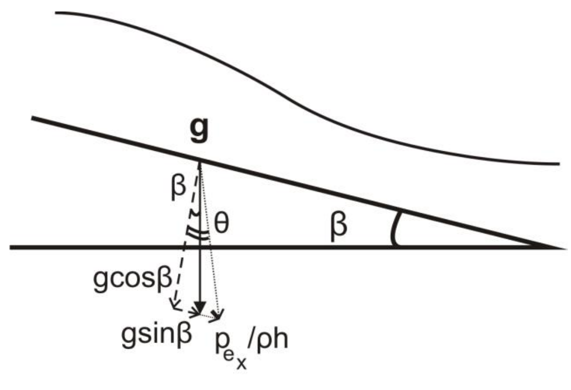

Figure 3: Scheme of fictitious θ when adding $p_{e_x}(\rho h)^{-1}$ to the longitudinal gravity component.

Conclusions

Combined with a numerical flux not depending on the possible bottom velocity, the conservative form of Saint-Venant equations expressed above turns to be very useful in many numerical experiences. Among the others, this model results to be the only consistent one to possess these two properties.

The results of the linear stability analysis for a liquid film sheared by a constant shear stress show that the effect of this stress consists in increasing the critical Reynolds number. By its definition, it is manifest that the presence of the shear stress stabilizes the film to large wavelength disturbances. This is not in agreement with the work of Smith. The difference comes out from the definition of the Reynolds number. In his work, it is defined as function of film thickness only. In our work, the Reynolds number depends on the average longitudinal velocity, which contains also the shear stress. In conclusion, his experience is obtained as the film thickness is kept fixed, whereas in our is the mass flow to be unchanged. This result is very important for academic experiments and numerical calculation, as also for industrial applications. Since the evident difficulties to express a unique stability condition for this experience, a critical condition of stability based on dimensionless thickness h has been developed.

The second main conclusion consists in the critical condition for stability when the film is driven by a pressure gradient too. The modification of the longitudinal component of gravity consists in a fictitious alteration of the inclination angle β and the gravity norm g. The expression of the critical Reynold number does not change its form, but the angle. Geometrical considerations drive to the conclusion that the Re_{cr} decreases compared to the case without pressure gradient. Its role is thus to destabilize the film, in contrast with the shear stress.

References