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The Stability of Differential Systems with Delay

by

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Abstract

A new criterion for the stability of differential delay systems is proved in terms of a matrix of parameters.

Keywords : Nonlinear Delay Systems, Stability, $\mathcal{M}_2$ spaces .

1 Introduction

The stability of differential delay systems has been investigated by many authors; see, for example, [4] and [5] . Recently, [1],[2] and [6] have obtained simple criteria for the stability of such systems. In this paper we shall prove a general criterion for stability and give an example the stability of which cannot be checked by the criteria of Amemiya or Mori.

In section 2 we shall prove a result in terms of the coefficients of certain matrices which define the system. This result will be applied in section 3 to obtain a result for an n^{th} order system in terms of the eigenvalues of the characteristic equation. An $\mathcal{M}_2$ space formulation will be given in section 4 and the example will follow in section 5.

2 The Main Result

Consider the system

$$\begin{aligned} \dot{x}_i(t) &= \sum_{j=1}^{n}[a_{ij}x_j(t) + b_{ij}x_j(t-\tau)] + f_i(t, x(t), x(t-\tau)) \qquad (2.1) \\ x_i(t) &= \theta_i(t) \ , \ t \in [-\tau, 0] \end{aligned}$$

$i = 1, \ldots, n$. The initial value functions θ_i , $1 \leq i \leq n$ are assumed to belong to $C[-\tau, 0; R]$. We shall seek solutions of (1) in $C[0, \infty; R^n]$.

We shall assume that the functions f_i satisfy the following condition:

$\mathcal{F}$: For any $t \geq 0$ and any function $x \in C[t-\tau, t]$, we have

$$| f_i(t, x(t), x(t-\tau)) | \leq \sum_{j=1}^{n} d_{ij} \sup_{t-\tau \leq \theta \leq t} | x_j(\theta) | .$$

where $d_{ij} \geq 0$ are constants.

In the following, $\rho(A)$ will denote the spectral radius of the matrix A. We can then state our main result:

Theorem 1 Let f satisfy condition $\mathcal{F}$. If there exist constants c_{ij} $(1 \leq i, j \leq n)$ such that $(a_{ii} - c_{ii}) < 0$, $1 \leq i \leq n$ and $\rho(\Omega) < 1$, where $\Omega = (\omega_{ij})$ and

$$\omega_{ij} = \frac{| a_{ij} - c_{ij} | v_{ij} + | b_{ij} + c_{ij} | + d_{ij}}{| a_{ii} - c_{ii} |} + 2 | c_{ij} | \tau$$

where $v_{ij} = 1 - \delta_{ij}$ (the Kronecker delta), then the zero solution of equation (2.1) is asymptotically stable.

Proof Let

$$\omega_{ij}(\epsilon) = \left(\frac{| a_{ij} - c_{ij} | v_{ij} + | b_{ij} + c_{ij} | + d_{ij} + | a_{ij} - c_{ij} | . | c_{ij} | \tau}{| a_{ii} - c_{ii} + \epsilon |} + \tau | c_{ij} | \right) e^{\epsilon\tau}$$

and

$$\Omega(\epsilon) = (\omega_{ij}(\epsilon)),$$

for any $0 < \epsilon < \max_{1 \leq i \leq n} | c_{ii} - a_{ii} |$. Since the eigenvalues of a matrix are continuously dependent on its elements, we have $\rho(\Omega(\epsilon)) < 1$ for sufficiently small ϵ. This follows from the assumption $\rho(\Omega) < 1$.

We can write the system (2.1) in the equivalent form

$$\frac{d}{dt}(x_i(t) \quad - \quad \sum_{j=1}^{n} c_{ij} \int_{t-\tau}^{t} x_j(\theta) d\theta)$$

$$= \sum_{j=1}^{n}[(a_{ij}-c_{ij})x_j(t)+(b_{ij}+c_{ij})x_j(t-\tau)]+f_i(t,x(t),x(t-\tau)), \tag{2.2}$$

for $t \geq 0$ and so by variation of parameters,

$$\begin{aligned} x_i(t) &= e^{(a_{ii}-c_{ii})t}\left[x_i(0)-\sum_{j=1}^{n}c_{ij}\int_{-\tau}^{0}x_j(\theta)d\theta\right] \\ &+ \sum_{j=1}^{n}\int_0^t e^{(a_{ii}-c_{ii})(t-s)}\{(a_{ij}-c_{ij})v_{ij}x_j(s)+(b_{ij}+c_{ij})x_j(s-\tau)\}ds \\ &+ \sum_{j=1}^{n}(a_{ii}-c_{ii})c_{ij}\int_0^t e^{(a_{ii}-c_{ii})(t-s)}\int_{s-\tau}^{s}x_j(\theta)d\theta.ds \\ &+ \sum_{j=1}^{n}c_{ij}\int_{t-\tau}^{t}x_j(\theta)d\theta+\int_0^t e^{(a_{ii}-c_{ii})(t-s)}f_i(s,x(s),x(s-\tau))ds \end{aligned}$$

for $t \geq 0$. Hence,

$$\begin{aligned} |\, x_i(t)\,| &\leq e^{(a_{ii}-c_{ii})t}M+\sum_{j=1}^{n}\tau\,|\,c_{ij}\,|\sup_{t-\tau\leq\theta\leq t}|\,x_j(\theta)\,| \\ &+ \sum_{j=1}^{n}(|\,a_{ij}-c_{ij}\,|\,v_{ij}+|\,b_{ij}+c_{ij}\,|+d_{ij}+|\,a_{ii}-c_{ii}\,|\,.\,|\,c_{ij}\,|\,\tau)\,. \\ &\int_0^t e^{a_{ii}-c_{ii})(t-s)}\sup_{s-\tau\leq\theta\leq s}|\,x_j(\theta)\,|\,ds \quad , \quad t\geq 0 \quad , \end{aligned}$$

where

$$M=\max_{1\leq i\leq n}\left[\sup_{-\tau\leq\theta\leq 0}|\,x_i(\theta)\,|\right].\left\{1+\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|\,c_{ij}\,|\right)\tau\right\}.$$

Let

$$\zeta_i(t)=\sup_{-\tau\leq\theta\leq t}[|\,x_i(\theta)\,|\,e^{\epsilon\theta}] \quad , \quad 1\leq i\leq n \quad , \quad t\geq -\tau\,.$$

Then

$$|\,x_i(t)\,|\,e^{\epsilon t} \quad \leq \quad e^{(a_{ii}-c_{ii})t+\epsilon t}M+\sum_{j=1}^{n}\tau\,|\,c_{ij}\,|\,e^{\epsilon\tau}.\sup_{t-\tau\leq\theta\leq t}[|\,x_j(\theta)\,|\,e^{\epsilon\theta}]$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}(\mid a_{ij}-c_{ij}\mid v_{ij}+\mid b_{ij}+c_{ij}\mid +d_{ij}+\mid a_{ii}-c_{ij}\mid . \mid c_{ij}\mid \tau)\,. \\
&e^{\epsilon\tau}\int_0^t e^{((a_{ij}-c_{ij})+\epsilon)(t-s)}\sup_{s-\tau\leq\theta\leq s}[\mid x_j(\theta)\mid e^{\epsilon\theta}]ds \\
\leq\;& M+\sum_{j=1}^{n}\tau\mid c_{ij}\mid e^{\epsilon\tau}.\zeta_j(t) \\
&+\sum_{j=1}^{n}\frac{\mid a_{ij}-c_{ij}\mid v_{ij}+\mid b_{ij}+c_{ij}\mid +d_{ij}+\mid a_{ii}-c_{ii}\mid . \mid c_{ij}\mid \tau}{\mid (a_{ii}-c_{ii})+\epsilon\mid}.e^{\epsilon\tau}\zeta_j(t) \\
=\;& M+\sum_{j=1}^{n}\omega_{ij}(\epsilon)\zeta_j(t)\;\;,\;\; t\geq 0\;.
\end{aligned}
$$

Since

$$\mid x_i(t)\mid e^{\epsilon t}\leq M\;\; for\;\; t\in[-\tau,0]$$

it follows that

$$\mid x_i(t)\mid e^{\epsilon t}\leq M+\sum_{j=1}^{n}\omega_{ij}(\epsilon)\zeta_j(t)\;\;,\;\; for\;\; t\geq -\tau \tag{2.3}$$

However, the right hand side is nondecreasing, so we obtain

$$\zeta_i(t)\leq M+\sum_{j=1}^{n}\omega_{ij}(\epsilon)\zeta_j(t)\;\;,\;\; t\geq -\tau\;.$$

i.e.

$$\zeta(t)\leq MI+\Omega(\epsilon)\zeta(t)\;\;,$$

where $\zeta=(\zeta_1,\ldots,\zeta_n)^T$ and I is the identity matrix. But $\rho(\Omega(\epsilon))<1$, by assumption, and so

$$\zeta(t)\leq M(I-\Omega(\epsilon))^{-1}\;.$$

i.e.

$$| x_i(t) | \leq K e^{-\epsilon t} \quad , \quad t \geq 0 \quad , \quad 1 \leq i \leq n \quad ,$$

for some constant $K \geq 1$. □

Corollary 1

If $a_{ii} < 0$ $(1 \leq i \leq n)$ and $\rho(\Omega) < 1$ where $\Omega = (\omega_{ij})$ and

$$\omega_{ij} = \frac{| a_{ij} | v_{ij} + | b_{ij} | + | d_{ij} |}{| a_{ij} |}$$

then the zero solution of (1) is asymptotically stable.

Proof Take $c_{ij} = 0$, $1 \leq i,j \leq n$. □

Corollary 2

If $(a_{ii} + b_{ii}) < 0$, $1 \leq i \leq n$ and $\rho(\Omega) < 1$, then the zero solution of (1) is asymptotically stable.

Proof Take $c_{ij} = -b_{ij}$, $1 \leq i,j \leq n$. □

In theorem 1 we have obtained conditions for stability of equation (2.1) in terms of the components a_{ij}, b_{ij} of matrices A and B, respectively. It is also possible to obtain basis-independent conditions by considering the equation (2.1) as a matrix system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + f(t, x(t), x(t-\tau)). \tag{2.4}$$

Assume that f satisfies the following condition

$\overline{\mathcal{F}}$: for any $t \geq 0$ and any function $x \in C[t-\tau, t]$ we have

$$||f(t, x(t), x(t-\tau))|| \leq d \sup_{t-\tau \leq \theta \leq t} ||x(\theta)|| \quad .$$

Theorem 2 Let f satisfy condition $\overline{\mathcal{F}}$. If there exists a matrix C such that

$$||e^{(A-C)t}|| \leq Ke^{-\delta t}$$

for some $K, \delta \geq 0$ and for which

$$\Gamma \triangleq \tau||C|| + \frac{K}{\delta}(||B+C|| + d + ||A-C||\ ||C||\delta) < 1 \quad ,$$

then the zero solution of the system (4) is asymptotically stable.

Proof From (4) we have

$$\frac{d}{dt}\left[x(t) - \int_{t-\tau}^{t} Cx(\theta)d\theta\right] = (A-C)x(t) + (B+C)x(t-\tau) + f(t, x(t), x(t-\tau))$$

for $t \geq 0$, and so

$$\begin{aligned} x(t) = \quad & e^{(A-C)t}\left[x(0) - \int_{-\tau}^{0} Cx(\theta)d\theta\right] \\ + \quad & \int_{0}^{t} e^{(A-C)(t-s)}(B+C)x(s-\tau)ds \\ + \quad & (A-C)C\int_{0}^{t} e^{(A-C)(t-s)}\int_{s-\tau}^{s} x(\theta)d\theta ds \\ + \quad & \int_{t-\tau}^{t} Cx(\theta)d\theta + \int_{0}^{t} e^{(A-C)(t-s)}f(s, x(s), x(s-\tau))ds. \end{aligned}$$

Hence,

$$\begin{aligned} ||x(t)|| \quad \leq \quad & Me^{-\delta t} + \tau||C|| \sup_{t-\tau\leq\theta\leq t} ||x(\theta)|| \\ & +(||B+C|| + d + ||A-C||\ ||C||\tau)K\int_{0}^{t} e^{-\delta(t-s)} \sup_{s-\tau\leq\theta\leq s} ||x(\theta)||ds \quad , \end{aligned}$$

$t \geq 0$, where

$$M = K\left(\sup_{-\tau\leq\theta\leq 0} ||x(\theta)||\right) . (1 + ||C||\tau) \quad .$$

Let $\zeta(t) = \sup_{-\tau \leq \theta \leq t} ||x(\theta)||$. Then

$$\begin{aligned} ||x(t)|| &\leq Me^{-\delta t} + \tau||C||\zeta(t) \\ &\quad +(||B + C|| + d + ||A - C||\ ||C||\tau)\frac{K}{\delta}.\zeta(t) \end{aligned}$$

i.e.

$$\zeta(t) \leq Me^{-\delta t} + \Gamma\zeta(t)$$

and so

$$\zeta(t) \leq \frac{M}{1-\Gamma}e^{-\delta t} \quad .$$

□

Corollaries 1 and 2 have obvious counterparts in the basis-independent case:

Corollary 3 If A is a stable matrix with $||e^{At}|| \leq Ke^{-\delta t}$ and

$$\Gamma \triangleq \frac{K}{\delta}(||B|| + ||A|| + d) < 1$$

then the zero solution of the system (2.4) is asymptotically stable. □

Corollary 4 If $A + B$ is a stable matrix with $||e^{(A+B)t}|| \leq Ke^{-\delta t}$ and

$$\Gamma \triangleq \tau||B|| + \frac{K}{\delta}(d + ||A + B||\ ||B||\tau) < 1$$

then the zero solution of the system (4) is asymptotically stable. □

3 Application to n^{th} Order Equations

In this section we shall apply the above theory to the n^{th} order equation

$$x^n(t) \quad + \quad \sum_{i=1}^{n}\{a_{n-i+1}x^{(n-i)}(t) + b_{n-i+1}x^{(n-i)}(t-\tau)\}$$

$$+ \quad f(t, x(t), \dot{x}(t), \ldots, x^{(n-1)}(t), x(t-\tau), \dot{x}(t-\tau), \ldots, x^{(n-1)}(t-\tau)) = 0 \ .$$

Define,as usual ,

$$x_1(t) = x(t), x_2(t), \ldots, x_n(t) = x^{(n-1)}(t) \ .$$

Then we have

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
\ldots\ldots &\ . \ \ldots\ldots \\
\dot{x}_n(t) &= -\sum_{i=1}^{n}(a_i x_i(t) + b_i x_i(t-\tau)) - f(t, x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau))
\end{aligned}$$

i.e.

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + F(t, x(t), x(t-\tau)) \tag{3.1}$$

where $x = (x_1, \ldots, x_n)^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & 1 \\ -a_1 & -a_2 & \ldots & \ldots & -a_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & 1 \\ -b_1 & -b_2 & \ldots & \ldots & -b_n \end{pmatrix}$$

$$F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ -f(t, x(t), x(t-\tau)) \end{pmatrix}$$

If f satisfies the condition

$$|f(t, x(t), x(t-\tau))| \leq d \sup_{t-\tau \leq \theta \leq t} ||x(\theta)|| \tag{3.2}$$

for all $x \in C[t-\tau, t]$, then we have

$$||F(t, x(t), x(t-\tau))|| \leq d \sup_{t-\tau \leq \theta \leq t} ||x(\theta)||$$

From corollaries 3 and 4, we therefore obtain

Theorem 3 If the eigenvalues of A have real parts bounded above by $-\delta$ and

$$\frac{1}{\delta}(||B|| + ||A|| + d) < 1$$

then the zero solution of system (3.1) is asymptotically stable. □

Theorem 4 If the eigenvalues of $A + B$ have real parts bounded above by $-\delta$ and

$$\tau||B|| + \frac{1}{\delta}(d + ||A + B|| \, ||B||\tau) < 1$$

then the zero solution of system (3.1) is asymptotically stable. □

4 $\mathcal{M}_2$-Space Formulation

In this section we shall extend the above results by using an $\mathcal{M}_2$ space formulation (see [3]). Consider again the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-\tau) + f(t, x(t), x_\tau(t)) \ , \qquad (4.1) \\ x(t) &= h(t) \ , \ t \in [-\tau, 0] \ , \end{aligned}$$

where $x_\tau(t)$ is the function defined by

$$x_\tau(t)(\theta) = x(t+\theta) \ , \quad -\tau \le \theta \le 0$$

Let $\mathcal{M}_2(-\tau, 0; R^n)$ denote the completion of $C(-\tau, 0; R^n)$ with respect to the inner product

$$\langle f, g \rangle_{\mathcal{M}_2} = \langle f(0), g(0) \rangle_{R^n} + \int_{-\tau}^{0} \langle f(t), g(t) \rangle_{R^n} dt$$

Clearly, $\mathcal{M}_2$ is isomorphic to $R^n \oplus L^2(-\tau, 0; R^n)$.

Now define the operator $\mathcal{A}$ on $\mathcal{M}_2(-\tau, 0; R^n)$ by

$$(\mathcal{A}z)(\theta) = \begin{cases} Az(0) + Bz(-\tau) & \text{if } \theta = 0 \\ \frac{dz}{d\theta} & \text{if } -\tau \le \theta < 0 \end{cases}$$

with domain $H^1[-\tau, 0; R^n]$. Then it can be shown that $\mathcal{A}$ generates a C^0 semigroup T_t given by

$$(T_t h)(\theta) = \begin{cases} x(t+\theta) & \text{if } t+\theta \ge 0 \\ h(t+\theta) & \text{if } t+\theta < 0 \end{cases}$$

for $-\tau \le \theta < 0$ where $x(t)$ is the solution of (4.1) with $f = 0$. We can now write equation (4.1) in an $\mathcal{M}_2$ space sense as follows:

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{F}(z(t)) \qquad (4.2)$$

where

$$\mathcal{F}(z(t))(\theta) = \begin{cases} f(t, z(t), z_\tau(t)) & \text{if } \theta = 0 \\ 0 & \text{if } -\tau \le \theta < 0 \end{cases}$$

By variation of parameters, we can write (2) in the form

$$z(t) = T_t z(0) + \int_0^t T_{t-s}\mathcal{F}(z(s))ds \tag{4.3}$$

Proposition 1 If there exists a matrix C such that

$$||e^{(A-C)t}|| \le Ke^{-\delta t}$$

for some $K, \delta \ge 0$ and for which

$$\tau||C|| + \frac{K}{\delta}(||B + C|| + ||A - C||\ ||C||\tau) < 1$$

then the semigroup T_t is exponentially stable on $\mathcal{M}_2$.

Proof By theorem 2, the conditions guarantee that the linear system

$$\dot{x}(t) = Ax(t) + Bx(t - \tau)$$

is exponentially stable, and so

$$||x(t)|| \le \overline{K}e^{-\overline{\delta}t} \quad ,$$

for some $\overline{K}, \overline{\delta} \ge 0$. Hence if we define $z(t) \in \mathcal{M}_2$ for each $t \ge 0$ by

$$z(t)(\theta) = x(t - \theta) \quad ,$$

then

$$\begin{aligned} ||z(t)||^2_{\mathcal{M}_2} &= ||x(t)||^2_{R^n} + \int_{-\tau}^0 ||x(t+\theta)||^2_{R^n} d\theta \\ &\le \overline{K}^2 e^{-2\overline{\delta}t} + \overline{K}^2 L e^{-2\overline{\delta}t} \end{aligned}$$

where $L = \frac{1}{2\bar{\delta}}(e^{2\bar{\tau}} - 1)$. □

We shall assume that $f : R^+ \oplus R^n \oplus \mathcal{M}_2(-\tau, 0; R^n) \longrightarrow R^n$ satisfies the condition

$$||f(t, z(t), z_\tau(t))|| \leq g(||z(t)||_{\mathcal{M}_2})||z(t)||_{\mathcal{M}_2} \tag{4.4}$$

for each function $z : R^+ \longrightarrow R^n$, where $g(x) \longrightarrow 0$ as $x \longrightarrow 0$, and such that g is monotonically increasing.

Theorem 4 Let $S = \{\alpha : g(\alpha) < 1\}$, and let $B = \{z \in \mathcal{M}_2 : ||z|| \in S\}$. Then, under the assumptions of proposition 1 and the above condition on f, the system (4.1) is asymptotically stable in B.

Proof B is an open ball in $\mathcal{M}_2$ with centre 0 and so by (4.4) and the definition of $\mathcal{F}$, we have

$$||\mathcal{F}(z)||_{\mathcal{M}_2} \leq ||z||_{\mathcal{M}_2}$$

if $z \in B$. By (4.4) and proposition 1, we have

$$||z(t)||_{\mathcal{M}_2} \leq K_1 e^{-\delta_1 t}||z(0)||_{\mathcal{M}_2} + \int_0^t K_1 e^{-\delta_1(t-s)}||z(s)||_{\mathcal{M}_2} ds \tag{4.5}$$

if $z(0) \in B$, for sufficiently small t. Applying Gronwall's inequality gives

$$||z(t)||_{\mathcal{M}_2} \leq ||z(0)||_{\mathcal{M}_2}$$

and so (4.5) is valid for all $t \geq 0$. Another application of Gronwall's inequality gives the result. □

5 Example

Consider the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -8 & 1 \\ -\frac{7}{8} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t-\frac{1}{8}) \\ x_2(t-\frac{1}{8}) \end{pmatrix}$$

Choosing $c_{11} = c_{12} = 0$, $c_{21} = -1$, $c_{22} = 1$ in theorem 1, we have

$$\omega_{11} = \omega_{22} = \frac{1}{4} \ , \ \omega_{12} = \omega_{21} = \frac{1}{2}$$

and $\rho((\omega_{ij})) < 1$ and so the zero solution of this system is asymptotically stable. However, the stability of the system cannot be checked by the criteria of Amemiya (1981.1983) or Mori et al (1981).

6 Conclusions

In this paper we have obtained a new criterion for the stability of differential delay systems. The main condition given in theorem 1 is given in terms of a matrix C of parameters, which makes the criterion particularly flexible, although the optimal choice of C will require further investigation.

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