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Analysis as a source of geometry: a non-geometric representation of the Dirac equation*

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Abstract

Consider a formally self-adjoint first order linear differential operator acting on pairs (two-columns) of complex-valued scalar fields over a four-manifold without boundary. We examine the geometric content of such an operator and show that it implicitly contains a Lorentzian metric, Pauli matrices, connection coefficients for spinor fields and an electromagnetic covector potential. This observation allows us to give a simple representation of the massive Dirac equation as a system of four scalar equations involving an arbitrary two-by-two matrix operator as above and its adjugate. The point of the paper is that in order to write down the Dirac equation in the physically meaningful four-dimensional hyperbolic setting one does not need any geometric constructs. All the geometry required is contained in a single analytic object—an abstract formally self-adjoint first order linear differential operator acting on pairs of complex-valued scalar fields.

Keywords: analysis of partial differential equations, gauge theory, Dirac equation

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1. Introduction

The paper is an attempt at developing a relativistic field theory based on the concepts from the analysis of partial differential equations as opposed to geometric concepts. The long-term goal is to recast quantum electrodynamics in curved spacetime in such ‘non-geometric’ terms. The potential advantage of formulating a field theory in ‘analytic’ terms is that there might be a chance of describing the interaction of different physical fields in a more consistent, and, hopefully, non-perturbative manner.

The current paper deals with the Dirac equation in curved spacetime, with the electromagnetic field appearing as a prescribed external covector potential. We expect to treat the Maxwell system in a separate paper.

Let M be a four-manifold without boundary and let m be the electron mass.

The traditional way of writing the massive Dirac equation is as follows. We equip our manifold M with a prescribed Lorentzian metric and a prescribed electromagnetic covector potential, and write the Dirac equation using the rules of spinor calculus, see appendix A. In the process of doing this one may encounter topological obstructions: not every four-manifold admits a Lorentzian metric and, even if it admits one, it may still not admit a spin structure.

We give now an analytic representation of the massive Dirac equation which, for parallelizable manifolds, turns out to be equivalent to the traditional geometric representation.

For the sake of clarity, prior to describing our analytic construction let us explain why we will not encounter topological obstructions related to the second Stiefel–Whitney class. We will work with operators satisfying the non-degeneracy condition (B.17) which is very natural from the analytic point of view as it is a generalization (weaker version) of the standard ellipticity condition (B.16). It turns out that the imposition of the non-degeneracy condition (B.17) has far reaching geometric consequences: it implies that our manifold M is parallelizable. Thus, in our construction we deal only with parallelizable manifolds, but we do not state the parallelizability condition explicitly because it is automatically encoded in the analytic non-degeneracy condition (B.17).

We assume that our four-manifold M is equipped with a prescribed positive density ρ which allows us to define an inner product on columns of complex-valued scalar fields, see formula (B.1), and, consequently, the concept of formal self-adjointness, see formula (B.2).

Let L be a first order linear differential operator acting on two-columns of complex-valued scalar fields over M . The standard invariant analytic way of describing this operator is by means of its principal symbol $L_{\text{prin}}(x, p)$ and subprincipal symbol $L_{\text{sub}}(x)$, see appendix B for details. Here $x = (x^1, x^2, x^3, x^4)$ are local coordinates on M and $p = (p_1, p_2, p_3, p_4)$ is the dual variable (momentum). It is known that L_{prin} and L_{sub} are invariantly defined 2×2 matrix-functions on T^*M and M respectively and that these matrix-functions completely determine the first order differential operator L .

Further on we assume that our differential operator L is formally self-adjoint and satisfies the non-degeneracy condition (B.17).

We now take an arbitrary matrix-function

$$Q: M \rightarrow GL(2, \mathbb{C}) \quad (1.1)$$

and consider the transformation of our differential operator

$$L \mapsto Q^*LQ. \quad (1.2)$$

The motivation for looking at such transformations is as follows. Let us write down the action (variational functional) associated with our operator, $\int_M v^*(Lv)\rho dx$, and let us perform an invertible linear transformation

$$v \mapsto Qv$$

in the vector space $V := \{v: M \rightarrow \mathbb{C}^2\}$ of two-columns of complex-valued scalar fields. Then the action transforms as

$$\int_M v^*(Lv)\rho dx \mapsto \int_M v^*(Q^*LQv)\rho dx.$$

We see that the transformation (1.2) of our differential operator describes the transformation of the integrand in the formula for the action. We choose to interpret (1.2) as a gauge transformation.

The transformation (1.2) of the differential operator L induces the following transformations of its principal and subprincipal symbols:

$$L_{\text{prin}} \mapsto Q^*L_{\text{prin}}Q, \tag{1.3}$$

$$L_{\text{sub}} \mapsto Q^*L_{\text{sub}}Q + \frac{i}{2} \left(Q_{x^\alpha}^* (L_{\text{prin}})_{\mathbb{R}} Q - Q^* (L_{\text{prin}})_{\mathbb{R}} Q_{x^\alpha} \right), \tag{1.4}$$

where the subscripts indicate partial derivatives. Here we made use of formula (9.3) from [5].

Comparing formulae (1.3) and (1.4) we see that, unlike the principal symbol, the subprincipal symbol does not transform in a covariant fashion due to the appearance of terms with the gradient of the matrix-function $Q(x)$. In order to identify the sources of this non-covariance we observe that any matrix-function (1.1) can be written as a product of three terms: a complex matrix-function of determinant one, a positive scalar function and a complex scalar function of modulus one. Hence, we examine the three gauge-theoretic actions separately.

Take an arbitrary scalar function

$$\psi: M \rightarrow \mathbb{R} \tag{1.5}$$

and consider the transformation of our differential operator

$$L \mapsto e^\psi L e^\psi. \tag{1.6}$$

The transformation (1.6) is a special case of the transformation (1.2) with $Q = e^\psi I$, where I is the 2×2 identity matrix. Substituting this Q into formula (1.4), we get

$$L_{\text{sub}} \mapsto e^{2\psi} L_{\text{sub}}, \tag{1.7}$$

so the subprincipal symbol transforms in a covariant fashion.

Now take an arbitrary scalar function

$$\phi: M \rightarrow \mathbb{R} \tag{1.8}$$

and consider the transformation of our differential operator

$$L \mapsto e^{-i\phi} L e^{i\phi}. \tag{1.9}$$

The transformation (1.9) is a special case of the transformation (1.2) with $Q = e^{i\phi} I$. Substituting this Q into formula (1.4), we get

$$L_{\text{sub}}(x) \mapsto L_{\text{sub}}(x) + L_{\text{prin}}(x, (\text{grad } \phi)(x)), \tag{1.10}$$

so the subprincipal symbol does not transform in a covariant fashion. We do not take any action with regards to the non-covariance of (1.10).

Finally, take an arbitrary matrix-function

$$R: M \rightarrow SL(2, \mathbb{C}) \tag{1.11}$$

and consider the transformation of our differential operator

$$L \mapsto R^*LR. \tag{1.12}$$

Of course, the transformation (1.12) is a special case of the transformation (1.2): we are looking at the case when $\det Q(x) = 1$. It turns out that it is possible to overcome the resulting non-covariance in (1.4) by introducing the *covariant subprincipal symbol* $L_{\text{csub}}(x)$ in accordance with formula

$$L_{\text{csub}} := L_{\text{sub}} - f(L_{\text{prin}}), \tag{1.13}$$

where f is a function (more precisely, a nonlinear differential operator) mapping a 2×2 non-degenerate Hermitian principal symbol $L_{\text{prin}}(x, p)$ to a 2×2 Hermitian matrix-function $(f(L_{\text{prin}}))(x)$. The function f is chosen from the condition that the transformation (1.12) of the differential operator induces the transformation

$$L_{\text{csub}} \mapsto R^*L_{\text{csub}}R \tag{1.14}$$

of its covariant subprincipal symbol and the condition

$$f(e^{2\psi}L_{\text{prin}}) = e^{2\psi}f(L_{\text{prin}}), \tag{1.15}$$

where ψ is an arbitrary scalar function (1.5).

The existence of a function f satisfying conditions (1.14) and (1.15) is a nontrivial fact, a feature specific to a system of two equations in dimension four. The explicit formula for the function f is formula (5.2).

Let us summarize the results of our gauge-theoretic analysis.

- Our first order differential operator L is completely determined by its principal symbol $L_{\text{prin}}(x, p)$ and covariant subprincipal symbol $L_{\text{csub}}(x)$.
- The transformation (1.2) of the differential operator induces the transformation (1.3) of its principal symbol.
- Transformations (1.6), (1.9) and (1.12) of the differential operator induce transformations

$$L_{\text{csub}} \mapsto e^{2\psi}L_{\text{csub}}, \tag{1.16}$$

$$L_{\text{csub}}(x) \mapsto L_{\text{csub}}(x) + L_{\text{prin}}(x, (\text{grad } \phi)(x)) \tag{1.17}$$

and (1.14) of its covariant subprincipal symbol.

We use the notation

$$L = \text{Op}(L_{\text{prin}}, L_{\text{csub}}) \tag{1.18}$$

to express the fact that our operator is completely determined by its principal symbol and covariant subprincipal symbol. The differential operator L can be written down explicitly, in local coordinates, via the principal symbol L_{prin} and covariant subprincipal symbol L_{csub} in accordance with formula (5.4), so formula (1.18) is shorthand for (5.4). We call (1.18) the *covariant representation* of the differential operator L .

Recall now a definition from elementary linear algebra. The *adjugate* of a 2×2 matrix is defined as

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \text{adj } P. \tag{1.19}$$

Using the covariant representation (1.18) and matrix adjugation (1.19) we can define the adjugate of the differential operator L as

$$\text{Adj } L := \text{Op}(\text{adj } L_{\text{prin}}, \text{adj } L_{\text{csub}}). \tag{1.20}$$

Note that in the case when the principal symbol does not depend on the position variable x (this corresponds to Minkowski spacetime, which is the case most important for applications) the definition of the adjugate differential operator simplifies. In this case the subprincipal symbol coincides with the covariant subprincipal symbol and one can treat the differential operator L as if it were a matrix: formula (1.20) becomes

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \mapsto \begin{pmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{pmatrix} = \text{Adj } L. \tag{1.21}$$

We define the Dirac operator as the differential operator

$$D := \begin{pmatrix} L & mI \\ mI & \text{Adj } L \end{pmatrix} \tag{1.22}$$

acting on four-columns v of complex-valued scalar fields. Here I is the 2×2 identity matrix. We claim that the system of four scalar equations

$$Dv = 0 \tag{1.23}$$

is equivalent to the Dirac equation in its traditional geometric formulation.

Examination of formula (1.22) raises the following questions.

- Where is the Lorentzian metric?
- Why don't we encounter topological obstructions?
- Where are the Pauli matrices?
- Where are the spinors?
- Where are the connection coefficients for spinor fields?
- Where is the electromagnetic covector potential?
- Where is Lorentz invariance?

These questions will be answered in sections 2–8. In section 9 we will collect together all the formulae from sections 2–8 and show, by direct substitution, that our equation (1.23) is indeed the Dirac equation (A.13). This fact will be presented in the form of theorem 9.1, the main result of our paper.

2. Lorentzian metric

Observe that the determinant of the principal symbol is a quadratic form in the dual variable (momentum) p :

$$\det L_{\text{prin}}(x, p) = -g^{\alpha\beta}(x) p_\alpha p_\beta. \tag{2.1}$$

We interpret the real coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, 2, 3, 4$, appearing in formula (2.1) as components of a (contravariant) metric tensor.

Lemma 2.1. *Our metric is Lorentzian, i.e. it has three positive eigenvalues and one negative eigenvalue.*

Proof. Decomposing $L_{\text{prin}}(x, p)$ with respect to the standard basis

$$s^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.2)$$

in the real vector space of 2×2 Hermitian matrices, we get

$$L_{\text{prin}}(x, p) = s^j c_j(x, p), \quad (2.3)$$

where the repeated index j indicates summation over $j = 1, 2, 3, 4$ and the $c_j(x, p)$ are some real-valued functions on T^*M . Each coefficient $c_j(x, p)$ is linear in p , so

$$c_j(x, p) = e_j^\alpha(x) p_\alpha, \quad (2.4)$$

where the repeated index α indicates summation over $\alpha = 1, 2, 3, 4$ and e_j is some real-valued vector field with components $e_j^\alpha(x)$. The quartet of real-valued vector fields e_j , $j = 1, 2, 3, 4$, is called the *frame*. Note that the non-degeneracy condition (B.17) ensures that the vector fields e_j are linearly independent at every point of our manifold M .

Substituting (2.2) and (2.4) into (2.3), we get

$$L_{\text{prin}}(x, p) = s^j e_j^\alpha(x) p_\alpha = \begin{pmatrix} e_4^\alpha p_\alpha + e_3^\alpha p_\alpha & e_1^\alpha p_\alpha - i e_2^\alpha p_\alpha \\ e_1^\alpha p_\alpha + i e_2^\alpha p_\alpha & e_4^\alpha p_\alpha - e_3^\alpha p_\alpha \end{pmatrix}. \quad (2.5)$$

Calculating the determinant of (2.5) and substituting the result into the lhs of (2.1), we get $g^{\alpha\beta} p_\alpha p_\beta = (e_1^\alpha p_\alpha)^2 + (e_2^\alpha p_\alpha)^2 + (e_3^\alpha p_\alpha)^2 - (e_4^\alpha p_\alpha)^2$. □

The proof of lemma 2.1 explains why we do not encounter topological obstructions: condition (B.17) implies that our manifold is parallelizable.

It is also easy to see that our frame defined in accordance with formula (2.5) is orthonormal with respect to the metric (2.1):

$$g_{\alpha\beta} e_j^\alpha e_k^\beta = \begin{cases} 0, & \text{if } j \neq k, \\ 1, & \text{if } j = k \neq 4, \\ -1, & \text{if } j = k = 4. \end{cases} \quad (2.6)$$

3. Geometric meaning of our transformations

In section 1 we defined four transformations of a formally self-adjoint 2×2 first order linear differential operator:

- conjugation (1.6) by a positive scalar function,
- conjugation (1.9) by a complex scalar function of modulus one,
- conjugation (1.12) by an $SL(2, \mathbb{C})$ -valued matrix-function and
- adjugation (1.20).

In this section we establish the geometric meaning of the transformations (1.6), (1.12) and (1.20). We do this by looking at the resulting transformations of the principal symbol.

We choose to examine the three transformations listed above in reverse order: first (1.20), then (1.12) and, finally, (1.6).

We know that L_{prin} can be written in terms of the standard basis (2.2) and frame e_j as (2.5). Similarly, $\text{adj } L_{\text{prin}}$ can be written as

$$\text{adj } L_{\text{prin}}(x, p) = s^j \tilde{e}_j^\alpha(x) \rho_\alpha = \begin{pmatrix} \tilde{e}_4^\alpha \rho_\alpha + \tilde{e}_3^\alpha \rho_\alpha & \tilde{e}_1^\alpha \rho_\alpha - i \tilde{e}_2^\alpha \rho_\alpha \\ \tilde{e}_1^\alpha \rho_\alpha + i \tilde{e}_2^\alpha \rho_\alpha & \tilde{e}_4^\alpha \rho_\alpha - \tilde{e}_3^\alpha \rho_\alpha \end{pmatrix}, \quad (3.1)$$

where \tilde{e}_j is another frame. Examination of formulae (1.19), (2.5) and (3.1) shows that the two frames, e_j and \tilde{e}_j , differ by spatial inversion:

$$e_j \mapsto -e_j, \quad j = 1, 2, 3, \quad e_4 \mapsto e_4. \quad (3.2)$$

The transformation (1.12) of the differential operator induces the following transformation of its principal symbol:

$$L_{\text{prin}} \mapsto R^* L_{\text{prin}} R. \quad (3.3)$$

If we recast the transformation (3.3) in terms of the frame e_j (see formula (2.5)), we will see that we are looking at a linear transformation of the frame

$$e_j \mapsto \Lambda_j^k e_k, \quad (3.4)$$

with some real-valued coefficients $\Lambda_j^k(x)$. The transformation of the principal symbol (3.3) preserves the Lorentzian metric (2.1), so the linear transformation of the frame (3.4) is a Lorentz transformation.

Of course, the transformation (3.2) is also a Lorentz transformation and it can be written in the form (3.4) with $\Lambda_j^k = \text{diag}(-1, -1, -1, +1)$. The difference between the two Lorentz transformations is that in the case of adjugation (1.20) we get $\det \Lambda_j^k = -1$, whereas in the case of conjugation (1.12) by an $SL(2, \mathbb{C})$ -valued matrix-function we get $\det \Lambda_j^k = +1$.

Finally, let us establish the geometric meaning of conjugation (1.6) by a positive scalar function. The transformation (1.6) of the differential operator induces the following transformation of its principal symbol:

$$L_{\text{prin}} \mapsto e^{2\psi} L_{\text{prin}}. \quad (3.5)$$

Comparing formulae (2.1) and (3.5) we see that we are looking at a conformal scaling of the metric

$$g^{\alpha\beta} \mapsto e^{4\psi} g^{\alpha\beta}. \quad (3.6)$$

Remark 3.1. We did not examine in this section the geometric meaning of the transformation (1.9). We did not do it because this transformation does not affect the principal symbol: one has to look at the subprincipal symbol to understand the geometric meaning of the transformation (1.9). We will do this later, in section 6: see formula (6.3).

4. Pauli matrices

The principal symbol $L_{\text{prin}}(x, p)$ of our operator L is linear in the dual variable p , so it can be written as

$$L_{\text{prin}}(x, p) = \sigma^\alpha(x) \rho_\alpha. \quad (4.1)$$

The four matrix-functions $\sigma^\alpha(x)$, $\alpha = 1, 2, 3, 4$, appearing in (4.1) are, by definition, our Pauli matrices.

The adjugate of the principal symbol can be written as

$$\text{adj } L_{\text{prin}}(x, p) = \tilde{\sigma}^\alpha(x) p_\alpha. \tag{4.2}$$

The matrices $\tilde{\sigma}^\alpha(x)$, $\alpha = 1, 2, 3, 4$, appearing in formula (4.2) are the adjugates of those from (4.1).

We have

$$\left[L_{\text{prin}}(x, p) \right] \left[\text{adj } L_{\text{prin}}(x, p) \right] = \left[\text{adj } L_{\text{prin}}(x, p) \right] \left[L_{\text{prin}}(x, p) \right] = -I g^{\alpha\beta} p_\alpha p_\beta, \tag{4.3}$$

where I is the 2×2 identity matrix and $g^{\alpha\beta}$ is the metric from formula (2.1). Formula (4.3) implies

$$\left[L_{\text{prin}}(x, p) \right] \left[\text{adj } L_{\text{prin}}(x, q) \right] + \left[L_{\text{prin}}(x, q) \right] \left[\text{adj } L_{\text{prin}}(x, p) \right] = -2I g^{\alpha\beta} p_\alpha q_\beta,$$

$$\left[\text{adj } L_{\text{prin}}(x, p) \right] \left[L_{\text{prin}}(x, q) \right] + \left[\text{adj } L_{\text{prin}}(x, q) \right] \left[L_{\text{prin}}(x, p) \right] = -2I g^{\alpha\beta} p_\alpha q_\beta.$$

Substituting (4.1) and (4.2) into the above formulae we arrive at (A.3) and (A.4). This means that our matrices $\sigma^\alpha(x)$ defined in accordance with formula (4.1) satisfy the abstract definition of Pauli matrices, definition A.1.

5. Covariant subprincipal symbol

Recall that we defined the covariant subprincipal symbol $L_{\text{csub}}(x)$ in accordance with formula (1.13). We need now to determine the function f appearing in this formula.

Let $R(x)$ be as in (1.11). Formulae (1.4) and (1.13) imply that the transformation (1.12) of the differential operator induces the following transformation of the matrix-function $L_{\text{csub}}(x)$:

$$\begin{aligned} L_{\text{csub}} \mapsto & R^* \left(L_{\text{csub}} + f(L_{\text{prin}}) \right) R - f \left(R^* L_{\text{prin}} R \right) \\ & + \frac{i}{2} \left(R_{x^\alpha}^* \left(L_{\text{prin}} \right)_{p_\alpha} R - R^* \left(L_{\text{prin}} \right)_{p_\alpha} R_{x^\alpha} \right). \end{aligned}$$

Comparing with (1.14) we see that our function f has to satisfy the condition

$$f \left(R^* L_{\text{prin}} R \right) = R^* f \left(L_{\text{prin}} \right) R + \frac{i}{2} \left(R_{x^\alpha}^* \left(L_{\text{prin}} \right)_{p_\alpha} R - R^* \left(L_{\text{prin}} \right)_{p_\alpha} R_{x^\alpha} \right) \tag{5.1}$$

for any non-degenerate 2×2 Hermitian principal symbol $L_{\text{prin}}(x, p)$ and any matrix-function (1.11). Thus, we are looking for a function f satisfying conditions (1.15) and (5.1).

Put

$$f \left(L_{\text{prin}} \right) := -\frac{i}{16} g_{\alpha\beta} \left\{ L_{\text{prin}}, \text{adj } L_{\text{prin}}, L_{\text{prin}} \right\}_{p_\alpha p_\beta}, \tag{5.2}$$

where subscripts p_α, p_β indicate partial derivatives and

$$\{F, G, H\} := F_{x^\alpha} G H_{p_\alpha} - F_{p_\alpha} G H_{x^\alpha} \tag{5.3}$$

is the generalized Poisson bracket on matrix-functions. Note that the matrix-function in the rhs of formula (5.2) is Hermitian.

Lemma 5.1. *The function (5.2) satisfies conditions (1.15) and (5.1).*

Proof. Substituting (3.5) into (5.2) we see that the terms with the gradient of the function $\psi(x)$ cancel out, which gives us (1.15). As to condition (5.1), the appropriate calculations are performed in appendix D. \square

It is interesting that the generalized Poisson bracket on matrix-functions (5.3) was initially introduced for the purpose of abstract spectral analysis, see formula (1.17) in [5]. It has now come handy in formula (5.2).

We will see later, in section 9, that the rhs of (5.2) is just a way of writing the usual, Levi-Civita, connection coefficients for spinor fields. More precisely, the rhs of (5.2) does not give each spinor connection coefficient separately, it rather gives their sum, the way they appear in the Dirac operator.

Remark 5.2. The function (5.2) is not a unique solution of the system of equations (1.15) and (5.1): one can always add $L_{\text{prin}}(x, A(x))$, where $A(x)$ is an arbitrary prescribed real-valued covector field. We conjecture that our solution (5.2) of the system of equations (1.15) and (5.1) is unique up to the transformation $f(L_{\text{prin}}) \mapsto f(L_{\text{prin}}) + L_{\text{prin}}(x, A(x))$. Unfortunately, we are currently unable to provide a rigorous proof of this conjecture. Moreover, even stating the uniqueness problem in a rigorous and invariant fashion is a delicate issue. Here the main difficulty is that our f is not a function in the usual sense, it is actually a nonlinear differential operator mapping a 2×2 non-degenerate Hermitian principal symbol $L_{\text{prin}}(x, p)$ to a 2×2 Hermitian matrix-function $(f(L_{\text{prin}}))(x)$.

Remark 5.3. If the conjecture stated in remark 5.2 is true, then the function (5.2) is singled out amongst all solutions of the system of equations (1.15) and (5.1) by the property that it does not depend on any prescribed external fields.

For the sake of clarity, we write down the differential operator L explicitly, in local coordinates, in terms of its principal symbol L_{prin} and covariant subprincipal symbol L_{csub} . Combining formulae (B.15), (1.13) and (5.2), we get

$$L = -\frac{i}{2\sqrt{\rho(x)}} \left(\left[(L_{\text{prin}})_{\ell} (x) \right] \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} \left[(L_{\text{prin}})_{\ell} (x) \right] \right) \sqrt{\rho(x)} - \frac{i}{16} \left(g_{\alpha\beta} \left\{ L_{\text{prin}}, \text{adj } L_{\text{prin}}, L_{\text{prin}} \right\}_{\ell} p_\beta \right) (x) + L_{\text{csub}}(x). \tag{5.4}$$

Here the covariant symmetric tensor $g_{\alpha\beta}(x)$ is the inverse of the contravariant symmetric tensor $g^{\alpha\beta}(x)$ defined by formula (2.1), $\{ \cdot, \cdot, \cdot, \cdot \}$ is the generalized Poisson bracket on matrix-functions defined by formula (5.3) and adj is the operator of matrix adjugation (1.19). See also remark B.1 which explains how to read formula (5.4) correctly.

6. Electromagnetic covector potential

The non-degeneracy condition (B.17) implies that for each $x \in M$ the matrices $(L_{\text{prin}})_{\ell}(x)$, $\alpha = 1, 2, 3, 4$, form a basis in the real vector space of 2×2 Hermitian matrices. Here and throughout the paper the subscript ℓ indicates partial differentiation.

Decomposing the covariant subprincipal symbol $L_{\text{csub}}(x)$ with respect to this basis, we get

$$L_{\text{csub}}(x) = \left(L_{\text{prin}} \right)_{\hbar} (x) A_{\alpha}(x) \tag{6.1}$$

with some real coefficients $A_{\alpha}(x)$, $\alpha = 1, 2, 3, 4$.

Formula (6.1) can be rewritten in more compact form as

$$L_{\text{csub}}(x) = L_{\text{prin}}(x, A(x)), \tag{6.2}$$

where A is a covector field with components $A_{\alpha}(x)$, $\alpha = 1, 2, 3, 4$. Formula (6.2) tells us that the covariant subprincipal symbol L_{csub} is equivalent to a real-valued covector field A , the electromagnetic covector potential.

It is easy to see that our electromagnetic covector potential A is invariant under Lorentz transformations (1.12) and conformal scalings of the metric (1.6), whereas formulae (1.17) and (6.2) imply that the transformation (1.9) of the differential operator induces the transformation

$$A \mapsto A + \text{grad } \phi. \tag{6.3}$$

7. Properties of the adjugate operator

In this section we list gauge-theoretic properties of operator adjugation (1.20).

Matrix adjugation (1.19) has the property

$$\text{adj}(R^*PR) = R^{-1}(\text{adj } P)(R^{-1})^* \tag{7.1}$$

for any matrix $R \in \text{SL}(2, \mathbb{C})$. It is easy to see that operator adjugation (1.20) has a property similar to (7.1):

$$\text{Adj}(R^*LR) = R^{-1}(\text{Adj } L)(R^{-1})^* \tag{7.2}$$

for any matrix-function (1.11).

It is also easy to see that operator adjugation (1.20) commutes with the transformations (1.6) and (1.9):

$$\text{Adj}(e^{\psi}Le^{\psi}) = e^{\psi}(\text{Adj } L)e^{\psi}, \quad \text{Adj}(e^{-i\phi}Le^{i\phi}) = e^{-i\phi}(\text{Adj } L)e^{i\phi}.$$

Finally, let us observe that the map (5.2) anticommutes with matrix adjugation (1.19)

$$\text{adj } f(L_{\text{prin}}) = -f(\text{adj } L_{\text{prin}}).$$

This implies that the full symbol of the operator $\text{Adj } L$ is not necessarily the matrix adjugate of the full symbol of the operator L .

In the special case when the principal symbol does not depend on the position variable x we get $f(L_{\text{prin}}) = f(\text{adj } L_{\text{prin}}) = 0$, so in this case the full symbol of the operator $\text{Adj } L$ is the matrix adjugate of the full symbol of the operator L . The definition of the adjugate operator then simplifies and becomes (1.21).

8. Lorentz invariance of the operator (1.22)

In this section we show that our Dirac operator (1.22) is Lorentz invariant. Recall that this operator acts on four-columns of complex-valued scalar fields.

Let $R(x)$ be as in (1.11). Define the 4×4 matrix-function

$$S := \begin{pmatrix} R & 0 \\ 0 & (R^{-1})^* \end{pmatrix}.$$

Then

$$S^*DS = \begin{pmatrix} R^*LR & mI \\ mI & R^{-1}(\text{Adj } L)(R^{-1})^* \end{pmatrix}. \tag{8.1}$$

The operator identity (7.2) tells us that the diagonal terms in (8.1) are adjugates of each other, so formula (8.1) can be rewritten as

$$S^*DS = \begin{pmatrix} R^*LR & mI \\ mI & \text{Adj } (R^*LR) \end{pmatrix}. \tag{8.2}$$

We see that the operator (8.2) has the same structure as (1.22), which proves Lorentz invariance.

9. Main result

Formulae (5.4), (4.1), (2.1), (5.3) (6.2), (1.19) and (1.20) allow us to rewrite our Dirac operator (1.22) in geometric notation—in terms of Lorentzian metric, Pauli matrices and electromagnetic covector potential. This raises the obvious question: what is the relation between our Dirac operator (1.22) and the traditional Dirac operator (A.11)? The answer is given by the following theorem, which is the main result of our paper.

Theorem 9.1. *Our Dirac operator (1.22) and the traditional Dirac operator (A.11) are related by the formula*

$$\rho^{1/2}D\rho^{-1/2} = |\det g_{\kappa\lambda}|^{1/4} D_{\text{trad}} |\det g_{\mu\nu}|^{-1/4}, \tag{9.1}$$

where ρ is the density from our inner product (B.1).

Here, of course, $\det g_{\kappa\lambda} = \det g_{\mu\nu}$. We used different subscripts to avoid confusion because tensor notation involves summation over repeated indices.

Proof of theorem 9.1. Proving the 4×4 operator identity (9.1) reduces to proving the following two separate 2×2 operator identities:

$$\rho^{1/2}L\rho^{-1/2} = |\det g_{\kappa\lambda}|^{1/4} \sigma^\alpha (-i\nabla + A)_\alpha |\det g_{\mu\nu}|^{-1/4}, \tag{9.2}$$

$$\rho^{1/2}(\text{Adj } L)\rho^{-1/2} = |\det g_{\kappa\lambda}|^{1/4} \tilde{\sigma}^\alpha (-i\tilde{\nabla} + A)_\alpha |\det g_{\mu\nu}|^{-1/4}. \tag{9.3}$$

Here σ^α are Pauli matrices (4.1), $\tilde{\sigma}^\alpha$ are their adjugates, and ∇_α and $\tilde{\nabla}_\alpha$ are covariant derivatives defined in accordance with formulae (A.6) and (A.7).

We shall prove the operator identity (9.2). The operator identity (9.3) is proved in a similar fashion.

In the remainder of the proof we work in some local coordinate system. The full symbols of the left- and right-hand sides of (9.2) read

$$\left(L_{\text{prin}}\right)_{\beta} \eta_{\alpha} - \frac{i}{2} \left(L_{\text{prin}}\right)_{x^{\alpha} \beta} - \frac{i}{16} g_{\alpha\beta} \left\{L_{\text{prin}}, \text{adj } L_{\text{prin}}, L_{\text{prin}}\right\}_{\beta} p_{\beta} + \left(L_{\text{prin}}\right)_{\beta} A_{\alpha}$$

and

$$\sigma^{\alpha} \eta_{\alpha} + \frac{i}{4} \sigma^{\alpha} \left(\ln |\det g_{\mu\nu}| \right)_{x^{\alpha}} + \frac{i}{4} \sigma^{\alpha} \tilde{\sigma}_{\beta} \left((\sigma^{\beta})_{x^{\alpha}} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^{\gamma} \right) + \sigma^{\alpha} A_{\alpha}$$

respectively, where $\left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\}$ denotes Christoffel symbols (A.8); see also formulae (B.3) and (B.4) for the definition of the full symbol of a differential operator. Comparing these with account of the fact that $(L_{\text{prin}})_{\beta} = \sigma^{\alpha}$, we see that the proof of the identity (9.2) reduces to the proof of the identity

$$\begin{aligned} & -\frac{i}{2} (\sigma^{\alpha})_{x^{\alpha}} - \frac{i}{16} g_{\alpha\beta} \left\{L_{\text{prin}}, \text{adj } L_{\text{prin}}, L_{\text{prin}}\right\}_{\beta} p_{\beta} \\ & = \frac{i}{4} \sigma^{\alpha} \left(\ln |\det g_{\mu\nu}| \right)_{x^{\alpha}} + \frac{i}{4} \sigma^{\alpha} \tilde{\sigma}_{\beta} \left((\sigma^{\beta})_{x^{\alpha}} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^{\gamma} \right). \end{aligned} \quad (9.4)$$

Using the standard formula $(\ln |\det g_{\mu\nu}|)_{x^{\alpha}} = 2 \left\{ \begin{matrix} \beta \\ \alpha\beta \end{matrix} \right\}$ we rewrite (9.4) as

$$\begin{aligned} & \frac{1}{2} g_{\alpha\beta} \left\{L_{\text{prin}}, \text{adj } L_{\text{prin}}, L_{\text{prin}}\right\}_{\beta} p_{\beta} \\ & = -2(2I g^{\alpha}_{\beta} + \sigma^{\alpha} \tilde{\sigma}_{\beta}) \left((\sigma^{\beta})_{x^{\alpha}} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^{\gamma} \right). \end{aligned} \quad (9.5)$$

Finally, using formula (D.1) we rewrite (9.5) as

$$\begin{aligned} & (\sigma^{\alpha})_{x^{\gamma}} \tilde{\sigma}_{\alpha} \sigma^{\gamma} - \sigma^{\gamma} \tilde{\sigma}_{\alpha} (\sigma^{\alpha})_{x^{\gamma}} \\ & = -2(2I g^{\alpha}_{\beta} + \sigma^{\alpha} \tilde{\sigma}_{\beta}) \left((\sigma^{\beta})_{x^{\alpha}} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^{\gamma} \right). \end{aligned} \quad (9.6)$$

Thus, we have reduced the proof of the operator identity (9.2) to the proof of the identity (9.6) for Pauli matrices. Calculations proving (9.6) are performed in appendix E. \square

It remains only to note that formula (9.1) implies

$$D = \rho^{-1/2} \left| \det g_{\kappa\lambda} \right|^{1/4} D_{\text{trad}} \left| \det g_{\mu\nu} \right|^{-1/4} \rho^{1/2}. \quad (9.7)$$

We identify a four-column of complex-valued scalar fields v with a bispinor field ψ by means of the formula

$$v = \left| \det g_{\alpha\beta} \right|^{1/4} \rho^{-1/2} \psi. \quad (9.8)$$

Substituting (9.7) and (9.8) into (1.23) we get

$$\rho^{-1/2} \left| \det g_{\kappa\lambda} \right|^{1/4} D_{\text{trad}} \psi = 0. \quad (9.9)$$

Clearly, equation (9.9) is equivalent to equation (A.13).

Appendix A. Dirac equation in its traditional form

Before writing down the Dirac equation in its traditional form, let us make several general remarks on the notation that we will be using.

- The notation in this appendix originates from [1, 2]. Covariant derivatives of spinor fields are defined in accordance with formulae (24) and (25) from [3]. The difference with [1–3] is that in the current paper we enumerate local coordinates with indices 1, 2, 3, 4 rather than 0, 1, 2, 3. Also, the difference with [1, 3] is that in the current paper we use opposite Lorentzian signature.
- The construction in this appendix is a generalization of that from appendix A of [6]: in [6] we dealt with the massless Dirac operator in dimension three.
- We will write the Dirac equation in its *spinor representation* as opposed to its *standard representation*, see appendix B in [4] for details. The spinors ξ^a and η_b that we will be using will be Weyl spinors, i.e. left-handed and right-handed spinors. Let us note straight away that the 4×4 matrix differential operator in the lhs of formula (B6) from [4] appears to have a structure different from (1.22). However, it is easy to see that the representation (B6) from [4] reduces to (1.22) if one multiplies by the constant 4×4 matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ from the left.

The construction presented below is local, i.e. we work in a neighbourhood of a given point of a four-manifold M without boundary. We have a prescribed Lorentzian metric $g_{\alpha\beta}(x)$, $\alpha, \beta = 1, 2, 3, 4$, and a prescribed electromagnetic covector potential $A_\alpha(x)$, $\alpha = 1, 2, 3, 4$. The metric tensor is assumed to have three positive eigenvalues and one negative eigenvalue.

Consider a quartet of 2×2 Hermitian matrix-functions $\sigma^\alpha_{\dot{a}b}(x)$. Here the Greek index $\alpha = 1, 2, 3, 4$ enumerates the matrices, whereas the Latin indices $\dot{a} = \dot{1}, \dot{2}$ and $b = 1, 2$ enumerate elements of a matrix. Here and throughout the appendix the first spinor index always enumerates rows and the second columns. We assume that under changes of local coordinates our quartet of matrix-functions transforms as the four components of a vector. Throughout this appendix we use Greek letters for tensor indices and we raise and lower tensor indices by means of the metric.

Define the ‘metric spinor’

$$\epsilon_{ab} = \epsilon_{\dot{a}b} = \epsilon^{ab} = \epsilon^{\dot{a}b} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.1}$$

We will use the rank two spinor (A.1) for raising and lowering spinor indices. Namely, given a quartet of 2×2 Hermitian matrix-functions $\sigma^\alpha_{\dot{a}b}(x)$ we define the quartet of 2×2 Hermitian matrix-functions $\tilde{\sigma}^{a\dot{b}}(x)$ as

$$\tilde{\sigma}^{a\dot{b}} := -\epsilon^{ab} \epsilon^{\dot{a}b} \sigma^\alpha_{\dot{a}b}. \tag{A.2}$$

Note the order of spinor indices in the matrix-functions $\tilde{\sigma}^{a\dot{b}}(x)$: we choose it to be opposite to that in [3] but in agreement with that in [2].

Examination of formulae (A.1) and (A.2) shows that the 2×2 matrices $\sigma^\alpha_{\dot{a}b}$ and $\tilde{\sigma}^{a\dot{b}}$ are adjugates of one another, see formula (1.19) for definition of matrix adjugation. Hence, we could have avoided the use of the ‘metric spinor’ in our construction of the Dirac equation, using the mathematically more sensible concept of matrix adjugation instead. The only reason we introduced the ‘metric spinor’ is to relate the notation of the current paper to that of [1–3].

Further on in this appendix we use matrix notation. This means that we hide spinor indices and write the matrix-functions $\sigma^\alpha_{\dot{a}b}(x)$ and $\tilde{\sigma}^{a\dot{b}}(x)$ as $\sigma^\alpha(x)$ and $\tilde{\sigma}^\alpha(x)$ respectively.

Definition A.1. We say that the 2×2 Hermitian matrix-functions $\sigma^\alpha(x)$ are *Pauli matrices* if these matrix-functions satisfy the identity

$$\sigma^\alpha \tilde{\sigma}^\beta + \sigma^\beta \tilde{\sigma}^\alpha = -2I g^{\alpha\beta}, \tag{A.3}$$

where I is the 2×2 identity matrix and the tilde indicates matrix adjugation.

Remark A.2. The identity (A.3) is, of course, equivalent to

$$\tilde{\sigma}^\alpha \sigma^\beta + \tilde{\sigma}^\beta \sigma^\alpha = -2I g^{\alpha\beta}. \tag{A.4}$$

Further on we assume that our $\sigma^\alpha(x)$ are Pauli matrices.

Consider a pair of spinor fields which we shall write as two-columns

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{A.5}$$

Using matrix notation, we define the covariant derivatives of these spinor fields as

$$\nabla_\alpha \xi := \frac{\partial \xi}{\partial x^\alpha} - \frac{1}{4} \tilde{\sigma}_\beta \left((\sigma^\beta)_{, \alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) \xi, \tag{A.6}$$

$$\tilde{\nabla}_\alpha \eta := \frac{\partial \eta}{\partial x^\alpha} - \frac{1}{4} \sigma_\beta \left((\tilde{\sigma}^\beta)_{, \alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \tilde{\sigma}^\gamma \right) \eta \tag{A.7}$$

respectively, where

$$\left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} := \frac{1}{2} g^{\beta\delta} \left(\frac{\partial g_{\gamma\delta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\delta}}{\partial x^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial x^\delta} \right) \tag{A.8}$$

are the Christoffel symbols.

Formulae (A.6) and (A.7) warrant the following remarks.

- The sign in front of the $\frac{1}{4}$ in formula (A.6) is the opposite of that in formula (24) of [3]. This is because in the current paper we use opposite Lorentzian signature.
- The rhs of formula (A.6) is a generalization of the expression appearing in the rhs of formula (A.3) from [6]. This follows from the observation that the adjugate of a trace-free 2×2 matrix σ_β is $-\sigma_\beta$.
- If we multiply formula (A.6) from the left by the ‘metric spinor’ (A.1), apply complex conjugation and denote $e\tilde{\xi}$ by η , this gives us (A.7).

The massive Dirac equation reads

$$\sigma^\alpha (-i\nabla + A)_\alpha \xi + m\eta = 0, \tag{A.9}$$

$$\tilde{\sigma}^\alpha (-i\tilde{\nabla} + A)_\alpha \eta + m\xi = 0, \tag{A.10}$$

see formulae (B1) and (B2) from [4] or formulae (20.2) and (20.5) from [1].

We define the Dirac operator written in traditional geometric form as

$$D_{\text{trad}} := \begin{pmatrix} \sigma^\alpha (-i\nabla + A)_\alpha & mI \\ mI & \tilde{\sigma}^\alpha (-i\tilde{\nabla} + A)_\alpha \end{pmatrix} \tag{A.11}$$

and the bispinor field as the four-column

$$\psi := \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{A.12}$$

Formulae (A.9) and (A.10) can then be rewritten as

$$D_{\text{trad}} \psi = 0. \tag{A.13}$$

Appendix B. Basic notions from the analysis of PDEs

In this appendix we work with m -columns of complex-valued scalar fields over an n -manifold M without boundary. The main text of the paper deals with the special case $n = 4, m = 2$, but in this appendix n and m are arbitrary.

We assume that our manifold is equipped with a prescribed positive density ρ . This allows us to define an inner product on pairs v, w of m -columns of complex-valued scalar fields

$$\langle v, w \rangle := \int_M w^* v \rho dx, \tag{B.1}$$

where the star stands for Hermitian conjugation, $dx = dx^1 \dots dx^n$ and $x = (x^1, \dots, x^n)$ are local coordinates.

Given a differential operator L , we define its formal adjoint L^* by means of the formal identity

$$\langle Lv, w \rangle = \langle v, L^*w \rangle. \tag{B.2}$$

Consider now a first order differential operator L . In local coordinates it reads

$$L = P^\alpha(x) \frac{\partial}{\partial x^\alpha} + Q(x), \tag{B.3}$$

where $P^\alpha(x)$ and $Q(x)$ are some $m \times m$ matrix-functions and summation is carried out over $\alpha = 1, \dots, n$. The full symbol of the operator L is the matrix-function

$$L(x, p) := iP^\alpha(x) p_\alpha + Q(x). \tag{B.4}$$

Working with the full symbol is inconvenient because the full symbol of a formally self-adjoint operator is not necessarily Hermitian. The standard way of addressing this issue is as follows. We decompose the full symbol into components homogeneous in p , $L(x, p) = L_1(x, p) + L_0(x)$, where

$$L_1(x, p) := iP^\alpha(x) p_\alpha, \quad L_0(x) := Q(x), \tag{B.5}$$

and define the principal and subprincipal symbols as

$$L_{\text{prin}}(x, p) := L_1(x, p), \tag{B.6}$$

$$L_{\text{sub}}(x) := L_0(x) + \frac{i}{2} (L_{\text{prin}})_{x^\alpha p_\alpha}(x) + \frac{i}{2} L_{\text{prin}}(x, \text{grad}(\ln \rho(x))), \tag{B.7}$$

where ρ is the density from (B.1). It is known that L_{prin} and L_{sub} are invariantly defined matrix-functions on T^*M and M respectively, see section 2.1.3 in [8] for details.

Let us explain why the formula for the subprincipal symbol has the particular structure (B.7). Firstly, using formulae (B.5) and (B.6) we rewrite (B.7) as

$$L_{\text{sub}} = Q - \frac{1}{2}(P^\alpha)_{x^\alpha} - \frac{1}{2}P^\alpha(\ln \rho)_{x^\alpha}. \quad (\text{B.8})$$

Here and further on in this paragraph we drop, for the sake of brevity, the dependence on x . The advantage of representing the subprincipal symbol in the form (B.8) is that the rhs is written explicitly in terms of the matrix-valued coefficients P^α and Q of the differential operator (B.3). Let us now substitute (B.3) into the lhs of (B.2), use the formula for our inner product (B.1) and perform integration by parts. We arrive at the expression for the adjoint operator in local coordinates

$$L^* = \widehat{P}^\alpha \frac{\partial}{\partial x^\alpha} + \widehat{Q}, \quad (\text{B.9})$$

where

$$\widehat{P}^\alpha = -(P^\alpha)^*, \quad \widehat{Q} = Q^* - [(P^\alpha)^*]_{x^\alpha} - (P^\alpha)^*(\ln \rho)_{x^\alpha}. \quad (\text{B.10})$$

We then calculate the subprincipal symbol of L^* using formula (B.8) and replacing matrix-valued coefficients accordingly, compare formulae (B.3) and (B.9). We get

$$(L^*)_{\text{sub}} = \widehat{Q} - \frac{1}{2}(\widehat{P}^\alpha)_{x^\alpha} - \frac{1}{2}\widehat{P}^\alpha(\ln \rho)_{x^\alpha}. \quad (\text{B.11})$$

Substitution of (B.10) into (B.11) gives us

$$(L^*)_{\text{sub}} = Q^* - \frac{1}{2}[(P^\alpha)^*]_{x^\alpha} - \frac{1}{2}(P^\alpha)^*(\ln \rho)_{x^\alpha}. \quad (\text{B.12})$$

Comparing formulae (B.8) and (B.12) we conclude that

$$(L^*)_{\text{sub}} = (L_{\text{sub}})^*. \quad (\text{B.13})$$

Thus, the whole point of introducing the two correction terms in (B.7) (last two terms in the rhs) is to ensure that we get the identity (B.13). Had we defined the subprincipal symbol as $L_{\text{sub}} := L_0$ we would not have the identity (B.13).

The definition of the subprincipal symbol (B.7) originates from the classical paper [7] of Duistermaat and Hörmander: see formula (5.2.8) in this paper. Unlike [7], we work with matrix-valued symbols, but this does not affect the formal definition of the subprincipal symbol. What affects the definition of the subprincipal symbol is the fact that we consider operators acting on columns of scalar fields rather than operators acting on columns of half-densities and this leads to the appearance of the $\text{grad} \ln \rho$ term in (B.7). Here we had to make a difficult decision: analysts prefer to work with operators acting on half-densities because this simplifies formulae, however the concept of a half-density is not commonly used in the mathematical physics and theoretical physics communities. We chose to avoid the use of the notion of a half-density at the expense of having an extra correction term in (B.7).

For the principal symbol things are much easier and, obviously, we have an analogue of formula (B.13):

$$(L^*)_{\text{prin}} = (L_{\text{prin}})^*. \quad (\text{B.14})$$

Examination of formulae (B.3)–(B.7) shows that L_{prin} , L_{sub} and ρ uniquely determine the first order differential operator L . Thus, the notions of principal symbol and subprincipal symbol provide an invariant way of describing a first order differential operator.

For the sake of clarity, we write down the differential operator L explicitly, in local coordinates, in terms of its principal and subprincipal symbols:

$$L = -\frac{i}{2\sqrt{\rho(x)}} \left(\left[(L_{\text{prin}})_{\mathbb{R}}(x) \right] \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} \left[(L_{\text{prin}})_{\mathbb{R}}(x) \right] \right) \sqrt{\rho(x)} + L_{\text{sub}}(x). \tag{B.15}$$

Remark B.1. In writing formula (B.15) we used the convention that both operators of partial differentiation $\frac{\partial}{\partial x^\alpha}$ act on all terms which come (as a product) to the right, including the m -column of complex-valued scalar fields v which is present in (B.15) implicitly. Thus, a more explicit way of writing formula (B.15) is

$$Lv = -\frac{i(L_{\text{prin}})_{\mathbb{R}}}{2\sqrt{\rho}} \frac{\partial(\sqrt{\rho}v)}{\partial x^\alpha} - \frac{i}{2\sqrt{\rho}} \frac{\partial\left((L_{\text{prin}})_{\mathbb{R}}\sqrt{\rho}v\right)}{\partial x^\alpha} + L_{\text{sub}}v.$$

Formulae (B.14) and (B.13) tell us that a first order differential operator is formally self-adjoint if and only if its principal and subprincipal symbols are Hermitian matrix-functions.

We say that a formally self-adjoint first order differential operator L is *elliptic* if

$$\det L_{\text{prin}}(x, p) \neq 0, \quad \forall (x, p) \in T^*M \setminus \{0\}, \tag{B.16}$$

and *non-degenerate* if

$$L_{\text{prin}}(x, p) \neq 0, \quad \forall (x, p) \in T^*M \setminus \{0\}. \tag{B.17}$$

The ellipticity condition (B.16) is a standard condition in the spectral theory of differential operators, see, for example, [5]. Our non-degeneracy condition (B.17) is weaker and is designed to cover the case of hyperbolic operators. In order to highlight the difference between the ellipticity condition (B.16) and the non-degeneracy condition (B.17) we consider two special cases.

Special case 1: $n = 3, m = 2$ and $\text{tr } L_{\text{prin}}(x, p) = 0$. In this case conditions (B.16) and (B.17) are equivalent.

Special case 2: $n = 4$ and $m = 2$. The proof of lemma 2.1 shows that for each $x \in M$ there exists a $p \in T_x^*M \setminus \{0\}$ such that $\det L_{\text{prin}}(x, p) = 0$, so it is impossible to satisfy the ellipticity condition (B.16). However, it is possible to satisfy the non-degeneracy condition (B.17). Indeed, consider the quantity (density to the power -1) $\det e_j^\alpha(x)$, where e_j is the frame from formula (2.5). It is easy to see that the non-degeneracy condition (B.17) is equivalent to the condition $\det e_j^\alpha(x) \neq 0, \forall x \in M$. In other words, the non-degeneracy condition (B.17) means that the vector fields $e_j, j = 1, 2, 3, 4$, encoded within the principal symbol in accordance with formula (2.5) are linearly independent at every point of our manifold M .

Appendix C. Additional properties of Pauli matrices

Throughout this appendix $\sigma^\alpha, \alpha = 1, 2, 3, 4$, are Pauli matrices and $\tilde{\sigma}^\alpha$ are their adjugates, see definition A.1.

Lemma C.1. *If P is a 2×2 matrix then*

$$\sigma_\alpha P \tilde{\sigma}^\alpha = -2(\text{tr } P)I, \tag{C.1}$$

$$\sigma_\alpha P \sigma^\alpha = 2 \text{adj } P. \tag{C.2}$$

Proof. Formulae (2.5), (3.1), (4.1) and (4.2) imply

$$\sigma^\alpha = s^j e_j^\alpha, \quad \tilde{\sigma}^\alpha = s^j \tilde{e}_j^\alpha, \tag{C.3}$$

where the matrices s^j are defined in accordance with (2.2) Substituting (C.3) into (C.1) and (C.2) and using the identities (2.6) and (3.2), we get

$$\sigma_\alpha P \tilde{\sigma}^\alpha = -s^1 P s^1 - s^2 P s^2 - s^3 P s^3 - s^4 P s^4,$$

$$\sigma_\alpha P \sigma^\alpha = s^1 P s^1 + s^2 P s^2 + s^3 P s^3 - s^4 P s^4.$$

The rest is a straightforward calculation. □

Note that an alternative way of proving formula (C.1) is by means of formula (1.2.27) from [2].

Appendix D. Technical calculations I

In this appendix we show that the function (5.2) satisfies the condition (5.1).

Formulae (5.3), (4.1) and (4.2) give us

$$\frac{1}{2} g_{\alpha\beta} \left\{ L_{\text{prin}}, \text{adj } L_{\text{prin}}, L_{\text{prin}} \right\}_{l_\alpha p_\beta} = (\sigma^\alpha)_{x^\gamma} \tilde{\sigma}_\alpha \sigma^\gamma - \sigma^\gamma \tilde{\sigma}_\alpha (\sigma^\alpha)_{x^\gamma}. \tag{D.1}$$

Note also that if we transform Pauli matrices σ^α as

$$\sigma^\alpha \mapsto R^* \sigma^\alpha R, \tag{D.2}$$

where $R(x)$ is as in (1.11), then the adjugate Pauli matrices $\tilde{\sigma}^\alpha$ transform as

$$\tilde{\sigma}^\alpha \mapsto R^{-1} \tilde{\sigma}^\alpha (R^{-1})^*, \tag{D.3}$$

see formula (7.1).

Substituting formulae (5.2), (4.1) and (D.1)–(D.3) into (5.1) we rewrite the latter as $Q + Q^* = 0$, where

$$Q := -\frac{i}{8} \left[R^* \sigma^\alpha R_{x^\gamma} R^{-1} \tilde{\sigma}_\alpha \sigma^\gamma R - R^* \sigma^\gamma \tilde{\sigma}_\alpha \sigma^\alpha R_{x^\gamma} \right] + \frac{i}{2} R^* \sigma^\alpha R_{x^\alpha}. \tag{D.4}$$

Hence, in order to prove (5.1) it is sufficient to prove

$$Q = 0. \tag{D.5}$$

Formula (A.4) implies that $\tilde{\alpha}_\alpha \sigma^\alpha = -4I$, so formula (D.4) becomes

$$Q = -\frac{i}{8} R^* \sigma^\alpha R_{x^\gamma} R^{-1} \tilde{\sigma}_\alpha \sigma^\gamma R. \tag{D.6}$$

The matrix-functions $R_{x^\gamma} R^{-1}$ are trace-free, so, by formula (C.1)

$$\sigma^\alpha R_{x^\gamma} R^{-1} \tilde{\sigma}_\alpha = 0. \tag{D.7}$$

Formulae (D.6) and (D.7) imply (D.5).

Appendix E. Technical calculations II

In this appendix we prove the identity (9.6).

Let us fix an arbitrary point $P \in M$ and prove the identity (9.6) at this point. As the left- and right-hand sides of (9.6) are invariant under changes of local coordinates x , it is sufficient to prove the identity (9.6) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the point P , the metric at $x = 0$ is Minkowski and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the identity we are proving involves only first partial derivatives, we may assume, without loss of generality, that the metric is Minkowski for all x in some neighbourhood of the origin.

Further on we assume that the metric is Minkowski. We need to prove

$$Q = 0, \quad (\text{E.1})$$

where

$$Q := (\sigma^\alpha)_{x^\gamma} \tilde{\sigma}_\alpha \sigma^\gamma - \sigma^\gamma \tilde{\sigma}_\alpha (\sigma^\alpha)_{x^\gamma} + 2(2I g^\alpha_\beta + \sigma^\alpha \tilde{\sigma}_\beta) (\sigma^\beta)_{x^\alpha}. \quad (\text{E.2})$$

Formula (E.2) can be rewritten in more compact symmetric form

$$Q = (\sigma^\alpha)_{x^\gamma} \tilde{\sigma}_\alpha \sigma^\gamma + \sigma^\gamma \tilde{\sigma}_\alpha (\sigma^\alpha)_{x^\gamma} + 4(\sigma^\alpha)_{x^\alpha}. \quad (\text{E.3})$$

Using formulae (A.3), (A.4) and the fact that the metric is Minkowski we can now rewrite (E.3) as

$$\begin{aligned} Q &= (\sigma^\alpha)_{x^\gamma} (-2g_\alpha{}^\gamma - \tilde{\sigma}^\gamma \sigma_\alpha) + (-2g^\gamma{}_\alpha - \sigma_\alpha \tilde{\sigma}^\gamma) (\sigma^\alpha)_{x^\gamma} + 4(\sigma^\alpha)_{x^\alpha} \\ &= -(\sigma^\alpha)_{x^\gamma} \tilde{\sigma}^\gamma \sigma_\alpha - \sigma_\alpha \tilde{\sigma}^\gamma (\sigma^\alpha)_{x^\gamma} = -(\sigma^\alpha)_{x^\gamma} \tilde{\sigma}^\gamma \sigma_\alpha - \sigma^\alpha \tilde{\sigma}^\gamma (\sigma_\alpha)_{x^\gamma} \\ &= \sigma^\alpha (\tilde{\sigma}^\gamma)_{x^\gamma} \sigma_\alpha - (\sigma^\alpha \tilde{\sigma}^\gamma \sigma_\alpha)_{x^\gamma}. \end{aligned} \quad (\text{E.4})$$

Formula (C.2) allows us to rewrite formula (E.4) in the form

$$Q = 2 \left[\text{adj} \left((\tilde{\sigma}^\gamma)_{x^\gamma} \right) - (\text{adj } \tilde{\sigma}^\gamma)_{x^\gamma} \right].$$

As the operations of matrix adjugation (1.19) and partial differentiation commute, we arrive at (E.1).

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