The Combinatorics of Adinkras

by

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Abstract

Adinkras are graphical tools created to study representations of supersymmetry algebras. Besides having inherent interest for physicists, the study of adinkras has already shown nontrivial connections with coding theory and Clifford algebras. Furthermore, adinkras offer many easy-to-state and accessible mathematical problems of algebraic, combinatorial, and computational nature. In this work, we make a self-contained treatment of the mathematical foundations of adinkras that slightly generalizes the existing literature. Then, we make new connections to other areas including homological algebra, theory of polytopes, Pfaffian orientations, graph coloring, and poset theory. Selected results include the enumeration of odd dashings for all adinkraizable chromotopologies, the notion of Stiefel-Whitney classes for codes and their vanishing conditions, and the enumeration of all Hamming cube adinkras up through dimension 5.

Thesis Supervisor: Richard Stanley
Title: Professor

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Chapter 1

Introduction

In a series of papers starting with [15], different subsets of the "DFGHILM collaboration" (Doran, Faux, Gates, Hübsch, Iga, Landweber, Miller) have built and extended the machinery of *adinkras.* Following the ubiquitous spirit of visual diagrams in physics, adinkras are combinatorial objects that encode information about the representation theory of supersymmetry algebras. Adinkras have many intricate links with other fields such as graph theory, Clifford theory, and coding theory. Each of these connections provide many problems that can be compactly communicated to a (non-specialist) mathematician. This work is a humble attempt to bridge the language gap and generate communication.

In short, adinkras are *chromotopologies*, a class of edge-colored bipartite graphs, with two additional structures, a *dashing* condition on the edges and a *ranking* condition on the vertices. In this chapter, we give the preliminaries, including a discussion of physical motivation in Section 1.4. We then redevelop the foundations in Chapter 2 in a slightly different way from the existing literature, leading to a cleanly-nested set of classifications (Theorems 2.3.1, 2.3.2, and 2.3.3).

After this semi-expository portion, we look at the two aforementioned conditions, dashings and rankings, separately. For each condition, we extract the purely combinatorial problem, make connections with different area of mathematics, and generalize the corresponding notion to wider classes of graphs.

In Chapter 3 we use homological algebra to study dashings; our main result is the enumeration of odd dashings for any chromotopology. We also make a connection with the theory of Pfaffian orientations. In Chapter 4, we introduce the idea of Stiefel-Whitney classes of codes, a concept inspired by the combination of dashings and topology.

After dashings, we study rankings. In Chapter 5, we use the theory of posets to put a lattice structure on the set of all rankings of any bipartite graph (including chromotopologies); we also count Hamming cube rankings up through dimension 5. After these enumerative results, we introduce the strongly-related concept of *discrete Lipschitz functions* and make some connections between rankings and the theory of polytopes using them.

These chapters focus on combinatorics and not as much on the foundational problems of adinkras from the physics literature. We return to these roots in Chapter 6, ending with a quick survey of recent developments and some original observations.

1.1 Posets

We assume basic notions of graphs. For a graph G, we use $E(G)$ to denote the edges of G and $V(G)$ to denote the vertices of G. We also assume most basic notions of posets (there are many references, including [38]).

One slight deviation from literature is that we consider the Hasse diagram for a poset as a directed graph, with $x \to y$ an edge if y covers x. Thus it makes sense to call the maximal elements (i.e. those x with no $y > x$) *sinks* and the minimal elements *sources*.

For this work, a *ranked* poset is a poset A equipped with a rank function $h\colon A \to \mathbf{Z}$ such that for all x covering y we have $h(x) = h(y)+1$. There is a unique rank function h_0 among these such that 0 is the lowest value in the range of h_0, so it makes sense to define the *rank* of an element v as $h_0(v)$. The largest element in the range of h_0 is then the length of the longest chain in A; we call it the *height* of A. We remark that such a poset is often called a *graded* poset, though there are similar but subtly different uses of that name. Thus, we use *ranked* to avoid ambiguity.

1.2 Chromotopologies and Adinkras

An n-dimensional *chromotopology* is a finite connected simple graph A such that the following conditions hold:

- A is n-regular (every vertex has exactly n incident edges) and bipartite. Respecting the physics literature, we call the two sets in the bipartition of $V(A)$ *bosons* and *fermions*. As the actual choice is mostly arbitrary for our purposes, we will usually not explicitly include this data.

- The elements of $E(A)$ are colored by n colors, which are elements of the set $[n] = \{1, 2, \ldots, n\}$ unless denoted otherwise, such that every vertex is incident to exactly one edge of each color.

- For any distinct i and j, the edges in $E(A)$ with colors i and j form a disjoint union of 4-cycles.

We now introduce the key example of a chromotopology. Define the n-dimensional *Hamming cube* I^n to be the graph with 2^n vertices labeled by the n-codewords, with an edge between two vertices if they differ by exactly one bit. It is easy to see that I^n is bipartite and n-regular. Noticing that $I = I^1$ is just a single edge, our exponentiation I^n is justified as a cartesian product. Now, if two vertices differ at some bit i, $1 \leq i \leq n$, color the edge between them with the color i. The 2-colored 4-cycle condition holds, so we get a chromotopology that we call the *n-cubical chromotopology* I_c^n. Figure 1-1 shows I_c^3.

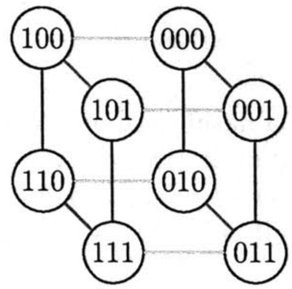

Figure 1-1: The 3-cubical chromotopology I_c^3. We can take the bosons to be either $\{000, 011, 101, 110\}$ or $\{001, 010, 100, 111\}$ and take the fermions to be the other set.

We now define two structures we can put on a chromotopology.

1. let a *ranking* of a bipartite graph (in particular, any chromotopology) A be a map $h : V(A) \to \mathbf{Z}$ that gives A the additional structure of a ranked poset on A via h as the rank function. By this, we mean that we identify A with the Hasse diagram of the said ranked poset with rank function h. We consider two rankings equivalent if they differ only by a translate, as the resulting ranked posets would then be isomorphic. Given a ranking h of A, we say that A is *ranked* by h.

   In this work, such as in Figure 1-3, we will usually represent ranks via vertical placement, with higher values of h corresponding to being higher on the page. The vertices at the odd ranks and the vertices at the even ranks naturally form the bipartition of $V(A)$.

   Any bipartite graph (and thus any chromotopology) A can be ranked as follows: take one choice of bipartition of $V(A)$ into bosons and fermions. Assign the rank function h to take values 0 on all bosons and 1 on all fermions, which creates a ranked poset with 2 ranks. We call the corresponding ranking a *valise*. Because we could have switched the roles of bosons and fermions, each bipartite graph gives rise to exactly two valises. For an example, see Figure 1-2.

   We remark that in the existing literature, such as [9], posets are never mentioned and the following equivalent definition is used, under the names *engineerable* or *non-escheric*: give A the structure of a directed graph, such that in traversing the boundary of any (non-directed) loop with a choice of direction, the number of edges oriented along the direction equals the number of edges oriented against the direction. This is easily seen to be equivalent to our definition.

2. let a *dashing* of a bipartite graph A be a map $d\colon E(A) \to \mathbf{Z}_2$ (in this work, we will always use $\mathbf{Z}_k$ as shorthand to denote $\mathbf{Z}/k\mathbf{Z}$). Given a dashing d of A, we say that A is *dashed* by d. We visually depict a dashing as making each edge $e \in E(A)$ either *dashed* or *solid*, corresponding to $d(e) = 1$ or 0 respectively. We will slightly abuse notation and write $d(v, w)$ to mean $d((v, w))$, where (v, w) is an edge from v to w.

   For a chromotopology A, a dashing is an *odd dashing* if the sum of $d(e)$ as e runs over each 2-colored 4-cycle (that is, a 4-cycle of edges that use a total of 2 colors) is $1 \in \mathbf{Z}_2$ (alternatively, every 2-colored 4-cycle contains an odd number of dashed edges). If A is dashed by an odd dashing d, we say that A is *well-dashed.*

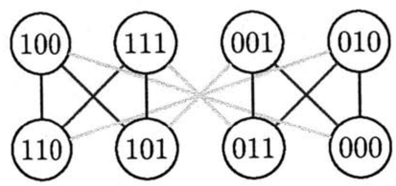

Figure 1-2: A valise; one possible ranking of the chromotopology I_c^3.

An *adinkra* is a ranked well-dashed chromotopology. We call a graph that can be made into an adinkra *adinkraizable.* Since any chromotopology, being a bipartite graph, can be ranked, adinkraizability is equivalent to the condition of having at least one odd dashing. A well-dashed chromotopology is just an adinkraizable chromotopology equipped with this dashing.

We frequently use some forgetful functions in the intuitive way: for example, given any (possibly ranked and/or well-dashed) chromotopology A, we will use "the chromotopology of A" to mean the underlying edge-colored graph of A, forgetting the ranking and the dashing.

Many of our proofs involve algebraic manipulation. To make our treatment more streamlined, we now set up algebraic interpretations of our definitions.

- The condition of A being a chromotopology is equivalent to having a map $q_i\colon V(A) \to V(A)$ for every color i that sends each vertex v to the unique vertex connected to v by the edge with color i, such that the different q_i commute (equivalently, the q_i generate a $\mathbf{Z}_2^n$ action on $V(A)$. The well-definedness of the q_i corresponds to the edge-coloring condition and the commutation requirement corresponds to the 4-cycle condition. Note that q_i is an involution, as applying q_i twice simply traverses the same edge twice. Furthermore, q_i sends any boson to a fermion, and vice-versa.

- The condition of a chromotopology A being well-dashed (with dashing function d) is equivalent to having the maps $\overline{q}_i$ anticommute, where we define $\overline{q}_i\colon \mathbf{R}[V(A)] \to \mathbf{R}[V(A)]$ for every color i by $\overline{q}_i(v) = d(v, q_i(v))q_i(v)$.

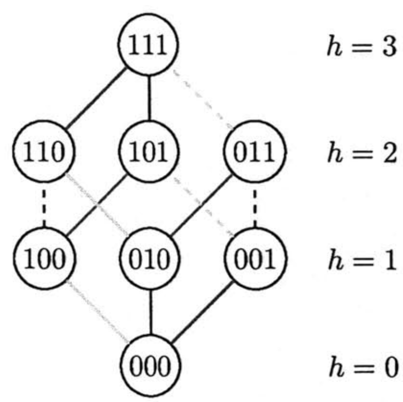

Figure 1-3: An adinkra with the chromotopology I_c^3.

1.3 Multigraphs

It seems natural to extend our definition to multigraphs. Let a *multichromotopology* be a generalization of *chromotopology* where we relax the condition that the graph be simple and now allow loops and multiple-edges. The n-regular condition remains, but is reinterpreted so that a loop counts as degree 1 as opposed to 2. The algebraic condition is still that the q_i must commute. However, the combinatorial version of the rule (that the union of edges of different colors i and j form a disjoint union of 4-cycles) must be extended to allow degenerate 4-cycles that use double-edges or loops. Define the *well-dashed* and *ranked* properties on multichromotopologies analogously, again extending our condition for 2-colored 4-cycles to allow double-edges and loops.

These generalizations exclude each other in a cute way:

- While there are ranked multichromotopologies with double-edges, no well-dashed multichromotopology can have a double-edge because a double-edge immediately gives a degenerate 2-colored 4-cycle, and it is impossible for the sum of dashes over a degenerate 2-colored 4-cycle to be odd.

- The loops have the opposite problem: they allow new well-dashed multichromotopologies, but none of these multichromotopologies can be ranked because bipartite graphs cannot have loops.

See Figure 1-4 for some examples.

The above discussion shows that the natural definition of *multiadinkras*, that is, well-dashed ranked multichromotopologies, does not give us any new objects that are not already adinkras. However, multigraphs naturally appear in our classification paradigm in Chapter 2, so they are still a nice notion to have for our work.

1.4 Physical Motivation

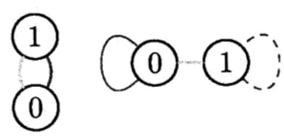

Figure 1-4: A ranked multichromotopology (with double-edge) that cannot be well-dashed, and a well-dashed multichromotopology (with loops) that cannot be ranked.

The reader is equipped to understand the rest of the paper (with the exception of Chapter 6) with no knowledge from this section. However, we hope our brief outline will serve as enrichment that may provide some additional intuition, as well as provide a review of the original problems of interest (where much remains to be done). While knowledge of physics will help in understanding this section, it is by no means necessary. We have neither the space nor the qualification to give a comprehensive review, so we encourage interested readers to explore the original physics literature.

The physics motivation for adinkras is the following: "we want to understand off-shell representations of the N-extended Poincaré superalgebra in the 1-dimensional worldline." There is no need to understand what all of these terms mean[1] to appreciate the rest of the discussion; we now sketch the thinking process that leads to adinkras.

Put simply, we are looking at the representations of the algebra $\mathfrak{po}^{1|N}$ generated by $N+1$ generators $Q_1, Q_2, \ldots, Q_N$ (the *supersymmetry generators*) and $H = i\partial_t$ (the *Hamiltonian*), such that

$$\{Q_I, Q_J\} = 2\delta_{IJ}H,$$
$$[Q_I, H] = 0.$$

Here, δ is the Kronecker delta, $\{A, B\} = AB + BA$ is the anticommutator, and $[A, B] = AB - BA$ is the commutator. We can also say that $\mathfrak{po}^{1|N}$ is a *superalgebra* where the Q_i's are *odd* generators and H is an *even* generator. Since H is basically a time derivative, it changes the *engineering dimension* (physics units) of a function f by a single power of time when acting on f.

Consider $\mathbf{R}$-valued functions $\{\phi_1, \ldots, \phi_m\}$ (the *bosonic fields* or *bosons*) and $\{\psi_1, \ldots, \psi_m\}$ (the *fermionic fields* or *fermions*), collectively called the *component fields.* The fact that the two cardinalities match come from the physics assumption that the representations are *off-shell*; i.e. the component fields do not obey other differential equations. We want to understand representations of $\mathfrak{po}^{1|N}$ acting on the following infinite basis:

$$\{H^k\phi_I, H^k\psi_J \mid k \in \mathbf{N}; I, J \leq m\}.$$

There is a subtlety here, as these infinite-dimensional representations are frequently called "finite-dimensional" by physicists, who would just call the $\{\phi_I\}$ and the $\{\psi_I\}$ as the "basis," emphasizing the finiteness of m. A careful treatment of this is given in [11].

In particular, we want to understand representations of $\mathfrak{po}^{1|N}$ satisfying some physics restrictions (most importantly, having the supersymmetry generators send bosons to only fermions, and vice-versa; this kind of "swapping symmetry" is what supersymmetry tries to study). We restrict our attention to representations where

[1] The author certainly does not.

the supersymmetry generators act as permutations (up to a scalar) and also possibly the Hamiltonian $H = i\partial_t$ on the basis fields: we require that for any boson ϕ and any Q_I,

$$Q_I\phi = \pm(-iH)^s\psi = \pm(\partial_t)^s\psi,$$

where $s \in \{0,1\}$, the sign, and the fermion ψ depends on ϕ and I. We enforce a similar requirement

$$Q_I\psi = \pm i(-iH)^s\phi = \pm i(\partial_t)^s\phi$$

for fermions. We call the representations corresponding to these types of actions *adinkraic representations.* For each of these representations, we associate an *adinkra.* We now form a correspondence between our definition of adinkras in Section 1.2 and our definition for adinkraic representations.

adinkras	representations
vertex bipartition	bosonic/fermionic bipartition
colored edges and q_I	action of Q_I without the sign or powers of $(-iH)$
dashing / sign in $\overline{q}_I$	sign in Q_I
change of rank by q_I and $\overline{q}_I$	powers of $(-iH)$ in Q_I
ranking	partition of fields by engineering dimension

To summarize,

An adinkra encodes a representation of $\mathfrak{po}^{1|N}$. An adinkraic representation is a representation of $\mathfrak{po}^{1|N}$ that can be encoded into an adinkra.

Our main problem is then the following:

Problem 1. What are all the adinkraic representations of $\mathfrak{po}^{1|N}$?

One motivation for the restriction of study to adinkraic representations is having an easily-visualized graphical tool to study these representations; another is that the set of adinkraic representations is already rich enough to contain representations of interest. When the poset structure of our adinkra A is a boolean lattice, we get the classical notion of *superfield* introduced in [35] by Salam and Strathdee. When the poset of A is a height-2 poset (in which case we say that A is a *valise*), we get [12]'s *Clifford supermultiplet.* By direct sums, tensors, and other operations familiar to the Lie algebras setting, it is possible to construct many more representations, a technique that has been extended to higher dimensions in [26].

Chapter 2

The Classification Theorems

We now classify multichromotopologies, chromotopologies, and adinkraizable chromotopologies; we also note the pleasant connections with codes and Clifford algebras that make adinkras fascinating. Compared to the relevant sections of the original literature, our approach is more general and at times more compact, though we owe most ideas in this chapter to the original work.

2.1 Graph Quotients and Codes

In this section, we recover the main result (Theorem 2.3.3) classifying adinkraizable chromotopologies from the existing literature. However, our more general approach (using multigraphs) gives the benefit of easily obtaining classification theorems of multichromotopologies and chromotopologies that are very analogous in flavor.

We now give a quick review of codes (there are many references, including [27]). An *n-codeword* is a vector in $\mathbf{Z}_2^n$, which we usually write as $b_1 b_2 \cdots b_n$, $b_i \in \mathbf{Z}_2$. We distinguish two n-codewords $\overrightarrow{1_n} = 11\ldots1$ and $\overrightarrow{0_n} = 00\ldots0$, and when n is clear from context we suppress the subscript n. The number of 1's in a codeword v is called the *weight* of the string, which we denote by $\mathrm{wt}(v)$. We use $\overline{v}$ to denote the bitwise complement of v, which reverses 0's and 1's. For example, $\overline{00101} = 11010$. An (n,k)*-linear binary code* (for this work, we will not talk about any other kind of codes, so we will just say *code* for short) is a k-dimensional $\mathbf{Z}_2$-subspace of codewords. A code is *even* if all its codewords have weight divisible by 2 and *doubly even* if all its codewords have weight divisible by 4.

Consider the n-cubical chromotopology I_c^n. For any linear code $L \subset \mathbf{Z}_2^n$, the quotient $\mathbf{Z}_2^n/L$ is a $\mathbf{Z}_2$-subspace. Using this, we define the map p_L, which sends I_c^n to the following multichromotopology, which we call the *graph quotient* (or *quotient* for short) I_c^n/L:

- let the vertices of I_c^n/L be labeled by the equivalence classes of $\mathbf{Z}_2^n/L$ and define $p_L(v)$ to be the image of v under the quotient $\mathbf{Z}_2^n/L$. When L is an (n,k)-code, the preimage over every vertex in I_c^n/L contains 2^k vertices, so I_c^n/L has 2^{n-k} vertices.

- let there be an edge $p_L(v,w)$ in I_c^n/L with color i between $p_L(v)$ and $p_L(w)$ in I^n/L if there is at least one edge with color i of the form (v',w') in $\mathbf{Z}_2^n$, with $v' \in p_L^{-1}(v)$ and $w' \in p_L^{-1}(w)$.

Every vertex in I_c^n/L still has degree n (counting possible multiplicity) and the commutivity condition on the q_i's is unchanged under a quotient, so I_c^n/L is indeed a multichromotopology. Denote its underlying multigraph by I^n/L. We now prove some properties of the quotient.

Proposition 2.1.1. *The following hold for $A = I^n/L$:*

1. *A has a loop if and only if L contains a codeword of weight 1; A has a double edge if and only if L contains a codeword of weight 2. Thus, A is a simple graph if and only if L contains only codewords of weight 3 or greater.*

2. *A can be ranked if and only if A is bipartite, which is true if and only if L is an even code.*

Proof. 1. Suppose A has a loop. This means some edge (v,w) in I_c^n has both endpoints v and w mapped to the same vertex in the quotient. Equivalently, $(v-w) \in L$. However, v and w differ by a codeword of weight 1. Suppose A has a double edge (v,w) with colors i and j. Since $q_1(q_2(v)) = v$ in $A = I_c^n/L$, for some $v' \in p_L^{-1}(v)$, we must have in I_c^n that $q_1(q_2(v')) - v'$ is in L. But this is a weight 2 codeword with support in i and j. The logic is reversible in both of these situations.

2. Suppose A were not bipartite, then A has some odd cycle. One of the preimages of this cycle in I_c^n is a path of odd length from some v to some w that both map to the same vertex under the quotient (i.e. $v - w \in L$). Since each edge changes the weight of the vertex by $1 \pmod 2$, $v-w$ must have an odd weight. Since $v - w \in L$, L cannot be an even code. For the other direction, note that if L were an even code such odd cycles cannot occur.

   Recall that any bipartite graph can be ranked by making a valise. If A can be ranked via a rank function h, the sets $\{v \in V(A) \mid h(v) \cong 0 \pmod 2\}$ and $\{v \in V(A) \mid h(v) \cong 1 \pmod 2\}$ must be a bipartition of A because all the edges in A change parity of h. □

The most difficult condition to classify is being well-dashed, which is intricately connected with Clifford algebras. We focus on them in the next section.

2.2 Clifford Algebras and Codes

For our purposes, the *Clifford algebra* is an algebra $\mathbf{Cl}(n)$ with generators $\gamma_1, \ldots, \gamma_n$ and the anticommutation relations

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j} \cdot 1.$$

The Clifford algebra can be defined for any field, but we will typically assume $\mathbf{R}$. There are also more general definitions than what we give, though we won't need them for our paper. For references, see [1] or [7].

We can associate an element of the Clifford algebra to any n-codeword $b = b_1 b_2 \cdots b_n$, by defining

$$\operatorname{clif}(b) = \prod_i \gamma_i^{b_i},$$

where the product is taken in increasing order of i. Call these elements *monomials*. The 2^n possible monomials form a basis of $\mathbf{Cl}(n)$ as a vector space, and the 2^{n+1} signed monomials $\pm \operatorname{clif}(b)$ form a multiplicative group $\mathbf{SMon}(n)$, or just $\mathbf{SMon}$ when the context is clear. It is easy to see that two signed monomials of degrees a and b commute if and only if $ab = 0 \pmod 2$, and one could equivalently define Clifford algebras as commutative superalgebras with odd and even parts generated by the odd and even degree monomials, respectively.

The following facts are needed for Proposition 2.2.4, where we defined the notion of an *almost doubly-even code* as a code with all codewords having weight 0 or 1 (mod 4).

Lemma 2.2.1. *For any two codewords w_1 and w_2 in an almost doubly-even code, we have*

$$(w_1 \cdot w_2) + \operatorname{wt}(w_1)\operatorname{wt}(w_2) = 0 \pmod 2,$$

where the first term is the dot product in $\mathbf{Z}_2^n$.

Proof. Given an almost doubly-even length n code L, introduce 3 new bits in the code to construct a doubly-even length $(n+3)$ code L' via the following map g: for $w \in L$, let $g(w \in L)$ be the concatenation $w|111$ if w has odd weight and $w|000$ if w has even weight. Since the weight parity of two codewords are additive modulo 2, $g(v+w) = g(v) + g(w)$ and L' is linear. Our construction also clearly ensures that L' is doubly-even. It is well known (see, for example, [27]) that doubly-even codes are self-orthogonal, so $(g(w_1) \cdot g(w_2)) = 0 \pmod 2$ for all w_1 and w_2 in L. But $(g(w_1) \cdot g(w_2)) - (w_1 \cdot w_2)$ is $0 \pmod 2$ when either w_1 or w_2 has even weight (because the additional bits 000 cannot affect the dot product) and is $1 \pmod 2$ exactly when both w_1 and w_2 have odd weights. This is equivalent to the condition we want to prove. □

Lemma 2.2.2. *The image of a code L under* clif *is commutative if and only if for all $a, b \in L$,*

$$(a \cdot b) + \operatorname{wt}(a)\operatorname{wt}(b) = 0 \pmod 2.$$

In particular, an almost-doubly-even code satisfies this property.

Proof. Finally, consider $\text{clif}(a) = \gamma_{a_1} \dots \gamma_{a_r}$ and $\text{clif}(b) = \gamma_{b_1} \dots \gamma_{b_s}$, where $r = \text{wt}(a)$ and $s = \text{wt}(b)$. Note we can get from $\text{clif}(a)\,\text{clif}(b)$ to $\text{clif}(b)\,\text{clif}(a)$ in $\text{wt}(a)\,\text{wt}(b)$ transpositions, where we move, in order $\gamma_{b_1}, \cdots, \gamma_{b_s}$ through $\text{clif}(a)$ to the left, picking up exactly $\text{wt}(a)$ powers of (-1). However, we've also overcounted once for each time a and b shared a generator γ_i, since γ_i commutes with itself. Therefore, we have exactly $(a \cdot b) + \text{wt}(a)\,\text{wt}(b)$ powers of (-1). The condition for commutativity is then that this quantity be even for all pairs a and b. □

These concepts of almost-doubly-even codes and commuting codewords have appeared independently in the undergraduate thesis work of Ratanasangpunth [34], where the commutation condition defines a class of codes called *Clifford codes.* Ratanasangpunth goes forth to prove further structural and classification theorems for these codes.

Proposition 2.2.3. *A code L is an almost doubly-even code if and only if L has the property that for a suitable sign function $s(v) \in \{\pm 1\}$ with $s(\overrightarrow{0}) = 1$, the set* $\mathbf{SMon}_L = \{s(v)\,\text{clif}(v) \mid v \in L\}$ *form a subgroup of* $\mathbf{SMon}$.

Proof. Without loss of generality, say $\text{clif}(v) = s(v) \prod_{i=1}^{k} \gamma_i$. Then

$$\begin{aligned} \text{clif}(v)^2 &= (\gamma_1\gamma_2 \cdots \gamma_k)(\gamma_1\gamma_2 \cdots \overline{q}_k) \\ &= (-1)^{(k-1)}(\gamma_2\gamma_3 \cdots \gamma_k)(\gamma_1)(\gamma_1\gamma_2 \cdots \gamma_k) \\ &= (-1)^{k(k-1)/2}. \end{aligned}$$

Suppose s exists. Then, we must not have $(-1) \in \mathbf{SMon}_L$ (since we already have $1 \in \mathbf{SMon}_L$. Therefore, it is necessary to have the last quantity equal 1, which happens exactly when $\text{wt}(v) = 0$ or $1 \pmod 4$ for all $v \in L$.

If L were an almost-doubly even code, then pick a basis $l_1, \dots, l_k$ of L and assign $s(l_i) = 1$ for all i. Note by the above equations $\text{clif}(l_i)^2 = 1$ for all i. The linear independence of the l_i is equivalent to the condition that no group axioms are broken by this choice of l_i. Now, we can extend the definition to

$$s(\prod_{i \in I} \text{clif}(l_i)) = \prod_{i \in I} s(\prod(\text{clif}(l_i)),$$

which is well-defined and closed under multiplication since the l_i commute and square to 1. □

Let an *almost-doubly even code* be a code where all codewords in the code have weight 0 or 1 (mod 4). The following result extends the ideas used in proving [12, Theorem 4.4].

Proposition 2.2.4. *The multichromotopology $A = I_c^n/L$ can be well-dashed if and only if L is an almost doubly-even code.*

Proof. Given a codeword $v = v_1v_2\cdots v_n$, define $\overline{q}_v$ to be the map $\overline{q}_n^{v_n}\cdots\overline{q}_1^{v_1}$.

Suppose we have an odd dashing. Let v and w be codewords in L. Both $\overline{q}_v$ and $\overline{q}_w$ take any vertex to itself in $\mathbf{R}[V(A)]$ with possibly a negative sign, since the $\overline{q}_i$ are basically the q_i with possibly a sign, and following a sequence of q_i corresponding to a codeword is a closed loop in I_c^n/L. This means $\overline{q}_v$ and $\overline{q}_w$ must commute; furthermore, $\overline{q}_v^2$ must be the identity map for any $v \in L$. By Lemma 2.2.2, this is exactly the condition required for L being an almost-doubly even code.

Now, suppose L were an almost doubly-even code. Then by Proposition 2.2.3, we can find a sign function s such that $\{s(v)\,\mathrm{clif}(v) \mid v \in L\}$ form a subgroup $\mathbf{SMon}_L \subset \mathbf{SMon}$, the signed monomials of $\mathbf{Cl}(n)$. This gives a well-defined action of $\mathbf{SMon}$ on $\mathbf{SMon}/\mathbf{SMon}_L$ via left multiplication while possibly introducing signs. The cosets of $\mathbf{SMon}_L$ under $\mathbf{SMon}$ naturally correspond to $V(A)$, so we can define $\overline{q}_i(v)$ to introduce the same sign as γ_i on $\mathrm{clif}(v) \in \mathbf{SMon}/\mathbf{SMon}_L$. Since we have a Clifford algebra action, we get the desired anticommutation relations between $\overline{q}_i$ and thus an odd dashing. □

2.3 The Classifications

In this section, we give three nested classification theorems for three nested types of codes.

To start, recall that quotients of I_c^n are multichromotopologies. Surprisingly, the converse is also true, which gives us our main classification:

Theorem 2.3.1. *Multichromotopologies are exactly quotients I_c^n/L for some code L.*

Proof. Take a multichromotopology A. Consider the abelian group G acting on $V(A)$ generated by the q_i. The elements of G can be written as $g = q_1^{s_1}q_2^{s_2}\cdots q_n^{s_n}$, where $s_i \in \mathbf{Z}_2$ for all i. Consider the isomorphism $\phi\colon G \to L$ which sends such a g to the n-codewords $s_1s_2\cdots s_n \in \mathbf{Z}_2^n$. Take any vertex $v_0 \in V(A)$ and consider its stabilizer group H under G. $\phi(H)$ is a subspace of $\mathbf{Z}_2^n$ and thus must be some code L. Any vertex v is equal to $g(v_0)$ for some $g \in G$, so we may label v with the coset $\phi(g) + L$. It is easy to check the resulting multichromotopology is exactly the one produced by the quotient I_c^n/L. □

Combining Proposition 2.1.1 and Theorem 2.3.1 immediately gives the classification of all chromotopologies and adinkraizable chromotopologies:

Theorem 2.3.2. *Chromotopologies are exactly quotients I_c^n/L, where L is an even code with no codeword of weight* 2.

Theorem 2.3.3 (DFGHILM, basically [12, Theorem 4.4])**.** *Adinkraizable chromotopologies are exactly quotients I_c^n/L, where L is a doubly even code.*

Thanks to Theorem 2.3.1, we can assume the following:

From now on, any multichromotopology (including chromotopologies) A we discuss comes from some (n,k)-code $L(A) = L$. If L is an (n,k)-code, we say that the corresponding A is an (n,k)-multichromotopology (or chromotopology).

An $(n,0)$-multichromotopology must be the n-cubical chromotopology, corresponding to the trivial code $\{\overrightarrow{0}\}$. The smallest non-cubical adinkraizable chromotopology, shown in Figure 2-1, is the quotient I_c^4/L for $L = \{0000, 1111\}$, the smallest non-trivial doubly-even code. Its underlying graph is the complete bipartite graph $K_{4,4}$.

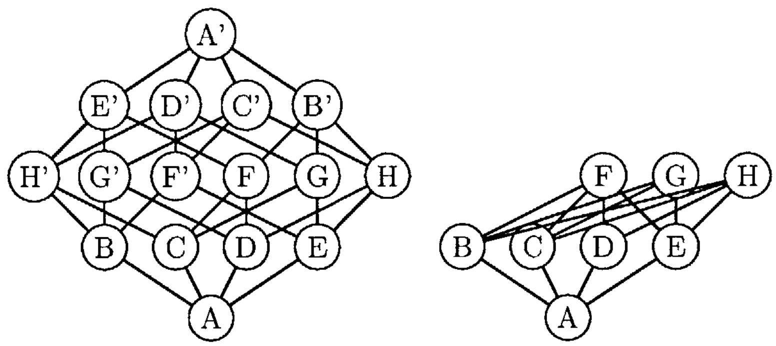

Figure 2-1: The graphs I^4 and $I^4/\{0000, 1111\}$. Labels with the same letter are sent to the same vertex.

While the problem of classifying adinkraizable chromotopologies reduces to that of classifying doubly-even linear codes, the theory of these codes is very rich and nontrivial. Computationally, [32] contains the current status of the classification through an exhaustive search. We invite the reader to explore the other connections between adinkras and coding theory (for example, the irreducible adinkraic representations correspond to the self-dual codes) from the original sources, such as [12].

Finally, we remark that studying well-dashed chromotopologies is basically equivalent to studying Clifford algebras. Some of this intuition is suggested by the proof of Proposition 2.2.4. We discuss this further in Section 6.3.

Chapter 3

Dashings

Given a chromotopology A, define $o(A)$ to be the set of odd dashings of A. Thus, adinkraizable chromotopologies are exactly those A with $|o(A)| > 0$.

Problem 2. What are the enumerative and algebraic properties of $o(A)$?

In this section, we first generalize the concept of odd dashings to other graphs using the homological language we introduced in Section 3.1. We make some observations of this generalized setting from other parts of mathematics, the closest link being that of noneven graphs from the study of Pfaffian orientations.

Most of our results are on the enumeration of dashings of adinkraizable chromotopologies, which is basically settled in this chapter. We introduce the concept of *even dashings* and relate them to odd dashings, showing that not only do the even dashings form a more convenient model for calculations, there is a bijection between the two types of dashings. We then count the number of (odd or even) dashings of I_c^n via a cute application of linear algebra and decompositions. Finally, we generalize our formula to all chromotopologies with a homological algebra computation.

3.1 A Homological View

From the theorems in Chapter 2, we see that the n-cubical chromotopologies I_c^n look like universal covers. We make this intuition rigorous in this section. We appeal to only basic techniques in homological algebra (any standard introduction, such as [24], is more than sufficient), but having another point of view greatly enriches the study of dashings on adinkras, as we will see in Section 3.5.

We work over $\mathbf{Z}_2$. Construct the following 2-dimensional complex $X(A)$ from a chromotopology A. Let C_0 be formal sums of elements of $V(A)$ and C_1 be formal sums of elements of $E(A)$. For each 2-colored 4-cycle C of A, create a 2-cell with C as its boundary as a generator in C_2, the boundary maps $\{d_i\colon C_i \to C_{i-1}\}$ are the natural choices (we do not worry about orientations since we are using $\mathbf{Z}_2$), giving homology groups $H_i = H_i(X(A))$. The most important observation about our complex $X(A)$ is the following, which we return to in Section 3.5.

Proposition 3.1.1. *Let A be an (n, k)-adinkraizable chromotopology with $L(A) = L$. Then $X(A) = X(I_c^n)/L$ as a quotient complex, where L acts freely on $X(I_c^n)$. We have that $X(I_c^n)$ is a simply-connected covering space of $X(A)$, with L the group of deck transformations.*

Proof. The fact that $X(A)$ is a quotient complex is already evident from the construction of the graph quotient, since we have restricted to simple graphs (recall that adinkraizable chromotopologies have simple graphs). It is easy to check that such an action is free for all i-dimensional cells if the codewords have minimal weight greater than i, and the minimal possible weight of our codes is 4.

A quick way to see that $X(I_c^n)$ is simply connected is note that $X(I_c^n)$ is the 2-skeleton of the n-dimensional (solid) Hamming cube D^n. Thus, $X(I^n)$ and D must have matching H_1 and π_1, but D is obviously simply-connected. □

We have learned that there is some independent work by the original authors of the adinkras literature [10] using similar homological techniques. They work with the full complex and not just the 2-skeleton, which has some added benefits but also loses some properties we enjoy (for example, our L is free on the 2-skeleton for doubly-even codes L; this freeness is lost for higher-dimensional cells). We do, however, revisit their construction in Chapter 4 when we discuss Stiefel-Whitney classes of codes.

3.2 Generalizations and Sightings of Odd Dashings

In Section 3.1, we created a (cubical) complex $X(A)$ from a chromotopology A by using A as the 1-skeleton and then attaching 2-cells for every 2-colored 4-cycle. One interpretation of this process is that we are distinguishing certain cycles (whose parities we care about when we dash A) and marking them via 2-cells.

Thus, an obvious generalization of odd dashings is the following: given a graph G and a set of distinguished cycles C in G, call a dashing $d\colon E(G) \to \mathbf{Z}_2$ an *odd dashing* of (G, C) if for all cycles $c \in C$, the sum of $d(e)$ for e in c is odd. We then call the set of odd dashings $o(G, C)$. We can construct a 2-dimensional cell complex $X(G, C)$ by attaching 2-cells for every $c \in C$. When $G = I_c^n/L$ and C is the set of 2-colored 4-cycles in G, we recover our original definitions. Since we will always only have one C to consider at a time, we can suppress this notation and just write $o(G)$ instead of $o(G, C)$ (and similarly for $X(G)$) once C is fixed.

The benefit of this generalization is that the odd dashings have a natural cohomological interpretation: associate a dashing of (G, C) with an element of $H^1(X(G, C), \mathbf{Z}_2)$ in the natural way by sending each edge e to $d(e)$. Then the distinguished element in $H^2(X(G, C))$ obtained by sending each 2-cell to 1 vanishes in cohomology if and only if there is an odd dashing. This suggests that the homological approach may make some proofs easier, as we will exhibit in Section 3.5 (as all our calculations are done in homology, we do not directly use the cohomological observation we just made in this work. However, we would like to note that it plays a critical role in [10]).

Remark 3.2.1. We want to remark that not only do these generalized odd dashings come up naturally in the study of adinkras, they have appeared elsewhere in mathematics under different disguises. A frivolous example is that making the signs of a total differential consistent in a double complex requires changing signs in the grid graph of differential maps such that every square has an odd number of sign changes (see, for example, [5]). A more sophisticated example is in [2, Lemma 10.4], where the authors needed to exhibit an odd dashing on a poset structure of the Weyl group of a Lie algebra $\mathfrak{g}$ in the construction of a $\mathfrak{g}$-module resolution. A uniform study of these occurrences would be very interesting.

3.3 Even Dashings

Let an *even dashing* be a way to dash $E(G)$ such that every 2-colored 4-cycle contains an even number of dashed edges, and let $e(G, C)$ (and again, just $e(G)$ for short) be the set of even dashings. We have the following nice fact:

Lemma 3.3.1. *If $|o(G, C)| > 0$, then we have $|o(G, C)| = |e(G, C)|$.*

Proof. Let $l = |E(G)|$. We may consider a dashing of G as a vector in $\mathbf{Z}_2^l$, where each coordinate corresponds to an edge and is assigned 1 for a dashed edge and 0 for a solid edge. There is an obvious way to add two dashings (i.e. addition in $\mathbf{Z}_2^l$) and there is a zero vector d_0 (all edges solid), so the family of all dashings (with no restrictions) form a vector space $V_{\text{free}}(n)$ of dimension l.

Observe that $e(G)$ is a subspace of $V_{\text{free}}(n)$. To see this, we can directly check that adding two even dashings preserve the even parity of each 2-colored 4-cycle and that d_0 is an even dashing. Alternatively, we can note the restriction of a dashing d having a particular cycle with an even number of dashes just means the inner product of d and some vector with four 1's as support is zero, so such dashings are exactly the intersection of $V_{\text{free}}(n)$ and a set of hyperplanes, which is a subspace.

Unlike the even dashings $e(G)$, the odd dashings $o(G)$ do not form a vector space; in particular, they do not include d_0. However, adding an even dashing to an odd dashing gives an odd dashing and the difference between any two odd dashings gives an even dashing. Thus, $o(A)$ is a coset in $V_{\text{free}}(n)$ of $e(G)$ and must then have the same cardinality as $e(G)$ given that at least one odd dashing exists. □

Corollary 3.3.2. *For any adinkraizable chromotopology A and C being the set of 2-colored 4-cycles, we have $|o(A, C)| = |e(A, C)|$.*

The proof of Lemma 3.3.1 hints that working with the even dashings may be slightly easier than the odd dashings, thanks to their vector space structure. Structurally, the intuition we gain is that the odd dashings form a *torsor* for the even dashings. Here is a linear algebraic explanation of these concepts that offers another (basically equivalent) proof of Lemma 3.3.1.

Consider the vector space V over $\mathbf{Z}_2$ indexed by the edges. We can then represent each $c \in C$ by a vector with 1's on the edges in c and 0's elsewhere. Define $M(C)$

to be the $|C| \times |E(G)|$ matrix where each row corresponds to a different cycle in C. Note we can represent a dashing by a column vector in $\mathbf{Z}_2^{|E(G)|}$. With this notation, even dashings are simply vectors that are annhilated by $M(C)$ and odd dashings are vectors that map to the all 1's vector. It is clear from this formulation that the cardinality of the latter set is either 0 (if the said vector has no preimage) or equal to the former set.

It is clear from this setup that our enumeration of even and odd (when they exist) dashings is equivalent to counting the dimension of the cycle space of cycles spanned by C. We give this formulation in the following Proposition, though we will not use it explicitly later.

Proposition 3.3.3. *The number of even dashings, $|e(G, C)|$, is equal to $2^{r(M(C))}$, where $r(M)$ denotes the rank of M. The number of odd dashings, $o(G, C)|$, is either 0 or equal to $e(G, C)$.*

Proof. This is basically obvious from the proof of Lemma 3.3.1 and the observation that $r(M(C))$ is exactly the dimension of the space of cycles generated by C. □

3.4 Decomposition and Dashings on Hamming Cubes

In this section, we resolve Problem 2 for Hamming cubes $A = I_c^n$, with C being the set of 2-colored 4-cubes. We will generalize the result later in Section 3.5, but we can gain some insights from this special case, the main observation being that dashings behave extremely well under a concept of *decomposition.*

Call a graph G *decomposable* if $G = H \times I$, the cartesian product of some graph G and a single edge. Say that a color i *decomposes* a chromotopology A if removing all edges of color i splits A into 2 separate connected components. In this case, we must have the underlying graph of A be decomposable. Our definition was inspired by observations in [12], where certain adinkras were called 1*-decomposable.* As Greg Landweber pointed out to us, the concept corresponding to decomposition in coding theory is *punctured codes* [27].

Lemma 3.4.1. *The color i decomposes the chromotopology A if and only if for all $d \in L(A)$, the i-th bit of d is 0.*

Proof. This is a very straightforward verification. We leave the proof as an exercise to the reader. □

Corollary 3.4.2. *Every color in $[n]$ decomposes I_c^n.*

In the situation where Lemma 3.4.1 holds, we say that i decomposes A into A_0 and A_1, or $A = A_0 \amalg_i A_1$, if removing all edges with color i creates two disjoint chromotopologies A_0 and A_1, which are labeled and colored in a natural fashion, equipped with an inclusion inc on their vertices that map into $V(A)$. Formally:

- $V(A)$ can be partitioned into two sets $V(A|0)$ and $V(A|1)$, where vertices in $V(A|0)$ have 0 in the i-th bit (by Lemma 3.4.1, this is a well-defined notion) and vertices in $V(A|1)$ have 1 in the i-th bit. Furthermore, all edges between $V(A|0)$ and $V(A|1)$ are of color i.

- define A_0 to be isomorphic to the edge-colored graph induced by vertices in $V(A|0)$, where any codeword $v = (b_1 b_2 \cdots b_n)$ in the vertex label class of $v' \in V(A|0)$ is sent to the $(n-1)$-codeword $(b_1 b_2 \cdots \widehat{b_i} \cdots b_n)$, where we remove the bit b_i. Color the edges analogously with colors in $\{1, 2 \cdots, \widehat{i}, \cdots, n\}$. Define A_1 in the same way with $V(A|1)$ instead of $V(A|0)$.

- define the maps $\mathrm{inc}(b_1 b_2 \dots b_{n-1}, j \to i) = b_1 \dots b_{i-1} j b_i \dots b_{n-1}$, which inserts j into the i-th place of an $(n-1)$-codeword to create an n-codeword. If $A = A_0 \amalg_i A_1$, let $\mathrm{inc}(v)$ send a vertex $v \in A_j$ to $\mathrm{inc}(v, j \to i)$ for $j \in \{0, 1\}$. Lemma 3.4.1 gives that the union of the image of $V(A_0)$ and $V(A_1)$ under inc is exactly $V(A)$.

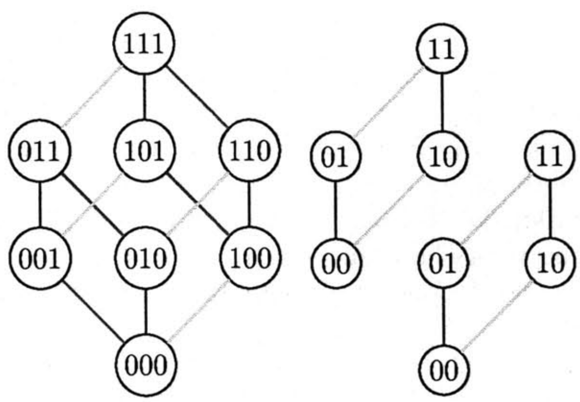

Figure 3-1: The color 3 decomposes a ranked chromotopology A (with chromotopology I_c^3) as $A = A_0 \amalg_3 A_1$. Each A_i has chromotopology I_c^2.

Proposition 3.4.3. *Let $A = A_0 \amalg_i A_1$, where A is an (n,k)-chromotopology. Then A_0 and A_1 are $(n-1,k)$ chromotopologies, isomorphic as graphs.*

Proof. The image of q_i on $V(A_0$ is exactly $V(A_1)$ and q_i is an involution, so we have a bijection between the vertices. If $q_j(v_1) = v_2$ in A_0, the 4-cycle condition on $(v_1, q_i(v_1), q_i(v_2), v_2)$ gives that $(q_i(v_1), q_i(v_2))$ is also an edge of color j in A_1, so the bijection between the vertices extends to a bijection between A_0 and A_1 as edge-colored graphs, and thus chromotopologies. Each of these chromotopologies has 2^{n-1} vertices and is $(n-1)$-regular, so by Proposition 2.1.1 they must be $(n-1,k)$-chromotopologies. $\square$

We now prove our main idea, which tells us that dashings of decomposable graphs are easy to compute recursively.

Lemma 3.4.4. *If A has l edges colored i and $A = A_0 \amalg_i A_1$, then each even (resp. odd) dashing of the induced graph of A_0 and each of the 2^l choices of dashing the i-colored edges extends to exactly one even (resp. odd) dashing of A.*

Proof. Without loss of generality, we can take $i = 1$, so A_0 contains equivalence classes of codewords with first bit 0 and A_1 contains those with first bit 1.

After an even dashing of A_0 and an arbitrary dashing of the i-colored edges, note the remaining 2-colored 4-cycles are of exactly two types:

1. the 4-cycles in A_1;

2. the 4-cycles of the form (u, v, w, x), where (u, v) is in A_0, (w, x) is in A_1, and (v, w) and (x, u) are colored i.

Note that in all the cycles (u, v, w, x) of the second type, (w, x) is the only one we have not selected. Thus, there is exactly one choice for each of those edges to satisfy the even parity condition. Since there is exactly one such cycle for every edge in A_1, this selects a dashing for all the remaining edges, and the only thing we have to check is that the 4-cycles of the first type, the ones entirely in A_1, are evenly dashed.

Now, a 4-cycle of this type is of form $(1a_1, 1a_2, 1a_3, 1a_4)$, which is a face of a 3-cube with vertices $(0a_1, 0a_2, 0a_3, 0a_4, 1a_1, 1a_2, 1a_3, 1a_4)$. There are 5 other 4-cycles in this Hamming cube which have all been evenly dashed (the $0a_i$ vertices form a cycle in A_0 and the other 4 cycles are evenly dashed by our previous paragraph). Thus, we have that:

$$\begin{aligned}
d(0a_1, 0a_2) + d(0a_2, 0a_3) + d(0a_3, 0a_4) + d(0a_4, 0a_1) &= 0;\\
d(0a_1, 0a_2) + d(0a_2, 1a_2) + d(1a_2, 1a_1) + d(1a_1, 0a_1) &= 0;\\
d(0a_2, 0a_3) + d(0a_3, 1a_3) + d(1a_3, 1a_2) + d(1a_2, 0a_2) &= 0;\\
d(0a_3, 0a_4) + d(0a_4, 1a_4) + d(1a_4, 1a_3) + d(1a_3, 0a_3) &= 0;\\
d(0a_4, 0a_1) + d(0a_1, 1a_1) + d(1a_1, 1a_4) + d(1a_4, 0a_4) &= 0.
\end{aligned}$$

Adding these equations in $\mathbf{Z}_2$ gives:

$$d(1a_1, 1a_2) + d(1a_2, 1a_3) + d(1a_3, 1a_4) + d(1a_4, 1a_1) = 0.$$

Thus, we have constructed an even dashing. The analogous result for odd dashings follow if we replace 0's on the right sides of the above equations by 1's. $\square$

Remark 3.4.5. This proof is easily seen to generalize to dashings of $(A \times I, C)$, where the cycles of C occur in a "mirrored" fashion in the two halves of the graph, and all 4-cycles that use the edges corresponding to I belong to C.

Proposition 3.4.6. *The number of even (or odd) dashings of I_c^n is*

$$|e(I_c^n)| = |o(I_c^n)| = 2^{2^n-1}.$$

Proof. A convenient property of Hamming cubes is that every 4-cycle is a 2-colored 4-cycle. Thus, we get to just say "4-cycles" instead of "2-colored 4-cycles" in this proof.

We prove our result by induction. The base case is easy: for $n = 1$ (a single edge), there are exactly 2 even dashings, since there is no 4-cycle. Suppose our result were true for every $k < n$. We will now show it is also true for n. Recall from Corollary 3.4.2 that every color decomposes I_c^n into two smaller I_c^{n-1}'s. Since we have 2^{n-1} edges with color 1, by Lemma 3.4.4 we get the recurrence

$$|e(I_c^n)| = 2^{2^{n-1}}|e(I_c^{n-1})|.$$

With the initial case $|e(I_c^1)| = 2$, we get $|e(I_c^n)| = 2^{2^{n-1}+2^{n-2}+\cdots+1} = 2^{2^n-1}$, as desired. The result for $|o(I_c^n)|$ is immediate by Lemma 3.3.1. $\square$

Note that $|o(I_c^n)| = 2^{2^n}/2$. This suggests that, besides a single factor of 2, each of the 2^n vertices gives exactly one "degree of freedom" for odd dashings. We will justify this hunch in the following discussion, in particular with Proposition 3.5.1.

3.5 Dashings on Adinkras

In this section, we generalize Proposition 3.4.6 to all adinkraizable chromotopologies, where A is some adinkraizable chromotopology I_c^n/L and C is again the set of 2-colored 4-cycles. We will use the idea of vertex switching and some homological algebra.

In [13], Douglas, Gates, and Wang examined dashings from a point of view inspired by Seidel's *two-graphs* ([36]). Define the *vertex switch* at a vertex v of a well-dashed chromotopology A as the operation that produces the same A, except with all edges adjacent to v flipped in parity (sending dashed edges to solid edges, and vice-versa). It is routine to verify that a vertex switch preserves odd dashings (in fact, parity in all 4-cycles), so the odd dashings of A can be split into orbits under all possible vertex switchings, which we will call the *labeled switching classes* (or *LSCs*) of A. We emphasize the adjective "labeled" because the term *switching class* in [13] refers to equivalence classes not only under vertex switchings, but also under different types of vertex permutations.

In the representation theory interpretation of adinkras (see Section 1.4), a vertex switch corresponds to adding a negative sign in front of a component field, which gives an isomorphic representation. Thus, it is natural to think about equivalence classes under these transformations. The following computation will not only be useful to study switchings, but will also justify our hunch about the "degrees of freedom" from Proposition 3.4.6.

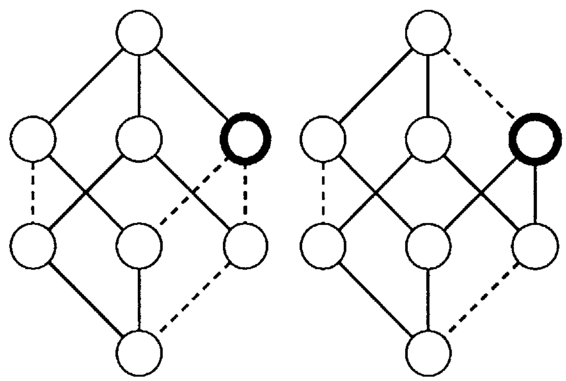

Figure 3-2: Before and after a vertex switch at the outlined vertex.

Proposition 3.5.1. *In an adinkraizable (n, k)-chromotopology A, there are exactly $2^{2^{n-k}-1}$ dashings in each LSC.*

Proof. Vertex switches commute and each vertex switch is an order-2 operation, so they form a $\mathbf{Z}_2$-vector space, which we may index by subsets of the vertices. Consider a set of vertex switches that fix a dashing. Then, each edge must have its two vertices both switched or both non-switched. This decision can only be made consistently over all vertices if all vertices are switched or all vertices are non-switched. Thus, the 2^{n-k} sets of vertex switches generate a $\mathbf{Z}_2$-vector space of dimension $2^{n-k} - 1$. This proves the result. □

Corollary 3.5.2. *The cubical chromotopology I_c^n has exactly one labeled switching class.*

Proof. This is immediate from Proposition 3.5.1 and Proposition 3.4.6, with the substitution $k = 0$. Alternatively, this is also evident from [13, Lemma 4.1]. □

Finally, we combine several ideas (even dashings, vertex switchings, and our cell complex interpretation of chromotopologies) to generalize Proposition 3.4.6.

Proposition 3.5.3. *Let A be an adinkraizable (n, k)-chromotopology. Then there are 2^k LSCs in A.*

Proof. First, vertex switchings preserve parity of all 4-cycles, so counting orbits of odd dashings (LSCs) under vertex switchings is equivalent to counting orbits of even dashings.

An even dashing can also be thought of as a formal sum of edges over $\mathbf{Z}_2$ (we dash an edge if the coefficient is 1 and do not otherwise), which is precisely a 1-chain of $X(A)$ over $\mathbf{Z}_2$. Second, the even dashings are defined as dashings where all 2-colored 4-cycles have sum 0. Since these 4-cycles, as elements of C_1, are exactly the boundaries of C_2, the even dashings are exactly the orthogonal complement of $\mathrm{Im}(d_2)$ inside of C_1 by the usual inner product. Thus, the even dashings have $\mathbf{Z}_2$-dimension:

$$\begin{aligned}
\dim((\mathrm{Im}(d_2)^\perp) &= \dim(C_1) - \dim(\mathrm{Im}(d_2)) \\
&= (\dim(\ker(d_1)) + \dim(\mathrm{Im}(d_1))) - \dim(\mathrm{Im}(d_2)) \\
&= \dim(H_1) + \dim(\mathrm{Im}(d_1)) \\
&= \dim(H_1) + \dim(C_0) - \dim(H_0).
\end{aligned}$$

However, note that $\dim(C_0) - \dim(H_0) = 2^{n-k} - 1$, which is exactly the dimension of the vector space of the vertex switchings for a particular LSC from Proposition 3.5.1. Since the product of the number of LSCs and the number of vertex switchings per LSC is the total number of even dashings, dividing the number of even dashings by $2^{2^{n-k}-1}$ gives that the dimension of switching classes is precisely $\dim(H_1)$.

By Proposition 3.1.1 and basic properties of universal covers and fundamental groups, $\pi_1(X(A)) = L$, the quotient group, which in this case is the vector space $\mathbf{Z}_2^k$. Also, $H_1 = \mathbf{Z}_2^k$ since H_1 is the abelianization of π_1. Thus, we have 2^k switching classes. □

Propositions 3.5.3 and 3.5.1 immediately give:

Theorem 3.5.4. *The number of even (or odd) dashings of an adinkraizable (n, k)-chromotopology A is*

$$|e(A)| = |o(A)| = 2^{2^{n-k}+k-1}.$$

A surprising but neat consequence of this result is that the number of dashings does not depend on the actual code $L(A)$, rather just on its dimension. This is very nonintuitive to see with elementary combinatorial methods. In either case, this section shows that the dashing enumeration problem on adinkras is mostly understood.

3.6 A Generalization of Dashings and Noneven Graphs

We started out with a notion of odd dashings for adinkras. In Section 3.2 we generalized this notion to odd (and even) dashings on general undirected graphs.

The concept of *noneven graphs* from the theory of Pfaffian orientations (for a reference, see [39]) is similar to odd dashings, but are defined for directed graphs instead of undirected graphs. In this section, we develop a generalization of noneven graphs and odd dashings on adinkras. The reader should take care to note that this is an independent direction of generalization as the generalization in Section 3.2.

For an undirected graph G and a set of even-lengthed cycles C_G, let an *(edge) orientation* of G be a directed graph with the underlying undirected graph G. Now, define an *odd orientation (relative to C_G)* to be an orientation of G where each cycle in C_G has an odd number of edges oriented in either direction (equivalently, both directions) around the cycle. As before, define *even orientation* analogously. Note that it is important that these C_G have even length; otherwise the parities of the number of edges oriented in the two directions would be different.

The following bijection shows us that odd (and even) dashings in adinkras can be seen as a special case of odd (and even) orientations.

Lemma 3.6.1. *There is a bijection between odd (even) dashings of an undirected bipartite graph G relative to a set of even cycles C_G and odd (even) orientations of G relative to C_G if all cycles in C_G have lengths divisible by* 4.

Proof. Pick a bipartition $O \cup E$ of G; let G_0 be the orientation of G such that all edges point from O to E. Now, for an odd (even) dashing d of G, define an orientation $G(d)$ to have an edge (u, v) be oriented in the same direction as in G if $d(u, v) = 0$ and in the reverse direction as in G if $d(u, v) = 1$.

Since all cycles in C_G have lengths divisible by 4, G_0 is an even orientation. Thus, $G(d)$ becomes an odd orientation if and only if for each cycle in C_G there is an odd number of reversals, which happens exactly when d is an odd dashing. □

Because adinkraizable chromotopologies are bipartite graphs and all the 2-colored 4-cycles have length 4, the bijection in Lemma 3.6.1 holds, so we can choose to think of dashings as orientations. Now, a noneven graph is exactly a graph G with an odd orientation relative to C_G being the set of all even cycles. These two facts together show that the concept of odd (and even) orientations generalize both concepts. It is also important to note that this generalization cannot be made to subsume the earlier generalization of dashings to general graphs because of the modulo 4 condition, though they coincide for adinkraizable chromotopologies.

Chapter 4

Stiefel-Whitney Classes of Codes

Doran et. al. [10] consider quotients I_f^n/L of the full Hamming cube I_f^n, which, unlike our $X(I^n)$, is homeomorphic to the unit ball and have cells of all dimensions up to n. They define homology classes $\omega_i \in H^i(I_f^n/L, \mathbf{Z}_2)$ to be the sum of all the i-cells in I_f^n/L. They make the following observation based on the adinkra $A(L)$:

Proposition 4.0.1 ([10]). *We have that $\omega_1 = 0$ if and only if $A(L)$ is bipartite and $\omega_2 = 0$ if and only if $A(L)$ has an odd dashing (which requires $\omega_1 = 0$).*

They conjecture that this may be related to Stiefel-Whitney classes, in the sense that when we have a manifold M with the tangent bundle $TM \to M$, the Stiefel-Whitney classes $\omega_k(TM)$ behave similarly: $\omega_1(TM) = 0$ if and only if M is orientable and $\omega_2(TM) = 0$ if and only if M has a discrete spin structure. In this chapter, we give a rigorous connection between codes and Stiefel-Whitney classes, which inspires some topological techniques applied to coding theory.

Remark 4.0.2. Alert readers may realize that for closed manifolds, these sums of i-cells are Poincaré dual to Stiefel-Whitney classes [23], however, this does not seem to immediately explain the actual observations that Doran et. al. [10]. For starters, RP^∞, the space we use, is different from I_f^n/L. Furthermore, I_f^n/L may have boundary and thus cannot be a closed manifold, so there are boundary terms to consider and the above sum does not seem to immediately have a Stiefel-Whitney interpretation. It is definitely desirable to reconcile these two viewpoints, however.

It is important to note that adinkras, while mostly absent from this chapter, were invaluable in motivating this connection due to the existence of so many ways (in this case, codes) of examinging them. We thank Josh Batson for a discussion that generated most of the material in this section.

4.1 Seeing Codes as Maps

To find something resembling Stiefel-Whitney classes for an adinkra, we need a real vector bundle. The obvious choices, such as taking the tangent bundle of $X_I(L)$ or

$X(L)$, do not seem to work. The key insight is that we can translate our adinkra into a code, think of the code as a map, and then produce a bundle (and the Stiefel-Whitney classes) from the map.

Recall (for basics of vector bundles and characteristic classes, see e.g. Hatcher [25]) that $EG \to BG$ is a principal G-bundle associated with any group G, where BG denotes the classifying space of G and EG its universal cover. The associated bundle to $EF_2^n \to BF_2^n$ is a real rank-m vector bundle (since $F_2 = O(1)$, $BF_2 = \mathbb{RP}^\infty$ is the classifying space of real line bundles, so our bundle over BF_2^n is no more than a direct sum of m real line bundles).

If we have a group G, a map $f\colon G \to F_2^n$ induces a pullback bundle over BG since f induces a homotopy class of maps from BG to BF_2^n. The Stiefel-Whitney classes of this bundle are elements of $H^*(BG)$; they will be the images of the universal Stiefel-Whitney classes in $H^*(BF_2^n)$ under the map in cohomology induced from f.

Thus, instead of looking for a manifold or a real bundle directly, we look for a map into BF_2^n. We have defined an (n,k)-linear code L as a k-dimensional subspace of F_2^n; an equivalent definition is thinking of L as a homomorphism $f_L\colon F_2^k \to F_2^n$. We can simply use this map as f!

Furthermore, in our situation, this map is very simple. It is well-known that the cohomology ring $H^*(BF_2^n)$ is a free algebra $F_2[x_1, \ldots, x_n]$, with one generator for each dimension; similarly, $H^*(BF_2^k) = F_2[y_1, \ldots, y_k]$ for some y_i. On the generators, the induced map in cohomology f_L^* is simply a linear map represented by the $k \times m$ matrix, with each row corresponding to a codeword; this is just the generator matrix A_L.

We may then obtain the Stiefel-Whitney classes in $H^*(BF_2^k)$ by computing the $\omega_j = f_L^*(\overline{\omega_j})$. Here, the universal Stiefel-Whitney classes $\overline{\omega_j}$ are the elementary symmetric polynomials $e_j(x_1, \ldots, x_n)$ in $H^*(BF_2^n)$. For shorthand, let us denote $f_L^*(x_i)$ by v_i, then $\omega_i = e_i(v_1, \ldots, v_n)$. Equivalently, we can take the image $\omega = f^*(\omega)$ of the total Stiefel-Whitney class $\overline{\omega} = (1+x_1)(1+x_2)\cdots(1+x_n)$ after the map and look at the graded components of ω.

Example 4.1.1. Say we have $L = \{111\cdots 1\} \in F_2^n$. This gives a map $f_L\colon F_2 \to F_2^n$. The dual f_L^* has the matrix $\begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix}$. It sends each generator x_i (corresponding to the unit vector in the i-th coordinate) to y_1, the unique generator of $H^*(BF_2) = F_2[y_1]$, so $v_i = f^*(x_i) = y$ for all i. Here, the total Stiefel-Whitney class ω is just the image of $(1+x_1)(1+x_2)\cdots(1+x_n)$ under this map, which is just $(1+y_1)^n$. Thus, $\omega_i = \binom{n}{i} y_1^i$. Note that $\omega_1 = ny_1$ vanishes exactly when n is even. Furthermore, given that ω_1 vanishes, $\omega_2 = \binom{n}{2} y_1^2$ vanishes exactly when n is divisble by 4. This calculation coincides with our expected intuition! For codes L of this type, L is even if and only if n is even and doubly-even if and only if n is divisible by 4.

Example 4.1.2. Consider $L = \{111100, 001111\}$. Here the matrix corresponding to f_L^* is $\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0. \end{bmatrix}$. Letting y_1 and y_2 be the generating classes of $H^*(BF_2^2)$.

We are then interested in the image of $(1+x_1)\cdots(1+v_x)$ under this map:

$$f_L^*((1+x_1)\cdots(1+x_6)) = (1+y_1)^2(1+y_2)^2(1+y_1+y_2)^2 \qquad (4.1.3)$$
$$= (1+y_1^2)(1+y_2^2)(1+y_1^2+y_2^2). \qquad (4.1.4)$$

This polynomial is even, so ω_1 vanishes. $\omega_2 = 2y_1^2 + 2y_2^2 = 0$, so it vanishes as well. As this is a doubly-even code, our intuition is again supported.

Our main problem, then, is the following:

Problem 3. Fix an (n,k)-code L. For any $m \leq n$, suppose the first m Stiefel-Whitney classes $\omega_1, \ldots, \omega_m$ all vanish in $F_2[x_1, \ldots, x_k]$. What constraints are then placed on L?

4.2 Preliminaries

Before going on, we need to review some elementary but useful facts from symmetric functions and number theory.

Recall that the *Newton identities* are the following: (recall that the power-sum symmetric functions are $p_i(x_1, \ldots,) = x_1^k + x_2^k + \cdots)$

$$p_1 = e_1$$
$$p_2 = e_1p_1 - 2e_2$$
$$p_3 = e_1p_2 - e_2p_1 + 3e_3$$
$$\cdots$$

Suppose our generator matrix A_L has entries $\{a_{ij}\}$. Define the *t-weights* of L as the numbers $\sum_s a_{i_1s}\cdots a_{i_ts}$ as we run through all different t-tuples $\{i_1, \ldots, i_t\}$ of numbers in $\{1, \ldots, k\}$. We will denote the t-weights of L by the set $c_t(L)$. We say that L satisfies the *t-weight condition* if $2^{t+1-j}|c_j$ for all j.

Example 4.2.1. Here are two examples that show this definition to be very natural: the 1-weights are just the sums of the parities in each codeword, which are the bona fide *weights* of the generators modulo 2. The 2-weights are the $\binom{k}{2}$ inner products between the generators. Note that even codes satisfy the 1-weight condition and doubly-even codes (because they are are self-orthogonal) satisfy the 2-weight condition.

Define the *type* of a monomial to be the partition of its powers. So the monomial $y_2y_3^2y_4$ has type (211). A recurring tool we will use is:

Lemma 4.2.2. *For a given partition ρ of m with p parts, the set of coefficients of type ρ in $p_m(v_1, \ldots, v_n)$ consists exactly of identical copies of p-weights, each number multiplied by $\binom{m}{\rho_1\cdots\rho_m}$.*

Proof. We have that $p_m = (a_{11}y_1 + \ldots + a_{k1}y_k)^m + (a_{12}y_1 + \ldots + a_{k2}y_k)^m + \cdots$. For the r-th term, there are $\binom{m}{\rho_1 \cdots \rho_m}$ ways of getting a particular term of form $y_{t_1}^{\rho_1} \cdots y_{t_p}^{\rho_p}$, which gets the coefficient $a_{t_1 r}^{\rho_1} \cdots a_{t_p r}^{\rho_p} = a_{t_1 r} \cdots a_{t_p r}$ (here is where we use the very simple but useful observation is that since we are over F_2, we always have $a_{ij}^k = a_{ij}$), which is a corresponding term in the p-weight. Since we sum over all r, we pick up exactly $\binom{m}{\rho_1 \cdots \rho_m}$ times the p-weight. By symmetry, each p-weight appears an equal number of times this way. $\square$

Corollary 4.2.3. *$p_m(v_1, \ldots, v_n)$ vanishes* $\pmod N$ *if and only if all elements in $\{\binom{m}{\rho_1 \cdots \rho_p} c_p\}$ do.*

Example 4.2.4. As an example, for the D_3 code we have

$$A_L = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

For $m = 3$ and $\rho = (2,1)$. The 2-weights of D_3 are $(2,2,0)$. The part of terms of type $(2,2)$ in $p_3 = 2y_1^3 + 2(y_1 + y_2)^3 + 2(y_2 + y_3)^3 + y_3^3$ is $6y_1^2y_2 + 6y_2^2y_1 + 6y_2^2y_3 + 6y_3^2y_2 + 0y_1^2y_3 + 0y_3^2y_1$, so the set of coefficients are $(6,6,0,6,6,0)$, which is just two copies of the 2-weights, scaled by $3 = \binom{3}{2,1}$, as desired.

We now introduce some notation to deal with all the binary arithmetic that we will end up doing. Let $\nu(n)$ be the highest power of 2 that divides n and let $o(n)$ be the number of 1's in the binary expansion of n. The following is a well-known fact [42].

Proposition 4.2.5. *$\nu_2(\binom{n}{\rho_1, \ldots, \rho_t})$ is equal to the number of carries made base 2 when the ρ_i are added to produce n.*

Finally, we will also need the following claim:

Lemma 4.2.6. *If $o(n) = m$, then adding $\rho_1 + \cdots + \rho_{m+s} = n$ requires at least s carries. Furthermore, this is achievable.*

Proof. We do this by induction. The base case $s = 0$ is trivial. Now, suppose this were true up to $s > 0$. Consider a partition $\rho_1, \cdots, \rho_{m+s+1}$. If adding two of the parts requires a carry, we are done by the inductive assumption. So adding no two parts requires a carry. However, every part needs at least one 1 in its base-2 expansion and n only has m; so if no carries were allowed there can only be at most m parts, a contradiction since $s > 0$.

The achievability is shown by just greedily breaking up powers of 2 into smaller powers of 2; each such step requires exactly one carry. $\square$

Now we are equipped to prove the main theorems in this chapter.

4.3 Interpretation of Vanishing Stiefel-Whitney Classes

Here are some small examples. The alert reader should be able to guess the pattern:

- w_1 vanishes if and only if the code is even.
- If $w_1 = 0$, $w_2 = 0$ if and only if the code is doubly even.
- If $w_1 = w_2 = 0$, then $w_3 = 0$.
- If $w_1 = w_2 = w_3 = 0$, then $w_4 = 0$ if and only if the 1-weights are 0 (mod 8), the 2-weights are 0 (mod 4), and the 3-weights are 0 (mod 2).

The following two structural lemmas relate the (m)-weight conditions and p_n.

Lemma 4.3.1. *If we have the (m)-weight condition, where $m \geq o(n) - 1$, then $2^{m-o(n)+1}|p_n$.*

Proof. It suffices to show that $2^{m-o(n)+1}|\binom{n}{\rho_1 \cdots \rho_j}c_j$ for all such partitions. For $j > m$, by Proposition 4.2.5 we know that the multinomial coefficient is divisible by $2^{j-o(n)}$, which implies divisibility by $2^{m-o(n)+1}$ since $j \geq m+1$. Every time we decrease j by 1 starting from $m+1$, we lose one power of 2 but gain one from the (m)-weight condition. Thus we will always have enough power of 2. □

An important intuition that comes with this result is that each higher weight-condition than the $(o(n)-1)$-st "activation energy" makes p_n one power of 2 more divisible. So, for example, 5 has 2 1's in its binary representation. Then the 1-weight condition gives no information, the 2-weight condition gives $2|p_5$, etc.

Lemma 4.3.2. *If $p_{2^t} = 0 \pmod{2^{t+1}}$, then we have the $(t+1)$-weight condition.*

Proof. This means, 2^{t+1} divides any number in $\binom{t}{\rho_1 \cdots \rho_j}c_j$ for any allowed j. By Lemma 4.2.6, since t has exactly one 1 in its base-2 expansion,there is a way of splitting into j parts with $j-1$ carries. Thus, by Proposition 4.2.5, $2^{t+1}|2^{j-1}c_j$. This means $2^{t+2-j}|c_j$ for all j, which is exactly the condition we needed. □

Proposition 4.3.3. *Let $2^{m+1} > n \geq 2^m$. If $n_1 + n_2 = n$, then we have the inequality*

$$(m+1) + o(n) \geq o(n_1) + o(n_2) + \nu(n_1).$$

Furthermore, if $n = 2^m$, then this bound is strict.

Proof. So we have $m+1$ bits. Think of the $(m+1)$ on the left as each bit contributing 1 to the LHS. Consider the rightmost $\nu(n_1)$ of them. Here there is no carry, so every 1 bit in n_2 contributes exactly 1 to both $o(n_2)$ and $o(n)$, and every 0 bit contributes to neither. Each one of these bits also contributes 1 to $\nu(n_1)$ and 1 to the $(k+1)$ sum, so we can deduce that the rightmost $\nu(n_1)$ bits do not affect the inequality at all and may be removed. Thus we may assume n_1 were odd and only consider the inequality

$$(m+1)+o(n) \geq o(n_1)+o(n_2).$$

Going through the addition base-2, it is apparent that the only local addition that can make this inequality more false is when we have added $1+1=10$ (with no carry) and created a carry, in which case we contribute 2 to the RHS and only 1 to the LHS. However, we know by the time we get to the left the carry must be "resolved" by a $0+0=1$ (with carry) somewhere, which contributes 2 to the LHS and none to the right. Thus, this inequality always holds.

Furthermore, the above argument shows that when a carry actually happens, we have actually gained an extra 1 for the inequality. When $n=2^m$ the final addition must be of such a form, so we actually have a strict inequality and thus may replace (m) by $(m+1)$. □

We are now ready to prove the main structural result:

Proposition 4.3.4. *For $2^{t+1} > n \geq 2^t$, if $w_1 = \cdots = w_n = 0 \pmod 2$, then the following are true:*

(A_n) *the $(t+1)$-weight condition holds.*

(B_n) *we have that the m-weight condition implies $2^{m+1-o(n)} | ne_n$ for all $k \geq (o(n)-1)$. In particular, the $(t+1)$-weight condition implies $2|e_n$, and thus $w_n = 0 \pmod 2$.*

Proof. We show this by induction. For the base case, it is trivial that $n=1=2^0$, $w_n = 0 \pmod 2$ if and only if the 1-weight condition holds. Furthermore, the m-weight condition implies $2^m|e_1$ for all m.

Now, assume the statement holds through $(n-1)$ and we are verifying n. Newton gives us $p_n = e_1 p_{n-1} + \cdots + e_{n-1}p_1 + ne_n$. Consider each $e_{n_1}p_{n-n_1}$ and let $n_2 = n - n_1$. We have two cases:

1. If $n = 2^t$ is a power of 2, then by induction, we know that the (t)-weight condition holds and we have $2^{t+1-o(n_1)}|n_1 e_{n_1}$. We also know that $2^{t+1-o(n_2)}|p_{n_2}$. Thus, we know that $2^{2t+2-o(n_1)-o(n_2)}|n_1 e_{n_1} p_{n_2}$. However, by Proposition 4.3.3, we have that

$$\begin{aligned} 2t+2-o(n_1)-o(n-2) &\geq t+2+\nu(n_1)-o(n) && (4.3.5)\\ &= t+1+\nu(n_1). && (4.3.6)\end{aligned}$$

thus, we have have $2^{t+1+\nu_2(n_1)}|n_1 e_{n_1} p_{n_2}$, or equivalently $2^{t+1}|e_{n_1}p_{n_2}$. Since this is true for all $n > n_1 > 1$, we know that $p_n = ne_n \pmod{2^{t+1}}$. Since $e_n = 0 \pmod 2$, we actually obtain that these are both $0 \pmod{2^{t+1}}$. By Lemma 4.3.2, we obtain the $(t+1)$-weight condition, or A_n.

We now prove statement B_n by a induction inside our main induction. We already know that the t-weight condition implies $2^t|2^t e_n = ne_n$ trivially. Note that in the expansion of Newton's identity, each time we already have the $(t+s)$-weight condition and gain the $(t+s+1)$-weight condition, all the e_i where $i < n$, increases divisibility of 2 by at least 1 power (thanks to previous instances of B), and the p_n term on the left also increases divisibility of 2 by 1 power thanks to Lemma 4.3.1. Thus an extra power of 2 must be required to divide e_n. Thus, this induction is complete and we get B_n.

2. If n is not a power of 2, then by induction we already know that the $(t+1)$-weight condition holds, as the highest power of 2 dividing n did not change from $(n-1)$ to n. Thus, it suffices to show B_n.

   Similar to previously, since we have the $(t+1)$-condition in this case, we know that $2^{t+2-o(n_1)}|n_1 e_{n_1}$ and $2^{t+2-o(n_2)}|p_{n_2}$. By Proposition 4.3.3, we have that

$$2t+4-o(n_1)-o(n-2) \geq t+3+\nu(n_1)-o(n),$$

   so we know that $2^{t+3-o(n)}$ divides $e_{n_1}p_{n_2}$ for all $1 < n_1 < n$. Thus we have $p_n = ne_n \pmod{2^{t+3-o(n)}}$. We also know from the $(t+1)$-condition that $2^{t+2-o(n)}$ divides p_n, so $2^{t+2-o(n)}$ must divide ne_n as well. This is the base case of statement B_n. The induction step is identical to the $n = 2^t$ case: any additional weight-condition we gain will increase the known-divisibility by all the p_i's in the Newton identity by 1 and thus also the ne_n term.

   To see the final claim in the clause of B_n, note that the $(t+1)$-condition gives $2^{t+2-o(n)-\nu(n)}|e_n$, since $o(n)+\nu(n)$ is at most $(t+1)$, the number of digits of n, we get at least one power of 2.

□

Thus, we obtain the following pattern: each time a new w_{2^t} is enforced to be even, we get the $(t+1)$-weight condition, and this immediately forces w_{2^t+1} through $w_{2^{t+1}-1}$ to be even.

Now, the alert reader may have realized that we have not used anything specific to linear codes so far, because we have been dealing with just the generating matrix. The final missing ingredient, the fact that we could have chosen any set of generators, makes the t-weight condition a particularly good one in our setting of linear codes. Note that if we have $4|c_1$, we don't immediately get $2|c_2$ – but we do if we extend the $4|c_1$ condition to any list of generators of the code! The reason we get $2|c_2$ is linearity. We generalize this concept with the main theorem:

Theorem 4.3.7. *The (t)-weight condition of a linear code L holds if and only if $2^t|c_1$ holds for every set of generators of L.*

Proof. Consider what happens when we replace a single codeword g_i by $g_i + g_j$, where g_j is another codeword. Let us call this operation S_{ij}. It is easy to see that a generating set will remain a generating set and the set of all generating sets is transitive under this action. Knowing this fact, the "only if" condition is easy. Thus, we will focus on the "if" condition.

For any given bit, consider the pair (a, b) of the values of that bit in c_i and c_j, respectively. When we perform S_{ij}, $(0,0)$ and $(0,1)$ will remain invariant und $(1,0)$ switches with $(1,1)$.

We will prove by induction that $2^{t-j+1}|c_j$ for any set of generators in C. The base case ($j = 1$) is given. Suppose we have this statement true for j and are trying to prove the statement for $(j+1)$. Consider the columns of A_C. For all m-words $a_1 \cdots a_m$, $a_i \in \{0,1\}$, define the 2^m quantities $\chi(a_1 \cdots a_m)$ to be the number of column vectors equivalent to $(a_1, \ldots, a_m)$. Then, the statement that $2^{t-j+1}|c_j$ is equivalent to

$$\sum_{a_1=1, a_2=1, \ldots, a_j=1} \chi(a_1 \cdots a_m) = 0 \pmod{2^{t-j+1}},$$

where the sum holds any j of the a_i fixed and allow any of the other $(m-j)$ a_i's to take either 0 or 1. Without loss of generality, we assumed that $a_1, a_2, \ldots, a_j$ are fixed and the other variables are free; but this statement will hold for all such j-element sets.

Now, the above sum can be partitioned into the two sets

$$\sum_{a_1=1, a_2=1, \ldots, a_j=1, a_{j+1}=1} \chi(a_1 \cdots a_m)$$

and

$$\sum_{a_1=1, a_2=1, \ldots, a_j=1, a_{j+1}=0} \chi(a_1 \cdots a_m).$$

However, $S_{j+1,j}$ is an involution that sends any column with the $(j+1)$st bit 0 to 1 and vice-versa (given that the j-th bit is 1), so these two sets must have the same size. In particular, the first set must be $0 \pmod{2^{t-j}}$. However, the first set is just a set where $(j+1)$ indices are fixed to be 1; by symmetry, all such sets must be $0 \pmod{2^{t-j}}$, which is the desired statement for $(j+1)$. By induction, we get all prerequisites for the t-weight condition. □

So to recap, a natural way of defining Stiefel-Whitney classes for codes gave a list of interesting conditions on generator matrices that are simple to state as Theorem 4.3.7. Thus, Problem 3 is completely answered.

One interesting consequence of this result is philosophical: since Stiefel-Whitney classes are very natural constructs, these weight conditions should correspond to natural families of codes. Sure enough, the first two such families are exactly the even and doubly-even codes, which are venerated special families of codes for study. But what about higher families? Sure enough, Betsumiya and Munemasa [3] have recently given a study of *triply-even codes*, which arose serendipitously from the study of vertex operator algebras. As this work is extremely recent, maybe the relevance of codes that satisfy higher-order weight conditions are only beginning to be untapped.

Chapter 5

Rankings

We now turn our study to rankings. Unlike dashings, it is easy to generalize rankings immediately to all connected bipartite graphs: define an *ranking* of a connected bipartite graph G to be equivalence classes of maps (up to translation) $h\colon V(G) \to \mathbf{Z}$, where adjacent vertices in G take adjacent values in $\mathbf{Z}$. Call the set of all rankings of G the *rank family* $R(G)$ of G. Note that the rank family has finite cardinality because of the connectedness of G. Figure 5-1 shows the rank family of I^2.

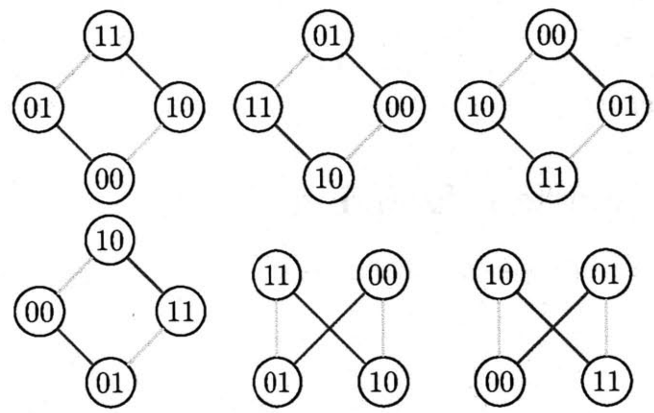

Figure 5-1: The rank family of I^2.

Problem 4. Fix a bipartite graph G. What are the enumerative and algebraic properties of $R(G)$?

Two more equivalent ways of thinking about rankings are graph homomorphisms and graded poset structures.

- A *graph homomorphism* from a graph G into another H is a map $G \to H$ that sends adjacent vertices into adjacent vertices (in our case, H is the infinite line graph). Graph homomorphisms generalize many other concepts, including graph coloring (when $H = K_n$, the complete graph). For a survey on graph homomorphisms, see [4]. Some particularly relevant papers to our work that have studied homomorphisms involving the Hamming cube are [20] and [14].

- Rankings are also equivalent to graded poset structures on graphs, which is a very natural combinatorial object. The study of poset structures on graphs is the study of *acyclic orientations*, so one can think of rankings as the subproblem of studying distinguished acyclic orientations as well. For a review of acyclic orientations, see [37].

All of these equivalent definitions suggest that rankings are a nice object of study and a good opportunity to combine ideas from different parts of mathematics.

After a short survey of the preliminaries (mostly developed in [9]) in Section 5.1, we give some original results using the language of posets and lattices in Section 5.2. In particular, we put a structure of a distributive lattice on $R(G)$ with Theorem 5.2.3. These results explicitly prove some observations made but not proven in [9, Section 8]. With the help of decomposition and a computer algorithm, we enumerate the rankings for I_c^n with $n \leq 5$ in Section 5.3.

However, we will not be done with rankings after these results – there are many flavorful connections between rankings, discrete Lipschitz functions, graph colorings, and even statistical mechanics. We make these connections clear in Sections 5.4, 5.5, and 5.7. We even give a strange connection between discrete Lipschitz functions and dashings in Section 5.6 that can potentially be used for further computation.

5.1 Preliminaries: Hanging Gardens

The main structural theorem for rankings is the following theorem. Let $D(v, w)$ be the graph distance between v and w.

Theorem 5.1.1 (DFGHIL, [9, Theorem 4.1]). *Fix a bipartite graph G. Let $S \subset V(G)$ and $h_S \colon S \to \mathbf{Z}$ satisfy the following properties:*

1. *h_S takes only odd values on one side of the biparition of $V(G)$ and only even values on the other.*

2. *For every distinct s_1 and s_2 in S, we have $D(s_1, s_2) \geq |h_s(s_1) - h_s(s_2)|$.*

Then, there exists a unique ranking h of G, such that h agrees with h_S on S and h's set of sinks is exactly S. By symmetry, there also exists a unique ranking of G whose set of sources is exactly S.

In other words, any ranking of G is determined by a set of sinks (or sources) and the relative ranks of those sinks/sources. We can visualize such a choice thus: pick some nodes as sinks and "pin" them at acceptable relative ranks, and let the other nodes naturally "hang" down. Thus, Theorem 5.1.1 is also called the "Hanging

Gardens" Theorem. Figure 5-2 shows an example. If we chose sources instead of sinks, we can imagine pinning down those nodes and having the other nodes "floating" up; the name "Floating Gardens" evokes an equally pleasant image.

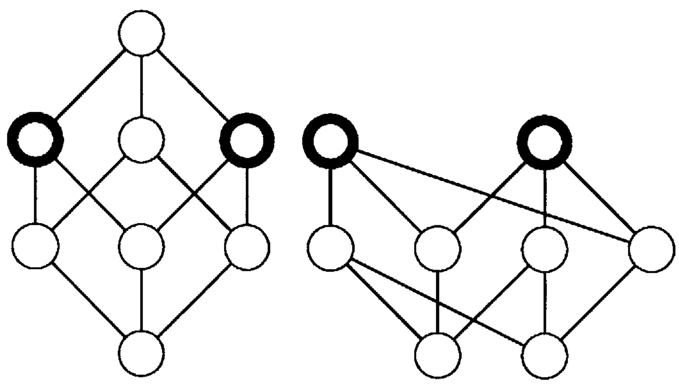

Figure 5-2: Left: I^3. Right: Hanging Gardens on I^3 applied to the two outlined vertices.

In particular, note that we can pick the set of sinks to contain only a single element, which defines a unique ranking. Thus, for any vertex v of a bipartite graph G, by Theorem 5.1.1 we can get a ranking G^v defined by its only having one sink v (visually, G^v "hangs" from its only sink v). We call G^v the *v-hooked ranking* and all such rankings *one-hooked.* By symmetry, we can also define the *v-anchored* ranking G_v, which "floats" from its only source v. For example, Figure 1-3 is both the 111-hooked ranking G^{111} and the 000-anchored ranking G_{000} of I^3.

Now, we introduce two operators on $R(G)$. Given a ranking B in $R(G)$ (with rank function h) and a sink s, we define D_s, the *vertex lowering on s*, to be the operation that sends B to the ranking h' where everything is unchanged except $h'(s) = h(s) - 2$ (visually, we have "flipped" s down two ranks and its edges with it). Observe that since s was a sink, this operation retains the fact that all covering relations have rank difference 1 and thus we still get a ranking. We define U_s, the *vertex raising on s*, to be the analogous operation for s a source. We call both of these operators *vertex flipping* operators.

Proposition 5.1.2 (DFGHIL, [9, Theorem 5.1, Corollary 5.2]). *Let G be a ranking. For any vertex v:*

1. *we can obtain G^v from G (or G from G_v) via a sequence of vertex lowerings;*
2. *we can obtain G from G^v (or G_v from G) via a sequence of vertex raisings.*

Furthermore, in any of these sequences we do not need to ever raise or lower v.

Proof. The main idea of the proof is again visually intuitive: starting with any G, "pin" v to a fixed rank and let everything else fall down by gravity (slightly more formal: greedily make arbitrary vertex lowerings, except on v, until it is no longer possible). The result is easily seen to be G^v. The other claims follow by symmetry. □

Corollary 5.1.3. *Any two rankings with the same graph G can be obtained from each other via a sequence of vertex-raising or vertex-lowering operations.*

Corollary 5.1.3 shows that there exists a connected graph H with $V(H) = R(G)$ and $E(H)$ corresponding to vertex flips. In the literature (say [9]), $R(G)$, equipped with this graph structure, is called the *main sequence.*

5.2 The Rank Family Lattice

Consider a bipartite graph G. We know from the discussion in the previous section that its rank family has the structure of a graph. In this section, we show that it actually has much more structure.

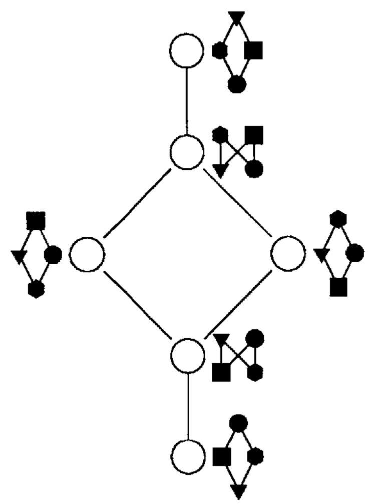

Figure 5-3: The rank family poset for $P_v(I^2)$, where next to each node is a corresponding ranking. The rankings are presented as miniature posets, with the black dots corresponding to v, the vertex at which we are not allowed to raise.

Theorem 5.2.1. *For a bipartite graph G and any vertex v of G, there exists a poset $P_v(G)$ such that:*

1. *$R(G) = V(P_v(G))$;*
2. *$P_v(G)$ is a symmetric ranked poset, with exactly one element in the top rank and exactly one element in the bottom rank;*

3. *each covering relation in $P_v(G)$ corresponds to vertex-flipping on some vertex $w \neq v$.*

Proof. Construct $P_v(G)$, as a ranked poset, in the following way: on the bottom rank 0 put G^v as the unique element. Once we finish constructing rank i, from any element B on rank i, perform a vertex-raising on all sources (except v) to obtain a set of rankings $S(B)$. Put the union of all $S(B)$ (as B ranges through the elements on rank i) on rank $i+1$, adding covering relations $C > B$ if we obtained C from B via a vertex-raising. It is obvious from this construction that the covering relations in $P_v(G)$ are exactly the vertex flippings on vertices that are not v.

We stop this process if all the elements of rank i have no sources besides v to raise. By Theorem 5.1.1, this is only possible for a single ranking, namely G_v. Thus, there is exactly one element in the top rank of $P_v(G)$ as well. By Proposition 5.1.2, we can get from G^v to any element of $R(G)$ by vertex-raising only, without ever raising v. This means that every element of $R(G)$ has appeared exactly once in our construction.

Now, consider the map ϕ that takes a ranking B of rank k to the ranking B', in which each any v with rank i in B has rank $k-i$ in B'. Since ϕ takes G_v to G^v, and vice versa, the top and bottom ranks are symmetric. However, ϕ also switches covering relations of vertex-raisings to vertex-lowerings. Thus, we can show that for every i the i-th ranks and the $(k-i)$-th ranks are symmetric by induction on i. This makes $P_v(G)$ into a symmetric poset as desired. □

For a ranking $B \in R(G)$, we now explicitly define $D_s(B)$ or $U_s(B)$ to 0 if the corresponding flip is not allowed. This allows us to consider D_s and U_s as operators $F[R(G)] \to F[R(G)]$, where F is an arbitrary field and $F[R(G)]$ are formal sums of rankings over F. Define $U(G)$ to be the operator algebra generated by all U_s with $s \in V(G)$.

Corollary 5.2.2. *The image of G^v under the action of the quotient $U(G)/U_v$ is* $\mathbf{R}[R(G)]$.

Proof. This is immediate from the construction in Theorem 5.2.1, where we started with a single ranking G^v. Taking the image under vertex raisings is exactly taking the image of the $U(G)$-action on G^v. Forbidding the vertex raising at v is exactly restricting this action to the quotient $U(G)/U_v$. □

The authors of [9] noted that the rank family is reminiscent of a Verma module. Corollary 5.2.2 is an algebraic realization of this observation. The ranking G^v takes the role of the *lowest-weight vector.* If we allowed vertex raisings at v, we would have obtained an infinite repeating family of rankings, as in Figure 5-4. When we strip the redundant rankings by forbidding U_v, we leave ourselves with a finite $R(G)$.

In fact, we can put even more structure on $P_v(G)$ with the language of *lattices.* For a review of lattices, see any standard reference such as [38]. We first construct an auxiliary poset $E_v(G)$, which we call the *v-elevation poset* of G:

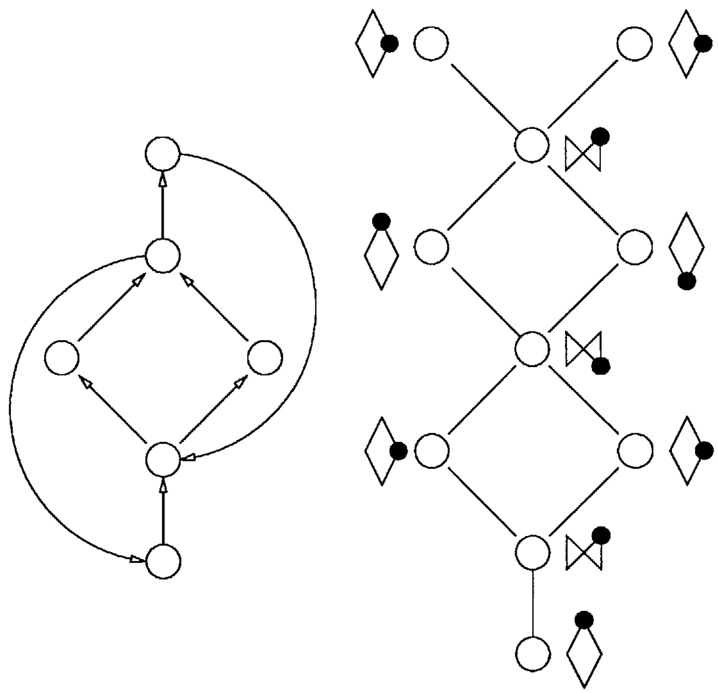

Figure 5-4: Left: if we were to allow vertex raising at v, we no longer get a poset since we introduce cycles. Right: we can also think of this situation by an infinite poset leading upwards.

- let the vertices of $E_v(G)$ be ordered pairs (w, h), where $w \in G$, $w \neq v$, and $h \in \{1, 2, \ldots, D(w, v)\}$.
- whenever $D(w_1, w_2) = 1$ and $D(w_1, v)+1 = D(w_2, v)$, have (w_1, h) cover (w_2, h) and $(w_2, h+1)$ cover (w_1, h).

Our main result of this section is the following theorem.

Theorem 5.2.3. *The v-elevation poset and the v-rank family poset are related by*

$$P_v(G) = J(E_v(G)).$$

Thus, the rank family poset $P_v(G)$ is a finite distributive lattice.

Proof. We show that there is a bijection between order ideals of $E_v(G)$ and elements of $P_v(G)$. The second claim in the theorem follows immediately from the Fundamental Theorem for Finite Distributive Lattices.

Each ideal I of $E_v(G)$ gives exactly one ranking in $P_v(G)$, as follows: for every vertex $w \in G$, take the maximum $h \in \mathbf{Z}$ such that $(w, h) \in I$, taking $h = 0$ if no (w, h) appears in I. Now assign to w the rank $2h - D(w, v)$. In other words, h

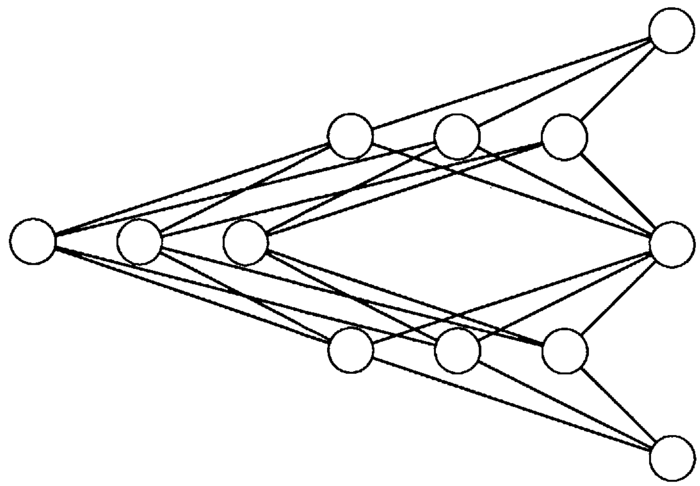

Figure 5-5: The elevation poset $E_{000}(I_c^3)$. Nodes (w, h) on each vertical line have the same w-value. In order (left to right), the w-values are: $100, 010, 001, 110, 101, 011, 111$.

indexes the elevation of w by counting the number of total times we flip w up from the initial state of the v-hooked ranking (which corresponds to the empty ideal), justifying the name *elevation poset*. It remains to check that this map is a bijection.

Take a ranking B in $P_v(G)$. For any $w \in B$, we have $h(w) = 2h - D(w, v)$ for some $0 \leq h \leq D(w, v)$. Define $I \subset V(E_v(G))$ to contain all (w, h'), possibly empty, with $h' \leq h$. The property of B being a ranking is equivalent to the condition that for every pair of neighbors w_1 and w_2 with $D(v, w_2) = D(v, w_1) + 1$, we have $|h(w_1) - h(w_2)| = 1$. However, this in turn is equivalent to the condition that the maximal h_1 and h_2 such that (w_1, h_1) and (w_2, h_2) appear (as before, define one of them to be 0 if no corresponding vertices exist in I) in I satisfy either $h_1 = h_2$ or $h_1 = h_2 + 1$, which is exactly the requirement for I to be an order ideal. Thus, our bijection is complete. $\square$

The proof of Theorem 5.2.3 gives another interpretation of Theorem 5.1.1. Consider the order ideals of $E_v(G)$. Each such order ideal corresponds to an antichain of maximal elements, which is some collection of $(w_i, h(w_i))$. It can be easily checked that in the corresponding element of $P_v(G)$, the w_i are exactly the sinks, placed at rank $2h_i - D(w_i, v)$.

Even though Theorem 5.2.3 gives us more structure on the rank family, it is very difficult in general to count the order ideals of an arbitrary poset. The typical cautionary tale is the case of the (extremely well-understood) Boolean algebra B_n, where the problem of counting the order ideals, known as Dedekind's Problem, has resisted a closed-form solution to this day, with answers computed up to only $n = 8$ (see [41]).

5.3 Counting Rankings of the Hamming Cube

For a general bipartite graph G, counting the cardinality of $R(G)$ seems to be difficult, even when $G = I_n^c$. We perform an algorithmic attack with the help of decomposition.

Suppose $G = G_0 \amalg_i G_1$. Now, let $z_0 = \text{inc}(\overrightarrow{0_{n-1}}, 0 \to i)$ and $z_1 = \text{inc}(\overrightarrow{0_{n-1}}, 1 \to i)$ be elements in $V(G)$. Since they are adjacent, their rank functions must differ by exactly 1; that is, $|h(z_0) - h(z_1)| = 1$. We denote $G = G_0 \nearrow_i G_1$ in the case $h(z_1) = h(z_0) + 1$ and $G = G_0 \searrow_i G_1$ otherwise. With this notation, we reinterpret Figure 3-1 as $G_0 \nearrow_3 G_1$, where we have $\nearrow$ because $000 > 001$ in G.

For any $A \in R(I_c^n)$, the color n must decompose A uniquely into $A_0 \nearrow_n A_1$ or $A_0 \searrow_n A_1$, where each of A_0 and A_1 is a ranking in $R(I_c^{n-1})$. Thus, we can iterate over potential pairs of rankings (A_0, A_1) and see if each of them could have come from some A. It suffices to check that each pair of vertices $\text{inc}(c, 0 \to n)$ and $\text{inc}(c, 1 \to n)$, where $c \in \mathbf{Z}_2^{n-1}$, has rank functions differing by exactly 1. However, this requires 2^{n-1} comparisons for each pair of ranking in $R(I^{n-1})$. The following lemma cuts down the number of comparisons.

Lemma 5.3.1. *For an (n,k)-ranking A and $(n-1,k)$-rankings A_0 and A_1, we have $A = A_0 \nearrow_n A_1$ if and only if the colors and vertex labelings of the three rankings are consistent and the following condition is satisfied: for each $c \in \mathbf{Z}_2^{n-1}$ and the pair of corresponding vertices $s_0 = \text{inc}(c, 0 \to n)$ and $s_1 = \text{inc}(c, 1 \to n)$ such that at least one of s_0 or s_1 is a sink (in A_0 or A_1, respectively), we have $|h(s_0) - h(s_1)| = 1$.*

Proof. These are clearly both necessary conditions. It obviously suffices if the adjacency condition $|h(s_0) - h(s_1)| = 1$ were checked over all c for all pairs s_0 and s_1 in A corresponding to the same c. It remains to show that checking the situations where at least one s_i is a sink (in their respective A_i) is enough.

Suppose we had a situation where checking just these pairs were not enough. This means for all pairs of vertices corresponding to the same c (where at least one vertex is a sink in its respective A_i) we meet the adjacency condition, but for some such pair where neither are sinks, we have $|h(s_0) - h(s_1)| \neq 1$. Let $(s_0^{(0)} = s_0, s_1^{(0)} = s_1)$ be such a pair. Without loss of generality, assume $h(s_0) > h(s_1)$. Since s_0 is not a sink in A_0, there is some $s_0^{(1)}$ covering $s_0^{(0)}$ by an edge with some color $i \neq n$. Similarly, let $s_1 = s_1^{(0)}$ be connected to $s_1^{(1)}$ via color i. Continuing this process, we eventually must come to a pair of vertices $s_0^{(l)}$ and $s_1^{(l)}$ where at least one is a sink. However, $h(s_0^{(l)}) = h(s_0) + l > h(s_1^{(l)})$. Since we assumed that $h(s_0) > h(s_1)$ and $|h(s_0^{(l)}) - h(s_1^{(l)})| = 1$, the only way for these conditions to hold is if for each $i < l$, we had $h(s_1^{(i+1)}) = h(s_1^{(i)}) + 1$. But this meant $|h(s_0^{(0)}) - h(s_1^{(0)})| = 1$ in the first place, a contradiction. Thus, these situations do not exist, and it suffices to only check pairs where at least one vertex is a sink. $\square$

Lemma 5.3.1 makes the following algorithm possible:

n	dashings	rankings	adinkras
1	2	2	4
2	8	6	48
3	128	38	4864
4	32768	990	32440320
5	2147483648	395094	848457904422912

Table 5.1: Enumeration of dashings, rankings, and adinkras with chromotopology I_c^n.

1. For the data structure, represent all rankings A by a set of sinks $S(A)$ and their ranks as in Theorem 5.1.1.
2. Start with $R(I_c^1)$, which is a set of 2 rankings.
3. Given a set rankings in $R(I_c^{n-1})$, iterate over all pairs of (possibly identical) rankings (A, B) in $R(I_c^{n-1}) \times R(I_c^{n-1})$. For each pair (with rank functions h_A and h_B, respectively),
   (a) Consider the ranking B' which is identical to B except with the rank function $h_{B'}(\overrightarrow{0}) = h_B(\overrightarrow{0}) + 1$.
   (b) For each sink $s \in S(A) \cup S(B')$, check that $|h_A(s) - h_{B'}(s)| = 1$.
   (c) If the above is satisfied for all s, put $A \nearrow_n B'$ in $R(I_c^n)$.

We used the above algorithm to compute the results for $n \leq 5$, which we include in Table 5.1 along with the counts of dashings (recall this is $o(n) = 2^{2^n - 1}$) and adinkras (which we obtain by multiplying $|R(I_c^n)|$ and $o(n)$ as the dashings and rankings are independent). Finding the answer for $n = 6$ seems intractible with an algorithm that is at least linear in the number of solutions.

For bipartite graphs other than $R(I_c^n)$ that can be decomposed (more generally, bipartite graphs of the form $A \times I$), Lemma 5.3.1 allows us to perform some similar computations.

5.4 Grid Graphs and Eulerian Orientations

While adinkras come from Hamming cubes and their quotients, counting rankings of other families of graphs make for interesting problems. The most interesting family would be the *grid graph* $G_{m,n}$, which define as the graph with mn vertices obtained by the Cartesian product of two path graphs of m and n vertices respectively. The grid graph serves as a nice approximation of the plane. Rankings of the square graph is well-studied, though in a disguised form: the "square ice" problem of Lieb ?? basically comes down to counting rankings of the square grid. We show the connection here, using the idea of Eulerian orientations.

A recurring problem in statistical mechanics is to calculate the entropy of a system via combinatorial approximations of the system: often we want to calculate the number of configurations of a graph satisfying certain combinatorial constraints. Define the *boundary-free grid graph* $G'_{m,n}$ as the graph with $((m+1)(n+1)-4)$ vertices obtained by removing the 4 corner vertices and all edges on the boundary of $G_{m+1,n+1}$. These are the "lattices of coordination number 4" that Lieb ?? was interested in. We wish to count orientations of the edges of $G'_{m,n}$ satisfying the property that each internal vertex (i.e. the 4-valent vertices) must have outdegree and indegree equal to 2. Call these configurations *square ice* configurations.

Now, recall that an *Eulerian orientation* of a graph G is a choice of direction on every edge of G such that each vertex has equal indegree and outdegree, which allows an Eulerian circuit, a cycle that uses each edge of G exactly once. An Eulerian orientation exists if and only if all vertices of G has even degree.

Lemma 5.4.1. *Square ice configurations of $G'_{m,n}$ are in bijection with Eulerian orientations of the dual of $G_{m,n}$.*

Proof. Take the boundary-free grid graph $G'_{m,n}$ and join all the vertices of degree 1 into a single vertex v_∞. This creates a planar graph $G''_{m,n}$. It is clear from inspection that $G''_{m,n}$ is the planar dual of $G_{m,n}$. Now, an edge orientation is Eulerian if and only if each vertex has equal indegree and outdegree. Thus, an Eulerian orientation of $G''_{m,n}$ satisfies the square ice configuration. It suffices to show that the indegree and outdegree of v_∞ are equal in a square ice configuration. However, if all vertices but one have equal indegree and outdegree, the remaining vertex must also have this property, becausing summing the quantity of indegree minus the outdegree over all vertices must obtain zero. □

Proposition 5.4.2. *The Eulerian orientations of a planar graph G are in bijection with rankings of its dual G'.*

Proof. Consider an orientation of G. For each edge e with face f_1 on its left and f_2 on its right, orient its dual edge in G' from f_1 to f_2. Now consider a minimal cycle in G'. The vertices in the cycle correspond to the faces around a vertex in G. The orientation on G is Eulerian if and only if the outdegree and indegree of each vertex match, which corresponds to there being an equal number of edges going one way as the other way on the cycle in G'. However, this is precisely what is required for the orientation on G' to be a ranking. □

Since the grid graph is planar, we immediately obtain the following result.

Corollary 5.4.3. *There is a bijection between square ice configurations of $G'_{m,n}$ and rankings of $G_{m,n}$.*

Lieb [30] attacks the square ice configuration enumeration problem with the transfer-matrix method. Asymptotically, the number of square ice configurations thus corresponds to powers of the highest eigenvalue of the transfer matrix with the rows and columns corresponding to the 2^n different possible configurations of vertical edges in each column of $G'_{m,n}$. Klein [28] has obtained some specific enumerations of rankings of the rectangular grid of small width and also enumerated rankings for other fundamental classes of graphs, including paths, trees, and cycles.

The grid graph has some particularly nice properties and we will revisit it after Theorem 5.7.1 is proved, due to its connection with 3-colorings.

5.5 Rankings and Discrete Lipschitz Functions

In this section, we present the concept of discrete Lipschitz functions, which turns out to be intricately related to rankings. We then construct some mathematical objects that connect them (and in turn rankings) to the theory of hyperplane arrangements and the theory of polytopes.

Let us define a *discrete Lipschitz function (DLF)* on a graph G as a function $f\colon V(G) \to \mathbf{Z}$ where for every edge (i,j) we have $|f(i) - f(j)| \leq 1$. As in rankings, we consider two functions to be equivalent if they differ only by translation in $\mathbf{Z}$. We denote the set of DLF's of G by $DLF(G)$.

While Lipschitz functions form a fundamental class of functions in analysis and our definition of DLF's seems to be the most natural discrete analogue, they have appeared in what seems to be isolated discussion in various fields, such as in the context of the No Free Lunch Theorem [6]. We came upon them as a method to study rankings; the following result is the main connection.

Theorem 5.5.1. *For any graph G, we have a 2-to-1 map identifying pairs of elements of $R(G \times I)$ and elements of $DLF(G)$.*

Proof. Suppose $|V(G)| = n$. Let v_0 be a distinguished vertex of $V(G)$. Let $DLF(G)$ be formed of representatives where v_0 takes value 0. Each $v \in V(G)$ corresponds to a pair of vertices $u(v) = (v', v'')$ in $V(G \times I)$. We give a bijection between $DLF(G)$ and elements h of $R(G \times I)$ where $h(v_0') = 0$ and $h(v_0'') = 1$. Note that by symmetry this accounts for exactly half of the rankings of $R(G \times I)$, since we can reverse all the signs to get a different ranking that has $h(v_0') = 0$.

Now take an element $h \in R(G \times I)$ where $h(v_0) = 0$ and $h(v_0'') = 1$. Construct a function $f\colon V(G) \to \mathbf{Z}$ as follows: assign $f(v)$ the value $(1/2)(h(v') + h(v'') - 1)$. In particular, $f(v_0) = 0$. Suppose $(v, w) \in E(G)$. Then because the adjacent pairs $(h(v'), h(w'))$ and $(h(v''), h(w''))$ both differ by exactly 1, the difference of $f(v)$ and $f(w)$ can take values exactly in $\{-1, 0, 1\}$, so $f \in DLF(G)$. Furthermore, it is easy to see that there is exactly one way to realize any of these three differences, as illustrated by Figure 5-6. This implies that this process is reversible so we in fact have a bijection. $\square$

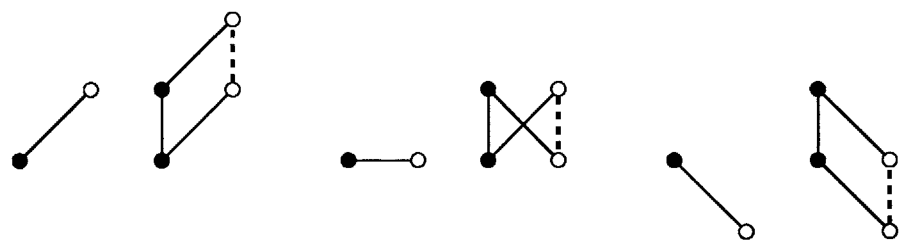

Figure 5-6: In each of these three cases, the vertical heights corresponds to the values of the relevant functions, the left graph represents an edge in an element of $DLF(G)$, and the right graph corresponds to the 4 related vertices in $G \times I$.

It has come to our attention that this result is just a slight generalization of the "Yadin Bijection" found in Peled's work [33] for Hamming cubes. In Section 5.3, we counted rankings of a specific graph, the Hamming cube. It is important to point out that if we were interested in asymptotic results instead of exact enumeration, the rankings of a Hamming cube (and related objects) are fairly well-understood thanks to Peled and some precursors, such as Galvin [20]. In particular, Peled [33] uses this bijection to reduce thinking about discrete Lipschitz functions to thinking about rankings.

In our context of adinkras, where the main objects are the Hamming cubes, Theorem 5.5.1 is especially nice because it reduces studying the number of rankings of I^n to studying DLF's of I^{n-1}, the Hamming cube of dimension one less:

Corollary 5.5.2. *We have* $2|DLF(I^n)| = R(I^{n+1})$.

We now introduce another connection between DLF's and rankings with the theory of polytopes. Take any (not necessarily bipartite) connected graph G. If $|V(G)| = n$, identify the vertices with $[n]$ and consider pairs of hyperplanes $x_i - x_j = \pm 1$ for each $(i, j) \in E(G)$. This gives a particular hyperplane arrangement. Now, intersect this arrangement with the hyperplane $x_n = 0$ (this corresponds to the fact that the interpretation of rankings as equivalence classes under translation is equivalent to the concept of assigning a fixed value to a particular vertex) to obtain an arrangement in $\mathbf{R}^{n-1}$ with a bounded central region, which we denote DLP_G, the *discrete Lipschitz polytope* of G. Now, we show that the name *discrete Lipschitz polytope* is appropriate via the following result.

Theorem 5.5.3. *For a connected graph G, the discrete Lipschitz polytope DLP_G satisfies the following:*

1. *The lattice points in DLP_G are in bijection with $DLF(G)$.*

2. *If G is bipartite, the vertices of DLP_G are in bijection with $R(G)$.*

Proof. Let the lattice point $(x_1, \ldots, x_{n-1})$ correspond to the ranking where vertices $i < n$ are assigned the value x_i and vertex n is assigned the value 0. It is then clear from the definitions of the hyperplanes given that the edge constraint on the edge (i, j) necessary for a ranking is satisfied exactly when $|x_i - x_j| \leq 1$. Since the point belongs to the bounded central region of the arrangement exactly when all these inequalities are satisfied, the lattice points exactly correspond to the discrete Lipschitz functions. We now focus on the second part of the theorem statement.

It will help us to imagine a point $x = (x_1, \ldots, x_{n-1})$ in DLP_G as a point $x' = (x_1, \ldots, x_{n-1}, x_n = 0)$ in the ambient space $\mathbf{R}^n$, and imagine DLP_G being embedded as the slice DLP'_G in $\mathbf{R}^n$, also with the last coordinate equal to 0. Now, x is a vertex of DLP_G if and only if there is no open line segment around x' that lies within DLP'_G. Equivalently, for any nonzero vector $e = (e_1, \ldots, e_n)$, one of $x' \pm \epsilon e$ is outside of DLP'_G for any positive ϵ. For a function f on $V(G)$, call an edge *rigid* if the values of f at those vertices differ exactly by 1. Let a connected component of vertices connected by rigid edges be called a *rigid cluster*. In particular, a ranking has a rigid cluster of the entirity of $V(G)$, and every edge is rigid in a ranking.

Now, note that if two vertices i and j are connected by a rigid edge, then if we have a nonzero e such that $e_i \neq e_j$, at least one of $x' \pm \epsilon e$ must leave DLP'_G since the difference $|x_i - x_j|$ will either strictly increase or decrease. Therefore, given a rigid cluster C, any nonzero e that do not assign the same e_i for $i \in C$ will force one of $x' \pm \epsilon e$ to leave the polytope. This is the main observation we will use.

Suppose we have a ranking, we now show it is a vertex of DLP_G. In a ranking, all edges are rigid. Thus, any perturbation vector e will leave the polytope unless $e = (\epsilon, \epsilon, \ldots, \epsilon)$. However, we need the last coordinate of e to be 0 to stay within DLP'_G, so e must be the zero vector, a contradiction. So all rankings are vertices.

Given a vertex x of DLP_G. We claim that $V(G)$ is a rigid cluster. Suppose not. Then there exists some rigid cluster C not including vertex n. Thus, we may shift the value of x' on all vertices in C simultaneusly in either direction by some small $\epsilon > 0$, corresponding to a vector e which has value ϵ in the coordinates corresponding to C and 0 otherwise. Since no vertices in C are at distance exactly 1 from the vertices outside of C, we may pick ϵ sufficiently small such that $x' \pm e$ stay within DLP'_G, which corresponds to an open segment around x in DLP_G. Therefore $V(G)$ must be a rigid cluster, and x' actually takes on only integral values on $V(G)$. This means that adjacent vertices must differ by either 0 or 1.

Now, because G is connected and $V(G)$ is a rigid cluster, we can take a spanning tree T of G that uses only rigid edges, rooted at n. Starting from the root (remember that $x_n = 0$ is enforced), going down the branches of T, we see that all nodes of even distance from n in T must be assigned even values and all nodes of odd distance from n must be assigned odd values. Since G is bipartite, all edges must actually be between odd and even values, and thus differ by 1 instead of 0. Therefore, all edges are actually rigid, and we have a ranking, as desired. $\square$

Theorem 5.5.3 give another connection between DLF's and rankings. Furthermore, DLP_G simply is a very nice object to study. Its dual also seems interesting: recall the polar P^* of a polytope P containing the origin in its interior is the set

$\{x|\langle x,y\rangle \leq 1$ for all $y\}$. Define GR_G, the *graphical root polytope*, to be DLP_G^*. Since DLP_G was defined via pairs of hyperplanes that straddle the origin, GR_G can be defined by the convex hull of vertices that correspond to the duals of these hyperplanes. As before, identify the vertices of G with $[n]$ and distinguish the vertex labelled 1. Let e_i be the unit vector in the direction of i. Then GR_G is just the convex hull of:

1. $\pm(e_i - e_j)$, for $(i,j) \in E(G)$ where $i,j \neq 1$.

2. $\pm(e_i)$, for $(i,1) \in E(G)$.

It follows that GR_G is the generalized root polytope (as defined in [31]) corresponding to G. For example, when G is the complete graph K_n, we get exactly the root polytope for A_n.

As is the case with rankings, counting DLF's for other families of graphs seems like an interesting problem, especially if we compare their asymptotic growth to that of rankings. We give an example computation here where we have both an algebraic expression and an asymptotic estimation (using Laplace's Method. For a primer, see, e.g., Flajolet and Sedgewick [18]). Klein [28] has some concrete computations for various other graphs.

Proposition 5.5.4. *The number of DLF's of the cyclic graph C_{2n} is*

$$\sum_{k\geq 0} \binom{2n}{k,k,2n-2k}.$$

Asymptotically, this grows as $\alpha 3^{2n}/\sqrt{n}$, where α is a constant.

Proof. After selecting a distinguished vertex to take value 0, each clockwise move around C_{2n} must change the value by $+1$, 0, or -1, ending with a net change of 0. We must thus pick an equal number of $(+1)$'s as (-1)'s. Thus, the sum expression above is exactly the quantity we desire.

To get an asymptotic approximation, we will use Laplace's method and some generating function manipulations.

Consider the function

$$f_n(z) = \sum_{k=0}^{n} \binom{2n}{k,k,2n-2k} z^k.$$

We wish to obtain $f_n(1)$ as $n \to \infty$. To do this, we invoke the auxiliary generating function

$$
\begin{aligned}
g(t,z) &= \sum_{n\geq 0} f_n(z)t^{2n} \\
&= \sum_k \Big(\sum_{n\geq k} \binom{2n}{2k} t^{2n}\Big)\binom{2k}{k} z^k \\
&= \frac{1}{2}\sum_k \Big[\frac{t^{2k}}{(1-t)^{2k+1}} + \frac{t^{2k}}{(1+t)^{2k+1}}\Big]\binom{2k}{k} z^k \\
&= \frac{1}{2}\sum_k \Big[\frac{(t^2z)^k}{(1-t)^{2k+1}} + \frac{(t^2z)^k}{(1+t)^{2k+1}}\Big]\binom{2k}{k},
\end{aligned}
$$

where in the third line we used the well-known fact

$$\sum_{n\geq k} \binom{n}{k} t^n = \frac{t^k}{(1-t)^{k+1}}.$$

Now, by differentiating the generating function for Catalan numbers (or direct power series expansion) we can obtain the generating function

$$\frac{1}{\sqrt{1-4z}} = \sum_n \binom{2n}{n} z^n.$$

Using this gives

$$
\begin{aligned}
g(t,z) &= \frac{1}{2(1-t)}\frac{1}{\sqrt{1-4t^2z/(1-t)^2}} + \frac{1}{2(1+t)}\frac{1}{\sqrt{1-4t^2z/(1+t)^2}} \\
&= \frac{1}{2}\Big[\frac{1}{\sqrt{1-2t+t^2-4t^2z}} + \frac{1}{\sqrt{1+2t+t^2-4t^2z}}\Big].
\end{aligned}
$$

Since we are interested in $z=1$, we have

$$g(t,1) = \frac{1}{2}\Big[\frac{1}{\sqrt{(1-3t)(1+t)}} + \frac{1}{\sqrt{(1+3t)(1-t)}}\Big].$$

We want the t^{2n} term of this generating function, so we want to evaluate the contour integral on a small loop around 0 of $t^{-2n+1}g(t,1)$. For the first term (the second term is analogous), substituting $y = \sqrt{-3(1-3t)(1+t)}$ gives the integral (now around $y=\sqrt{3}i$ instead of $t=0$):

$$\sqrt{3}i\int (4+y^2)^{-1/2}\Big(\frac{3}{\sqrt{4+y^2}-1}\Big)^{2n+1}dy$$

Now, the function in the parentheses has a unique critical point at $y=0$ and its double derivative is nonzero. By Laplace's method, this integral is dominated asymptotically by $1/\sqrt{n}$ times the evaluation of the integrand, evaluated at $y = 0$. We thus obtain that $f_{2n}(1) \sim \alpha 3^{2n}/\sqrt{n}$ for some nonzero constant α. $\square$

5.6 Discrete Lipschitz Functions and Dashings

While having connections between DLF's and rankings make some sense, there is a connection between DLF's and dashings, which is strange as dashings seem to be entirely independent from rankings.

Given a dashing d on G, denote by $\mathrm{Con}(G, d)$ the contraction of G by all solid edges in d. We then have the following result:

Proposition 5.6.1. *Let G be a bipartite graph. We have that*

$$|DLF(G)| = \sum_{d \in e(G, C_G)} |R(\mathrm{Con}(G, d))|.$$

Proof. For each DLF, consider the edges where the two vertices are given the same value. It is clear that contracting the edge gives a DLF on the reduced graph. If we make all such edges solid and contract them, since there is no edge connecting two vertices with the same value, the resulting DLF on the reduced graph must actually be a ranking. So it is clear that

$$|DLF(G)| = \sum_{d} |R(\mathrm{Con}(G, d))|,$$

where we sum over all rankings d. However, for each cycle in C_G, we need exactly an even cycle to remain for a ranking to exact after the contractions. This means that there must be an even number of dashes allocated to each cycle, and thus we only care about terms with an even dashing. $\square$

In principle, this may lead to some sort of computation of $|R(I^6)| = 2|DLF(I^5)|$. However, as there are now 2^{2^5-1}, or about 10^9 terms, in the sum, we would still need a couple of optimizations to make this breakthrough.

5.7 Rankings and Colorings

There is another very nice connection involving rankings and graph colorings that happen to work very well for graphs arising from adinkras. This section draws upon joint work with Aaron Klein from the MIT PRIMES high school program [28].

Define *squarely generated graphs* to be graphs where the cycle space has a basis consisting of 4-cycles or 2-cycles. Note that a squarely generated graph must be bipartite. Suppose we have a graph G with a 2-cycle (i.e. a multiedge). Note that replacing the multiedge by a single edge does not change the squarely generated property. Thus, studying squarely generated graphs is equivalent to thinking about simple graphs with a basis of the cycle space consisting of 4-cycles.

The main result relating rankings and colorings is the following result.

Theorem 5.7.1. *For any G, we have $|R(G) \leq (1/3)\chi_G(3)$. Furthermore, if G is squarely generated, then*

$$|R(G)| = (1/3)\chi_G(3),$$

where χ_G is the chromatic polynomial of G.

Proof. Let the vertices of G be labeled with $[n]$ and rankings having the property that the vertex n is assigned 0 in the ranking. Label the 3 colors with members of $\mathbf{Z}_3$. We now show that every ranking of G gives a different 3-coloring such that the vertex n is assigned the color 0. The number of such colorings are exactly the $(1/3)\chi_G(3)$ on the right-hand-side of the statement.

To see this, consider a ranking h. Now, assign every vertex v the color $c(v) = h(v) \pmod 3$. Now, since $|h(v) - h(w)| = 1$ for adjacent v and w, we must have $c(v) \neq c(w)$, giving a proper coloring. We definitely have $c(n) = 0$ as we wanted; furthermore, this map is obviously one-to-one, so the inequality holds.

When G is squarely generated, we now show that this map is surjective as well. Starting with a coloring $c\colon V(G) \to \mathbf{Z}_3$, we seek to create a ranking h by first assigning the rank 0 to the vertex n, recalling that we have $c(n) = 0$. Now, we greedily assign ranks adjacent to vertices that we have already assigned in the following manner: suppose w was assigned the value $h(w)$ and unassigned v is adjacent to w; now, assign $h(v) = h(w) + q(c(v) - c(w))$, where $q(t) \in \mathbf{Z}$ for $t \in \mathbf{Z}_3$ is the integer corresponding to the representative of t in $\{-1, 0, 1\}$. This guarantees $h(w)$ to be a ranking because in the proper coloring c we must have $c(v) - c(w)$ equal to 1 or -1 $\pmod 3$.

The only trouble we would run into is if v were adjacent to two w_1 and w_2 that want to assign different values of $c(v)$ via the above algorithm. Since this assignment process keeps the vertices that are assigned values already connected, w_1 and w_2 must already be connected, creating a cycle through v. Since G is squarely generated, the minimal such cycle must have length 4; in particular, this means both w_1 and w_2 must be adjacent to some w. It is easy to check from the definition of our algorithm that the value of $c(v)$ that w_1 wants to assign must be equal to the value of $c(v)$ that w_2 wants to assign modulo 3. Nowever, as we have completed a cycle of length 4, these two potential values must actually be equal to the sum of 4 elements taken from $\{-1, 1\}$, since each adjacent edge corresponds to the change of value by 1. The only way for this to be possible is if the two assigned values to be the same, so there is no ambiguity in assigning $c(v)$. Thus, there is a consistent method to obtain a ranking h, which is easily seen to give c back via the previously described method of taking $c(v) = h(v) \pmod 3$. Thus, we have equality of the two quantities for squarely generated graphs, as desired. $\square$

The result most relevant for us for adinkras is:

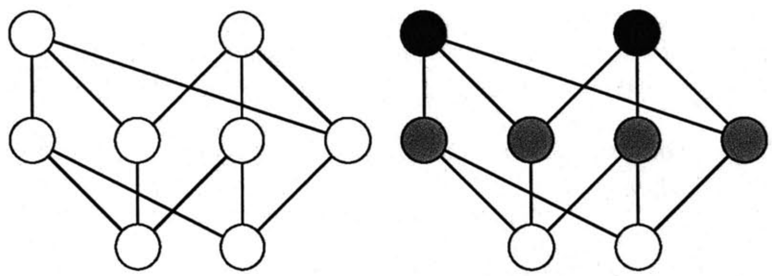

Figure 5-7: Example of turning a ranking into a 3-coloring.

Proposition 5.7.2. *The graph I^n/L is squarely generated if L has a basis of weights 4 or 2. In particular, the Hamming cubes I^n are squarely generated.*

Proof. First, we show that I^n is squarely generated. For this we again use some topology: because we have a 2-cell for all the 4-cycles and there are no 2-cycles, I^n is squarely generated if every cycle is a boundary, or if the first homology vanishes. However, I^n is the two-skeleton of the unit-cube, which has trivial H_1.

Now, suppose that L has a basis of weights 4 or 2. Take a cycle C in I^n/L. It lifts to a path $x \to y$ in I^n where $y - x \in L$. Since I^n is squarely generated, we may write $y - x$ as a sum of weight-2 and weight-4 codewords, which becomes a sum C' of 2− and 4− cycles under the quotient. Since $C + C'$ must then lift to a cycle (which in turn quotients to a sum of weight-2 and 4 codewords), C itself can be written as a sum of weight-2 and 4 codewords, so I^n/L is squarely generated. □

A neat corollary of the above is that studying rankings of certain adinkras (including the Hamming cube) is equivalent to studying 3-colorings.

For the Hamming cube [33] and the square grid [30], the equality part of Theorem 5.7.1 seems to have already been known in literature. Theorem 5.7.1 is a generalization of both of these observations. As a bonus, the inequality portion of the statement shows that this is the sharpest possible statement about when such equalities can exist. It is remarkable that Theorem 5.7.1 bears some resemblance to Stanley's result [37], where the number of acyclic orientations is related to $\chi_A(-1)$.

The grid graph from Lieb's work is also squarely generated. It was mentioned in Lieb [30] that the problem was also effectively counting 3-colorings. We can interpret this as a combination of Proposition 5.4.2 and Theorem 5.7.1. The square grid is an especially nice graph, being self-dual, squarely generated, and effectively (again, modulo boundary conditions) Eulerian. Thus, it serves as an intersection where the different conditions required for our results all apply and where rankings, Eulerian orientations, and 3-colorings all appear.

Chapter 6

Physics and Representation Theory

The study of adinkras originated from physics, in particular supersymmetric representation theory. The main purpose of our work has been to extract combinatorics problems from adinkras and showcase their connections to other parts of mathematics.

In this chapter, we end our work with some discussion more relevant to the original physics context. We survey the recent papers and make some observations that would hopefully be useful to a reader interested in these problems. We believe each of these topics is a resource for rich mathematical discussion.

6.1 Constructing Representations

Take an adinkra A, and consider the component fields (the bosons ϕ and the fermions ψ) as a basis. Then, consider a set of matrix generators $\{\rho(Q_i)\}$ in that basis, where $\rho(Q_i)$ is the adjacency matrix of the subgraph of A induced by the edges of color i. If we order all the ϕ to come before all the ψ in the row/column orderings, these matrices are block-antisymmetric of the form

$$\rho(Q_i) = \begin{pmatrix} 0 & L_i \\ R_i & 0 \end{pmatrix},$$

where the L_i and R_i are [21]'s *garden matrices.* For the adinkra in Figure 6-1, we have the following matrices, where the row/column indices are in the order $00, 11, 10, 01$.

$$\rho(Q_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \rho(Q_2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

So far, we have encoded the graph and the dashing into the matrices, but we do not yet have a representation of the supersymmetry algebra $\mathfrak{po}^{1|N}$. In fact, the

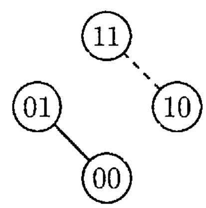

Figure 6-1: An adinkra with chromotopology I^2.

$\rho(Q_i)$ form a representation of the Clifford algebra $\mathbf{Cl}(N)$, which we discuss further in Section 6.3. The missing information (up to scalars) is the ranks of the vertices, which we can add into these matrices by adding the Hamiltonian operator H's to appropriate entries (recall Section 1.4 for details). In this sense, we are partitioning the infinite-dimensional basis of the representation into finite-dimensional "slices," each slice corresponding to a single finite-dimensional representation corresponding to our finite-dimensional matrices.

An obvious question to consider whenever we study representations is the following:

Problem 5. Which adinkraic representations are irreducible?

In the valise case, this is well-understood (see [12]) with a surprising answer. If L were not a maximal subspace inside $\mathbf{Z}_2^n$, we may quotient I_c^n/L further to give a subrepresentation. Thus, irreducible valise adinkraic representations must have maximal doubly-even codes, which are self-dual. There seems to be no good general method for other rankings. The intuition of the obstruction is that this method of creating subrepresentations require the vertices in each coset to come from the same rank, corresponding to the same engineering dimension. This kind of physics constraint is intricately connected to the selection of the right notion of isomorphism for adinkraic representations, which we now discuss.

6.2 When are Two Adinkras Isomorphic?

The instinct for the choice of the definition of "isomorphism" seems to be completely intuitive for the authors of the literature (see [21] and [13]), but we think it is important to have a formal discussion since the existing literature is somewhat cavalier about these definitions.

We usually consider two representations isomorphic if they are conjugate by some change of basis. However, because of our physics context we need more restrictions. To find the right notion, we now recall/define three types of transformations and discuss what it means for them to give the "same" adinkra.

- Recall that a *vertex switching* changes the dashing of all edges adjacent to a vertex. This corresponds to simply changing the sign (as a function) of the component field corresponding to that vertex, or equivalently, conjugation of the representation by a diagonal matrix of all 1's except for a single (-1). It is reasonable to consider this move as an operation that preserves isomorphism.

- Let a *color permutation* permute the names of the colors (in the language of codes, it is a simultaneous column permutation of the codewords corresponding to each vertex). In our situation, this is just a shuffling of the generators, so at first glance it is reasonable to consider this operation to preserve isomorphism. The existing literature, e.g. [12], seems to do so as well. However, this is not quite what we want in a natural definition, where we need to consider the base ring fixed. By analogy, consider the $k[x, y]$-modules $k[x, y]/(x)$ and $k[x, y]/(y)$, which may look "equivalent" (they are indeed isomorphic as algebras) but are not isomorphic as modules. They should not be: we really want $A \oplus B$ to be isomorphic to $A \oplus B'$ if B and B' were isomorphic, but the direct sums $k[x, y]/(x) \oplus k[x, y]/(x)$ and $k[x, y]/(x) \oplus k[x, y]/(y)$ are not isomorphic in any reasonable way.

  In fact, the existing adinkra literature notices this problem when considering disconnected adinkras (i.e. adinkras with topology of a disconnected graph). These graphs correspond to direct sums of representations of single adinkras. However, since a color permutation is done over all disjoint parts simultaneously, if we consider color permutations as operations that preserve isomorphism, we obtain situations where $A \cong C$ and $B \cong D$, but $A \oplus B \not\cong C \oplus D$. The literature deals with this situation by calling color permutations *outer isomorphisms*. We believe the correct thing to do is to just to **not** consider these situations isomorphic and treat them as a separate kind of similarity.

- Let a *vertex permutation* permute the vertex labels of an adinkra A. This corresponds to conjugating the matrices $\rho(Q_i)$ by permutation matrices. Here it makes sense to impose further physics constraints: we want these transformations to preserve the engineering dimensions of the component fields. This prevents us from allowing arbitrary vertex permutations and simply considering two adinkraic representations isomorphic if they're conjugate. On the adinkras side, this corresponds to us enforcing that the rank function of A be preserved under any vertex permutation (in particular, bosons must go to bosons, and fermions to fermions). Happily, this neatly corresponds to the natural definition of isomorphism for ranked posets.

Problem 6. What is the right definition of "isomorphism" for two adinkraic representations? How does it relate to the combinatorics of adinkras?

- Following physics requirements, we define two adinkras A and B to be *isomorphic* if there is some matrix R that transforms each generator $\rho(Q_I)$ of A to the corresponding $\rho(Q_I)$ in B via conjugation, with the stipulation that such a conjugation preserves the ranks of the component fields. To be explicit, let the component fields be partitioned into $P_1 \cup P_2 \cup \cdots$, where each P_i contains all ψ_j or ϕ_j of some rank (equivalently, engineering dimension). We require M to be block-diagonal with respect to this partition.

- Following combinatorial intuition, we define two adinkras to be *C-isomorphic* if there is a sequence from one to the other via only vertex switchings or ranked poset isomorphisms. The discussion in Appendix **??** shows that both of these operations preserve isomorphism, so C-isomorphism is more restrictive than isomorphism.

In a perfect world, these two notions would coincide. We are not so lucky here: the underlying graph is an invariant of the operations in the definition of C-isomorphism, but there are adinkras with different graphs that correspond to isomorphic representations [8]. We still do not have a complete picture of the nuances between the two definitions. C-isomorphism is studied in more detail in [13] (where it is simply called "isomorphism"; this is an example of why a formal discussion would be good to avoid accidental overlap of different concepts), which gives a deterministic algorithm to determine if two adinkras are C-isomorphic. Similar discussion relevant to isomorphism (even though it was not defined as such) can be found in [21] and [12].

A lot of interesting mathematics remain in the area. As an example, both [21] and [13] distinguish adinkras with the help of what amounts to the trace of the matrix

$$\rho(Q_1)\rho(Q_2)\cdots\rho(Q_N)$$

after multiplying by the matrix $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. We would like to point out that this is precisely the well-known *supertrace* from the theory of superalgebras.

6.3 Clifford Representations

In Section 1.4, we called the adinkraic representations arising from valise adinkras "Clifford supermultiplets." This is no big surprise – when we ignore the Hamiltonian H in the defining relations

$$\{Q_I, Q_J\} = 2\delta_{IJ}H,$$

we get precisely the Clifford algebra relations

$$\{Q_I, Q_J\} = 2\delta_{IJ}.$$

In other words, when we forget about the rank of an adinkra and look at only the well-dashed chromotopology (alternatively, the valise, where no bosons or fermions are privileged by rank from the other fields of the same type), we are really looking at a Clifford algebra representation, something that we saw in 6.1 and in the proof of Theorem 2.3.3. Therefore, we can think of adinkraic representations as extensions of representations of the Clifford algebra; [11] makes this analogy more rigorous by realizing adinkraic representations as filtered Clifford supermodules.

While Clifford algebras and their representations are well-understood (see [1] or [29]), the following is a natural question to ask:

Problem 7. Can adinkras give us better intuition (organizational or computational) about the theory of Clifford representations?

In [12], each valise adinkra with the I_c^n chromotopology is used to explicitly construct a representation of the Clifford algebra $\mathbf{Cl}(n)$. This introduces a plethora of representations with lots of isomorphisms between them – after all, there are at most 2 irreducible representations for each Clifford algebra over $\mathbf{R}$. Even for a beginner, the visually appealing aspect of adinkras (more precisely, well-dashed chromotopologies) may be the foundation of an easier mental model of thinking about Clifford algebras.

6.4 Extensions

As we brushed over in Section 1.4, the adinkraic representations correspond to the 1-dimensional (more precisely, $(1,0)$-dimensional) worldline situation with N supercharge generators. We now discuss the more general context. Helpful expositions of related concepts are [19] and [40].

In general, we are interested in some $(1+q)$-dimensional vector space over $\mathbf{R}$ with Lorentzian signature $(1,q)$. Besides our $(1,0)$ situation, some examples are $(1,1)$ (*worldsheet*) and $(1,3)$ (*Lorentzian spacetime*). We can write this more general situation as $(1,q|N)$-supersymmetry[1] We would then call the corresponding superalgebra $\mathfrak{po}^{1+q|N}$, which specializes to the particular superalgebra $\mathfrak{po}^{1|N}$ we have been working with when $q=0$.

In the case where $(1+q) = 2, 6 \pmod 8$, we actually get two different types of supercharge generators (this again corresponds to the fact that there are two Clifford algebra representations over $\mathbf{R}$ in those situations), so we can partition $N = P + Q$ and call these situations $(1,q|P,Q)$-supersymmetry.

Problem 8. What happens when we look at $q > 0$? What kind of combinatorial objects appear? Will the machinery we developed for adinkraic representations in the wordline case be useful?

[22] examines the $(1,1)$-case, where the combinatorics get more complex. The reader may have gotten the intuition that the dashings and rankings are fairly independent conditions of the adinkra. This is true for the $(1,0)$-case but no longer

[1]Here is another unfortunate source of language confusion: for physicists, N means the number of supersymmetry generators, whereas mathematicians would instead count the total number of dimensions and write dN instead of N, where d is the real dimension of the minimal spin-$(1/2)$ representation of $\mathbf{R}^{1+q}$. These minimal dimensions are $1,1,2,4,8,\ldots$ starting with $n=1$. Luckily, for most of our work, we have $d=1$ and thus no problems. A clear explanation is given in [19].

holds for the $(1,1)$-case, where certain forbidden patterns arise that depend both on the dashings and rankings. [26] creates $(1,1|P,Q)$ representations by tensoring and quotienting worldline representations, similar in spirit to the construction of representations of semisimple Lie algebras.

In a different direction, [16] and [17] examine which worldline representations can be "shadows" of higher-dimensional ones and give related consistency-tests and algorithms. As worldline representations are involved, the 1-dimensional theory already built plays an instrumental role.

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