ON THE INVERSE SPECTRAL PROBLEM FOR POLYGONAL DOMAINS

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Thesis Abstract

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This thesis uses the solution operator U for the wave equation to prove several results in the inverse spectral theory of polygonal domains. Because a polygonal domain is completely specified by a finite number of geometric constants, the inverse spectral problem of extracting geometric information about a region from its spectrum becomes finite dimensional when the region is restricted to the class of polygonal domains. In this context a polygonal domain X is a bounded, connected, open subset of $\mathbf{R}^2$ whose boundary is a finite union of disjoint polygons. The spectrum under examination is the spectrum of the Laplacian on X. The technique of extracting information from the spectrum by examining the trace of the solution operator of the wave equation has been successful in other settings and is applicable to polygonal domains. In this method the study of the trace of the solution operator centers on the geometric information carried by the singularities.

One main result of this thesis is the proof of the Poisson relation for polygonal domains. Conjectured by F. G. Friedlander, the relation states that the singularities of the trace operator U are confined to the length spectrum of closed geodesics of X:

$$\text{sing supp trU} \subseteq \{l : l = 0 \text{ or } |l| \text{ is the length of a closed geodesic of X}\}.$$

The appropriate definitions of each side of the Poisson relation are provided by F. G. Friedlander's work [F] analyzing the Sommerfeld kernel for the initial boundary value problem of the wave operator on wedges. Using the Sommerfeld kernel Friedlander constructed Green's function $U(t,x,t',x')$ for the wave equation on a polygonal domain. In this notation trU is the distribution on $\mathbf{R}_t$ given by the following integral:

$$\int_X U(t,x,0,x)\,dx$$

Due to the finite propagation speed, for small time intervals Green's function on a polygonal domain is locally equal to the Sommerfeld kernel on a corresponding wedge. The known form of the Sommerfeld kernel dictates that the set of geodesics for a polygonal domain includes two classes of geodesics, reflective and diffractive. A *reflective geodesic* is a billiard ball trajectory confined to the interior of X, the edges of ∂X, and those vertices of ∂X at which the edges meet to form an angle π/N, $N \in \mathbf{Z}$. At the edges of X the trajectory is reflected according to Snell's Law.

At the allowed vertices, reflection is according to a generalization of Snell's Law. A *diffractive geodesic* is a billiard ball trajectory in $X \cup \partial X$ which includes at least one vertex for which the the associated angle is not π/N. Such trajectories obey the reflection laws governing reflective geodesics, while at the vertices not of π/N type diffraction occurs and the trajectory emerges from the vertex at an arbitrary angle.

The proof of the Poisson relation proceeds in several steps. The main lemma, proved in this thesis, is that the wavefront set of U away from the vertices is restricted to points of the form $(t,x,t',x',\tau,\xi,\tau',\xi')$ for which a geodesic of length $t-t'$ joins x and x' while ξ and ξ' are the directions of the geodesic as it passes through x and x' respectively. Further, the following quantities must be equal: $\tau^2, \tau'^2, |\xi|^2$, and $|\xi'|^2$. This relationship is shown using the known form of the wavefront set for small time intervals together with a series expansion for the Sommerfeld kernel which gives control of the kernel as x and x' approach the vertex. This done, the proof proceeds by considering ϕU and $(1-\phi)U$ where ϕ is a smooth function in x supported away from the vertices. Standard wavefront analysis then shows that $\mathrm{tr}\,\phi U$ satisfies the Poisson relation. The proof of the Poisson relation for $\mathrm{tr}(1-\phi)U$ hinges on the existence of a trace class operator closely related to $\mathrm{tr}(1-\phi)U$. Manipulation of this operator reduces the proof to a manageable question of the regularity of U on smooth initial data with appropriately bounded behavior near the vertices.

One extension of this result is the solution of the inverse spectral problem for triangles. The crucial spectral invariant for the solution is the height of the triangle, which can now generally be found from the length of the shortest diffractive geodesic passing through a single corner.

[F] F. G. Friedlander, On the Wave Equation in Plane Regions with Polygonal Boundary, *Proceedings of the NATO ASI*, Castelvecchio-Pascoli (Lucca), Italy, 1985.

Contents

Chapter 1

Introduction

This thesis uses the solution operator for the wave equation to prove the Poisson relation for polygonal domains and to derive a spectral invariant of triangles. In this context a polygonal domain X is a bounded, open subset of R^2 whose boundary is a finite union of disjoint polygons. Denote the wave operator $(\frac{\partial}{\partial t})^2 - (\frac{\partial}{\partial x})^2 - (\frac{\partial}{\partial y})^2$ by P. The forcing problem for polygonal domains is stated as follows: given $f \in C_0^\infty(X)$, find u satisfying $Pu = f$, $u = 0$ on the edges of X, and $u = 0$ for $t \ll 0$. If additional assumptions are made about the regularity of u near the vertices of X this problem has a unique solution. F. G. Friedlander in [1] constructs the solution operator G for this problem from the Sommerfeld kernels for the angles involved. In this thesis the operator used is generally the solution operator for the initial value problem: given g_0 and $g_1 \in C_0^\infty(X)$ find $g(t,x)$ satisfying $Pg = 0$, $g(0,x) = g_0$, $\frac{\partial g}{\partial t}(0,x) = g_1$, and $g = 0$ on the edges of X. The solution operator $U(t)$ maps initial data (g_0, g_1) to $(g(t), \frac{\partial g}{\partial t}(t))$. The domain is $H_1^0 \oplus L^2$. Chapter 3 contains the construction of $U(t)$, its uniqueness, and its expression in terms of $G(t)$.

The operator $U(t)$ gives rise to a tempered distribution $tr\, U$ on R_t. The pairing $< tr\, U, f >$ is defined to be the trace of the operator $\int U(t)f(t)dt$. This distribution is determined by the spectrum of the Laplacian on X with Dirichlet conditions.

The first objective of this thesis is to prove the Poisson relation for $U(t)$. The Poisson relation states that the singular support of $tr\, U$ is confined to the length spectrum of closed geodesics of X:

$$singsupp\; tr\, U \subseteq \{l : l = 0 \text{or } |l| \text{ is the length of a closed geodesic of } X\}$$

Here the appropriate definition for a geodesic of X is given in Friedlander[2]. Both reflective and diffractive geodesics must be taken into account. A reflective geodesic is a closed billiard ball trajectory confined to X, the edges of X, and the vertices of X at which the edges meet to form an interior angle $\frac{\pi}{N}$, $N \in \mathbf{Z}$. At the edges the trajectory is reflected according to Snell's law. At the allowed vertices, reflection is according to a generalization of Snell's law. A diffractive geodesic is one which includes at least one vertex for which the associated angle is not of the form $\frac{\pi}{N}$. Such trajectories obey the reflection laws above. In addition, at vertices not of $\frac{\pi}{N}$ type, diffraction occurs and the trajectory emerges at an arbitrary angle.

The wavefront set of U is the major element in the proof of the Poisson relation. In Chapter 4 it is shown that the wavefront set of U is restricted to points $(t, z, z', \tau, \zeta, \zeta') \in T^*(R \times X \times X)$ such that a geodesic of length $|t|$ joins z and z'. The direction from z' to z along the geodesic is given by $sign(t)sign(\tau)\xi'$ and the direction from z to z' is given by $sign(t)sign(\tau)\xi$. The proof proceeds by examining the propagation of singularities by $U(t)$, using induction on the range of t for which the propagation is along geodesics. The construction of U guarantees propagation along geodesics for small t. Extension to larger t is accomplished by cutting off the solution $g(t)$ by some $\psi(t) \in C_\infty(R_t)$ supported in the region where the propagation property holds. Then applying the forcing kernel to $P\psi g$ extends g. The forcing kernel for small time intervals has known wavefront set, therefore this also extends the propagation property to a larger range of t.

The Poisson relation follows from this and the regularity of $U(t)$ and $G(t)$ on data which is smooth in in an appropriate sense. Chapter 5 is a proof of the Poisson relation for polygonal domains. The approach is to split $tr\, U$ into a sum

$$tr\, U[1-\phi] + tr\, U(t-\epsilon)[\phi]U(\epsilon)[\phi] + tr\, U(t-\epsilon)[\phi]U(\epsilon)[1-\phi]$$

where ϕ equals 1 in a neighborhood of each vertex, and ϕ vanishes outside a neighborhood which is small compared to ϵ. Then the terms $tr\, U[1-\phi]$ and $tr\, U(t-\epsilon)[\phi]U(\epsilon)[1-\phi]$ can be handled by standard microlocal analysis. The term $tr\, U(t-\epsilon)[\phi]U(\epsilon)[\phi]$ is smooth for all t, because $[\phi]U(\epsilon)[\phi]$ is smooth, dies off appropriately toward the vertices, and satisfies $P[\phi]U(\epsilon)[\phi] = 0$ on a neighborhood of the vertices.

One application of this result is to the inverse spectral problem for triangles. The area and perimeter are well known spectral invariants. If the height could be determined from the singularities of $tr\, U$, this would show that the triangle was determined by its spectrum. For triangles with one height shorter than the others, the height is available from $tr\, U$.

Suppose that a triangle X has a unique shortest height, h. Then there is a closed geodesic of length $2h$ lying along the perpendicular dropped from the largest angle α to the opposite side. This produces a singularity of $tr\, U$ at $t = 2h$. The singularity is of order -1 if the triangle is a right triangle, and of order $-\frac{1}{2}$ otherwise. This is proved in Chapter 6 by reducing the terms $tr\, U(t)[1-\phi]$ and $tr\, U(t-\epsilon)[\phi]U(\epsilon)[1-\phi]$ to expressions roughly of the form $tr\, F\, U_s(t, \rho(z), z')[\tilde{\phi}]$. Here F is a pseudodifferential operator acting basically as a cutoff function, and $\tilde{\phi}$ is a function supported along the height geodesic away from the vertex. The operator U_s is the solution operator for the sector corresponding to the angle α. The operator ρ is reflection across the side opposite α.

The reduction is straightforward for $tr\, U(t)[1-\phi]$. For $tr\, U(t-\epsilon)[\phi]U(\epsilon)[1-\phi]$ the key fact is that only singularities a distance $\frac{2}{3}\epsilon$ to $\frac{4}{3}\epsilon$ survive the cutoff by $[\phi]$. Of these, the singularities closest to the corner are propagated through the corner by $U(\epsilon)$. Then $U(t-\epsilon)$ acts on these like $V(t-\epsilon)$. Consequently $tr\, U(t-\epsilon)[\phi]U(\epsilon)[1-\phi]$ has the same leading singularity at $2h$ as $-tr\, V(t-\epsilon)[\phi]V(-t+\epsilon)U_s(t, \rho(z), z')$. Here $V(t)$ is the solution operator for the half plane containing X with boundary along the side opposite α. Similar equivalences are shown for the rest of the singularities in the range.

Using the information from these singularities, a variety of inverse spectral results can be stated. One of the simplest consequences is that the class of obtuse triangles is spectrally determined, and that within this class no two triangles are isospectral.

Chapter 2

Forcing Problem Solution Operators

2.1 Introduction

The Sommerfeld kernel for a sector preserves regularities of the forcing function and propagates the singularities of the forcing function in ways that are inherited by the forcing kernel for a polygonal domain. This chapter recapitulates the description of the Sommerfeld kernel and the construction of the forcing kernel for polygonal domains given in Friedlander[1]. Aspects of each kernel that are important to the computation of the trace of $U(t)$ are emphasized.

The Sommerfeld kernel is the solution operator for a boundary value problem of the wave operator on a sector. Let S be a sector of angle $\alpha < 2\pi$ in R^2, given in polar coordinates by $\{(r, \theta) : r > 0 \text{ and } \theta \in (0, \alpha)\}$. The wave operator in these coordinates is

$$P = (\frac{\partial}{\partial t})^2 - (\frac{\partial}{\partial r})^2 - r^{-2}(\frac{\partial}{\partial \theta})^2 - r^{-1}\frac{\partial}{\partial r}.$$

A basic statement of the forward forcing problem for S is the following. Given a function f in $C_0^\infty(R_t \times S)$, find a function u in $C^\infty(R_t \times S)$ satisfying

$$Pu = f$$

$$u(t, r, 0) = u(t, r, \alpha) = 0$$

$$u = 0 \text{ if } t \ll 0.$$

The Sommerfeld kernel is a distribution $F(t,t',r,\theta,r',\theta')$ satisfying

$$P_{t,r,\theta}F = \frac{1}{r'}\,\delta(t-t')\,\delta(r-r')\,\delta(\theta-\theta')$$

with $F=0$ for $t<0$. That is, F is a forward fundamental solution for the forcing problem. The function $u(t,r,\theta) = < F, f >$ is in fact a solution of the problem given. This particular kernel does better than that. For any integers $k,l,m \geq 0$ the function $r^{-1/2}(r\frac{\partial}{\partial r})^k(\frac{\partial}{\partial \theta})^l(\frac{\partial}{\partial t})^m u(t,r,\theta)$ is bounded on any set $(t_1,t_2)\times(0,r_1)\times(0,\alpha)$ for $-\infty < t_1, t_2 < \infty$ and $0 < r_1 < \infty$. These properties specify u, and consequently F, uniquely.

This sort of boundedness together with smoothness in the interior provides a useful definition of smoothness of functions on the sector.

Definition 2.1 *Let O be an open set in R^n and $u(z,r,\theta)$ a function defined on points in $O\times S$. Then u is* smooth on $O\times S$ with boundary *if*
i. the function $\tilde{u}(z,r,\theta)$ periodic in θ with period 2α defined by

$$\tilde{u}(z,r,\theta) = \begin{cases} u(z,r,\theta) & 0 \leq \theta \leq \alpha \\ -u(z,r,\theta) & -\alpha \leq \theta \leq 0 \end{cases}$$

is smooth on $O\times(0,\infty)\times R_\theta$, and
ii. the function $r^{-1/2}(r\frac{\partial}{\partial r})^k(\frac{\partial}{\partial \theta})^l(\frac{\partial}{\partial z})^\beta u(z,r,\theta)$ is bounded on any set $V\times(0,r_1]\times[0,\alpha]$ compactly contained in O and r_1 finite. Here k and l are non-negative integers and β is a multi index of non-negative integers.

This is extended to functions with two arguments in sectors by

Definition 2.2 *Let O be an open set in R^n and let S and S' be sectors with angles α and α', respectively. Then a function $u(z,r,\theta,r',\theta')$ on $O\times S\times S'$ is* smooth on $O\times S\times S'$ with boundary *if*
i. the function $\tilde{u}(z,r,\theta,r',\theta')$ defined on $O\times R^+\times R_\theta\times R^+\times R_{\theta'}$ by the conditions that it is equal to u on $O\times R^+\times(0,\alpha)\times R^+\times(0,\alpha)$, odd in R_θ and $R_{\theta'}$, and periodic of period 2α in R_θ and $2\alpha'$ in $R_{\theta'}$ is a smooth function, and
ii. the function $(rr')^{-1/2}(r\frac{\partial}{\partial r})^k(\frac{\partial}{\partial \theta})^l(r'\frac{\partial}{\partial r'})^{k'}(\frac{\partial}{\partial \theta'})^{l'}(\frac{\partial}{\partial z})^\beta\tilde{u}$ is bounded on any set $V\times(0,r_1]\times R_\theta\times(0,r_2]\times R_{\theta'}$. Once again, V is compactly contained in O and r_1 and r_2 are finite. The exponents k,l,k' and l' are non-negative integers and β is a multi index of non-negative integers.

Several explicit expressions for the Sommerfeld kernel are given in Friedlander[1]. In the case of a sector with angle α equal to π/N for some positive integer N, the kernel is simply a sum of reflections and rotations of the standard kernel on R^2. The basic properties of this sector kernel are well understood on the basis of the properties of the kernel in free space. The Sommerfeld kernel for a sector S with angle $\alpha \neq \pi/N$ is more complicated. Consequently it is worthwhile examining the forward fundamental solution on a related manifold. The kernel for the sector is obtained from the kernel on the manifold and inherits many of the properties of the manifold kernel . In addition, the manifold kernel provides a sensible way to talk about the wavefront set of the sector kernel on the edges of $R \times S \times S$.

2.2 The conic manifold solution

Let S be a sector with angle α. The manifold M associated with the forcing problem on $R_t \times S$ is given by

$$M = R_t \times R^+ \times R \bmod 2\alpha\mathbf{Z}.$$

The manifold has canonical coordinates (t, r, θ) and measure $r\, dr\, d\theta\, dt$. The integration in these coordinates is taken over $R_t \times R^+ \times (a, a + 2\alpha)$ for any a. The wave operator on M is still given by

$$P = (\frac{\partial}{\partial t})^2 - (\frac{\partial}{\partial r})^2 - r^{-2}(\frac{\partial}{\partial \theta})^2 - r^{-1}\frac{\partial}{\partial r}.$$

The forward fundamental solution on M with the required regularity is equal to $E(t - t', r, r', \theta - \theta')$ for $E(t, r, r', \theta)$ given as follows:

$$E(t, r, r', \theta)\,|_{t>r+r'} = (2\pi)^{-1}H(t) \sum_{n=-\infty}^{\infty} \chi(\theta - 2n\alpha)(t^2 - r^2 - r'^2 + 2rr'\cos(\theta - 2n\alpha))_+^{-1/2}$$

$$E(t, r, r', \theta)\,|_{t<r+r'} = (2\pi)^{-1} \int_0^\infty K(\eta, \theta)(2rr'\cosh\eta + r^2 + r'^2 - t^2)_+^{-1/2}\, d\eta.$$

Here H is the Heaviside function. The function χ is the characteristic function of the interval $[-\pi, \pi]$. The expression for K is in the terms of $\kappa = \pi/\alpha$. The value of $K(\eta, \theta)$ is

$$\frac{1}{\alpha}\left(1 + 2\sum_{m=0}^{\infty} e^{-m\kappa\eta}\cos(m\kappa\eta)\cos(m\kappa\theta)\right).$$

One immediate and important (if unsurprizing) fact is that $supp\, E(t, r,', \theta)$ is contained in $\{(t, r, r', \theta) : t > |r - r'|\}$.

The operator with kernel $E(t-t', r, r', \theta-\theta')$, call it E, maps forcing functions which are essentially "smooth on M with boundary" to functions of the same regularity. Made precise, this is

Theorem 2.3 *(Friedlander[1]) Let $f \in C^\infty(M)$ be supported in $[t_0, \infty) \times R^+ \times R \bmod 2\alpha\mathbf{Z}$, and odd in θ. Suppose also that for all $k, l, m \geq 0$, $r > 0$ and s, s' finite, the function*

$$r^{-1/2}(r\frac{\partial}{\partial r})^k(\frac{\partial}{\partial \theta})^l(\frac{\partial}{\partial t})^m f$$

is bounded on the set $[s, s'] \times (0, r_0] \times [0, 2\alpha]$. Then Ef solves the forcing problem with forcing term f. Further $Ef \in C^\infty(M)$ is also odd in θ, and

$$r^{-1/2}(r\frac{\partial}{\partial r})^k(\frac{\partial}{\partial \theta})^l(\frac{\partial}{\partial t})^m Ef$$

is bounded on amy set $[s, s'] \times (0, r_0] \times [0, 2\alpha]$.

The bound guaranteed by Theorem 2.3 can be computed in terms of the bounds on f.

Theorem 2.4 *Let f be a function satisfying the hypotheses of Theorem 2.3. Let D be the set $\{(t, r, \theta) : r + t < r_1 + t_1,\ t_0 < t < t_1\}$, with $t_0 < t_1 < \infty$ and $0 < r_1 < \infty$. Then there exists a constant C depending only on D such that*

$$|r^{-1/2}Ef| \leq C\max\{\sup_{\mathrm{D}}|r^{-1/2}g|, \sup_{\mathrm{D}}|r^{-1/2}(\frac{\partial}{\partial \theta})^2 g|\}.$$

The propagation of singularities by E is closely related to the propagation of singularities by the sector kernel. One expects points $(t, t'r, r', \theta, \theta', \tau, \tau', \rho, \rho', \Theta, \Theta')$ in $WF\,E$ to be the points related by a geodesic. Indeed $WF\,E$ is the union of three sets of points in $T^*(M \times M)$ which have a natural interpretation in terms of geodesics.

The first of these is just $N^*(\Delta)\backslash 0$, the conormal bundle over the diagonal in $M \times M$, excluding the zero section. This represents the "trivial geodesics" of zero length.

The points in the set

$$\Gamma = \{(m, m') \in M \times M : (t-t')^2 - r^2 - r'^2 + 2rr'\cos(\theta - \theta' - 2n\alpha) = 0,$$
$$|\theta - \theta' - 2n\alpha| < \pi,\ 0 < t - t' < r + r',\ n \in \mathbf{Z}\}$$

are essentially pairs $(r, \theta)\,(r', \theta')$ a distance $t - t'$ apart along a path which does not intersect the boundary of M. Denote the conormal bundle over Γ less the zero section by $N^*(\Gamma) \setminus 0$.

Finally, those points corresponding to diffractive geodesics may be included. Let B equal $\{(m, m') \in M \times M : |t - t'| = |r + r'|\}$.

Theorem 2.5 *(Friedlander[1]) If M is of the form $R_t \times R^+ \times R \bmod \pi/N\mathbf{Z}$ then the wavefront set of $E(t-t', r, r', \theta-\theta')$ is equal to the closure of*

$$N^*(\Delta) \setminus 0 \bigcup N^*(\Gamma) \setminus 0.$$

If M is of the form $R_t \times R^+ \times \text{mod} 2\alpha\mathbf{Z}$, $\alpha \neq \pi/N$ then the wavefront set of $E(t-t', r, r', \theta, \theta')$ is equal to

$$N^*(\Delta) \setminus 0 \bigcup N^*(\Gamma) \setminus 0 \bigcup N^*(B) \setminus 0.$$

There is another expression for E which shows what type of singularity occurs on the set B, at least at points satisfying the condition that $|\theta - \theta'|$ is not equal to π modulo $2\alpha\mathbf{Z}$. The kernel of E is given by $E(t-t', r, r', \theta-\theta')$ for

$$E(t,r,r',\theta) = (2\pi)^{-1} H(t) \sum_{n=-\infty}^{\infty} \chi(\theta - 2n\alpha)\,(t^2 - r^2 - r'^2 + 2rr'\cos(\theta - 2n\alpha))_+^{-1/2}$$

$$+ (2\pi)^{-1} \int_0^\infty L(\eta,\theta)\,(t^2 - r^2 - r'^2 - 2rr'\cosh\eta)_+^{-1/2}\, d\eta$$

where

$$L(\eta,\theta) = \frac{-1}{2\alpha}\left(\frac{\sin\kappa(\pi+\theta)}{\cosh\kappa\eta - \cos\kappa(\pi+\theta)} + \frac{\sin\kappa(\pi-\theta)}{\cosh\kappa\eta - \cos\kappa(\pi-\theta)}\right)$$

with κ defined as in other kernel.

One can verify that this version of E basically has a jump discontinuity across the set B. Each piece $\chi(\theta - \theta' - 2n\alpha)(t^2 - r^2 - r'^2 + 2rr'\cos(\theta - \theta' - 2n\alpha)$ is smooth on a sufficiently small neighborhood of a point of B which also satisfies the condition $|\theta - \theta' - 2n\alpha| \neq \pi$. Next consider

$$\int_0^\infty L(\eta, \theta-\theta')\,(t^2 - r^2 - r'^2 - 2rr'\cosh\eta)_+^{-1/2}\, d\eta$$

in a sufficiently small neighborhood of a point $(m, m') \in B$, $|\theta - \theta'| \neq \pi$. In such a neighborhood the function on $M \times M$ given by the integral can be written as a product $\phi(t - t' - r - r')f(m, m')$. Here $f(m, m')$ is smooth and ϕ is the characteristic function of the interval $[0, \infty)$.

2.3 The sector solution, F

Let β be the identification of $R_t \times S$ with an open set in M, $\beta : (r,\theta,t) \in S \to (r,\theta,t) \in M$. The sector kernel $F(t-t',r,r',\theta,\theta')$ is simply $E(t-t',r,t',\theta-\theta')-E(t-t',r,r',\theta+\theta')$, though for F the integration extends only over θ in the interval $(0,\alpha)$. One intepretation of this presents $F(f)$ as a pull back by β of E applied to a specially modified forcing function $\tilde{f}$. Let f be a function which is smooth on $R \times S$ with boundary, with $f(t) = 0$ $t \ll 0$. Define $\tilde{f}(t,r,\theta)$ on M by

$$\tilde{f}(t,r,\theta) = \begin{cases} f(t,r,\theta) & 0 < \theta < \alpha \\ -f(t,r,-\theta) & -\alpha < \theta < 0. \end{cases}$$

Then $F(f) = E\tilde{f} \circ \beta$.

As a consequence of this relationship, one obtains from Theorem 2.3 the following

Theorem 2.6 *(Friedlander[1]) Let f be smooth on $R_t \times S$ with boundary. Suppose the support of f is contained in $[t_0,\infty) \times S$. Then $F(f)$ is smooth on $R_t \times S$ with boundary, and $F(f)$ is supported in $[t_0,\infty) \times S$.*

As a consequence of Theorem 2.4 and the commutation relations $[\frac{\partial}{\partial t}, E] = 0$, $[\frac{\partial}{\partial \theta}, E] = 0$ and $[r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t}, E] = 2E$, one can draw conclusions about the behavior of F on smooth families of forcing functions. In particular,

Theorem 2.7 *Let S_1 and S_2 be sectors. Let $f(t,r',\theta',r'',\theta'')$ be smooth on $R_t \times S_1 \times S_2$ with support in $t > t_0$. Consider f as a family of forcing functions parametrized by r'',θ''. Then $Ff(t,r,\theta,r'',\theta'')$ is also smooth on $R_t \times S_1 \times S_2$.*

The kernel for F has an asymptotic expansion for $t - t' > r + r'$ which is actually a little smoother than the one which can be wrung from E.

Theorem 2.8 *On any neighborhood $(t_0,t_1) \times (0,r_0) \times (0,r_1) \times (0,\alpha) \times (0,\alpha)$, r_0,r_1 sufficiently small, $F(t,r,r',\theta,\theta')$ has an asymptotic expansion*

$$\sum_{m=1}^{\infty} (2rr')^{mk} f_m(\theta,\theta',r,r',t).$$

Each f_m is smooth on $R_\theta \times R_{\theta'} \times (-r_0,r_0) \times (-r_1,r_1) \times R_t$, and odd and periodic of period 2α in R_θ and $R_{\theta'}$. The derivatives $D^\beta f_m$ are bounded by $C_1(C_2)^{mk}m^\beta$, C_1 and C_2 independent of m.

In fact $\phi F(t,r,r',\theta,\theta')$ is smooth on $R_t \times S \times S$ with boundary if ϕ is a smooth cutoff function radial in S and S' supported in $t > r + r'$.

From the definition of F comes a natural extension of F across the θ and θ' boundaries of $R_t \times S \times R_t \times S$. Simply extend the identification β to a sector neighborhood S' of S in $R_t \times R^2$, and let $\beta \times \beta$ map $(R_t \times S') \times (R_t \times S') \to M \times M$. Then the extension of F, $\tilde{F}$ is just

$$\tilde{F}(t-t', r, r', \theta, \theta') = E(t-t', r, r', \theta - \theta') \circ \beta \times \beta - E(t-t', r, r', \theta + \theta') \circ \beta \times \beta.$$

Note that this is the odd extension of F across the lines $\theta = 0$, $\theta = \alpha$, $\theta' = 0$. That is, if θ and θ' are in $(0, \alpha)$ then

$$\tilde{F}(t-t', r, r', \theta, \theta') = -\tilde{F}(t-t', r, r', \theta, -\theta') = -\tilde{F}(t-t', r, r', \theta, 2\alpha - \theta'),$$

and likewise for θ.

The wavefront set of E dictates the wavefront set of F. The extension of F to $\tilde{F}$ gives a way to interpret the wavefront set of F at the edges of $(R_t \times S) \times (R_t \times S)$. The significance of the wavefront set is clearer if S is expressed in rectangular coordinates. Also, it is convenient to have the following

Definition 2.9 *A point* $(t, z, z', \tau, \zeta, \zeta')$ satisfies a geodesic relation *if the following conditions are met:*
i. a geodesic, diffractive of reflective, of length $|t|$ *joins* z *to* z'*, and*
ii. the direction of the geodesic in i. at z *toward* z' *is* $sign(t)sign(\tau)\zeta$*. The direction from* z' *toward* z *is* $sign(t)sign(\tau)\zeta'$*.*

In these terms one obtains the following description of the wavefront set of F.

Theorem 2.10 *A point* $p = (t, z, t', z', \tau, \zeta, \tau', \zeta')$ *in* $T^*[(R \times S) \times (R \times S)]$ *is in the wavefront set of* F *only if either*
i. p *is in the conormal bundle of the diagonal,*
or
ii. $t - t' > 0$ *and* $(t-t', z, z', \tau, \zeta, \zeta')$ *satisfies a geodesic relation.*

Let ρ_1 be a reflection in R^2 across the x axis. Let ρ_2 be the reflection across the line $re^{i\alpha}$, and let ρ_0 be the identity. Let Q be the sector $\{(r, \theta) : \theta \in [-\epsilon, \alpha + \epsilon]\}$ where ϵ is sufficiently small that this is not equal to $R^2 \backslash (0,0)$. To a point $p = (t, z, t', z', \tau, \zeta, \tau', \zeta')$ in $T^*(R \times Q \times R \times Q)$ associate the set

$$U = \{p_{ij} : p_{ij} = (t, \rho_i(z), t', \rho_j(z'), \tau, \rho_i(\zeta), \tau', \rho_j(\zeta'))\ i, j = 0, 1, 2\}$$

There is finally enough terminology to state

Theorem 2.11 *The wavefront set of $\tilde{F}$ is confined to the set of points p such that*
i. some element of U is in the conormal bundle to the diagonal in $(R_t \times Q \times R_t \times Q)$,
or
ii. some element of U, $p_{ij} = (t, z, t', z', \tau, \zeta, \tau', \zeta')$, *is such that* $(t - t', z, z', \tau, \zeta, \zeta')$ *satisfies a geodesic relation.*

2.4 The polygonal domain solution, G

Before proceeding to construct the forcing kernel for a polygonal domain from the Sommerfeld kernel, it is convenient to establish some details of terminology connected with a polygonal domain X. In this thesis X is a bounded open region in R^2. The boundary of X is a finite union of disjoint polygons. An edge of one of these polygons is an edge of X. Likewise, a vertex of one of these polygons is a vertex of X. An edge is not considered to contain the vertices which are its endpoints.

It is also helpful to associate to X a constant describing the distances between edges and vertices. Call an open set in X a pure interior set if its closure is contained in X. An open set in X is a pure edge set if its closure intersects exactly one edge and no vertices. A pure corner set is one with closure intersecting the boundary of X in two edges which have a common vertex. In this case, the closure is allowed to contain that common vertex. If d is a sufficiently small positive number, then for any point z in X the neighborhood of z of radius $3d$ falls into one of these three categories.

Definition 2.12 *Given a polygonal domain X, $d(X)$ is the supremum of all $d \in R$ with the property that a ball of radius $3d$ about any point in X is a pure interior, pure edge or pure corner neighborhood.*

The notion of smoothness on the product of an open set in R^n and a sector, and smoothness on the product of an open set in R^n and two sectors carry over to X.

Definition 2.13 *Let O be an open set in R^n. A function $f(w, z)$ is* smooth on $O \times X$ with boundary *if it meets all the following conditions.*
i. The function $f(w, z)$ is smooth in the interior of $O \times X$
ii. Let z_0 be a point on an edge Σ of X. Let ρ be a reflection across that edge. Define $\tilde{f}(w, z)$ in a neighborhood of z_0 in R^2 by

$$\tilde{f}(w,z) = \begin{cases} f(w,z) & z \in X \\ -f(w,\rho(z)) & \rho(z) \in X \\ 0 & z \in \Sigma \end{cases} .$$

Then $\tilde{f}(w,z)$ is smooth in a neighborhood of z_0.
iii. Let x_0 be a vertex of X. Let $\phi \in C_0^\infty(R^2)$ be a radial function in polar coordinates centered at x_0. Suppose that ϕ is constant on a neighborhood of x_0 and that $supp\phi \cap X$ is contained in a pure corner neighborhood of x_0. Then considered as a function on the product of O and the sector S determined by x_0 and the adjacent edges, $\phi(z)f(w,z)$ is smooth on $O \times S$ with boundary.

Definition 2.14 *Let O be an open set in R^n. A function $f(w,z,z')$ is* smooth on $O \times X \times X$ with boundary *if each of the following conditions is satisfied.*
i. The function $f(w,z,z')$ is smooth on $O \times X \times X$.
ii. Let z_1 be a point on an edge Σ of X. Let ρ be reflection across that edge and let O_1 be a small neighborhood of z_1 in R^2. Then $\tilde{f}(w,z,z')$ defined on $O \times O_1 \times X$ by

$$\tilde{f}(w,z,z') = \begin{cases} f(w,z,z') & z \in X \\ -f(w,\rho(z),z') & \rho(z) \in X \\ 0 & z \in \Sigma \end{cases}$$

is smooth. Likewise, odd extension across an edge in the z' variable is smooth.
iii. Let z_1 be a point on the edge Σ_1 and z_2 a point on the edge Σ_2. Then in a small enough neighborhood of $O \times z_1 \times z_2$ in $O \times R^2 \times R^2$ the function the function defined by odd extension across both edges is smooth.
iv. Let x_1 and x_2 be vertices of X corresponding to sectors S_1 and S_2. Let ϕ_i, $i = 1,2$ be smooth radial functions supported in pure corner neighborhoods of x_1 and x_2 resfpectively. Suppose that ϕ_1 is constant in a neighborhood of x_i. Then $\phi_1\phi_2 f$ is smooth on $O \times S_1 \times S_2$ with boundary.

These definitions of smoothness are valuable because they are preserved by a forcing operator for X which is built from the Sommerfeld kernels of the sectors involved in X

Theorem 2.15 *(Friedlander[1]) There is a unique operator G which maps any function f which is smooth on $R_t \times X$ with boundary and supported in $[t_0, \infty) \times X$ to a function Gf with the same properties satisfying*

$$((\frac{\partial}{\partial t})^2 - \Delta)Gf = PGf = f.$$

Sketch of Proof: For t near t_0 Gf is constructed by cutting up X. The space X has a finite covering by disks of radius $d(X)$. Let $\phi_1, \dots \phi_n$ be a partition of unity subordinate to this cover. Require the functions ϕ_i to have the property that if f is smooth on $O \times X$ with boundary then so is $\phi_i f$. Under these conditions, each ϕ_i is supported in a disk O_i' of radius $d(x)$. Let O_i be a disk of radius $2d(X)$ centered around O_i'. Then O_i is a pure neighborhood of interior, edge or corner type. Let F_i be the forward fundamental solution for R^2, the half plane, or the sector corresponding to the pure neighborhood O_i. Then for $t - t_0 < d(X)$ set Gf equal to $\sum_{i=1}^{n} F_i(\phi_i f)$. Where defined, this satisfies the conditions of the theorem.

The function Gf can be extended to larger t by a repetition of the construction for small $t - t_0$. Simply let $\psi(t)$ be a smooth cutoff function equal to one for $t < t_0 + d(X) - 2\epsilon$ and equal to zero for $t > t_0 + d(X) - \epsilon$. Set Gf equal to

$$\psi Gf - G((1-\psi)f - 2\frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}(Gf) - \frac{\partial^2\psi}{\partial t^2}).$$

This genuinely gives an extension of G because each of the new terms is supported in $t > t + d(X) - 2\epsilon$. Note also that each of the terms in the extension has the same regularity as f.

Iterating this process produces a function Gf smooth on $R \times X$ with boundary satisfying $PGf = f$.

Definition 2.16 *The kernel of the operator G in Theorem 2.15 is $G(t - t', z, z')$.*

One immediate corollary of the construction is

Corollary 2.17 *(Friedlander[1]) Let $f(t, z', z'')$ be supported in $[0, \infty) \times X \times X$ and smooth on $R_t \times X \times X$ with boundary. View f as a family of forcing functions parametrized by z''. Then $Gf(t, z', z'')$ is smooth on $R_t \times X \times X$ with boundary and it is supported in $[t_0, \infty) \times X \times X$.*

While this result is global in R_t, most of the immediate facts about the regularity of $G(t, z, z')$ are confined to $(-\infty, d(X)) \times X \times X$. Much of Chapter 3 is devoted to extending local properties which follow to constructions about $R_t \times X \times X$.

Corollary 2.18 *Let t be an element of $O = (t_0, t_1)$ with $0 < t_0 < t_1 < d(X)$. Let x_0 be a vertex of X, and let S be the sector determined by x_0 and the adjacent edges. Give $G(t, z, z')$ near $O \times x_0 \times x_0$ in polar coordinates r, θ and r', θ' centered at x_0. For any $\phi(r) \in C^\infty(R)$ and small $\epsilon > 0$ with $\phi(r) = 1$ for $r < d(X) - 2\epsilon$ and $\phi(r) = 0$ for $r > d(X) - \epsilon$, the function $\phi(r + r')G(t, r, \theta, r', \theta')$ is smooth on $O \times X \times X$ with boundary.*

In a neighborhood of $r = r' = 0$, $G(t,r,\theta,r',\phi\theta')$ has an asymptotic expansion of the form

$$\sum_{m=1}^{\infty}(2rr')^{m\kappa} f_m(t,r,\theta,r',\theta')$$

as in Theorem 2.8.

Corollary 2.19 *The wavefront set of G in $(-\infty, d(X)) \times X \times X$ is contained in the union of the conormal bundle to $\{0\} \times$ (the diagonal of $X \times X$) and the set of points $(t,z,z',\tau,\zeta,\zeta')$ satisfying a geodesic relation in X.*

There is an interpretation of the wavefront set of G on the edges of $(-\infty, d(X)) \times X \times X$ which is still related to the geodesics of X. Let z_1 and z_2 be points on the edges Σ_1 and Σ_2, respectively. Denote reflections across Σ_i by ρ_i, $i = 1,2$. Take ρ_0 to be the identity operator on R^2. Let O_1 and O_2 be neighborhoods of z_1 and z_2 in R^2 that satisfy the condition that $\rho_i(O_i)$ is equal O_i and the condition that O_i intersects the boundary of X only in Σ_i. Then $G(t,z,z')$ can be extended to $(-\infty, d(X)) \times X \cup O_1 \times X \cup O_2$ by

$$\tilde{G}(t,z,z') = \begin{cases} G(t,z,z') & \text{if } z, z' \in X \\ -G(t,\rho_1(z),z') & \text{if } \rho_1(z), z' \in X \\ -G(t,z,\rho_2(z')) & \text{if } z, \rho_2(z'), \in X \\ G(t,\rho_1(z),\rho_2(z')) & \text{if } \rho_1(z), \rho_2(z') \in X \\ 0 & \text{if } z \text{ or } z' \in \Sigma_i \end{cases} .$$

Corollary 2.20 *A point $(t,z,z',\tau,\zeta,\zeta')$ is in the wavefront set of $\tilde{G}$ only if either*
i. $(t,\rho_i(z),\rho_j(z'),\tau,\rho_i(\zeta),\rho_j(\zeta'))$ is in the conormal bundle to $\{0\} \times$ (diagonal in $X \cup O_1 \times X \cup O_2$) for some $0 \geq i,j \geq 2$,
or
ii. $(t,\rho_i(z),\rho_j(z'),\tau,\rho_i(\zeta),\rho_j(\zeta'))$ satisfies a geodesic relation in X for some $0 \geq i,j \geq 2$

Chapter 3

Solution of the IVP

3.1 Construction of U

The approach here to proving the Poisson relation for a polygonal domain X is through the solution operator for the initial value problem associated with the wave operator. A basic statement of this problem follows. Let g_0 and g_1 be smooth, real-valued, compactly supported functions on X. Find $g(t)$ smooth on $R_t \times X$ satisfying

$$Pg = 0, \quad g(0,z) = g_0(z), \quad \frac{\partial g}{\partial t}(0,z) = g_1(z) \quad \text{and } g = 0 \text{ on the edges of } X.$$

For fixed t_0 one can consider the operator $U(t_0)$ mapping (g_0, g_1) to $(g(t_0,\cdot)\frac{\partial g}{\partial t}(t_0,\cdot))$. This point of view produces a whole family of operators $\{U(t)\}_{t\in R}$. The basic properties of $\{U(t)\}$ are the concern of the chapter.

The purpose of the first section is to show that $\{U(t)\}$ can be extended to a unitary group acting on a Hilbert space $H_1^0 \oplus L^2$. A distribution solution $g(t,z)$ to the initial value problem with data $g_0 \oplus g_1$ in $H_1^0 \oplus L^2$ results in from application of $\{U(t)\}$. This solution is in $H_1^{loc}(R_t \times X)$. It also has the property that the restriction to fixed t of $(g(t,\cdot), \frac{\partial g}{\partial t}(t,\cdot))$ is in $H_1^0 \oplus L^2$. Section 2 shows that g is the unique solution to the initial value problem having these properties. The final section characterizes $\{U(t)\}$ in terms of the operator $G(t)$ from Chapter 2.

The underlying set of $H_1^0 \oplus L^2$ and the inner product structure must be specified. The space H_1^0 is the closure of $C_0^\infty(X)$ in the norm

$$||g_0|| = \left(\int_X \nabla g \cdot \nabla g \, dz\right)^{1/2}$$

Becase the space X is bounded, this norm is equivalent to the more customary Sobolev norm

$$||g_0||_1 = \sum_{|\alpha|\leq 1}(\int(|D^\alpha g|)^2\,dz)^{1/2}$$

The second space L^2 in the direct sum is the usual space of real-valued square-integrable functions on X. The inner product used here to make $H_1^0 \oplus L^2$ a Hilbert space acts on pairs by

$$\langle g_0 \oplus g_1, f_0 \oplus f_{f1}\rangle = \int_X \nabla g_0 \cdot \nabla f_0 + g_1 f_1\,dz.$$

The corresponding norm is denoted by the energy norm. Throughout the rest of this discussion, π_1 will be the projection of $H_1^0 \oplus L^2$ to H_1^0.

The action of the operators $\{U(t)\}$ on $H_1^0 \oplus L^2$ will be defined by continuity from the action of $\{U(t)\}$ on $C_0^\infty(X) \oplus C_0^\infty(X)$. The first proposition of this section shows that $\{U(t)\}$ is well defined on $C_0^\infty(X) \oplus C_0^\infty(X)$. In order to do this, one shows that each operator $U(t_0)$ preserves the energy norm.

Proposition 3.1 *Let $g_0 \oplus g_1$ be an element of $C_0^\infty(X) \oplus C_0^\infty(X)$. There exists a unique function g which is smooth on $R_t \times X$ and which is a solution of the initial value problem. In addition, for any t_0, t_1 the function g satisfies the equality*

$$\int \nabla g \cdot \nabla g(t_0, z) + (\frac{\partial g}{\partial t}(t_0, z))^2\,dz = \int \nabla g \cdot \nabla g(t_1, z) + (\frac{\partial g}{\partial t}(t_1, z))^2\,dz.$$

Proof: For a given $g_0 \oplus g_1 \in C_0^\infty(X) \oplus C_0^\infty(X)$, the solution of the initial value problem on the whole plane provides $g(t)$ when the absolute value of t is less than the distance from the support of g_i to the boundary of X for $i = 1, 2$. Say this gives $g(t)$ for $t \in (-t', t')$.

To extend $g(t)$ to all t, choose $\psi(t) \in C^\infty(R)$ with $\psi(t)$ equal to one when $|t| < t'/4$ and equal to zero when $|t| > t'/2$.

Then $P\psi(t)g(t)$ is equal to the sum of two functions, $f_- \in C_0^\infty((-\infty, 0) \times X)$ and $f_+ \in C_0^\infty((0, \infty) \times X)$. Apply the backward fundamental solution tor the forcing problem , G_-, to f_- and the forward fundamental solution G to f_+. Then set $g(t)$ equal to $\psi(t)g(t) - G_- f_- - G f_+$. This $g(t)$ is a solution of the initial value problem. The regularity of G_- and G guarantees that g is smooth on $R_t \times X$.

For any function g smooth on $R_t \times X$ and satisfying $Pg = 0$ the following computation can be carried out.

$$0 = \int_X \int_{t_0}^{t_1} \frac{\partial}{\partial t} g(t,z) P g(t,z) =$$

$$1/2 \left(\int_X \nabla g \cdot \nabla g(t,z) + (\frac{\partial g}{\partial t}(t,z))^2 \, dz - \int_X \nabla g \cdot \nabla g(t_0,z) - (\frac{\partial g}{\partial t}(t_0,z))^2 \, dz \right)$$

The uniqueness of g follows from this.

Another consequence of this is that if $\{g_{0,n} \oplus g_{1,n}\}_{n=1}^{\infty}$ is a sequence in $C_0^\infty(X) \times C_0^\infty(X)$ which converges in the energy norm to $g_0 \oplus g_1$ then $\{U(t)(g_{0,n} \oplus g_{1,n})\}$ is a Cauchy sequence in the energy norm uniformly in t. Thus the action of $\{U(t)\}$ is well-defined.

Definition 3.2 *Let $g_0 \oplus g_1$ be an element of $H_1^0 \oplus L^2$ and let $\{g_{0,n} \oplus g_{1,n} \in C_0^\infty(X) \oplus C_0^\infty(X)\}_{n=1}^{\infty}$ be a sequence converging to $g_0 \oplus g_1$ in the energy norm. Then*

$$U(t) g_0 \oplus g_1 := \lim_{n \to \infty} U(t) g_{0,n} \oplus g_{1,n}.$$

The group property of the family of operators $\{U(t) : t \in R\}$ will figure prominently in the proof of the Poisson relation.

Proposition 3.3 *$\{U(t) : t \in R\}$ is a unitary group of operators on $H_1^0 \oplus L^2$ in the energy norm.*

First establish that for the initial data $g_0 \oplus g_1 \in C_0^\infty(X) \oplus C_0^\infty(X)$ the equality

$$U(t)U(s) g_0 \oplus g_1 = U(t+s) g_0 \oplus g_1$$

is valid.

Denote by $g(r) \oplus \frac{\partial}{\partial t} g(t)$ the element $U(r) g_0 \oplus g_1$. Considered as a function on $R \times X$, $g(t+s)$ solves the initial value problem

$$(\frac{\partial^2}{\partial t^2} - \Delta) g(t+s,z) = 0$$

$$g(t+s,z)\,|_{t=0} = g(s,z)$$

$$\frac{\partial}{\partial t} g(t+s,z)\bigg|_{t=0} = g(s,z).$$

The function $g(t+s,z)$ is smooth on $R\times X$ with boundary. There exists a sequence of functions in $C_0^\infty(X)\times C_0^\infty(X)$, $\{h_{0,n}\oplus h_{1,n}\}$, converging to $g(s)\oplus\frac{\partial}{\partial t}g(s)$ in the energy norm. The function $h_n(t)=\pi_1 U(t)h_{0,n}\oplus h_{1,n}$ solves the initial value problem

$$(\frac{\partial^2}{\partial t^2}-\Delta)h_n(t,z)=0$$

$$h_n(t,z)\,|_{t=0}=h_{0,n}$$

$$\frac{\partial}{\partial t}h_n(t,z)\Big|_{t=0}=h_{1,n}$$

Each function $h_n(t,z)$ is smooth on $R\times X$ with boundary.

Consequently $h_n-g(t,s)$ is sufficiently regular that, in the energy norm one has

$$||h_n(t)\oplus\frac{\partial}{\partial t}h_n(t)-g(t+s)\oplus\frac{\partial}{\partial t}g(t+s)||=$$

$$||h_{0,n}\oplus h_{1,n}-g(s)\oplus\frac{\partial}{\partial t}g(s)||.$$

Taking the limit as $n\to\infty$, obtain

$$||U(t)g(s)\oplus\frac{\partial}{\partial t}g(s)-g(t+s)\oplus\frac{\partial}{\partial}g(t+s)||=0.$$

This proves the group property for smooth, compactly supported initial data. By continuity, the equality

$$U(t)U(s)g_0\oplus g_1=U(t+s)g_0\oplus g_1$$

holds for general $g_0\oplus g_1$ in $H_1^0\oplus L^2$.

The remaining requirement for $\{U(t)\}$ to be a unitary group is that $\lim_{t\to 0}||U(t)g_0\oplus g_1-g_0\oplus g_1||$ is equal to zero. This is easily seen to be true for smooth, compactly supported intial data. The fact that $C_0^\infty(X)\times C_0^\infty(X)$ is dense in $H_1^0\oplus L^2$ and the fact that $\{U(t)\}$ is unitary imply the limit for arbitrary $g_0\oplus g_1\in H_1^0\oplus L^2$.

3.2 Uniqueness of U

Consider the distribution $g(t,z)$ obtained from the initial data $g_0\oplus g_1\in H_1^0\oplus L^2$ by defining $g(t,z)=\pi_1 U(t)g_0\oplus g_1$. This distribution is an element of $H_1^{loc}(R\times R^2$ and

$g(t,z)\oplus\frac{\partial}{\partial t}g(t,z)$ is an element of $H_1^0\oplus L^2$ for each fixed t. In addition, the energy norm of $g(t,z)\oplus\frac{\partial}{\partial t}g(t,z)$ is bounded for all values of t in some compact set. In fact, $g(t,z)$ is the unique such solution of the initial value problem

$$Pg=0,\quad g(0,z)=g_0(z),\quad \text{and}\quad \frac{\partial g}{\partial t}(0,z)=g_1(z).$$

This uniqness is an immediate corollary of the following

Proposition 3.4 *Let v be smooth on $R\times X$ with boundary. Suppose v satisfies $Pv=0$. Let $g(t,z)$ be defined on $(-a,T+a)\times X$ for some $a>0$. Suppose*
i. $g(t,z)\in H_1^{loc}([-a,T+a]\times X)$
ii. $Pg=0$ where defined
iii. $g(t_0,z)\oplus\frac{\partial}{\partial t}g(t_0,z)\in H_1^0\oplus L^2$ for all $t_0\in(-a,T+a)$
iv. $g(0,z)\oplus\frac{\partial}{\partial t}g(0,z)=0\oplus 0$. Then

$$0=\int_0^T\int_X Pv\,g\,dz\,dt=\int_X(\frac{\partial}{\partial t}v)g-v(\frac{\partial}{\partial t}g)\,dz\Big|_0^T.$$

Proof: Essentially one approximates $v(t,z)$ and $g(t,z)$ by functions which are sufficiently regular to allow the integration by parts applied above to be carried out rigorously. The first step is to approximate $v(t,z)$ by functions $v_n(t,z)$ supported away from the vertices of X. Then on the support of v_n the function g can be approximated by functions $g_{nj}(t,z)$ which are smooth on $(-a/2,T+a/2)$ with boundary. Finally one must show that the resulting expressions in terms of the approximating functions converge to the desired expressions in $v(t,z)$ and $g(t,z)$.

There exists a sequence of smooth functions $\{\phi_n(z)\}$ such that each ϕ_n is supported away from the corners of X and such that $\phi_n(z)v(t,z)$ converges to $v(t,z)$ in H_1^0 uniformly for t in any fixed compact set. One can verify that if $\{\phi_n\}$ is produced according to the construction which follows then $\{\phi_n\}$ has the necessary properties.

Let $f:R\rightarrow[0,1]$ be a smooth function of R satisfying the restrictions

$$f(r)=\begin{cases}0 & r<d(X)/4\\ 1 & r>d(X)/2\end{cases}.$$

Let $\phi_n(z)$ be equal to $f(nr)$ in polar coordinates centered at each vertex x_i if z is within $d(X)/2$ of x_i. Let ϕ_n equal one otherwise. That is, ϕ_n vanishes in a $d(X)/4n$ disk around each vertex. Note the equality

$$\int_0^T\int_X P\phi_n v g\,dz\,dt=\int_0^T\int_X P\phi_n v\phi_{3n}g\,dz\,dt.$$

Now the object is to replace $\phi_{3n}g$ with a sequence of functions g_{nj} which are smooth on $(-a/2, T + a/2) \times X$ with boundary. Let u_n be an H_1 extension of $\phi_{3n}g$ to a neighborhood of $[-a/2, T + a/2] \times X$ such that u_n is locally odd across the edges of X and u_n vanishes outside $[-a/2, T + a/2] \times R^2$. Let $\{\psi_j\}$ be a smooth approximation ofthe identity on $R_t \times R_z^2$ such that each ψ_j is a radial function in the space variables. Require also that the support of each ψ_j in the space variables is sufficiently small that $\psi_j * u_n$ vanishes at the edges of X, vanishes in a neighborhood of each vertex, and satisfies $P(\psi_j * u_n) = 0$ on a neighborhood of $[0, T] \times supp\phi_n$. Let g_{nj} be the restriction of u_{nj} to X in the space variables.

With these definitions $\{g_{nj}\}$ converges to $\psi_{3n}g$ in $H_1^{loc}([-a/2, T + a/2] \times X)$. Consequently $g_{nj}\,|_{t=t_0}$ converges to $\phi_{3n}g|_{t=t_0}$ in $H_{1/2}(X)$.

The equation

$$\int_0^T \int_X P\phi_n v g_{nj}\, dz\, dt = \int_X (\frac{\partial}{\partial t}\phi_n v) g_{nj} - \phi_n v (\frac{\partial}{\partial t} g_{nj})\, dz \Big|_0^T$$

is valid. The limit of this equation as j approaches infinity is

$$\int_0^T \int_X P\phi_n v g\, dz\, dt = \int_X (\frac{\partial}{\partial t}\phi_n v) g - \phi_n v (\frac{\partial}{\partial t} g)\, dz \Big|_0^T \tag{3.1}$$

For the terms $\int_0^T \int_X P\phi_n v g\, dz\, dt$ and $\int_X (\frac{\partial}{\partial t}\phi_n v) g\, dz\Big|_0^T$, the limit can be understood by examining the Sobolev spaces in which the convergence $g_{nj} \to g_n$ takes place. To prove

$$\lim_{j\to\infty} \int \phi_n v \frac{\partial}{\partial t} g_{nj}\, dz \Big|_0^T = \int \phi_n v \frac{\partial}{\partial t} g\, dz \Big|_0^T,$$

it suffices to show that $\frac{\partial}{\partial t} u_{nj}\Big|_{t=t_0}$ converges to $\frac{\partial}{\partial t} u_n$ in $H_{-1}(R^2)$.

To prove the claim consider the Fourier transform, in the space variables, of the function $\frac{\partial}{\partial t}(u_{nj} - u_n)\Big|_{t=t_0}$. Provided no particular property of $t = 0$ is used, one can, without loss of generality, restrict attention to $\frac{\partial}{\partial t}(u_{nj} - u_n)\Big|_{t=0}$. The claim is then equivalent to the statement that the functions

$$\int (\hat{\psi}_j - 1)\tau \hat{u}_n(\tau, \eta)(1 - |\eta|^2)^{-1/2}\, d\tau$$

comprise a sequence of L^2 functions converging to zero as j approaches infinity.

The extra factor of τ in $\int (\hat{\psi}_j - 1)\tau \hat{u}_n(\tau, \eta)(1 - |\eta|^2)^{-1/2}\, d\tau$ is actually harmless. This follows because the wavefront set of u_n is contained in $\{(t, z, \tau, \eta) : \ |\tau| = |\eta|\}$. Thus

$(1-|\eta|^2)^{-1/2}$ is bounded on a neighborhood of the set on which $\hat{u}_n$ fails to be rapidly decreasing. Since $\hat{u}_n$ is an element of $H_1(R_t \times R^2)$ this implies that $(1+|\eta|^2)^{-1/2}\tau\hat{u}_n$ is the Fourier transform of a function $h_n(t,z) \in H_1(R_t \times R^2)$.

The functions $\psi_j * h$ converge to h in H_1, so $\psi_j * h|_{t=0}$ converges to $h|_{t=0}$ in $H_{1/2}(R^2) \subset L^2(R)$. Thus $(\psi_j * h|_{t=0})\hat{}\,(\eta)$ converges to $(h|_{t=0})\hat{}\,(\eta)$ in $L^2(R^2)$. This proves the claim because

$$(\psi_j * h|_{t=0})\hat{}\,(\eta) - (h|_{t=0})\hat{}\,(\eta)$$

is equal to

$$\int (\hat{\psi}_j - 1)\tau\hat{u}_n(\tau,\eta)(1+|\eta|^2)^{-1/2}\, d\tau$$

To complete the proof of the proposition, note that the limit as n approaches infinity of (3.1) is

$$0 = \int_0^T \int_X Pvg\, dz\, dt = \int_X \frac{\partial}{\partial t} vu - v\frac{\partial}{\partial t} u\, dz \Big|_0^T .$$

This proposition does imply the uniqueness of g satisfying conditions i-iv because $v(T)$ and $\frac{\partial}{\partial t}v(T)$ can be chosen arbitrarily from functions in $C_0^\infty(X)$. Thus

Corollary 3.5 *The function $g(t,z) = \pi_1 U(t) g_0 \oplus g_1$, $g_0 \oplus g_1 \in H_1^0 \oplus L^2$, is the unique solution to the initial value problem*

$$Pg = 0 \quad g|_{t=0} = g_0 \quad \frac{\partial}{\partial t} g \Big|_{t=0} = g_1$$

which also satisfies conditions i-iii of the previous proposition.

3.3 U in terms of G

The operators $U(t)$ have another characterizaton in terms of the forward fundamental solution for the forcing problem, $G(t-t',z,z')$, constructed by Friedlander.

Proposition 3.6 *Let*

$$\begin{bmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{bmatrix} = U(t)$$

and let $f(t,z)$ be an arbitrary function in $C_0^\infty(R_t \times X)$. Then

$$\int_{-\infty}^{t} \int_X U_{12}(t-t',z,z') f(t',z')\, dz'\, dt' = \int_{-\infty}^{\infty} \int_X G(t-t',z,z') f(t',z')\, dz'\, dt'.$$

Proof: The function $u(t,z) = \int_{-\infty}^{\infty}\int_X G(t-t',z,z')f(t',z')\,dz'\,dt'$ is the unique function which is smooth on $R_t \times X$ with boundary and satisfying

$$Pu(t,z) = f(t,z) \quad \text{and} \quad u(t,z) = 0 \ \text{ for } \ t \ll 0.$$

Proceed by showing that $v(t,z) = \int_{\infty}^{t}\int_X U_{12}(t-t',z,z')f(t',z')\,dz'\,dt'$ has the same regularity as u and satisfies the same conditions. View $f(t',z)$, $t' \in R$ as a smooth family of initial conditions $0 \oplus f(t',z)$ parametrized by t'. Let $g(t,t',z)$ equal the projection of $U(t)(0 \oplus f(t',z))$ onto the first component. Note that $g(0,t',z) = 0$ and $(\frac{\partial}{\partial t}g)(0,t',z) = f(t',z)$. There exists $\epsilon > 0$ such that $g(t,t',z)$ is in $C^\infty([-\epsilon,\epsilon] \times R_{t'} \times X)$ with support contained in a compact set in X. This follows from the fact that for t sufficiently small $g(t,t',z)$ is given by the solution of the Cauchy problem on the plane.

To extend $g(t,t',z)$ to arbitrary t use the method of construction for $U(t)$. That is, let $\psi(t)$ be a cutoff function in $C_0^\infty(R)$ with

$$\psi(t) = \begin{cases} 1 & |t| < \epsilon/4 \\ 0 & |t| > \epsilon/2 \end{cases}.$$

The function $(\frac{\partial^2}{\partial t^2} - \Delta)(\psi(t)g(t,t',z))$ is the sum of two functions $h_+(t,t',z)$ supported in $(\epsilon/4,\epsilon/2) \times R \times X$ and $h_-(t,t',z)$ supported in $(-\epsilon/2,-\epsilon/4) \times R \times X$. The function $g(t,t',z)$ for arbitrary time equals

$$\psi(t)g(t,t',z) + \int\int G(t-t'',z,z')h_+(t'',t',z)\,dz\,dt'' + \int\int G(t''-t,z,z')h_-(t'',t',z')\,dz'\,dt''.$$

The functions h_+ and h_- are smooth on $R_t \times R_{t'} \times X$ with boundary, thus by Theorem 2.15 $g(t,t',z)$ is smooth on $R_t \times R_{t'} \times X$ with boundary. The same is true of $g(t-t',t',z)$. The fact that $v(t,z) = \int_0^t g(t-t',t',z)\,dt'$ is smooth on $R \times X$ with boundary follows from this. Computation of $Pv(t,z)$ gives

$$Pv(t,z) = \frac{\partial}{\partial t}(g(0,t,z)) + (\frac{\partial}{\partial t}g)(0,t,z) + \int_{-\infty}^{t} Pg(t-t',t',z)\,dt' = f(t,z)$$

Clearly $v(t,z)$ vanishes for all t sufficiently negative. Thus, by the uniqueness of the solution to the forcing problem $u(t,z) = v(t,z)$.

Corollary 3.7

$$U(t) = \begin{cases} \begin{bmatrix} \frac{\partial}{\partial t}G(t) & G(t) \\ \frac{\partial^2}{\partial t^2}G(t) & \frac{\partial}{\partial t}G(t) \end{bmatrix} & t > 0 \\ \begin{bmatrix} \frac{\partial}{\partial t}G(-t) & -G(-t) \\ -\frac{\partial^2}{\partial t^2}G(-t) & \frac{\partial}{\partial t}G(-t) \end{bmatrix} & t < 0 \end{cases}$$

Chapter 4

The Wavefront Set of U, G

The main result of this section is the conclusion that the wavefront set of $G(t,z,z')$ in the polygonal domain X is confined to points $(t,z,z',\tau,\zeta,\zeta')$ for which $|t|$ is the length of a geodesic joining z to z' which passes through z in the $sign(t)sign(\tau)\zeta$ direction and through z' in the $-sign(z)sign(\tau)\zeta'$ direction. This restriction on the wavefront set of G enables one to use standard techniques of wavefront analysis to conclude that if $(1-\phi)$ is a smooth cutoff function with support bounded away from the vertices of X then the integral $\int_X(1-\phi)G(t,z,z')dz$ is a smooth function of t away from the length spectrum of X. Essentially, the main result is proved by showing that $G(t,z,z')$ as an operator on R^2 propagates singularities along geodesics. More specifically, given initial data $0\oplus g_1$ supported away from the vertices of X, let $g(t,z)$ equal $\pi_1(U(t)0\oplus g_1)$, which is equal to $\int G(t,z,z')g_1(z')dz'$ as a function of z and t. Then to any point (t,z,τ,ζ) in the wavefront set of g there corresponds a point (z',ζ') in the wavefront set of g_1 and a geodesic of length $|t|$ joining z to z' and passing through each point in the directions $sign(t)sign(\tau)\zeta$ and $sign(t)sign(\tau)\zeta'$ respectively. This is shown first for the initial data $0\oplus g_1$ for which g_1 is an L^2 function. The result is then extended to allow g_1 to be an arbitrary distribution by noting that the regularity of the data is improved by repeated applications of the inverse of the Laplacian. Special notice of the edges of the region must be taken. The indicator of singularity most useful in these arguments is the wavefront set of an extension of the function which is locally odd across the edges of X. This is made rigorous in

Definition 4.1 *A point (y,z,η,ζ) is in the* extended wavefront set *of a function f : $(R^k\times X)\to R$, denoted EWF, if*
i. z is in the interior of X and (y,z,η,ζ) is in the wavefront set of f, or
ii. z is in an edge of X, ρ is reflection across the edge containing z, and (y,z,η,ζ)

is in the wavefront set of f extended to a neighborhood of z by setting $f(x)$ equal to $-f(\rho(x))$ if $\rho(x)$ is in the interior of X.

Using this, the following theorem on the propogation of singularities for $H_1^0 \oplus L^2$ initial data is stated:

Theorem 4.2 *Let $g_0 \oplus g_1$ be initial data in $H_1^0 \oplus L^2$ which is identically zero in a neighborhood of each vertex. Let $g(t,z)$ equal $\pi_1 U(t) g_0 \oplus g_1$ Then the following statements are true of g.*
i. If (t,z,τ,ζ) is an element of $EFWg$ then there exists a point (z',ζ') in $EFWg_0 \cup EFWg_1$ and a geodesic of length $|t|$ joining z to z' which passess through z in the direction $sign(t)sign(\tau)\zeta$ and through z' in the direction $sign(t)sign(\tau)\zeta'$.
ii. If for all points (z',ζ') in $EWFg_1 \cup EWFg_2$ the geodesic of length $|t|$ beginning at z' in the direction $\pm\zeta'$ does not pass through the vertex x_0 of X, then there is some neighborhood of (t,x_0) on which $g(t,z)$ is smooth on X with boundary.

Proof: The statements are true when $|t|$ is sufficiently small. This follows from the fact that for sufficiently small t, the function $g(t,z)$ results from the application of the solution operator on R^2 or the solution operator on a half plane, as appropriate, to each term of $\phi_i g_0 \oplus \phi_i g_1$ where $\{\phi_i\}_{i=1}^n$ is a suitable partition of unity. Assume that $g(t)$ satisfies i and ii for all t in the range $(0,T)$ (there is no loss of generality in working with $t>0$). The form of the forcing kernel for small time will guarantee that g satisfies the conditions for all t in the range $(0,T+b)$ for some $b>0$. Several cases must be examined.

First consider the case in which some point (z',ζ') in $EWFg_0 \cup EWGg_1$ and a vertex x_0 of X are joined by a geodesic of length T, but for some $a>0$ no geodesic with length in the range $(T,T+a]$ joins a point of $EWFg_0 \cup EWFg_1$ to x_0. The task is to show first that $EWFg$ has the required form for $t \in (T,T+b]$. One must also show that $g(t)$ is smooth on X with boundary in a neighborhood of each point (t,x_0), $t \in (T,T+b)$. This is done by cutting g off near $t=T$ by a function $\psi(t)$ and applying the short term forcing kernel separately to the terms $\phi_1 P\psi g$, $\phi_2 P\psi g$ and $\phi_3 P\psi g$ where $\{\phi_1,\phi_2,\phi_3\}$ is a carefully chosen partition of unity for X.

The relationships among the supports of ψ, ϕ_1, ϕ_2 and ϕ_3 are important. Pick $\epsilon>0$ which is small compared to $d(X)$. Let $\psi(t)$ be a smooth cutoff function satisfying the conditions

$$\psi(t) = \begin{cases} 1 & t < T-2\epsilon; \\ 0 & t > T-\epsilon. \end{cases}$$

Let the partition of unity $\{\phi_1, \phi_2, \phi_3\}$ be such that $supp\phi_1$ is contained in an $\epsilon/4$ ball about x_0, $supp\phi_2$ is contained in the set of points $\{z : \epsilon/8 \leq ||z - x_0|| \leq 4\epsilon\}$, and $supp\phi_3$ is contained in the set $\{z : ||z - x_0|| > 3\epsilon\}$.

For t in a sufficiently small neighborhood of T and z within a ball of radius $d(X)$ of x_0, the extended wavefront sets of the terms $\int G(t-t', z, z')\phi_i P\psi g(t,t')\,dt'\,dz'$ $i = 2, 3$ are confined to the desired points. In this range, the known form of $G(t-t', z, z')$ implies that the singularities of $\phi_2 P\psi g$ are propagated along the geodesics. The singularities of $\phi_3 P\psi g$ which are a small distance away from the other vertices are propagated along geodesics, while the forcing terms very near the other verticies do not affect this neighborhood of x_0. Moreover, if z is an edge point of X and ρ is reflection across the edge containing z, the odd extension of $G\phi_i P\psi g(t,z)$, $i = 2, 3$, across the edge has its wavefront set contained in (t, z, τ, ζ) : (t, z, τ, ζ) or $(t, \rho(z), \tau, \rho(\zeta))$ satisfies the relation in condition i. Thus $EWF\, G\phi_i P\psi g(t,z)$, $i = 2, 3$, satisfies condition i in the specified neigborhood of (T, x_0). Consider the term $\int G(t-t', z, z')\phi_1 P\psi g(t,z)\,dt'\,dz'$, $b \geq -\epsilon/2$, $t \in [T + b - \epsilon/16, T + b + \epsilon/16]$. It has support in the set $\{z : \ ||z - x_0|| < b + \epsilon/16 + 2\epsilon + \epsilon/4\}$ and is smooth on X with boundary in the set $\{z : \ ||z - x_0|| < b - \epsilon/16 + \epsilon - \epsilon/4\}$ by Theorem 2.8 Thus $EWF\, G\phi_1 P\psi g$ is empty outside the band $\{z : \ b + \frac{11}{16}\epsilon \leq ||z - x_0|| \leq b + \frac{37}{16}\epsilon\}$. However, this band in which $EWF\, G\phi_1 P\psi g$, and hence $EWFg$, is problematic can be moved entirely by replacing ϵ in the argument by $\epsilon_1 = \epsilon/8$ except in the specification of the neighborhood of T in question. In this second analysis, the troublesome region is $\{z : \ b + \epsilon/32 < ||z - x_0|| < b + \frac{11\epsilon}{32}\}$. Since the bands from the two arguments are disjoint, $EWFg$ satisfies condition i in a neighborhood of $t - T + b$.

In order to verify that condition ii is satisfied, let ψ and ϕ_1,ϕ_2,ϕ_3 be as above. If ϵ is taken to be small compared to $b > 0$ then $G\phi_1 P\psi g$ and $G\phi_2 P\psi g$ are both smooth on X with boundary in a neighborhood of $(T + b, x_0)$.

The final forcing term $f = \phi_3 P\psi g$ has the property that no point $(t, z, \tau, \zeta) \in EWF\, f$ gives rise to a geodesic reaching x_0 in the time interval $[T - 2\epsilon, T + a]$. Thus there is a neighborhood of x_0 such that if (t, z, τ, ζ) is an element of $EWF\, f$ then the geodesic passing through z at time t in the direction $-sign(\tau)\zeta$ does not enter that neighborhood of x_0 in the time interval $[T - 2\epsilon, T + a]$. There exists $r < \epsilon$ such that this neighborhood contains the set $\{z : \ ||z - x_0|| < r\}$. Consider $Gf(t,z)$ for $t \in (T - 2\epsilon, T + r)$. This function is supported away from x_0 and is smooth on X with boundary in the set $(T - 2\epsilon, T + r) \times \{z : \ ||z - x_0|| < r\}$. Choose a $\delta > 0$ which is very small compared to r. Let$\psi \in C^{\infty}(R)$ be a function satisfying

$$\psi(t) = \begin{cases} 1 & t < T + r - 2\delta; \\ 0 & t > T + r - \delta. \end{cases}$$

Pick $\phi(z) \in C^{\infty}(R^2)$ satisfying

$$\phi(z) = \begin{cases} 1 & \|z - x_0\| < r - \delta; \\ 0 & \|z - x_0\| > r + \delta. \end{cases}$$

Note that

$$Gf(t,z) = \psi Gf(t,z) + G[(1-\psi)f(t,z) - 2(\frac{\partial}{\partial t}\psi)(\frac{\partial}{\partial t}Gf) - (\frac{\partial^2\psi}{\partial t^2}Gf](t,z).$$

The function $\phi[(1-\psi)f(t,z) - 2(\frac{d\psi}{dt}(\frac{\partial}{\partial t}Gf) - \frac{d^2\psi}{dt^2}Gf]$ is smooth on X with boundary, and so the forcing kernel applied to this expression results in a function smooth on X with boundary. The term $\psi Gf(t,z)$ vanishes in a neighborhood of x_0. The remaining term in the expression for $Gf(t,z)$ is the forcing kernel applied to $f_1 = (1-\phi)[(1-\psi)f - 2\frac{d\psi}{dt}\frac{\partial}{\partial t}Gf - \frac{d^2\psi}{dt^2}Gf$. The function f_1 is supported in $\{z : \|z - x_0\| > r - \delta\}$ and has the property that none of the points in $EWF\,f_1$ is associated with a geodesic passing through $\{z : \|z - z_0\| < r\}$ in the time interval $[T - 2\epsilon, T + a]$. The preceding argument can essentially be iterated with f_1 in place f. If δ is taken to be sufficiently small, a finite number of iterations will show that Gf is smooth on X with boundary in $(T, T+a) \times \{z : \|z - z_0\| < r\}$. This completes the argument for the first case.

The remaining cases are considerably simpler. The case in which no geodesic arising from the initial data arrives at x_0 in a neighborhood of $t = T$ reduces to showing that if no geodesic arrives from the initial data to x_0 in the interval $[T - a, T + a]$ then conditions i and ii are satisfied for $t \in [T, T + a)$. This is shown by using the foregoing argument in the situation in which g is known to have the desired behavior in the interval $[0, T - a]$.

The final case, in which there is a geodesic from $EWF\,g_0 \cup EWF\,g_1$ arriving at x_0 for all $t \in [T, T+a]$ can be handled by using the first part of the argument for case 1. Note that in this final case condition ii is null.

This proves that g satisfies conditions i and ii in a ball of radius $d(X)$ about x_0 for $t \in [0, T + b]$. Since x_0 was arbitrary, this shows that g satisfies conditions i and ii in $d(X)$ neighborhoods of all the vertices of X. The proofs that if g satisfies i and ii for $t \in [0, T)$ then then g satisfies i and ii in a neighborhood of radius $d(X)$ contained in the interior and distance at least $d(X)/2$ from the vertices of X, or in a neighborhood of radius $d(X)$ of an edge point a distance at least $d(X)/2$ from any vertex, are much simpler and are omitted. Covering X by such neighborhoods proves the theorem.

The definition of $U(t)$ can be extended as follows. Let z_i, $i = 1, 2$, be an edge point of X. Let ρ_i be a reflection across the edge Σ_i containing z_i and let O_i be an open neighborhood of z_i in R^2 with the property that $O_i \cap \partial X$ is contained in Σ_i and $\rho_i(O_i) = O_i$. For any $g_0 \oplus g_1 \in H_2 \oplus L^2$ supported in O_1 the restriction to X of $g_0 \oplus g_1 - g_0 \circ \rho_1 \oplus g_1 \circ \rho_1$, call it $f_0 \oplus f_1$, is an element of $H_1^0 \oplus L^2$. Set $U(t)\, g_0 \oplus g_1$ equal to $U(t)\, f_0 \oplus f_1$ in the interior of X. In O_2 set $U(t)\, g_0 \oplus g_1$ equal to the odd extension of $U(t)\, f_0 \oplus f_1$ across the edge Σ_2. The result on the propagation of singularities by $U(t)$ gives

Corollary 4.3 *Let $U(t)\, g_0 \oplus g_1$ be defined as above and let g equal $\pi_1\, U(t)\, g_0 \oplus g_1$. Then $WF\, g \bigcap T^*(R_t \times O_1)$ is confined to points of the form (t, z, τ, ζ) for which there exists a point (z', ζ') in $WF\, g_0 \bigcap WF\, g_1$ and a geodesic of length $|t|$ in X*
joining z to z' with directions ζ and ζ' at the end points,
or joining z and $\rho_1(z')$ with directions ζ and $\rho_1(\zeta')$,
or joining $\rho_2(z)$ and z' with directions $\rho_2(\zeta)$ and ζ'
or, finally, joining $\rho_2(z)$ and $\rho_1(z')$ with directions $\rho_2(\zeta)$ and $\rho_1(\zeta')$.

The family $U(t)$ can also be viewed as acting on initial data $g_0 \oplus g_1$ for g_0 and g_1; both distributions supported in X away from the corners. This is done by increasing the regularity of the initial data by applying some power of the inverse of the Laplacian on R^2, Δ^{-1}. Consider first g_0 and g_1 which are supported in the interior of X. Take k such that $\Delta^{-k}\, g_0$ is an element of $H_1(R^2)$ and $\Delta^{-k}\, g_1$ is an element of $L^2(R^2)$. Let $\phi \in C_0^\infty(X)$ be equal to 1 in a neighborhood of $supp\, g_0 \cup supp\, g_1$. Define $g(t,z) = \pi_1 U(t) g_0 \oplus g_1$ by

$$g(t) = \Delta^k \pi_1 U(t)(\phi\Delta^{-k} g_0 \oplus \phi\Delta^{-k} g_1) - \pi_1 U(t)[\Delta^k \phi \Delta^{-k} g_0 \oplus \Delta^k \phi \Delta^{-k} g_1 - \phi g_0 \oplus \phi g_1].$$

The extended wavefront set of the first term arises from the wavefront set of g_0 and g_1 as described in the preceding theorem. The second term is the result of applying the solution operator to initial data which is in $C_0^\infty(X) \oplus C_0^\infty(X)$ and so this does not contribute to the extended wavefront set of $g(t,z)$.

Likewise if f_0 and f_1 are distributions supported in a small neighborhood in X of an edge point of X, and ρ is a reflection across that edge, let $\tilde{f}_0 \oplus \tilde{f}_1$ equal $f_0 \oplus f_1 - f_0 \circ \rho \oplus f_1 \circ \rho$ restricted to X. Define $f(t,z) = \pi_1 U(t) f_0 \oplus f_1$ by

$$f(t) = \Delta^k \pi_1 U(t)(\phi\Delta^{-k} \tilde{f}_0 \oplus \phi\Delta^{-k} \tilde{f}_1) - \pi_1 U(t)[\Delta^k \phi \Delta^{-k} \tilde{f}_0 \oplus \Delta^k \phi \Delta^{-k} \tilde{f}_1 - \phi \tilde{f}_0 \oplus \phi \tilde{f}_1].$$

Once again $f(t)$ has the desired extended wavefront set. Using a partition of unity one can make any initial data supported on X away from the corners into a sum terms of athe form $g_0 \oplus g_1$ or $f_0 \oplus f_1$ as above. This is summarized in

Corollary 4.4 *Let $g_0 \oplus g_1$ be initial data supported in $\overline{X}$ and supported away from the vertices. Let $g(t)$ equal $\pi_1 U(t) g_0 \oplus g_1$. Then a point (t, z, τ, ζ) is in $EWFg$ only if there is a point $(z', \zeta') \in EWFg_0 \cup EWFg_1$ and a geodesic of length $|t|$ joining z to z' which passes through z and z' in the directions $sign(t)sign(\tau)\zeta$ and $sign(z)sign(\tau)\zeta'$ respectively.*

The definition of $U(t)$ can now be extended to initial data comprised of distributions supported in a small neighborhood O_1 of a point on the edge Σ_1. Simply choose Δ^{-k} such that $\Delta^{-k} g_0 \oplus \Delta^{-k} g_1$ is in $H_1^{loc} \oplus L^2_{loc}$. Let ϕ be a cutoff function supported in O_1 equal to 1 on the support of g_0 and g_1, and symmetric with respect to Σ_1. Then $U(t) g_0 \oplus g_1$ is defined in terms of the extension in Corollary( ). Let $g(t) = \pi_1 U(t) g_0 \oplus g_1$ be given by

$$g(t) = \Delta^k \pi_1 U(t)(\phi \Delta^{-k} g_0 \oplus \phi \Delta^{-k} g_1) - \pi_1 U(t)[\Delta^k \phi \Delta^{-k} \phi g_0 \oplus \Delta^k \phi \Delta^{-k} g_1 - \phi g_0 \oplus \phi g_1].$$

Once again the second term on the right is smooth on X with boundary. Thus the conclusion of Corollary() is true when g_0 and g_1 are allowed to be distributions. Call this extension $\tilde{U}(t)$. Note that $\tilde{U}$ is locally an odd extension of U in that, for z' in $O_1 \cap X$ and z in $O_2 \cap X$ one has

$$U(t, z, z') = \tilde{U}(t, z, z') = -\tilde{U}(t, z, \rho_1(z'))$$
$$= -\tilde{U}(t, \rho_2(z), z') = U(t, \rho_2(z), \rho_1(z')).$$

The object of this examination of the propagation of singularities is to determine the wavefront sets of the elements of U and $\tilde{U}$. Each of these families of operators can be written as a matrix. The form of the matrix $U(t)$ was shown in Proposition to be

$$\begin{pmatrix} \frac{\partial G}{\partial t} & G \\ \frac{\partial^2 G}{\partial t^2} & \frac{\partial G}{\partial t} \end{pmatrix}.$$

The operator $\tilde{U}(t)$ mapping distributional initial data supported in O_1 to distributions defined on O_2 can be written

$$\begin{pmatrix} \frac{\partial \tilde{G}}{\partial t} & \tilde{G} \\ \frac{\partial^2 \tilde{G}}{\partial t^2} & \frac{\partial \tilde{G}}{\partial t} \end{pmatrix}$$

The manner in which G and $\tilde{G}$ propagate singularities is given by the propagation of singularities from initial data $0 \oplus g_1$ by the operators $\pi_1 U$ and $\pi_1 \tilde{U}$. The operator G can be considered as acting in distributions supported in X mapping them to distributions defined on $R_t \times X$. The operator $\tilde{G}$ maps distributions supported in in O_1 to distributions defined in $R_t \times O_2$. From this behavior and the fact that certain types of points are not in the wavefront set of G or $\tilde{G}$, the wavefront set of each operator can be restricted to points $(t, z, z', \tau, \zeta, \zeta')$ whose terms are related by a geodesic.

One useful observation is that both $WF\,G$, defined at points in $R_t \times X \times X$, and $WF\,\tilde{G}$ are contained in the set of points $\{(t, z, z', \tau, \zeta, \zeta') : \ |\tau| = |\zeta|\}$. This is just a consequence of the fact that if P is the wave opeator in the variables t and z then $PG = P\tilde{G} = 0$. A further assertion is possible. Neither WFG nor $WF\tilde{G}$ contains points of the form $(t, z, z', \tau, \zeta, 0)$ or $(t, z, z', 0, 0, \zeta')$. For G, this is a consequence of the fact that $G(t)$ can be written as $\frac{\partial}{\partial t}G(t-\epsilon) \circ G(\epsilon) + G(t-\epsilon) \circ \frac{\partial}{\partial t}G(\epsilon)$. Provided ϵ is sufficiently small the wavefront set $G(t, z, z')$ is known to consist entirely of points for which $|\tau| = |\zeta| = |\zeta'|$. The rules for the wavefront set of the composition of operators together with the fact that $|\tau|$ is equal to $|\zeta|$ for points in WFG imply the assertion. The wavefront set of $\tilde{G}$ can be analyzed similarly. Let

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be the standard solution operator for the initial value problem associated with the wave operator in $R_t \times R^2$. Then $\tilde{G}(t)$ is equal to $\frac{\partial}{\partial t}\tilde{G}(t-\epsilon) \circ A_{12}(\epsilon) + \tilde{G}(t-\epsilon) \circ A_{22}$. The argument proceeds as above.

Viewing the situation more abstractly, in each case, G and $\tilde{G}$, there is an operator K from distributions supported in an open region Z in R^n to distributions defined on an open region Y in R^m. Recall that a wavefront relation C' for K is a closed set of points $((y, \eta), (z, \zeta))$ in $T^*Y \times T^*Z$, such that a point (y_0, η_0) is in the wavefront set of Ku only if there is some point $((y_0, \eta_0), (z, \zeta))$ in C' for which (z, ζ) is in the wavefront set of u. In each case here, a wavefront relation is known. Further, no points of the form $((y, 0), (z, \zeta))$ or $((y, \eta), (z, 0))$ are elements of either the wavefront relation of G ofthe wavefront relation of $\tilde{G}$. Consequently the follwing theorem is applicable.

Theorem 4.5 *Let K be a continuous operator from distributions supported in a region Z in R^n to distributions defined on a region Y in R^m. Let C' be a wavefront relation for K. If there are no points of the form $((y, 0), (z, \zeta))$ or $((y, \eta), (z, 0))$ in C' then the wavefront set of the kernel of K is confined to the set*

$$\{(y, z, \eta, \zeta) : ((y, \eta), (z, -\zeta)) \in C'\}.$$

Applying this theorem to G gives the result that $WF\,G$ is restricted to points $(t, z, z', \tau, \zeta, \zeta')$ such that there is a geodesic of length $|t|$ connecting z and z', and the direction of the geodesic fron z to z' is $sign(t)sign(\tau)\zeta$, while the direction of the geodesic from z' to z is $sign(t)sign(\tau)\zeta'$. Likewise $WF\,\tilde{G}$ is restricted to points $(t, z, z', \tau, \zeta, \zeta')$ for which at least one of the following satisfies the geodesic relation for G given above:
$(t, z, z', \tau, \zeta, \zeta')$, $(t, \rho_2(z), z', \tau, \rho_2(\zeta), \zeta')$, $(t, \rho_2(z), \rho_1(z'), \tau, \rho_2(\zeta), \rho_2(\zeta'))$
or $(t, z, \rho_1(z'), \tau, \zeta, \rho_1(\zeta'))$. Again thinking of $G(t)$ as $\frac{\partial}{\partial t}G(t-\epsilon)\circ G(\epsilon) + G(t-\epsilon)\circ \frac{\partial}{\partial t}G(\epsilon)$ one can conclude that $EW\,G$ is further resticted to points $(t, z, z', \tau, \zeta, \zeta')$ satisfying the equalities $|\tau| = |\zeta| = |\zeta'|$. Likewise , for points in $WF\,\tilde{G}$ the equalities $|\tau| = |\zeta| = |\zeta'|$ are true.

Chapter 5

The Poisson Relation

The purpose of this section is the proof of the Poisson relation for $U(t)$. It has already been mentioned that the operator which maps a Schwartz function of t, $f(t)$ to the trace of the operator $\int U(t,z,z')f(t)dt$ is a tempered distribution. Call it $tr\,U$. The Poisson relation states that

Theorem 5.1 *The singular support of* $tr\,U$ *is contained in the set*

$$\{l : \pm l \text{ is zero or the length of a closed geodesic in } X\}$$

It is possible to show reasonably directly that $tr\,U(t)$ is smooth when t is not in the length spectrum of X. The necessary information is the form of the wavefront set of $G(t,z,z')$, the regularity of $U(t)$ applied to data which is smooth on $X \times X$ with boundary and supported away from the corners, and the regularity of the forcing kernel on forcing data which is smooth on $R \times X \times X$. The method exploits the group structure of $U(t)$ by examining separately the terms $tr\,U(t) \circ [1-\phi]$, $tr\,U(t-\epsilon[\phi]U(\epsilon)[\phi]$, and $tr\,U(t-\epsilon)[\phi]U(\epsilon)[1-\phi]$. Here ϕ is a smooth cutoff function supported in a neighborhood of the vertices of X. The operator $[\phi]$ on $H_2^0 \oplus L^2$ maps $g_0 \oplus g_1$ to $\phi g_0 \oplus \phi g_1$. This separation into three terms is justified by considering the distribution $tr\,U$. Certainly *trace* $\int U(t)f(t)dt$ is equal to *trace* $\int U(t)[\phi]f(t)dt$ + *trace* $\int U(t)[1-\phi]f(t)dt$. Also, the operator $\int U(t+s)[\phi]f(t)dt$ is equal to $U(s)\int U(t)[\phi]f(t)dt$. Thus the term *trace* $\int U(t)[\phi]f(t)dt$ is equal to *trace* $U(\epsilon)\int U(t-\epsilon)[\phi]f(t)dt$, which is equal to *trace* $\int U(t-\epsilon)[\phi]U(\epsilon)f(t)dt$.

In particular, for the purposes of proving Theorem 5.1 in a neighborhood of a fixed point t_0, choose ϵ and ϕ as follows. Let $\epsilon > 0$ be a number which is small compared to $d(X)$ and which satisfies the condition that the interval $(t_0 - 3\epsilon, t_0 + 3\epsilon)$ contains no elements of the length spectrum of X. Given this choice of ϵ, let $\phi \in C_0^\infty(R^2)$ be a function which is supported in the set of $\frac{\epsilon}{4}$ balls about each vertex of X. Require also that ϕ is a radial function in the $\frac{\epsilon}{4}$ ball about each vertex x_0 in the polar coordinates centered at x_0, and that ϕ is equal to one in the $\frac{\epsilon}{8}$ ball about each x_0. Given this choice of ϕ, and the manipulations above, the proof of Theorem 5.1 reduces to the following Lemmas:

Lemma 5.2 *The distribution tr* $U(t)[1-\phi]$ *is smooth in a neighborhood of any point* t_0 *not in the length spectrum of* X.

Lemma 5.3 *The distribution tr* $U(t-\epsilon)[\phi]U(\epsilon)[1-\phi]$ *is smooth in a neighborhood of any point* t_0 *not in the length spectrum of* X.

Lemma 5.4 *The distribution tr* $U(t-\epsilon)[\phi]U(\epsilon)[\phi]$ *is smooth on all* R.

Proof of Lemma 5.2: The trace of $\int U(t)[1-\phi]f(t)dt$ as an operator on $H_1^0 \oplus L^2$ is equal to the sum of the trace of the operator $\int \frac{\partial}{\partial t}G(t,z,z')(1-\phi(z'))f(t)dt$ from L^2 to L^2, and the trace of the same operator considered as a mapping from H_1^0 to H_1^0. Since these traces are equal, verifying that $tr\, U(t)[1-\phi]$ is smooth near t_0 reduces to checking that $\int \frac{\partial}{\partial t}G(t,z,z)(1-\phi(z))dz$ is smooth near t_0.

First consider the integral in the interior of X. Let ϕ_0, $\phi_1 \in C_0^\infty(X)$ be cutoff functions such that $supp\, \phi_0$ is contained in the set on which ϕ_1 is equal to one. Then $\phi_1(z)\frac{\partial}{\partial t}G(t,z,z')\phi_0(z')(1-\phi(z')$ is a distribution with known wavefront relation. Standard wavefront computations show that $\int \frac{\partial}{\partial t}G(t,z,z)\phi_0(z)(1-\phi(z))dz$, the result of restricting this distribution to the diagonal in the space variables and then integrating out the remaining space variable, is smooth in a neighborhood of t_0.

To understand the behavior near the edges, use $\tilde{G}(t,z,z')$. Let O be a sufficiently small neighborhood of a point z_0 on an edge Σ_0 of X. Then $\tilde{G}(t,z,z')$ is a well-defined operator from distributions supported in O to distributions defined in a neighborhood of O. Let $\phi_0,\phi_1 \in C_0^\infty(R^2)$ be cutoff functions supported in O and even across Σ_0. Suppose $supp\, \phi_0$ is contained in the set on which ϕ_1 is equal to one. Then $\phi_1\tilde{G}(t,z,z')\phi_0(z')(1-\phi(z'))$ is a distribution on $R^2 \times R^2$. As before, wavefront considerations imply that $\int \tilde{G}(t,z,z)\phi_0(z)(1-\phi(z))dz$ is smooth on a neighborhood of t_0.

The distribution $\tilde{G}(t,z,z)\phi_0(z)(1-\phi(z))$ is even across Σ_0. It is equal to $G(t,z,z')\phi_0(z)(1-\phi(z))$ on X. Thus

$$\frac{1}{2}\int \tilde{G}(t,z,z)\phi_0(z)(1-\phi(z))f(t)dzdt =$$

$$\int G(t,z,z)\phi_0(Z)(1-\phi(z))f(t)dzdt.$$

Using a partition of unity subordinate to a division of the closure of X into sets of the type of O and sets with closure contained in the interior of X completes the proof of Lemma 5.2.

The proof of Lemma 5.3 is very similar, though some technical complications arise due to the extra $[\phi]$ in the expression. The main complication is the absence of an extension of $U(t-\epsilon)[\phi]U(\epsilon)[1-\phi]$ corresponding to $\tilde{U}[1-\phi]$. However, one can define an extension of $U(t-\epsilon)[1-\phi]U(\epsilon)[1-\phi]$ just as $\tilde{U}$ was defined. This will suffice for the proof of the lemma.

To define the extension of $U(t-\epsilon)[1-\phi]U(\epsilon)[1-\phi]$, call it $B(t)$, proceed as follows. Let O_1 and O_2 be neighborhoods of the type specified in the definition of $\tilde{U}$, with corresponding edges Σ_1 and Σ_2, and reflections ρ_1 and ρ_2. The operator $B(t)$ acts on initial data $g_0 \oplus g_1$ supported in O_1. The action is specified on $O_2 \cap X$ by the equality

$$\pi_1 B(t) g_0 \oplus g_1 = \pi_1 U(t-\epsilon)[1-\phi]U(\epsilon)[1-\phi](g_0 \oplus g_1 - g_0 \circ \rho_1 \oplus g_1 \circ \rho_1).$$

Let $g(t)$ equal $\pi_1 B(t) g_0 \oplus g_1$.The function $g(t)$ is defined on all of O_2 by the fact that it is odd across the edge Σ_2.

The wavefront relation for $B(t)$ is the set of pairs $((t,z,\tau,\zeta),(z',\zeta'))$ for which $(t,z,z',\tau,\zeta,\zeta')$ or $(\pm|t-2\epsilon|,z,z',\tau,\zeta,\zeta')$ satisfies a geodesic relation. Note that $Pg=0$. Denote by $A(t)$ the family of solution operators for the initial value problem associated with the wave operator on R^2. If δ is sufficiently small, then

$$B(t) = \frac{1}{2}U(t-\epsilon)[1-\phi]U(\epsilon-\delta)A(\delta)[1-\phi].$$

Thus the same arguments that were used to compute the wavefront set of $\tilde{G}$ can be applied to $B(t)$. Conclude that if $B(t)$ is written

$$\begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$$

then the wavefront set of $B(t)$ is contained in the set of points $\{(t,z,z',\tau,\zeta,\zeta') : ((t,z,\tau,\zeta),(z',zeta'))$ is an element of the wavefront relation for $B\}$. Furthermore $B_{ij}(t,z,z') = -B_{ij}(t,z,\rho_1(z')) = -B_{ij}(t,\rho_2(z),z') = B_{ij}(t,\rho_2(z),\rho_1(z'))$. This completes the background necessary to treat $tr\, U(t-\epsilon)[1-\phi]U(\epsilon)[1-\phi]$ in essentially the same way as $U(t)[1-\phi]$. The result is

Proof of Lemma 5.3: Note that, in light of Lemma 5.2, it suffices to show that $tr\,U(t-\epsilon)[1-\phi]U(\epsilon)[1-f]$ is smooth in a neighborhood of t_0. To do this, let $\phi_0,\phi_1 \in C_0^\infty(X)$ be cutoff functions with $supp\,\phi_0$ contained in the set on which ϕ_1 is equal to one. The $\int \phi_1 B_{ii}(t,z,z)\phi_0 dz$ is smooth in a neighborhood of t_0, $i=1,2$.

Likewise, let ϕ_0 and ϕ_1 be smooth functions supported in O_1 that are even across the edgeΣ_1. If $supp\,\phi_0$ is contained in the set in which ϕ_1 is equal to one, then $\int \phi_1 B_{ii}(t,z,z)\phi_0 dz$ is smooth in a neighborhood of t_0.

As in the case of the previous lemma, these two facts suffice to show that $tr\,U(t-\epsilon)[1-\phi]U(\epsilon)[1-\phi]$ is smooth in a neighborhood of t_0.

The final case, Lemma 5.4, is substantially different. Here the fact that the elements of $\phi U(\epsilon)\phi$ are smooth in $O \times X \times X$ where O is a neighborhood of t_0 is crucial.
Proof of Lemma 5.4: Consider the matrix form of $[\phi]U(\epsilon)[\phi]$,

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

View this as two sets of initial data parametrized by z', $C_{11}(\epsilon,z,z') \oplus C_{21}(\epsilon,z,z')$, and $C_{12}(\epsilon,z,z') \oplus C_{22}(\epsilon,z,z')$. Claim that $\pi_1 C_{11}(\epsilon) \oplus C_{21}(\epsilon)$ and $\pi_1 C_{12}(\epsilon) \oplus C_{22}(\epsilon)$ are smooth on $R \times X \times X$ with boundary. Consider first

$$C_1(t+\epsilon; z,z') = \pi_1 U(t) C_{11}(\epsilon) \oplus C_{21}(\epsilon).$$

The function $C_1(t,z,z')$ can be obtained by considering

$$C_{11}(s) = \phi(z)\frac{\partial}{\partial t}G(s,z,z')\phi(z')$$

for s in the range $(\frac{3\epsilon}{4},\infty)$. Note that $\frac{\partial}{\partial s}C_{11}(\epsilon) = C_{12}(\epsilon)$. Basically, $C_{11}(s)$ comes very close to solving the initial value problem with initial values $C_{11}(\epsilon) \oplus C_{21}(\epsilon)$. The failure is just a smooth function supported away from the vertices. Consequently the solution to the initial value problem is smooth on $R \times X \times X$ with boundary.

To begin filling in the details of this argument, note that the wave operator in the s and z variables applied to C_{11} results in a function $f(s,z,z')$ which is smooth on $(\frac{3\epsilon}{4},\infty) \times X \times X$ with boundary. In fact the support in z remains outside the neighborhood of the vertices on which ϕ is equal to one. Let $\psi(s) \in C^\infty(R)$ be a function which is equal to one on $(\frac{31\epsilon}{32},\infty)$ and equal to zero on $(-\infty,\frac{30\epsilon}{32})$. Apply the forcing kernel to $\psi(s)f(s,z,z')$ to obtain a function $g(s,z,z')$ which is smooth on $R \times X \times X$ with boundary. The function $\tilde{C}_1(s,z,z') = C_{11}(s,z,z') - g(s,z,z')$ is then smooth on $(\frac{3}{4}\epsilon, \infty) \times X \times X$ with boundary. The wave operator applied to $\tilde{C}_1(s, z, z')$ in the s and z variables is equal to zero when s is greater than $\frac{31\epsilon}{32}$. Conclude that

$$\pi_1 \, U(t) \tilde{C}_1(\epsilon, z, z') \oplus \frac{\partial}{\partial s} \tilde{C}_1(\epsilon, z, z')$$

is equal to $\tilde{C}_1(t + \epsilon, z, z')$ for t greater than $-\frac{\epsilon}{32}$.

The function $\tilde{C}_1(s, z, z')$ solves an initial value problem very similar to the one with initial data $C_{11}(\epsilon) \oplus C_{21}(\epsilon)$. In fact, $\tilde{C}_1(\epsilon, z, z') \oplus \frac{\partial}{\partial s} \tilde{C}_1(\epsilon, z, z')$ is equal to $C_{11}(\epsilon) \oplus C_{21}(\epsilon)$ when z is in a ball of radius $\frac{1}{16}\epsilon$ around any vertex. This implies that

$$\pi_1 \, U(t) \, \left(\tilde{C}_1(\epsilon) \oplus \frac{\partial}{\partial s} \tilde{C}_1(\epsilon) - C_{11}(\epsilon) \oplus C_{21}(\epsilon) \right)$$

is smooth on $R \times X \times X$ with boundary.

Observe now that $C_1(t + \epsilon)$ is equal to

$$\tilde{C}_1(t + \epsilon) - \pi_1 \, U(t) \, \left(\tilde{C}_1(\epsilon) \oplus \frac{\partial}{\partial s} \tilde{C}_1(\epsilon) - C_{11}(\epsilon) \oplus C_{21}(\epsilon) \right)$$

for $t > -\frac{\epsilon}{32}$. This implies that $C_1(t + \epsilon)$ is smooth on $(\frac{31\epsilon}{32}, \infty) \times X \times X$ with boundary. To verify that $C_1(t + \epsilon)$ is actually smooth on $R \times X \times X$ with boundary, consider a smooth function $\tilde{\psi}(s)$ which is equal to one in a neighborhood of $[\epsilon, \infty)$ and equal to zero in a neighborhood of $(-\infty, \frac{31\epsilon}{32}]$. Apply the backward fundamental solution $\check{G}$ to $P\tilde{\psi}C_1$. Then $C_1(s, z, z')$ is equal to $\tilde{\psi}C_1(s, z, z') - \check{G}P\tilde{\psi}C_1(s, z, z')$, a difference of two functions both smooth on $R \times X \times X$ with boundary.

Define $C_2(t + \epsilon)$ to be $\pi_1 U(t) C_{12}(\epsilon) C_{22}(\epsilon)$. The argument above also serves to show that $C_2(t, z, z')$ is smooth on $R \times X \times X$ with boundary.

In these terms, $U(t - \epsilon)[\phi] U(\epsilon)[\phi]$ is equal to

$$\begin{pmatrix} C_1(t) & C_2(t) \\ \frac{\partial}{\partial t} C_1(t) & \frac{\partial}{\partial t} C_2(t) \end{pmatrix}.$$

Each of the distributions $tr\, C_1(t)$ and $tr\, \frac{\partial}{\partial t} C_2(t)$ can be computed by direct integration of smooth functions. This proves the Lemma.

As mentioned earlier, the Poisson relation follows from these three Lemmas. Thus the proof of the Poisson relation is now complete.

Chapter 6

Applications to Triangles

6.1 The inverse spectral problem

Specializing to the case in which the polygonal domain X is a triangle should make some strong inverse spectral statements possible. Area, A, and perimeter, p, are well known spectral invariants of triangles in particular, and much more complicated regions in general. For triangles, however, just one more piece of data is needed to determine the specific triangle up to rigid motion. For example, if the height is known the triangle can be determined.

Thus one might hope that the lengths of the shorter closed geodesics of a triangle together with area and perimeter would determine the triangle. Provided that the lengths of diffractive geodesics are distinguished from the lengths of reflective geodesics, this is the case. For obtuse triangles the shortest closed geodesic lies along the perpendicular dropped from the obtuse angle to the opposite side. It is a diffractive geodesic with length $2h$, twice the height. For right triangles this same geodesic is the shortest, but it is reflective. There is no diffraction at an angle of $\frac{\pi}{2}$.

The shortest geodesic for acute triangles is again reflective. It is formed by joining the feet of the perpendiculars dropped from each angle. It has length $2h \sin \alpha$, where α is one of the three angles and h is the length of the height dropped from that angle. The next closed geodesic in order of length for an acute triangle is diffractive. It lies along the height dropped from the largest angle.

F.G. Friedlander [3] has determined that it is not possible for a right triangle to be mistaken for an acute triangle given this data. That is, if X is an acute triangle with a given area and perimeter and a given value of $h \sin \alpha$, then there is no right triangle with the same area and perimeter having its shortest height equal to the value $h \sin \alpha$

of the acute triangle.

Consequently the folllowing process shows that the given data determines a triangle. If the shortest closed geodesic is diffractive, the triangle is obtuse. Its height is now known, and so X is determined. If the shortest closed geodesic is reflective and its length is twice the height of a right triangle with the given area and perimeter, then X is that right triangle. If the area, perimeter and the length of the shortest geodesic are not consistent with the area, perimeter, and the shortest height of a right triangle, then the triangle is acute. Its height is half the primitive length of the next closed geodesic in order of length. This determines X.

Much of the data mentioned is available from $tr\,U$. For a triangle with one height h shorter than the others, $tr\,U$ has a singularity of order $-1/2$ at $t = 2h$. This is the content of Theorem 6.17. The result of Guillemin and Melrose [1] for regions with smooth boundary imply that $tr\,U$ of an acute triangle has a singularity of order -1 at $t = 2h\sin\alpha$, the primitive length of the reflected height geodesic. Theorem 6.18 states this.

The missing element is the shortest height of acute isosceles triangles. In this case the diffracted wavefront set of U is tangent to the reflective wavefront set along the shortest height geodesic. This complication means that the method of Theorem 6.17 does not apply directly.

Even without this, a number of partial results can be stated. For example, as a consequence of the reasoning above, Theorem 6.17, and Theorem 6.18, the following theorem holds.

Theorem 6.1 *Obtuse triangles can be distinguished from arbitrary triangles by spectral information. In the class of obtuse triangles, a particular triangle is determined by its spectrum.*

6.2 $tr\,U$ in terms of the Sommerfeld kernel

Let X be a triangle. A height geodesic of X is a closed geodesic beginning at one vertex, traveling along the perpendicular dropped from that vertex to the opposite side, then reflecting back. If one of the height geodesics is shorter than the others, it is of particular interest. Say it has primitive length $2h$. For this geodesic the singularity of $tr\,U$ at $t = 2h$ can be computed by the extension of the method used to prove the Poisson relation. This computation is given in section 3. Note that the angle through which this height passes is either $\frac{\pi}{2}$ or not of the form $\frac{\pi}{N}$.

The computation of singularity at $2h$ involves a series of reductions to operators simpler than $U(t)$. At this point the hypothesis that h is the shortest height is unnessary. Instead assume that the height geodesic dropped from vertex x_0 with primitive length $2h$ is the only closed geodesic of that length. As in the proof of the Poisson relation, $tr\,U$ is treated in three pieces. Take $\phi \in C_0^\infty(R^2)$ to be equal to 1 in an $\epsilon - \delta$ ball around each vertex, $\epsilon \ll d(X)$, $0 < \delta \ll \epsilon$. Require that ϕ is supported within ϵ -neighborhood of each vertex, and that in the ϵ -neighborhood of each particular vertex ϕ is radial in polar coordinates centered at that vertex. As in Chapter 5 , $tr\,U$ is equal to

$$tr\,U(t)[1-\phi] + tr\,U(t-3\epsilon)[\phi]U(3\epsilon)[\phi] + tr\,U(t-3\epsilon)[\phi]U(3\epsilon)[1-\phi].$$

As a consequence of Lemma 5.4 the contribution of $tr\,U(t-3\epsilon)[\phi]U(3\epsilon)[\phi]$ is always smooth. Also, if ψ is smooth on X with boundary and supported away from the height geodesic in question then $tr\,U(t)[1-\phi][\psi]$ and $tr\,U(t-3\epsilon)[\phi]U(3\epsilon)[1-\phi][\psi]$ are also smooth near $t = 2h$. This reduces the problem to $tr\,U(t)[1-\phi][1-\psi]$ and $tr\,U(t-3\epsilon)[\phi]U(3\epsilon)[1-\phi][1-\psi]$ where $1-\psi$ may be taken to have support very near the height geodesic.

The goal is to give an explicit expression in terms of the Sommerfeld kernel which captures the singularity at $t = 2h$. Consider $U(t)$ applied to the initial data which has singularities propagating along the height geodesic. In some sense, under $U(t)$ the singularities of this form experience the Sommerfeld kernel in corner at x_0 just once. Otherwise they are propagated by the operator for free space or the operator for the edge. This is the idea behind the replacement of $U(t)$ by a Sommerfeld kernel.

In the case of $tr\,U(t)[1-\phi][1-\psi]$ this motivating idea applies fairly directly. Let ρ be reflection in R^2 across the edge opposite x_0. Say α is the angle formed by the edges meeting at x_0. Take U_s to be the solution operator for the initial value problem associated with the wave operator on the sector S with angle α. The operator U_s has the form

$$\begin{pmatrix} \frac{\partial F}{\partial t} & F \\ \frac{\partial^2 F}{\partial t^2} & \frac{\partial F}{\partial t} \end{pmatrix}$$

where $F(t-t', z, z')$ is the Sommerfeld kernel for the sector S. The thrust of Lemma 6.3 and Lemma 6.4 is that $tr\,U(t)[1-\phi][1-\psi]$ differs from

$$-tr\ [(U_s(t,\rho(z),z') + U_s(t,z,\rho(z')))[1-\phi][1-\psi]]$$

by a smooth function in a neighborhood of $t = 2h$. Since the formula for U_s is known explicitly, this makes the computations in section 3 possible for $tr\,U(t)[1-\phi][1-\psi]$.

More complicated arguments are needed to reduce the computation of the singularity of $tr\,U(t-3\epsilon)[\phi]U(3\epsilon)[1-\phi][1-\psi]$ to a computation of traces which essentially just involve the Sommerfeld kernel and reflection. These appear in Lemma 6.6, Lemma 6.7 and Lemma 6.8 in this section.

The reduction argument for $U(t)[1-\phi][1-\psi]$ breaks into two parts. First consider $\tilde{\phi} \in C_0^\infty(X)$ supported in a small neighborhood of a point on the height geodesic. The distribution $tr\,U(t)\tilde{\phi}$ is analysed in Lemma 6.3. In Lemma 6.4 the function $\tilde{\phi} \in C_0^\infty(X)$ is even with respect to ρ and supported near the foot of the perpendicular from x_0.

In these lemmas, the interest is in the singularity at $t = 2h$, so $tr\,U(t)\tilde{\phi}$ can be replaced by any distribution which is equal to $tr\,U(t)\tilde{\phi}$ plus a smooth function.

Definition 6.2 *Let u and v be distributions on R_t. Say u* is equivalent to *v if, in some neighborhood of $2h$, $u - v$ is smooth. Denote this relation by $u \approx v$.*

Lemma 6.3 *Let z_0 be a point on the height geodesic from the vertex x_0 to the edge Σ and back. There exists a neighborhood O of z_0 such that for any $\tilde{\phi} \in C_0^\infty(O)$, $tr\,U(t)\tilde{\phi}$ is equivalent to $tr\,\left((-U_s(t,z,\rho(z')) - U_s(t,\rho(z),z'))\,\tilde{\phi}\right)$.*

Proof: Denote by ζ_0 the vector $\frac{x_0-z_0}{\|x_0-z_0\|}$. Let $\tilde{\phi} \in C_0^\infty(X)$ be a cutoff function with support in a small neighborhood of z_0. Construct a pseudodifferential operator A which is a microlocal cutoff near the points $(z_0, \pm\zeta_0)$ in T^*X. Specifically, let $\tilde{\phi}_1 \in C_0^\infty(X)$ be equal to one on $supp\tilde{\phi}$ and have just slightly larger support than $\tilde{\phi}$. Take $a_0(\zeta) \in C^\infty(R^2)$ to be homogeneous of degree zero for large ζ. Require that a_0 is supported in a small conic neighborhood of $\{\zeta_0, -\zeta_0\}$ for large ζ_0, and that $a_0(\zeta_0)$ is equal to $a_0(-\zeta_0)$ which is equal to one. Then A is an operator with symbol $a(z,\zeta) = \tilde{\phi}_1(z)a_0(\zeta)$. Let $[A]$ be the operator on $H_1^0 \oplus L^2$ mapping $g_0 \oplus g_1$ to $Ag_0 \oplus Ag_1$. The distribution $tr\,U(t)[A][\tilde{\phi}]$ is equivalent to $tr\,U(t)[\tilde{\phi}]$.

Now let b be small compared to the distance from z_0 to the boundary of X. If $\tilde{\phi}$ and A are appropriately chosen then $WF\,U(b)[A][\tilde{\phi}]$ has two components. Basically, the singularities selected by A have moved distances $\pm b$ along the height geodesic. Thus it is possible to select functions $f_1, f_2 \in C_0^\infty(X)$ such that $WF\,[f_1]U(b)[A][\tilde{\phi}]$ is contained in a small conic neighborhood of $(z_0 + b\zeta_0, \zeta_0)$, and $WF\,[f_2]U(b)[A][\tilde{\phi}]$ is contained in a small conic neighborhood of $(z_0 - b, -\zeta_0)$, and $WF\,[1-f_1-f_2]U(b)[A][\tilde{\phi}]$ is empty. Then $tr\,U(t)[A][\tilde{\phi}]$ is equivalent to

$$tr\left(U(t-b)[f_1]U(b)[A][\tilde{\phi}]\right) + tr\left(U(t-b)[f_2]U(b)[A][\tilde{\phi}]\right).$$

Denote $[f_i]U(b)[A][\tilde{\phi}]$ by F_i. The term $U(t-b)F_1$ ultimately contributes the expression $-U_s(t,\rho(z),z')$ in the statement of the lemma. To show that $tr\,U(t-b)F_1$ is equivalent to $tr(-U_s(t,\rho(z),z')$ note first that $U(t)F_1$ differs from $U_s(t)[f_1]U_s(b)[A][\tilde{\phi}]$ by a smoothing operator for t near h. Let $c \in R$ be such that $c+b$ is greater than $|z_0 - x_0|$ but $c+b-|z_0-x_0|$ is much smaller than $d(X)$. Loosely, this means that at the time $c+b$ a geodesic starting near (z_0,ζ_0) has gone through the corner at x_0 but has not reached other corners.

There exists a pseudodifferential operator B such that $WF(U(c)F_1 - [B]U(c)F_1)$ does not intersect a conic neighborhood of $\{(z,-\zeta_0) : z \text{ is on the height geodesic}\}$. Then $tr\,U(t-b)F_1$ is equivalent to $tr\,U(t-b-c)[B]U(c)F_1$. Further $tr\,U(t-b-c)[B]U(c)F_1$ is equivalent to

$$tr(-U_s(t-b-c,\rho(z),z')[B]U_s(c)[f_1]U_s(b)[A][\tilde{\phi}])$$

which is equivalent to $-tr\,U_s(t,\rho(z),z')[\tilde{\phi}]$.

The argument showing that $tr\,U(t-b)F_2$ is equivalent to $-tr\,U_s(t,z,\rho(z'))$ is similar, and is omitted.

The set O can be taken to be any set contained in the region where the $\tilde{\phi}$ in the preceeding argument is equal to one. This completes the proof.

The set O above depends uniformly on z_0. Suppose z_0 is at least a distance d_0 from x_0 and from Σ, $d_0 \ll d(X)$. Then the ball of radius $\frac{d_0}{8}$ about z_0 is an acceptable O.

The next case to consider is that where z_0 is the foot of the perpendicular from x_0 to Σ. This is treated by reflecting $U(t)$ across Σ. That is, let Y be the polygon produced by reflecting X across Σ and taking the union of X and the reflection. Let $U_Y(t)$ be the operator $U(t)$ for Y to distinguish it from $U(t)$ for X. Consider the operator

$$V(t,z,z') = U_Y(t,z,z') - U_Y(t,z,\rho(z')).$$

By the uniqueness of $U(t)$, if z and z' are in X then

$$V(t,z,z') = U(t,z,z').$$

Now let $\tilde{\phi} \in C_0^\infty(R^2)$ be supported near $z_0 \in \Sigma$. If $\tilde{\phi} = \tilde{\phi} \circ \rho$, then $tr\,U(t)[\tilde{\phi}]$ is equal to $\frac{1}{2}tr\,V(t)[\tilde{\phi}]$. This is the key to the proof of

Lemma 6.4 *Let $z_0 \in \Sigma$ be the foot of the height geodesic from x_0 to Σ. Then there exists a neighborhood O of z_0 in R^2 such that for any $\tilde{\phi} \in C_0^\infty(O)$ as above, $tr\,U(t)[\tilde{\phi}]$ is equivalent to*

$$-\frac{1}{2}tr\left((U_s(t,z,\rho(z')) + U_s(t,\rho(z),z'))[\tilde{\phi}]\right).$$

Proof: This is the same as the claim that $tr\, V(t)[\tilde{\phi}]$ is equivalent to

$$-tr\left((U_s(t,z,\rho(z')) + U_s(t,\rho(z),z'))[\tilde{\phi}]\right).$$

This can be verified by a straightforward application of the ideas in Lemma 6.3

Given these two lemmas, $tr\, U(t)[1-\phi][1-\psi]$ can be computed from the Sommerfeld kernel. The support of $(1-\psi)$ can be restricted to a very narrow band around the height geodesic. There is a partition of unity $\{\phi_i\}_{i=1}^{k}$ on $supp(1-\phi)(1-\psi)$ with ϕ_i, $i > 1$ satisfying the hypothesis of Lemma 6.3 and ϕ_1 satisfying the hypothesis of Lemma 6.4. Thus $tr\, U(t)[1-\phi][1-\psi]$ is equivalent to

$$-\frac{1}{2}tr\left(U_s(t,z,\rho(z')) + U_s(t,\rho(z),z')\right)[\phi_1] - \sum_{i=2}^{k} tr\left(U_s(t,z,\rho(z')) + U_s(t,\rho(z),z')\right)[\phi_i]$$

The next objective is to produce an similar expression for

$$tr\, U(t-3\epsilon)[\phi]U(3\epsilon)[1-\phi][1-\psi].$$

The contribution to the singularity of this term comes from a rather small region in X. Let $f \in C_0^\infty(X)$ be equal to one on a neighborhood of

$$Q = \{z : z \text{ is on the height geodesic and } 2\epsilon - \delta < |z - x_0| < 4\epsilon + \delta\}$$

for some $0 < \delta \ll \epsilon$. Suppose f is also supported in a small neighborhood of Q. Then

$$tr\, U(t-3\epsilon)[\phi]U(3\epsilon)[1-f][1-\phi][1-\psi]$$

is smooth near $t = 2h$ because the operator is smoothing. No singularities survive all those cutoffs. Thus it suffices to consider $tr\, U(t-3\epsilon)[\phi]U(3\epsilon)[\phi_i]$ where $[\phi_i]_{i=1}^{3}$ is a partition of unity for a small neighborhood of Q.

In particular, it is convenient to take ϕ_1, ϕ_2, ϕ_3 as follows. The support of ϕ_1 is contained in a small neighborhood of

$$Q_1 = \{z \in Q : ||x_0 - z|| < 2.5 + \delta\}.$$

The support of ϕ_2 is restricted to a neighborhood of

$$Q_2 = \{z \in Q : 2.5 \leq ||x_0 - z|| \leq 3.5\}.$$

Finally the support of ϕ_3 is contained in a small neighborhood of

$$Q_3 = \{z \in Q : 3.5 < ||x_0 - z||\}.$$

The virtue of this partition is that $[\phi]U(3\epsilon)[\phi_i]$ is relatively simple for each ϕ_i. Consider initial data $g_0 \oplus g_1$ and the function $g(t) = \pi_1 U(t)(g_0 \oplus g_1)$. Points in $WFg_0 \cup WFg_1$ which are in a small conic neighborhood of $Q \times \{\zeta_0, -\zeta_0\}$ give rise to singularities of $g(t)$ which have already cleared the corner at $t = 3\epsilon$. Thus $U(t - 3\epsilon)$ acts on $[\phi]U(3\epsilon)[\phi_1]$ essentially as the free space operator with reflection across Σ on one space variable. The singularities of g due to points of $WFg_0 \cup WFg_1$ in a small conic neighborhood of $Q_2 \times \{\zeta_0, -\zeta_0\}$ are still in the region where $\phi$is equal to one at $t = 3\epsilon$. This indicates that $U(t - 3\epsilon)[\phi]U(3\epsilon)[\phi_2]$ is essentially $U(t)[\phi_2]$. Finally, singularities arising from points in a conic neighborhood of $Q_3 \times \{\zeta_0, -\zeta_0\}$ have not yet reached the corner at time 3ϵ. This means that $[\phi]U(3\epsilon)[\phi_3]$ is essentially the free space kernel sandwiched between $[\phi]$ and $[\phi_3]$. Lemma 6.6, Lemma 6.7 and Lemma 6.8 make these observations rigorous.

Some notation is needed for the free space kernel with reflection in various variables.

Definition 6.5 *Let $V(t, z, z')$ be the standard solution operator for the initial value problem associated to the wave operator in R^2. Then*

$$V_R(t, z, z') := V(t, z, \rho(z'))$$

and

$$_RV(t, z, z') := V(t, \rho(z), z').$$

For each of these Lemmas, let $\phi_0 \in C_0^\infty(X)$ be equal to one on

$$\{z \in Q : \delta < ||x_0 - z|| < \epsilon\}$$

and let $\phi_0' \in C_0^\infty(X)$ be equal to one on the support of ϕ_0. Let $\{\phi_1, \phi_2, \phi_3\}$ be a partition of unity as described above.

Lemma 6.6 *Let $\phi_1' \in C_0^\infty(X)$ be equal to 1 on the support of ϕ_1. The distribution $tr\, U(t - 3\epsilon)[\phi]U(3\epsilon)[\phi_1]$ is equivalent to*

$$-tr\, {}_RV(t - 3\epsilon)[\phi][\phi_0]V_R(-t + 3\epsilon)[\phi_1']U_s(t, \rho(z), z')[\phi_1]$$

Proof: This is simply a matter of chasing equivalences. In short,

$$tr\, U(t - 3\epsilon)[\phi]U(3\epsilon)[\phi_1]$$

$$\approx -tr\,_R V(t-3\epsilon)[\phi][\phi_0]U(3\epsilon)[\phi_1]$$

$$\approx -tr\,_R V(t-3\epsilon)[\phi][\phi_0']V_R(-t+3\epsilon)_R V(t-3\epsilon)[\phi_0]U(3\epsilon)[\phi_1]$$

$$\approx -tr\,_R V(t-3\epsilon)[\phi][\phi_0]V_R(-t+3\epsilon)[\phi_1']U_s(t,\rho(z),z')[\phi_1]$$

Lemma 6.7 *The distribution*

$$tr\, U(t-3\epsilon)[\phi]U(3\epsilon)[\phi_2]$$

is equivalent to

$$-tr\, U_s(t,\rho(z),z')[\phi_2].$$

Proof: This is a corollary of the proof of Lemma 6.3.

Lemma 6.8 *The distribution*

$$tr\, U(t-3\epsilon)[\phi]U(3\epsilon)[\phi_3]$$

is equivalent to

$$-tr\, V(3\epsilon)[\phi_3]V(-3\epsilon)[\phi_0']U(t,\rho(z),z')[\phi][\phi_0]$$

Proof: Again, one just chases equivalences, starting with

$$tr\, U(t-3\epsilon)[\phi]U(3\epsilon)[\phi_3] \approx tr\, U(t-3\epsilon)[\phi][\phi_0]V(3\epsilon)[\phi_3].$$

Then let $\phi_3' \in C_0^\infty(X)$ be equal to 1 on a neighborhood of $\{z \in Q : 3\epsilon - 2\delta < ||z - x_0|| < 4\epsilon + \delta\}$, and also on a neighborhood of $supp\phi_3$. Then the equivalences

$$tr\, U(t-3\epsilon)[\phi]V(3\epsilon)[\phi_3]V(-3\epsilon)V(3\epsilon)[\phi_3']$$

$$\approx tr\, V(3\epsilon)[\phi_3]V(-3\epsilon)V(3\epsilon)[\phi_3']U(t-3\epsilon)[\phi][\phi_0]$$

$$\approx -tr\, V(3\epsilon)[\phi_3]V(-3\epsilon)[\phi_0']U_s(t,\rho(z),z')[\phi][\phi_0]$$

are straightforward.

The advantage of the equivalent distribution over the original in these Lemmas is that, for these purposes, ${}_RV(t-3\epsilon)[\phi][\phi_0']V_R(-t+3\epsilon)$ and $V(3\epsilon)[\phi_3]V(-3\epsilon)$ are pseododifferential operators with simple principal symbol. The operator $U_s(t,\rho(z),z')$ is a matrix of conormal distributions on the regions in question. Thus the calculus of conormal distributions can be brought to bear on the problem of computing the singularity of $tr\,U$ at $t=2h$.

6.3 The singularity arising from the shortest height

In this section attention is restricted to triangles X having one height shorter than the others. The equivalent expressions for $tr\,U$ derived in the last section are used to compute the leading term of the singularity of $tr\,U$ at $t=2h$. Here $2h$ is the length of the shortest height geodesic. There are two cases in which X has a unique shortest height. Either the angle where the shortest height originates is $\frac{\pi}{2}$ or it is some α which is not of the form $\frac{\pi}{N}$. In fact, α is greater than $\frac{\pi}{3}$ because the unique shortest height must be dropped from the unique largest angle of the triangle X. Attention is restricted to the shortest height geodesic in order to ensure that $U_s(t)$ will have a convenient expression along the geodesic near $t=2h$. Here U_s is the solution operator for the sector S with angle α, the largest angle of X.

Let x_0 be the vertex of the largest angle. Denote by Σ the opposite edge. Once again, ρ is the reflection across the line containing Σ.

If the largest angle α is equal to $\frac{\pi}{2}$, then $U_s(t,\rho(z),z')$ is a sum of rotations and reflections of the standard solution operator in free space. From the explicit expressions for $U_s(t,\rho(z),z')$ the leading term of the singularity at $t=2h$ of $tr\,U(t,\rho(z),z')\tilde{\phi}$ is computed by the method of stationary phase. This is Lemma 6.11.

The situation is more subtle if α is not equal to $\frac{\pi}{2}$. In this case, position the triangle in R^2 with x_0 at the origin. Let one edge containing x_0 lie along the positive x-axis and the other along the line containing $(r\cos\alpha, r\sin\alpha)$. The shortest height geodesic forms an angle β with the x-axis. The fact that α is the unique largest angle of X guarantees that if θ and θ' are sufficiently near β then $|\theta\pm\theta'+2n\alpha|$ is not equal to π. Consider points z'' and $z'\in S$ in a small neighborhood of a point $z_0=(r_0\cos\beta, r_0\sin\beta)$ on the height geodesic. Then $\rho(z'')$ is equal to $(r\cos\theta, r\sin\theta)$ with θ near β and $r+r'$ near $2h$. Further, $|\theta\pm\theta'+2n\alpha|$ is not equal to π. Consider the Sommerfeld kernel $F(t,z,z')$ in a neighborhood of the point $(t,z,z')=(2h,\rho(z''),z')$. Recall that in such a neighborhood F can be written as the sum of a smooth function and a function χf_2. Here χ is the characteristic function for the set $\{(t,r,r') : t > r+r'\}$. Thus near $(2h,z'',z')$ the entries in the matrix $U_s(t, \rho(z''), z')$ are conormal distributions with kernels available explicitly from χf_2. Lemma 6.10 gives the highest order term of the singularity at $t = 2h$ of $tr\, U_s(t, \rho(z), z')\tilde{\phi}$ where $\tilde{\phi}$ is supported near the height geodesic. This is computed by applying the method of stationary phase to $\frac{\partial}{\partial t}F(t, \rho(z), z')\tilde{\phi}$, a conormal distribution on the region in question.

The rest of the effort in the section goes into showing that the terms

$$tr\, {}_RV(t-3\epsilon)[\phi][\phi_0']V_R(-t+3\epsilon)[\phi_1']U_s(t, \rho(z), z')[\phi_1]$$

and

$$tr\, V(3\epsilon)[\phi_3]V(-3\epsilon)[\phi_0']U_s(t, \rho(z), z')[\phi][\phi_0]$$

from section 6.2 have singular parts at $t = 2h$ much like those of $tr\, U_s(t, \rho(z), z')[\phi_1]$ and $tr\, U_s(t, \rho(z), z')[\phi_1\phi_0]$ respectively. In the regions under consideration, the operators ${}_RV(t-3\epsilon)[\phi\phi_0]V_R(-t+3\epsilon)$ and $V(3\epsilon)[\phi_3]V(-3\epsilon)$ are pseododifferential operators. Their principal symbols can be computed from Lemma 6.12. For the purposes of the trace computation, the principal parts of the two operators act like cutoff functions on the space variables. The actual computations of the leading term at $t = 2h$ for the cases $\alpha \neq \frac{\pi}{N}$ and $\alpha = \frac{\pi}{2}$ conclude the section.

The Sommerfeld kernel for the angle $\alpha \neq \frac{\pi}{2}$ has, as mentioned, singular part $\chi f_2(t, r, r', \theta, \theta')$ in a neighborhood of $(2h, \rho(z), z)$ if z is on the shortest height geodesic. Consequently the term $\frac{\partial}{\partial t}F(t, r, r', \theta, \theta')$ in such a neighborhood can be written

$$(2\pi)^{-1}\int e^{i\tau(t-r-r')}f_2(r+r', r, r', \theta, \theta')\,d\tau + \text{smoother terms}$$

Here

$$f_2(t, r, r', \theta, \theta') = (2\pi)^{-1}\int_0^\infty (L(\eta, \theta-\theta') - L(\eta, \theta+\theta'))(t^2 - r^2 - r'^2 - 2rr'\cosh\eta)_+^{-1/2}\,d\eta$$

with $L(\eta, \theta)$ as defined in section 2.1. The value of f_2 at $(2h, \rho(z), z)$ becomes important in later computations.

Lemma 6.9 *The value of* $f_2(2h, r, 2h-r, \beta, \beta)$ *is*

$$(2\pi)^{-1}(L(0,0) - L(0,2\beta))(2r(2h-r))^{-1/2}\lim_{a\to+1}\int_0^\infty (a-\cosh\eta)_+^{-1/2}\,d\eta.$$

The simplest term to compute in $tr\, U(t)$ at $t = 2h$ is the term $tr\, U(t)[\tilde{\phi}]$ where $\tilde{\phi}$ is smooth in a small neighborhood of a point on the shortest geodesic. Take X to be positioned as above. Then it makes sense to speak of $U_s(t, \rho(z), z')[\tilde{\phi}]$. The reflection ρ is, in these coordinates, reflection across the line $Re(ze^{-i\beta}) = 2h$.

Lemma 6.10 *The distributions* $tr\, U_s(t,\rho(z),z')[\tilde{\phi}]$ *and* $tr\, U_s(t,z,\rho(z'))$ *have a singularity at* $t = 2h$ *with leading term*

$$C \int \tilde{\phi}(r,\beta)\, dr\ h^{-1/2}(L(0,0) - L(0,2\beta))(t-2h)_+^{-1/2}$$

where $C \neq 0$ *is the same in both cases, independent of the geometry and independent of* $\tilde{\phi}$.

Proof: Consider first

$$\int e^{i\tau(t-r-r')} f_2(r+r', r, r', \theta, \theta')\tilde{\phi}(r',\theta')\, d\tau$$

restricted to $z = \rho(z')$. Let $(R(r,\theta), \Theta(r,\theta))$ be the point $\rho(r\cos\theta, r\sin\theta)$ in polar coordinates. Then

$$\int U_s(t,\rho(z),z)\tilde{\phi}(z)\, dz =$$

$$= \int\int e^{i\tau(t-R(r,\theta)-r)} f_2(r+R(r,\theta), R(r,\theta), r, \Theta(r,\theta), \theta)\tilde{\phi}(r,\theta)\, d\tau\, d\theta\, r dr.$$

Application of the method of stationary phase in θ results in the top order term

$$\int_0^\infty \left(\frac{2\pi}{\tau}\right)^{1/2} \left(\frac{2hr}{2h-r}\right)^{-1/2} f_2(2h, r, 2h-r, \beta, \beta)\tilde{\phi}(r,\beta) e^{-i\frac{\pi}{4}} e^{i\tau(t-2h)} d\tau\, r dr +$$

$$\int_0^\infty \left(\frac{2\pi}{\tau}\right)^{1/2} \left(\frac{2hr}{2h-r}\right)^{-1/2} f_2(2h, r, 2h-r, \beta, \beta)\tilde{\phi}(r,\beta) e^{i\frac{\pi}{4}} e^{-i\tau(t-2h)} d\tau\, r dr.$$

Substituting the value of f_2 from Lemma 6.9 proves the result for $U_s(t,\rho(z),z)$. The computation for $U(t,z,\rho(z))$ is the same.

The result in Lemma 6.11 is analogous and is proved in an analogous way. The Sommerfeld kernel for the sector with angle $\frac{\pi}{2}$ is

$$(2\pi)^{-1} \sum_{n=0,1} \left(\left(t^2 - r^2 - r'^2 + 2rr'\cos(\theta - \theta' - n\pi)\right)_+^{-1/2} - \right.$$

$$\left. \left(t^2 - r^2 - r'^2 + 2rr'\cos(\theta + \theta' - n\pi)\right)_+^{-1/2}\right)$$

The term producing the singularities corresponding to the height geodesic is

$$(2\pi)^{-1}(t^2 - r^2 - r'^2 + 2rr'\cos(\theta - \theta' - \pi))_+^{-1/2} = (2\pi)^{-1}(t^2 - ||z + z'||^2)_+^{-1/2}.$$

This follows from the fact that $||\rho(z) + z|| = 2h$ for any z along the height geodesic. This operator can be expressed as the sum of two Fourier integrals,

$A'_0 + B'_0$ where

$$A_n = \frac{(2\pi)^{-2}}{2}\int e^{i(<z-z',\xi>+t|\xi|)}(i|\xi|)^n\,d\xi$$

and

$$B_n = \frac{(2\pi)^{-2}}{2}\int e^{i(<z-z',\xi>-t|\xi|)}(i|\xi|)^n\,d\xi,$$

$n = -1, 0, 1$. By A'_0 denote A_0 with z' replaced by $-z'$. The operator B'_0 is B_0 with z' replaced by $-z'$.

Writing ξ in polar coordinates (τ, ψ) and applying the method of stationary phase to ψ yields

$$\frac{(2\pi)^{-2}}{2}\int \tau(\frac{2\pi}{\tau})^{1/2}|z + z'|^{-1/2}\left(e^{i\tau(t+|z+z'|)-\frac{i\pi}{4}} + e^{i\tau(t-|z-z'|)+\frac{i\pi}{4}}\right)\,d\tau$$

for the principal part of A'_0. The complex conjugate of this is the principal part of B'_0.

In this form A'_0 and B'_0 can be plugged into the method of Lemma 6.10. The result is

Lemma 6.11 *Let $\tilde{\phi}$ be as in Lemma 6.10, but let X be a right triangle and $\alpha = \frac{\pi}{2}$. Then $tr\,U(t, \rho(z), z')[\tilde{\phi}]$ and $tr\,U(t, z, \rho(z'))[\tilde{\phi}]$ have singularities at $t = 2h$. In both cases the leading term of the singularity is*

$$K\int \tilde{\phi}(r, \beta)\,dr\,((t - 2h + i0)^{-1} + (t - 2h - i0)^{-1}).$$

The constant K is equal for both cases. It is independent of $\tilde{\phi}$ and of the geometry of the situation.

The last two Lemmas amount to computations of the leading term in the singularities of $tr\,U(t)[1 - \phi][1 - \psi]$ and $tr\,U(t - 3\epsilon)[\phi]U(3\epsilon)[\phi_2]$ in the notation of section 6.2. The terms $tr\,U(t - 3\epsilon)[\phi]U(3\epsilon)[\phi_1]$ and $tr\,U(t - 3\epsilon)[\phi]U(3\epsilon)[\phi_3]$ necessitate a closer examination of the operator $[\psi_2]V(s)[\psi]V(-s)[\psi_1]$. Here V is the free space solution operator, as in the previous section. The variable s is taken to be in some range $[s_0, s_1]$

with $s_0 > 0$. Further, the support of $\psi \in C_0^\infty(R^2)$ is contained in a set with diameter less than s_0. Similarly, both ψ_1 and ψ_2 are supported in the same set O, also having diameter less than s_0. This guarantees that, under geodesic flow, points in O are mapped outside O by time $2s$.

Let A_n and B_n be as above. For $n \in \{-1, 0, 1\}$, the operators $A_n(t)$ and $B_n(t)$ are Fourier integral operators whose canonical relations $\chi_{A(t)}$ and $\chi_{B(t)}$ are homogeneous canonical graphs. For $s > 0$ the operator $V(s)$ is given by

$$\begin{pmatrix} (A_0 + B_0)(s) & (A_{-1} - B_{-1})(s) \\ (A_1 - B_1)(s) & (A_0 + B_0)(s) \end{pmatrix}.$$

The operator $V(-s)$ is

$$\begin{pmatrix} (A_0 + B_0)(s) & (-A_{-1} + B_{-1})(s) \\ (-A_1 + B_1)(s) & (A_0 + B_0)(s) \end{pmatrix}.$$

Egorov's Theorem, with parameters, cf Hörmander [1] applies to

$$[\psi_2]V(s)[\psi]V(-s)[\psi_1]$$

to give

Lemma 6.12 *The family of operators*

$$[\psi_2]V(s)[\psi]V(-s)[\psi_1]$$

is equal to a matrix family of pseudodifferential operators

$$[\psi_2]\begin{bmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{bmatrix}[\psi_1] + \begin{bmatrix} \text{a matrix of} \\ \text{smoothing operators} \end{bmatrix}$$

such that
i. The operators p_{ij} are smoothly parameterized by s.
ii. The principal symbol of p_{11} and p_{22} is

$$\frac{1}{2}(\psi \circ \chi_{A(s)} + \psi \circ \chi_{B(s)})$$

in rectangular coordinates $(z, \zeta) \in T^*(R^2)$.
iii. The principal symbol of p_{12} is

$$\frac{a(\zeta)}{2i|\zeta|}(\psi \circ \chi_{B(s)} - \psi \circ \chi_{A(s)}).$$

iv. The principal symbol of p_{21} is

$$a(\zeta)\frac{i|\zeta|}{2}(\psi \circ \chi_{B(s)} - \psi \circ \chi_{A(s)}).$$

Here $a(\zeta) \in C^\infty(R^2)$ is equal to zero in a neighborhood of $\zeta = 0$, and equal to one outside another neighborhood of $\zeta = 0$.

Extend the pseudodifferntial operators p_{ij} above to pseudodifferential operators on distributions of $R^2 \times R^2$ in the following manner. Let $a'(\zeta, \zeta') \in C^\infty(R^2 \times R^2)$ be homogeneous of degree zero for large (ζ, ζ'). Suppose $a'(\zeta, \zeta')$ is equal to zero in a conic neighborhood of $O' = \{(\zeta, \zeta') : \zeta = 0, \ |\zeta'| = 1\}$. Stipulate further that for large (ζ, ζ') the function a' is equal to one outside a small conic neighborhood of O'. Then multiplication of the symbol of p_{ij} by $a'(\zeta, \zeta')$ results in a symbol in (z, z', ζ, ζ'). Using this and mimicking the computation in Lemma 6.10 yields

Lemma 6.13 *Suppose the angle α is not equal to $\frac{\pi}{2}$. The leading term in the singularity at $t = 2h$ of*

$$tr\,_R V(t - 3\epsilon)[\phi\phi_0']V_R(-t + 3\epsilon)[\phi_1']U_s(t, \rho(z), z)[\phi_1]$$

is

$$C \int \psi_1(r)\, dr \ h^{-1/2}(t - 2h)_+^{-1/2}.$$

Here C is equal to the C in Lemma 6.10. The function $\psi_1(r)$ is equal to $\phi_1(r, \beta)$ for $r \in (2\epsilon + \delta, \infty)$ and equal to $\phi(3\epsilon - r, \beta)$ for $r \in (0, 2\epsilon + \delta)$.

Proof: Consider the matrix of pseudodifferntial operators given by

$$[\phi_1']_R V(t - 3\epsilon)[\phi\phi_0']V_R(-t + 3\epsilon)[\phi_1']$$

extended to act on $R^2 \times R^2$ by a'. Denote this by

$$P(t) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

The operator

$$P(t)U_s(t, \rho(z), z')[\phi_1]$$

differs by a smoothing operator from

$$_RV(t-3\epsilon)[\phi\phi_0']V_R(-t+3\epsilon)[\phi_1']U_s(t,\rho(z),z)[\phi_1]$$

because $|\zeta|$ is equal to $|\zeta'|$ in $WF\,U_s$.

This reduces the proof of to the computation of the leading terms of

$$tr\,(P_{11}(t)\frac{\partial}{\partial t}F(t,\rho(z),z')\phi_1+P_{12}(t)\frac{\partial^2}{\partial t^2}F(t,\rho(z),z')\phi_1)$$

and

$$tr\,(P_{21}(t)F(t,\rho(z),z')\phi_1+P_{22}(t)\frac{\partial}{\partial t}F(t,\rho(z),z')\phi_1$$

at $t=2h$. Here F is the Sommerfeld kernel for the sector S.

First compute $tr\,P_{11}(t)\frac{\partial}{\partial t}F(t,\rho(z),z')\phi_1$. This is equivalent to

$$tr\,P_{11}(t)\int e^{i\tau(t-R(r,\theta)-r')}f_2(R(r,\theta)+r',R,r',\Theta(r,\theta),\theta')\,d\tau.$$

Note that the restriction of the principal symbol of P_{11} to the conormal bundle associated with $\phi_1\frac{\partial}{\partial t}F(t,\rho(z),z')\phi_1$ is just $\phi\phi_0'\circ\chi_R$. Here χ_R is the relation

$$\{(z,\zeta,z',\zeta'):\text{ a geodesic of length } t \text{ joins } (z,\zeta) \text{ and } (\rho(z'),\rho(\zeta'))\}.$$

Restricting the variables to $r=r'$, $\theta=\theta'$ results in the expression

$$\int\frac{1}{2}\phi\phi_0'\circ\sigma(r,\theta)e^{i\tau(t-R(r,\theta)-r)}f_2(R+r,R,r,\Theta,\theta)\,d\tau\,d\theta\,rdr \tag{6.1}$$

for the leading term of the singularity of the trace. The map σ takes a point (r,θ) to $(R(r+t,\theta),\Theta(r+t,\theta)$. As in Lemma 6.10 applications of the method of stationary phase in the θ variable results in the leading term of the singularity. The leading term is equal to

$$\frac{1}{2}C\int\psi_1(r)\,dr\,h^{-1/2}(L(0,0)-L(0,2\beta))(t-2h)_+^{-1/2}.$$

The computation of $tr\,P_{22}(t)\frac{\partial}{\partial t}F(t,\rho(z),z')$ is identical to this. For the computations concerning $tr\,P_{12}(t)\frac{\partial^2}{\partial t^2}F(t,\rho(z),z')$ simply note that the restriction of the principal symbol of P_{12} to the conormal bundle associated with $\phi_1'\frac{\partial^2}{\partial t^2}F(t,\rho(z),z')\phi_1$ is $\frac{1}{2i\tau}\phi\phi_0'\circ\chi_R$. This reduces computation to the same integral 6.1.

Likewise, the restriction of the principal symbol of P_{21} to the conormal bundle associated with the conormal distribution $\phi_1'F(t,\rho(z),z')\phi_1$ is $\frac{i\tau}{2}\phi\phi v_0'\circ\chi_R$. Thus the leading term of $tr\, P_{21}F(t,\rho(z),z')\phi_1$ is, once again, given by 6.1. This completes the proof.

An entirely analogous result is true for the right triangle. The proof is virtually the same and so is omitted.

Lemma 6.14 *The leading term in the singularity at a $t = 2h$ of*

$$tr\,_R V(t-3\epsilon)[\phi\phi_0']V_R(-t+3\epsilon)[\phi_1']U_s(t,\rho(z),z')[\phi_1]$$

is

$$K\int \psi_1(r)\,dr\,((t-2h+i0)^{-1}+(t-2h-i0)^{-1}).$$

The computations for $tr\,U(t-3\epsilon)[\phi]U(3\epsilon)[\phi_3]$ are related to the ones above, though somewhat simpler. The statements of the results follow. The proofs are omitted.

Lemma 6.15 *If the angle of the sector S is not equal ot $\frac{\pi}{2}$, then the leading term in the singularity at $t = 2h$ of*

$$tr\,V(3\epsilon)[\phi_3]V(-3\epsilon)[\phi_0']U_s(t,\rho(z),z')[\phi\phi_0]$$

is

$$C\int \psi_3(r)\,dr\,h^{-1/2}(L(0,0)-L(0,2\beta))(t-2h)_+^{-1/2}.$$

The constant C is the same as in the previos Lemmas. The function $\psi_3(r)$ is equal to $\phi_3(r+3\epsilon,\beta)$ for $r\in(0,\epsilon/2+\delta)$, and equal to $\phi(r,\beta)$ for $r\in(\epsilon/2,\infty)$.

Lemma 6.16 *If α is equal to $\frac{\pi}{2}$, the singularity at $t = 2h$ of*

$$tr\,V(3\epsilon)[\phi_3]V(-3\epsilon)[\phi_0']U_s(t,\rho(z),z')[\phi\phi_0]$$

is

$$K\int \psi_3(r)\,dr\,((t-2h+i0)^{-1}+(t-2h-i0)^{-1}).$$

Note that these Lemmas can all be assembled to give the leading term of $tr\,U$ at $t = 2h$ for the triangle X. Recall that $tr\,U$ was equivalent to

$$tr\,(U(t)[1-\phi][1-\psi]+\sum_{i=1}^{3}U(t-3\epsilon)[\phi]U(3\epsilon)[\phi_i]$$

in the notation of section 6.2.

Theorem 6.17 *Suppose X is not a right triangle, and suppose X has one height h shorter than the others. Then the singularity of $tr\, U$ at $t = 2h$ has leading term*

$$-2C\, h^{1/2}\,(L(0,0) - L(0,2\beta))(t-2h)_+^{-1/2}.$$

Here C is a constant independent of X and $L(\eta,\theta)$ is the function from section 3.1.

Proof: Lemma 6.3, Lemma 6.4 and Lemma 6.10 imply the equality of the leading term of $tr\, U(t)[1-\phi][1-\psi]$ and of

$$-2C \int_0^h (1-\phi)(r,\beta)\, dr\;\; h^{-1/2}\,(L(0,0) - L(0,2\beta))(t-2h)_+^{-1/2}.$$

The equaliaty

$$\int \phi_2(r,\beta)\, dr + \int \psi_1(r)\, dr + \int \psi_3\, dr = 2\int \phi(r)\, dr$$

can be deduced from the definitions of ψ_1 and ψ_3 in Lemma 6.13 and Lemma 6.15. But $tr\, \sum_{i=1}^3 U(t-3\epsilon)[\phi]U(3\epsilon)[\phi_i]$ has the leading term

$$-C\int \phi_2(r,\beta) + \psi_1(r) + \psi_3(r)\, dr\;\; h^{-1/2}\,(L(0,0) + L(0,2\beta))(t-2h)_+^{-1/2}.$$

Combining this with the leading term of $tr\, U(t)[1-\phi][1-\psi]$ completes the proof of the Theorem.

The same manipulations of the terms contributing to the leading singularity of $tr\, U(t)$ proves the result when X is the right triangle.

Theorem 6.18 *Suppose X is a right triangle and h is the height from the right angle to the hypotenuse. The singularity of $tr\, U$ at $t = 2h$ has the leading term*

$$-2hK\,((t-2h+i0)^{-1} + (t-2h-i0)^{-1}).$$

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