Generalized Approach to Minimal Uncertainty Products

by

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Abstract

A general technique to construct quantum states that saturate uncertainty products using variational methods is developed. Such a method allows one to numerically compute uncertainties in cases where the Robertson-Schrodinger (RS) uncertainty approach fails. To demonstrate the limitations of the RS approach, the $(\Delta x^2)(\Delta p)$ relation is examined using both the variational and direct method.

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Chapter 1

Introduction

Minimal uncertainty quantum states have a wide variety of applications from optical physics to high energy physics. Coherent states, which are characterized by resembling the dynamics of classical physics at the quantum level, are states that minimize the position-momentum uncertainty. The uncertainty principle is one of the most important features of quantum mechanics; it states that there is a fundamental limit to the precision two incompatible observables[1] can be measured. The position-momentum Heisenberg's uncertainty principle is given by:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{1.1}$$

In this paper, we will approach the problem of finding wavefunctions that saturates general uncertainty relations by two methods: Using the Robertson-Schrodinger uncertainty relation constraint, discussed in chapter 2; and the variational approach, which will be discussed in chapter 3.

We will start by analyzing the position-momentum uncertainty using both methods. We will construct wavefunctions that saturate this uncertainty and show that both solutions agree with one another. Then in chapter 4, we will construct wavefunctions for a more complicated uncertainty, the $\Delta x^2 \Delta p$ relation, using the two different approaches. And chapter 5 will discuss the implications of both solutions and why

[1] A and B are incompatible observables if $[A, B] \neq 0$

they are incompatible.

Chapter 2 will introduce a constraint for a state that saturates uncertainty, which is given by:

$$(A - \langle A \rangle)|\psi\rangle = i\frac{\Delta A}{\Delta B}(B - \langle B \rangle)|\psi\rangle \tag{1.2}$$

It is found by using the RS-uncertainty relation. In Chapter 4, through the variational method, we will get another constraint which will be given by:

$$\left[\left(\frac{A - \langle A \rangle}{\Delta A}\right)^2 + \left(\frac{B - \langle B \rangle}{\Delta B}\right)^2\right]|\psi\rangle = 2|\psi\rangle \tag{1.3}$$

And later it will be shown that one is the *square root* of the other, in a similar way that the Klein-Gordon Equation is to the Dirac Equation.

Chapter 2

Robertson-Schrodinger Uncertainty Relation

2.1 Introduction

The Robertson-Schordinger uncertainty relation is a slight generalization to what is known as the "generalized uncertainty principle" which states:

$$(\Delta A)(\Delta B) \geq \frac{\langle [A,B] \rangle}{2i} \tag{2.1}$$

This chapter will focus on generalizing this expression and finding a constraint for the wavefunction that would make 2.1 an equality.

The RS relation considers an extra term which is dropped in 2.1 due to its positive-definite nature. The extra term is known in statistical terms as the covariance of two random variables. The RS relation is given by:

$$(\Delta A)(\Delta B) \geq \sqrt{\left(\frac{\langle \{A,B\} \rangle - 2\langle A \rangle \langle B \rangle}{2}\right)^2 + \left(\frac{\langle [A,B] \rangle}{2i}\right)^2} \tag{2.2}$$

and it will be proved in the next section. The term involving anti-commutators can be also written as $\{A - \langle A \rangle, B - \langle B \rangle\}/2$

2.2 Derivation

Starting from any two observables A and B, their uncertainties with respect to the state $|\psi\rangle$ are defined as:

$$\begin{aligned}(\Delta A)^2 &\equiv \langle\psi|(A-\langle\psi|A|\psi\rangle)^2|\psi\rangle = \langle\psi|A^2|\psi\rangle - (\langle\psi|A|\psi\rangle)^2 \\ (\Delta B)^2 &\equiv \langle\psi|(B-\langle\psi|B|\psi\rangle)^2|\psi\rangle = \langle\psi|B^2|\psi\rangle - (\langle\psi|B|\psi\rangle)^2\end{aligned} \tag{2.3}$$

For convenience we define the states $|X\rangle$ and $|Y\rangle$ as:

$$\begin{aligned}|X\rangle &\equiv (A-\langle A\rangle)|\psi\rangle \\ |Y\rangle &\equiv (B-\langle B\rangle)|\psi\rangle\end{aligned} \tag{2.4}$$

and the uncertainty product takes the simple form:

$$\begin{aligned}(\Delta A)^2(\Delta B)^2 &= \langle X|X\rangle\langle Y|Y\rangle \\ &= |\langle X|Y\rangle|^2 + \langle X|Y^\perp\rangle\langle Y|Y\rangle\end{aligned} \tag{2.5}$$

where $|Y^\perp\rangle = |X\rangle - \frac{\langle Y|X\rangle}{\langle Y|Y\rangle}|Y\rangle$ is the orthonormal projection of the state $|Y\rangle$. Since the second term is positive definite, it can be dropped from the equality leading to the Cauchy-Schwarz inequality. Furthermore, by expanding the remaining term into its real and imaginary part, 2.5 can be written as:

$$(\Delta A)^2(\Delta B)^2 \geq \left(\frac{\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle}{2}\right)^2 + \left(\frac{\langle[A,B]\rangle}{2i}\right)^2 \tag{2.6}$$

This is known as the Roberson-Schrodinger uncertainty relation. Equality holds if and only if $|Y^\perp\rangle = 0$, that is:

$$|X\rangle = \frac{\langle Y|X\rangle}{\langle Y|Y\rangle}|Y\rangle \tag{2.7}$$

It is clear that the Robertson-Schrodinger relation 2.6 reduces to the familiar uncertainty relation involving commutators when dropping the first term [1]

2.3 The Direct Method

Equation 2.7 gives a condition that makes 2.6 into an equality. However, we would like to construct states that minimize uncertainty. The direct method provides a constraint the wavefunction must satisfy in order to minimize an arbitrary uncertainty relation.

This section will discuss the formulation to saturate uncertainties given any two observables. According to this method, the uncertainty relation must reduce to $(\Delta A)(\Delta B) = \langle [A, B]\rangle/2i$. There are two equations that make this true:

$$\langle\{X, Y\}\rangle = 0 \tag{2.8}$$

$$X|\psi\rangle = \frac{\langle YX\rangle}{\langle Y^2\rangle} Y|\psi\rangle \tag{2.9}$$

where $X \equiv A - \langle A\rangle$ and $Y \equiv B - \langle B\rangle$. With the operators defined as such, the uncertainty could be written as $\langle X^2\rangle\langle Y^2\rangle = (\frac{\langle [X,Y]\rangle}{2i})^2$

By acting with $\langle\psi|X$ on 2.9 and using 2.8, it can be found that $\langle YX\rangle = \pm i\sqrt{\langle X^2\rangle\langle Y^2\rangle}$. The sign ambiguity is due to the square root, and it will make sense shortly.

Now we could replace the $\langle YX\rangle$ expressing into 2.9 and find that the wavefunction must satisfy:

$$(A - \langle A\rangle)|\psi\rangle = \pm i\frac{\Delta A}{\Delta B}(B - \langle B\rangle)|\psi\rangle \tag{2.10}$$

Note that the $\pm$ makes the equation symmetric. This must be true since it must be invariant under $A \leftrightarrow B$, i.e the uncertainty should be the same regardless of the order in which we choose the operators, since $\Delta A\Delta B = \Delta B\Delta A$. If we can find a wavefunction that satisfies 2.10, it will saturate the uncertainty product.

[1] This can always be done since it is the square of a real number by construction i.e. $Re[\langle X|Y\rangle] = \frac{\langle X|Y\rangle + \langle Y|X\rangle}{2} = \frac{\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle}{2}$

Wavefunctions of this form are called Intelligent States and are characterized by being the generalization of squeezed coherent states. For instance, if we consider the harmonic oscillator, this can easily be seen by using 2.10 with $A = \hat{x}$ and $B = \hat{p}$ and rearranging. The resulting equation takes the form[2] $(\hat{x} + i\lambda\hat{p})\psi = \mu\psi$ where λ and μ are constants involving uncertainties and expectations. This is exactly like the eigenvalue equation for a coherent state $a|\alpha\rangle = \alpha|\alpha\rangle$ up to a constant factor. Hence, 2.10 gives a simple differential equation the wavefunction must satisfy in order to minimize an uncertainty product.

2.3.1 Position-Momentum Minimal Uncertainty

In order to saturate the position-momentum uncertainty, we use 2.10 with $A = \hat{x}$ and $B = \hat{p}$ and taking the negative solution. It is possible to always make the momentum and position expectation values of a wavefunction vanish – the position expectation value $\langle x\rangle$ by making a shift and the momentum expectation $\langle p\rangle$ by introducing a phase factor of the form $e^{-i\langle p\rangle/\hbar}$. Hence, a wavefunction satisfying 2.10 can be written as:

$$\hat{x}|\psi\rangle = -i\frac{\Delta x}{\Delta p}\hat{p}|\psi\rangle \tag{2.11}$$

Note that the constant pre-factor in 2.11 multiplying the momentum are integrals over all space with respect to the wavefunction, making 2.11 an integro-differential equation. In the case of position and momentum, all the integrals can be absorbed in the constant factor $\frac{\Delta x}{\Delta p}$, which can be evaluated after the form of the solution is found. The fact that this can be done will become important when we start to consider more complicated uncertainty relations.

To solve the equation, we project 2.11 into position basis: $\hat{x} \to x$ and $\hat{p} \to \frac{\hbar}{i}\frac{d}{dx}$ and reduce the expression to a simple first-order differential equation with Gaussian form solutions. This is the standard traveling wave packet solution that saturates the position-momentum uncertainty relation $\Delta x \Delta p = \frac{\hbar}{2}$

[2] $(\hat{x} - i\lambda\hat{p})\psi = \mu\psi$ leads to non-normalizable solutions, so they are ignored

The most general solution is given by:

$$\psi(x) = \mathcal{N} e^{i\frac{\langle p \rangle x}{\hbar}} e^{-\frac{\Delta p}{\Delta x}(x-\langle x \rangle)/2\hbar} \tag{2.12}$$

which was obtained by shifting back the position and momentum expectation values.

Our current strategy for constructing uncertainty saturating states is by making most of the terms in 2.5 vanish. This method supplies us with equations that the wavefunction must satisfy. Soon we will discover this is not an effective method for constructing minimal uncertainty states, for observables with arbitrary commutation relations: the second term in 2.5 does not vanish.

Chapter 3

Variational Method Approach

3.1 Introduction

In this chapter we will review a general formulation for minimizing uncertainties using the variational method develop by R. Jackiw [1]. The general idea is to make an arbitrary variation to the uncertainty with respect to the wavefunction and find stationary points.

3.2 Variational Method

Given an unnormalized square-integrable wavefunction, the expectation values are given by $\langle A \rangle \equiv \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}$. This normalization condition will become important later when we discuss non-normalizable solutions for minimal uncertainty states. Starting with definitions given in 2.3, consider the variation $\langle\psi| \rightarrow \langle\psi| + \langle\delta\psi|$ in $(\Delta A)^2$.

$$\begin{aligned}
\delta(\Delta A)^2 &= \delta(\langle A^2\rangle - \langle A\rangle^2) \\
&= \delta\langle A^2\rangle - 2\langle A\rangle\delta\langle A^2\rangle \\
&= \frac{\langle\delta\psi|A^2|\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\psi|A^2|\psi\rangle}{\langle\psi|\psi\rangle^2}\langle\delta\psi|\psi\rangle - 2\langle A\rangle\left(\frac{\langle\delta\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle^2}\langle\delta\psi|\psi\rangle\right) \\
&= \langle\delta\psi|\left(\frac{A^2|\psi\rangle - 2\langle A\rangle A|\psi\rangle}{\langle\psi|\psi\rangle}\right) - (\langle A^2\rangle - 2\langle A\rangle^2)\frac{\langle\delta\psi|\psi\rangle}{\langle\psi|\psi\rangle} \\
&= \langle\delta\psi|\left[\frac{(A-\langle A\rangle)^2|\psi\rangle}{\langle\psi|\psi\rangle}\right] - \left[\langle A^2\rangle - \langle A\rangle^2\right]\frac{\langle\delta\psi|\psi\rangle}{\langle\psi|\psi\rangle} \qquad (3.1)
\end{aligned}$$

where the last line is a completion of the square. A similar calculation for evaluating $\delta(\Delta B)^2$ can be done, then combining with the stationary solution condition $\delta[(\Delta A)^2(\Delta B)^2 = 0$, the equation reduces to:

$$\langle\delta\psi|\left[\frac{(A-\langle\ A\rangle)^2}{2(\Delta A)^2} + \frac{(B-\langle B\rangle)^2}{2(\Delta B)^2}\right]|\psi\rangle = \langle\delta\psi|\psi\rangle \qquad (3.2)$$

Note that the $\langle\psi|\psi\rangle$ quantities simplify as long as the wavefunction is square-integrable. An undefined normalization condition would make 3.2 no longer valid.

Since 3.2 must vanish for any variation $\langle\delta\psi|$, it can be concluded that:

$$\left[\left(\frac{A-\langle\ A\rangle}{\Delta A}\right)^2 + \left(\frac{B-\langle B\rangle}{\Delta B}\right)^2\right]|\psi\rangle = 2|\psi\rangle \qquad (3.3)$$

We have obtained an Euler-Lagrange-type differential equation for a state $|\psi\rangle$ with minimal uncertainty.

3.2.1 Remarks

The normalization convention for expected values $\langle A\rangle = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}$ is crucial to obtain the term $2|\psi\rangle$ in 3.3, it ensures that the wavefunction is normalizable . An alternative derivation using Lagrange multipliers and $\langle A\rangle = \langle\psi|A|\psi\rangle$ is done in [1]

Since both the Direct and Variational methods saturate uncertainties, they should be equivalent. In this subsection we will show the equivalence of the two methods, as well as the larger range of validity for the variational method.

For simplicity define $X \equiv A - \langle A \rangle$ and $Y \equiv B - \langle B \rangle$, since $\langle X^2 \rangle = (\Delta A)^2$ and $\langle Y^2 \rangle = (\Delta B)^2$. With the substitutions equations 2.10 and 3.3 can be written as:

$$Y|\psi\rangle - i\sqrt{\frac{\langle Y^2\rangle}{\langle X^2\rangle}}X|\psi\rangle = 0 \tag{3.4}$$

$$\left[\frac{X^2}{\langle X^2\rangle} + \frac{Y^2}{\langle Y^2\rangle}\right]|\psi\rangle = 2|\psi\rangle \tag{3.5}$$

By multiplying 3.4 by $\frac{Y}{\langle Y^2\rangle} + i\frac{X}{\langle X^2\rangle}$ and simplifying:

$$\left[\frac{X^2}{\langle X^2\rangle} + \frac{Y^2}{\langle Y^2\rangle} + i\frac{[X,Y]}{\sqrt{\langle X^2\rangle\langle Y^2\rangle}}\right]|\psi\rangle = 0 \tag{3.6}$$

By construction, the state satisfying 3.4 minimizes the Roberson-Schrodinger uncertainty relation, turning the inequality into an equality with vanishing anti-commutator expectation. Hence, $\sqrt{\langle X^2\rangle\langle Y^2\rangle} = \frac{\langle [X,Y]\rangle}{2i}$, and 3.6 becomes:

$$\left[\frac{X^2}{\langle X^2\rangle} + \frac{Y^2}{\langle Y^2\rangle}\right]|\psi\rangle = 2\frac{[X,Y]}{\langle [X,Y]\rangle}|\psi\rangle \tag{3.7}$$

which is of the same form as 3.5. Furthermore, if we assume the commutator $[X,Y]$ is a c-number, $[X,Y] = \langle [X,Y]\rangle$ for a state normalized to unity, and 3.7 becomes identical to 3.4. Note that the constant commutator assumption was necessary to show that both methods are equivalent. This demonstrates that for an arbitrary commutation relation, the direct method might not minimize uncertainty. We will show this through and example in Chapter 4.

3.3 Variational Approach to Position Momentum Uncertainty

We will approach the problem of finding states with minimal position-momentum uncertainty using the variational method. By letting $A = \hat{x}$ and $B = \hat{p}$, equation 3.3

becomes:

$$\left[\left(\frac{\hat{x}-\langle x\rangle}{\Delta x}\right)^2+\left(\frac{\hat{p}-\langle p\rangle}{\Delta p}\right)^2\right]|\psi\rangle = 2|\psi\rangle \tag{3.8}$$

Now shifting the expectations by applying the transformations $\psi(x) \to e^{-i\langle p\rangle x/\hbar}\psi(x+\langle x\rangle)$. Furthermore, by implementing the canonical transformations:

$$\hat{x} \;\to\; \sqrt{\frac{\Delta x}{\Delta p}}\hat{x} \tag{3.9}$$

$$\hat{p} \;\to\; \sqrt{\frac{\Delta p}{\Delta x}}\hat{p} \tag{3.10}$$

equation 3.8 simplifies to:

$$\left[\frac{1}{2}\hat{p}^2+\frac{1}{2}\hat{x}^2\right]|\psi\rangle = \Delta x\Delta p|\psi\rangle \tag{3.11}$$

which is like a time-independent Schrodinger with harmonic oscillator potential and energy eigenvalue $\Delta x\Delta p$. Hence, stationary solutions[1]of the position-momentum uncertainty are bound states of the harmonic oscillator with $\omega = 1$.

Since our interest is to find states that saturate the uncertainty $\Delta x\Delta p$, we choose the ground state energy of the harmonic oscillator. Then $\Delta x\Delta p = \hbar/2$ and the solution to 3.11 is:

$$\psi(x) = \mathcal{N}e^{-i\frac{x^2}{2\hbar}} \tag{3.12}$$

Now we are ready to find the general wavefunction, which is the solution to 3.8, by changing back to the original variables[2] and making the transformation $\psi(x) \to e^{i\langle p\rangle x/\hbar}\psi(x-\langle x\rangle)$. The state with minimal uncertainty is given by:

$$\psi(x) = \mathcal{N}e^{i\langle p\rangle x/\hbar}e^{-i\frac{\Delta p}{\Delta x}(x-\langle x\rangle)^2/2\hbar} \tag{3.13}$$

[1]States for which $(\Delta x)^2(\Delta p)^2$ has critical points, since we are solving for $\delta[(\Delta x)^2(\Delta p)^2]=0$

[2]Physical position x and momentum p, since x and p in 3.11 are canonical variables

Chapter 4

$\Delta x^2 \Delta p$ uncertainty

4.1 Introduction

The application of both the direct and variational approach will be considered. We will find a state that saturates the uncertainty $\Delta x^2 \Delta p$ using both approaches.

4.2 Direct Approach to $\Delta x^2 \Delta p$

As we found out in chapter 2, the direct approach to saturating uncertainties is performed by expanding $\Delta A \Delta B$ in terms of expectations and having most of the terms vanish by considering a certain constraint on the wavefunction. It was shown that the condition to saturate uncertainty between observables A and B is given by:

$$(B - \langle B \rangle)|\psi\rangle = i\frac{\Delta B}{\Delta A}\,(A - \langle A \rangle)|\psi\rangle \tag{4.1}$$

where the positive overall factor, $\frac{\Delta B}{\Delta A}$, was chosen for convenience.

In order to find a state that saturates the uncertainty $\Delta x^2 \Delta p$, 4.1 must be solved with the substitution $A = \hat{x}^2$ and $B = \hat{p}$. To make the differential equation simpler, the transformation $|\psi\rangle \rightarrow e^{-i\frac{\langle p \rangle}{\hbar}}|\psi\rangle$ is used to vanish the momentum expectation value. Equation 4.1 can then be written in position basis as:

$$\psi'(x) = \left(\frac{1}{\hbar}\frac{\Delta p}{\Delta x^2}(x^2 - \langle x^2\rangle)\right)\psi(x) \tag{4.2}$$

A wavefunction $\psi(x)$ satisfying 4.2 will saturate $\Delta x^2 \Delta p$. To solve it, standard differential equation techniques can be employed. The solution is given by:

$$\psi(x) = \mathcal{N}\exp\left[\frac{1}{\hbar}\frac{\Delta p}{\Delta x^2}\left(\frac{x^3}{3} - \langle x^2\rangle x\right)\right] \tag{4.3}$$

However, this solution asymptotically has the form e^{x^3}, which becomes non-normalizable since $\psi(x) \to \infty$ as $x \to \infty$. This is an unphysical state since it is not square-integrable.

We could consider the negative solution of 4.1, but the same problem arises, this time with $\psi(x) \to \infty$ as $x \to -\infty$. This poses a deep problem in the direct method, since no physical solution exist to a problem that clearly must have a physical interpretation.

Another method to fix the normalizability condition is to consider the negative solution for $x > 0$ and positive for $x < 0$, and have it be continuous at $x = 0$. More explicitly:

$$\psi'(x) = \begin{cases} \frac{1}{\hbar}\frac{\Delta p}{\Delta x^2}(x^2 - \langle x^2\rangle)\psi(x) & x \leq 0 \\ -\frac{1}{\hbar}\frac{\Delta p}{\Delta x^2}(x^2 - \langle x^2\rangle)\psi(x) & x > 0 \end{cases} \tag{4.4}$$

Then the solution is given by:

$$\psi(x) = \mathcal{N}\begin{cases} \exp[\frac{1}{\hbar}\frac{\Delta p}{\Delta x^2}(x^3/3 - \langle x^2\rangle x)] & x \leq 0 \\ \exp[-\frac{1}{\hbar}\frac{\Delta p}{\Delta x^2}(x^3/3 - \langle x^2\rangle x)] & x > 0 \end{cases} \tag{4.5}$$

A plot is shown in figure 4-1. Although the equation is normalizable and continuous, it is not differentiable[1]. Hence, the momentum expectation value diverges.

[1] $\psi'(0^-) = -\frac{1}{\hbar}\frac{\Delta p}{\Delta x^2}\langle x^2\rangle$ and $\psi'(0^+) = \frac{1}{\hbar}\frac{\Delta p}{\Delta x^2}\langle x^2\rangle$. Differentiability condition is satisfied when momentum uncertainty vanishes. If so, position uncertainty would be infinite

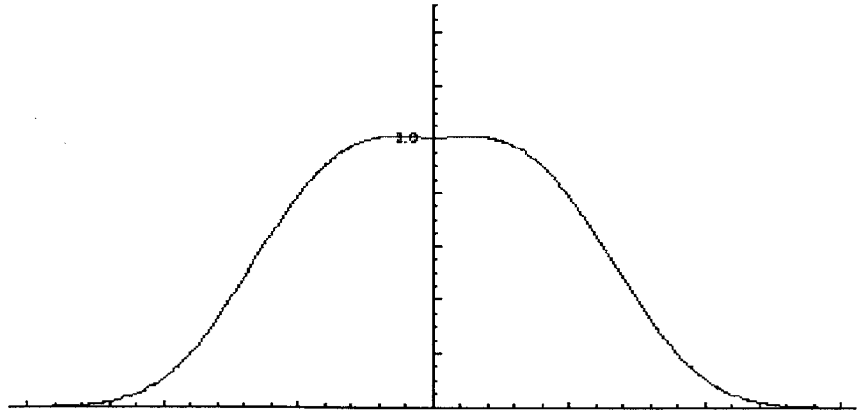


Figure 4-1: Plot illustrating graphical form of 4.5. The explicit functional form is given by $\exp[\pm(ax^3 + abx)]$ with $a = \frac{1}{\hbar}\frac{\Delta p}{\Delta x^2} = 0.654$ $b = \langle x^2 \rangle = 0.059$

It can be concluded that this is not a valid solution since $(\Delta x^2)(\Delta p)$ is infinite. The direct method does not seem to give any physical solutions to the saturation problem, instead, it gives infinite momentum uncertainty, addressing a severe problem in its formulation. First of all, the saturation condition 4.1 was obtained by forcing the uncertainty relation be proportional to the commutator of both observables. This condition does not guarantee the uncertainty would be minimized[2]. Second, a minimal uncertainty state in this formulation must have a vanishing anti-commutator expectation value. It could easily be the case that there are no normalizable solutions to 4.1 that satisfy $\langle\{x^2, p\}\rangle$ and $\Delta x^2 \Delta p = |\langle[x^2, p]\rangle|/2$ simultaneously. Therefore, no normalizable solutions satisfying the RS constraints exist.

4.3 Variational Approach

Now we will solve the same problem using the variational approach. It was shown in Chapter 3 that the wavefunction must satisfy:

$$\left[\left(\frac{A - \langle\, A\rangle}{\Delta A}\right)^2 + \left(\frac{B - \langle B\rangle}{\Delta B}\right)^2\right] |\psi\rangle = 2\psi\rangle \tag{4.6}$$

[2]It is only true if the commutator is a constant. Since the minimum possible value of an expression involving sums of state depend expectations and a constant term is the constant term by itself.

in order to saturate uncertainty.

Letting $A = \hat{p}$, $B = \hat{x}^2$ and making the transformation $\psi(x) \rightarrow e^{-i\frac{\langle p\rangle}{\hbar}}\psi(x)$, equation 4.6 becomes:

$$\left[\frac{\hat{p}^2}{2(\Delta p)^2} + \frac{x^4 - 2\langle x^2\rangle x^2 + \langle x^2\rangle^2}{2(\Delta x^2)^2}\right]\psi(x) = \psi(x)$$

then rearranging:

$$\left[\frac{\hbar^2}{2}\frac{d^2}{dx^2} - \frac{\langle x^2\rangle(\Delta p)^2}{(\Delta x^2)^2}x^2 + \frac{(\Delta p)^2}{2(\Delta x^2)^2}x^4\right]\psi(x) = (\Delta p)^2\left(1 - \frac{\langle x^2\rangle^2}{2(\Delta x^2)^2}\right)\psi(x) \quad (4.7)$$

This equation has the time-independent schrodinger's equation form. Therefore, the problem of solving for a minimal uncertainty states reduces to finding a bound state for a system with an effective potential given by:

$$V_{eff}(x) = -\omega x^2 + \lambda x^4 \quad (4.8)$$

where ω, λ are constant defined by $\omega = \langle x^2\rangle(\Delta p)^2/(\Delta x^2)^2$ and $\lambda = (\Delta p)^2/2(\Delta x^2)^2$

Furthermore, by defining $E = (\Delta p)^2\left(1 - \frac{\langle x^2\rangle^2}{2(\Delta x^2)^2}\right)$, equation 4.7 can be written as:

$$\left[\frac{\hat{p}^2}{2} + V_{eff}(x)\right]\psi(x) = E\ \psi(x) \quad (4.9)$$

The solution of this equation would saturate uncertainty. Unlike the direct method, this approach already seems to have normalizable solutions. Hence, the normalizability of the wavefunction will not be an issue. However, it is not clear what values should be assigned to ω and λ given our current Hamiltonian[3]. The parameters are dependent on the final state, since they involve expectation values.

The goal is to minimize the uncertainty $(\Delta x^2)(\Delta p)$, so we consider:

$$(\Delta x^2)^2(\Delta p)^2 = \frac{(E + \omega^2/4\lambda)}{2\lambda} \quad (4.10)$$

[3]The value of E is uniquely defined by finding normalizable bound states for the system given the effective potential

which is an expression that relates the variables from the effective potential to the actual uncertainty. A proper pair of ω and λ must be found in order to make the uncertainty the minimum possible. Note that E is constrained by the effective potential, since it must be the energy of the ground state.

If we could obtain the energy spectrum, $E(\omega, \lambda)$, of the family of effective potentials, $V_{eff}(x, \omega, \lambda)$, the problem would be much easier, we could take partial derivatives of 4.10 with respect to ω and λ and find a grobal minimum. Then these values, ω_0 and λ_0 would be used to solve equation 4.9. However, there is no known methods for finding explicit closed-form solutions to schrodinger's equation with potentials of the form given by 4.8. The problem must be solved numerically.

The procedure used to find an appropriate pair (ω_0, λ_0) is as follows:

1. Start with arbitrary ω and λ
2. Find numerically the bound state energy, $E(\omega, \lambda)$
3. Evaluate $(\Delta x^2)^2(\Delta p)^2 = \frac{(E+\omega^2/4\lambda)}{2\lambda}$
4. Find numerically the bound state energy $E(\omega + \Delta\omega, \lambda + \Delta\lambda)$
5. Reevaluate $(\Delta x^2)^2(\Delta p)^2 = \frac{(E+\omega^2/4\lambda)}{2\lambda}$ with $(\omega + \Delta\omega, \lambda + \Delta\lambda)$
6. Compare both uncertainty values, choose the minimum uncertainty and repeat steps 4 through 6
7. Stop when the global minimum is found

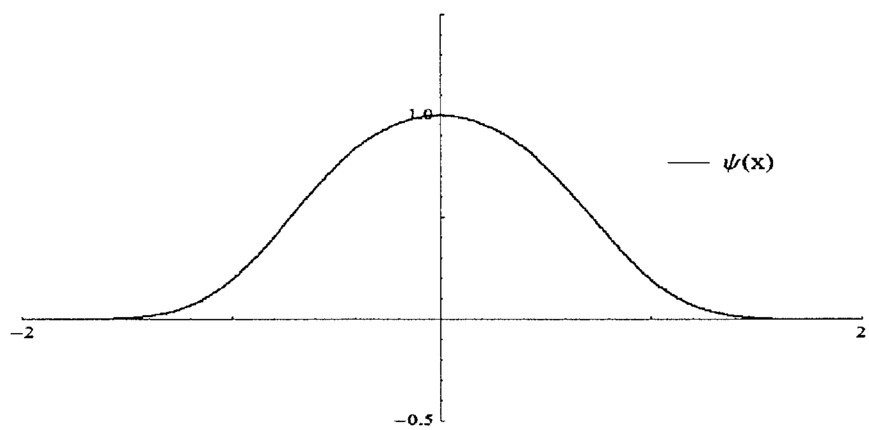
This procedure systematically searches the $(\Delta x^2)^2(\Delta p)^2$ global minimum in (ω, λ)-space.

When ω_0 and λ_0 are found, they must match with the respective combination of expectations they are defined to be; it is a test that proves the solution is self-consistent. Table 4.1 shows several (ω, λ) pairs for which bound states and expectations were calculated.

It was found that $\omega_0 \approx 3.1086$ and $\lambda \approx 9.4919$. Given those values, the state that saturates uncertainty is plotted in below.

Table 4.1: Effective Potential Parameters

E	λ	ω	$(\Delta x^2)^2(\Delta p)^2 = (E+\omega^2/4\lambda)^2/2\lambda$	$2\lambda\langle x^2\rangle = \omega_0$
0.30381	5.4074	4.3122	0.1251750	2.7519
0.28010	6.2222	4.8942	0.1240580	3.0596
0.86148	9.1526	3.5028	0.0782234	3.1383
0.903606	9.4855	3.3982	0.0769156	3.1709
0.944825	9.4868	3.1500	0.0766948	3.1168
0.948184	9.4879	3.1300	0.0766883	3.1128
0.953128	9.4883	3.1000	0.0766862	3.1065
0.985648	9.4883	2.9000	0.0768008	3.0648
0.969443	9.4883	3.0000	0.0767170	3.0855
0.953218	9.4896	3.1000	0.0766827	3.1067
0.953152	9.4896	3.1004	0.0766826	3.1069
0.953038	9.4896	3.1011	0.0766827	3.1070
0.951969	9.4919	3.1086	0.0766763	3.1089



And the effective potential and energy level are plotted in figure 4-2. The ground state solution resembles the harmonic oscillator ground state, since the saturation problem is similar[4]. However, due to the two wells on the effective potential, the wavefunction is more spread out than the harmonic oscillator ground state.

[4]Instead of finding the momentum position uncertainty, we are looking for the momentum-variantion of position uncertainty

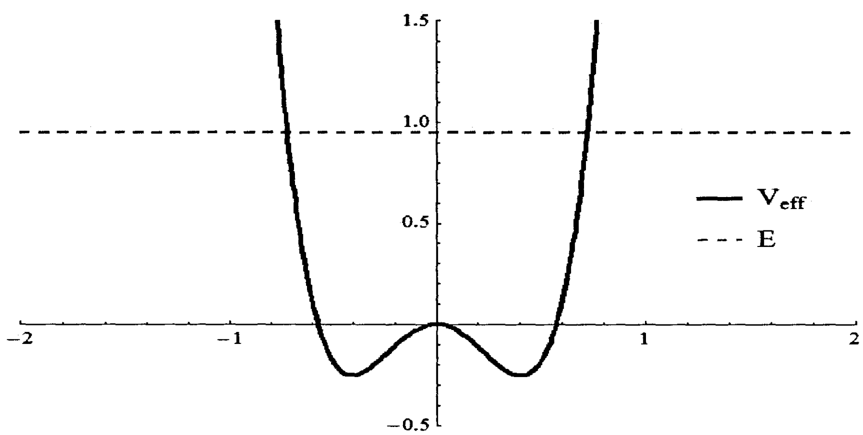


Figure 4-2: Effective potential and energy level for the ground state wavefunction shown in 4.3

Chapter 5

Concluding Remarks

We approached the problem of finding a minimal uncertainty state for $(\Delta x^2)(\Delta p)$ using two different techniques: the direct method and variational method. It was shown that the direct method fails to provide with a physical solution, due to its restrictions on the commutator relation conditions. On the other hand, the variational method presents no problem on finding a normalizable and differentiable wavefunction through numerical calculations.

The reason the direct method fails on the $(\Delta x^2)(\Delta p)$ uncertainty, but not in the $(\Delta x)(\Delta p)$ case, is because the commutator is not constant[1]. We showed in chapter 4 that the two methods are equivalent if the pair of observables, A and B, have a constant commutator relation. If $\langle [A, B] \rangle$ is not a constant, it does not necessarily imply the minimum uncertainty is reached when the remaining terms of the RS-relation vanish. However, the minimum is reached for a state that minimizes the whole $(\Delta A)(\Delta B)$ expansion given by 2.5. The direct method provides no systematic procedure to minimize the expression in this case, while the variational method does by finding stationary points of the uncertainty with respect to the wavefunction.

[1] $[\hat{x}^2, \hat{p}] = 2i\hbar\hat{x}$

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