Definable obstruction theory

Tomohiro Kawakami[1] and Ikumitsu Nagasaki[2]
[1]Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan
[2]Department of Mathematics, Kyoto Prefectural University of Medicine, 13 Nishi-Takatsukaso-Cho, Taishogun Kita-ku, Kyoto 603-8334, Japan

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Abstract

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field R. In this paper, we consider an obstruction theory in the definable category of $\mathcal{N}$.

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1 . Introduction.

Obstruction theory addresses several types of problems(see chap. 7 [2]). Let (X, A) be a CW pair and Y a topological space. One of these problems is Extension Problem.

Problem 1.1. *Suppose that $f : A \to Y$ is a continuous map. When does f extend to all of X?*

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure of a real closed field R. General references on o-minimal structures are [3], [5], see also [9]. Examples and constructions of them can be seen in [4], [6], [7].

In this paper, we consider an obstruction theory in the definable category of $\mathcal{N}$. Everything is considered in $\mathcal{N}$, a definable map is assumed to be continuous and $I = \{x \in R | 0 \le x \le 1\}$.

Theorem 1.2. *Let (X, A) be a relative definable CW complex, $n \ge 1$, and Y a definably connected n-simple definable set. Let $g : X_n \to Y$ be a definable map.*

(1) There exists a cellular cocycle $\theta(g) \in C^{n+1}(X, A, \pi_n(Y))$ which vanishes if and only if g extend to a definable map $X_{n+1} \to Y$.

(2) The cohomology class $[\theta(g)] \in H^{n+1}(X, A, \pi_n(Y))$ vanishes if and only if the restriction $g|X_{n-1} : X_{n-1} \to Y$ extend to a definable map $X_{n+1} \to Y$.

2 . Preliminaries.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f $(\subset X \times Y \subset R^n \times R^m)$ is a definable set. A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $h : Y \to X$ such that $f \circ h = id_Y, h \circ f = id_X$. A definable subset X of R^n is *definably compact* if for every definable map $f : (a, b)_R \to X$, there exist the limits $\lim_{x\to a+0} f(x), \lim_{x\to b-0} f(x)$ in X, where $(a,b)_R = \{x \in R | a < x < b\}, -\infty \le a < b \le \infty$. A definable subset X of R^n is definably compact if and only if X is closed and bounded ([8]). Note that if X is a definably compact definable set and $f : X \to Y$ is a definable map, then $f(X)$ is definably compact.

If R is the field $\mathbb{R}$ of real numbers, then for any definable subset X of $\mathbb{R}^n$, X is compact if and only if it is definably compact. In general, a definably compact definable set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \le x \le 1\}$ is definably compact but not compact.

Recall existence of definable quotient and properties of dimensions of definable sets.

Theorem 2.1. *(Existence of definable quotient (e.g. 10. 2.14 [3])). If X is a definable set and A is a definably compact definable subset of X, then the set obtained by collapsing A to a point exists a definable set.*

Proposition 2.2 (e.g. 4.1.3 [3])**.** *(1) If $X \subset Y \subset R^n$, then $\dim X \le \dim Y \le n$.*

(2) If $X \subset R^n, Y \subset R^m$ are definable sets and there is a definable bijection between X and Y, then $\dim X = \dim Y$.

Let $(X,A),(Y,B)$ be two pairs of definable sets. Two definable maps $f,h : (X,A) \to (Y,B)$ is *definably homotopic relative to A* if there exists a definable map $H : (X \times I, A \times I) \to (Y,B)$ such that $H(x,0) = f(x)$, $H(x,1) = g(x)$ for all $x \in X$ and $H(x,t) = f(x), (x,t) \in A \times I$. The *o-minimal homotopy set $[(X,A),(Y,B)]$ of (X,A) and (Y,B)* is the set of homotopy classes of definable maps from (X,A) to (Y,B). If $A = \emptyset, B = \emptyset$, then we simply write $[X,Y]$ instead of $[(X,A),(Y,B)]$.

Let $D^n = \{(x_1,\dots,x_n) \in R^n | x_1^2 + \cdots + x_n^2 \le 1\}, S^{n-1} = \{(x_1,\dots,x_n) \in R^n | x_1^2 + \cdots + x_n^2 = 1\}$. Then D^n is the closed unit disk of R^n and S^{n-1} is the unit sphere of R^n.

We now define relative CW complexs in the definable category. To reserve definablity, we consider the case where finitely many cells attached.

Definition 2.3. Let X be a definable set and A a definable closed subset of X. We say that *X is obtained from A by attaching n-cells $\{e_i^n\}_{i=1}^{k_n}$* if the following four conditions satisfy.

(1) For each i, e_i^n is a definable subset of X, called an *n-cell*.

(2) $X = A \cup \cup_{i=1}^{k_n} e_i^n$.

(3) Letting ∂e_i^n denote the intersection of e_i^n and A, $e_i^n - \partial e_i^n$ is disjoint from $e_j^n - \partial e_j^n$ for $i \ne j$.

(4) For each i, there exists a surjective definable map $\phi_i^n : (D^n, S^{n-1}) \to (e_i^n, \partial e_i^n)$, called the *characteristic map* of e_i^n, such that the restriction of ϕ_i of the interior Int D^n of D^n is a definable homeomorphism onto $e_i^n - \partial e_i^n$. The restriction of the characteristic map of S^{n-1} is the *attaching map* of e_i^n.

Definition 2.4. A *relative defianble CW complex* (X,A) is a definable set X, a definable closed set A and a sequence of definable closed subset X_n, $n = -1,0,1,2,\dots$ called the *relative n-skeleton* such that

(1) $X_{-1} = A$ and X_n is obtained from X_{n-1} by attaching n-cells.

(2) $X = \cup_{i=-1}^{\dim X} X_i$.

The smallest n such that $X = X_n$ is called the *dimension* $\dim(X,A)$ of (X,A). If A is a definable CW complex, we say that (X,A) is a *definable CW pair*. If $A = \emptyset$, then X is called a definable *CW complex*, and X_n is called the *n-skeleton* of X.

Remark that in Definition 2.4, the maximum dimension of attaching cells to A does not exceed $\dim X$ and $\dim A \le \dim X$ because Proposition 2.2.

Let Y be a definable set and $y_0 \in Y$. The *o-minimal homotopy group of dimension n*, $n \ge 1$ (see [1]) is the set $\pi_n(Y,y_0) = [(I^n, \partial I^n),(Y,y_0)] = [(S^n,x_0),(Y,y_0)]$, where ∂I^n denote the boundary of I^n and $x_0 = (0,,\dots,0,1)$. We define $\pi_0(Y,y_0)$ as the set of definably connected components of Y.

A definable set Y is *definably arcwise connected* if for every two points $x,y \in Y$, there exists a definable map $f : I \to Y$ such that $x = f(0)$ and $y = f(1)$. Note that Y is definably connected if and only if it is definably arcwise connected. In this case, for any $y_0, y_1 \in Y$ and $n \geq 1$, $\pi_n(Y, y_0)$ is isomorphic to $\pi_n(Y, y_1)$ and we denote it $\pi_n(Y)$.

For $n \geq 1$, a definably connected definable set is *definably n-connected* if $\pi_i(Y) = 0$ for each $1 \leq i \leq n$.

Lemma 2.5. *Let Y be a definably connected definable set. If $\pi_{n-1}(Y) = 0$, then for every definable map $h : S^{n-1} \to Y$, there exists a definable map $H : D^n \to Y$ with $H|S^{n-1} = f$.*

Proof. For $i \geq 1$, since Y is definably connected, $\pi_i(Y) \to [S^i, Y], [h] \to [h]$ is bijective. Thus h is definably homotopic to a constant map $C : S^{n-1} \to Y, C(x) = c$. Hence there exists a definable map $\phi : S^{n-1} \times I \to Y$ such that $\phi(x, 0) = c, \phi(x, 1) = h(x)$ for all $x \in S^{n-1}$. Collapsing $S^{n-1} \times \{0\}$ to a point, by Theorem 2.1, we have the cone CS^{n-1} which is definably homeomorphic to D^n and a definable map $H : D^n \to Y$ with $H|S^{n-1} = f$. □

Proposition 2.6. *If Y is definably $(n-1)$-connected, $f : A \to Y$ is a definable map, $\dim(X, A) \leq n$ and $n \geq 1$, then there exists a definable map $F : X \to Y$ with $F|A = f$.*

Proof. If $i = 0$, then we may assume that $X_0 = A \cup e^0_1 \cup \cdots \cup e^0_{r_0}$, $e^0_1, \ldots, e^0_{r_0}$ denote the 0-cells of (X, A). For each e^0_j, defining the image of e^0_j, there exists a definable map $f_0 : X_0 \to Y$ extending f.

We may assume that $X_i = X_{i-1} \cup e^i_1 \cup \cdots \cup e^i_{r_i}$, $e^i_1, \ldots, e^i_{r_i}$ denote the i-cells of (X, A). By assumption, there exists a definable map $h_j : \partial e^i_j \to Y$. Since ∂e^i_j is definably homeomorphic to S^{n-1} and by Lemma 2.5, we have a definable map $H_j : e^i_j \to Y$ with $H_j|\partial e^i_j = h_j$. Using H_j, we obtain a definable map F with $F|A = f$. □

Let X be a definably connected definable set and $n \geq 1$. As in the topological setting, $\pi_1(X)$ acts on $\pi_n(X)$. We say that a definably connected definable set X is *n-simple* if the $\pi_1(X)$ action on $\pi_n(X)$ is tirivial. Since the $\pi_1(X)$ action on $\pi_1(X)$ is $\pi_1(X) \times \pi_1(X) \to \pi_1(X), (h_1, h_2) \mapsto h_1 h_2 h_1^{-1}$, X is 1-simple if and only if $\pi_1(X)$ is abelian.

Let X be a definable CW complex, A a definable subcomplex of X, $n \geq 1$ and Y a definably connected n-simple definable set. We define the cohomology group $H^n(X, A, \pi_n(Y))$ as follows. Remark that $[S^n, Y] = \pi_n(Y)$ because Y is n-simple.

We define the n-dimensional chain complex $C_n(X, A)$ to be $H_n(X_n, X_{n-1})$. Let $i_{n-1} : X_{n-1} \to X_n, j_n : (X_n, \emptyset) \to (X_n, X_{n-1})$ be inclusions. As in the topological setting, we have an exact sequence

$$\cdots \to H_n(X_n, X_{n-1}) \overset{\partial'_n}{\to} H_{n-1}(X_{n-1}) \overset{i_{*n-1}}{\to}$$

$$H_{n-1}(X_n) \overset{j_{*n}}{\to} H_{n-1}(X_n, X_{n-1}) \to \cdots.$$

The boundary operator $\partial_n : H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$ is $j_{*n-1} \circ \partial'_n$. We define the n-dimensional cochain complex $C^n(X, A) = Hom_{\mathbb{Z}}(C_n(X, A), \pi_n(Y))$ and the coboundary operator $\delta_n : C^n(X, A) \to C^{n+1}(X, A), (\delta f)c = f(\partial c)$.

Let (X, A) be a relative definable CW complex, $n \geq 1$, and Y a definably connected n-simple definable set. Let $g : X_n \to Y$ be a definable map.

Let e^{n+1}_i be an $(n+1)$-cell and $\phi_i : (D^{n+1}, S^n) \to (e^{n+1}_i, \partial e^{n+1}_i) \subset (X_{n+1}, X_n)$ the characteristic map of e^{n+1}_i. Composing $f_i = \phi_i|S^n$ with $g : X_n \to Y$, we have an element $[g \circ f_i] \in [S^n, Y] = \pi_n(Y)$. We define the obstruction cochain $\theta^{n+1}(g) \in C^{n+1}(X, A, \pi_n(Y))$ on the basis of $(n+1)$-cells by the formula $\theta^{n+1}(g)(e^{n+1}_i) = [g \circ f_i]$ and extend by linearly.

In the rest of this section, we prove the o-minimal cellular approximation theorem

Theorem 2.7 (O-minimal cellular approximation theorem)**.** *Let (X, A), (Y, B) be definable CW pairs and $f : (X, A) \to (Y, B)$ a definable map. Then there exists a definable map $g : (X, A) \to (Y, B)$ such that f is definably homotopic to g relative to A and for any nonnegative integer n, $g(X'_n) \subset Y'_n$, where X'_n (resp. Y'_n) denotes the union of the n-skeleton X_n (resp. Y_n) of X (resp. Y) and A (resp. B).*

Lemma 2.8 (O-minimal homotopy extension lemma [1]). *Let X, Z, A be definable sets with $A \subset X$ closed in X. Let $f : X \to Z$ be a definable map and $H : A \times I \to Z$ a definable homotopy such that $H(x, 0) = f(x), x \in A$. Then there exists a definable homotopy $F : X \times I \to Z$ such that $F(x, 0) = f(x), x \in X$ and $F|A \times I = H$.*

By the above lemma, we have the following o-minimal homotopy extension theorem.

Theorem 2.9. *Let (X, A) be a definable CW pair. Let $f : X \to Y$ be a definable map and $H : A \times I \to Y$ a definable homotopy with $H(x, 0) = f(x), x \in A$. Then there exists a definable homotopy $F : X \times I \to Y$ such that $F(x, 0) = f(x), x \in X$ and $F|A \times I = H$.*

To prove Theorem 2.7, we prepare three claims.

Claim 2.10. *Let (Z, C) be a definable CW pair. For any definable map $g : D^q \to Z$ with $g(S^{q-1}) \subset \overline{Z^{q-1}}$, there exists a definable map $g' : D^q \to Z$ such that $g \simeq g'$ rel S^{q-1} and $g'(D^q) \subset \overline{Z^q}$, where $\overline{Z^{q-1}} = Z^q \cup C$.*

Proof. Let n be the maximum dimension of cells not contained in C. We may assume that $n > q$ and proceed by induction on the number of such n-cells. Let $\phi : (D^n, S^{n-1}) \to (Z, \overline{Z^{n-1}})$ be the characteristic map of an n-cell e. Let D^n_1, (D^n_2) be the closed ball of center 0 with radius $\frac{1}{3}$, $(\frac{2}{3})$, respectively. Put $U = \phi(D^n - D^n_1) \cup (Z - e), V = \phi(\text{Int } D^n_2), z_0 = \phi(0)$, where Int D^n_2 denotes the interior of D^n_2. Then $U \cup V = Z$. Taking a refinement of D^q, every simplex $|s|$ of it is contained in $g^{-1}(U)$ or $g^{-1}(V)$. Let $E_1 = \cup_{|s| \cap g^{-1}(z_0) \neq \emptyset} |s|, E_2 = \cup_{|s| \cap g^{-1}(z_0) = \emptyset} |s|$. Then $g(E_1) \subset V$, $g(E_1 \cap E_2) \subset V - \{z_0\}$. Thus we have a definable map $\phi^{-1} \circ g : E_1 \cap E_2 \to \text{Int } D^n_2 - \{0\}$. Since Int $D^n_2 - \{0\}$ is definably homotopy equivalent to S^{n-1} and S^{n-1} is $(n-2)$-connected, there exists a definable map $h : E_1 \to \text{Int } D^n_2 - \{0\}$ with $h|E_1 \cap E_2 = \phi^{-1} \circ g$. Define a definable homotopy $h_t : E_1 \to \text{Int } D^n_2$ by $h_t(x) = (1-t)\phi^{-1} \circ g(x) + th(x)$. Then h_t is a definable homotopy between $\phi^{-1} \circ g$ and h relative to $E_1 \cap E_2$. Define a definable homotopy $h'_t : D^q \to Z$ by $h'_t|E_1 = \phi^{-1} \circ g, h'_t|E_2 = g|E_2$. Then h'_t is a definable homotopy between g and h'_1 relative to S^{q-1} and $h'_1(D^q) \subset Z - \{z_0\}$. Taking a definable retraction $r : Z - \{z_0\} \to Z - e$, $h'_1 \simeq r \circ h'_1$ rel $S^{q-1} : D^q \to Z - \{z_0\}$. Let $g'' = r \circ h'_1$. Then $g \simeq g''$ rel $S^{q-1} : D^q \to Z, g''(D^q) \subset Z - e$. By the inductive hypothesis, there exists a definable map g' such that $g'' \simeq g'$ rel $S^{q-1} : D^q \to Z - e, g'(D^q) \subset \overline{Z^q}$. □

Claim 2.11. *For any definable map $f : (\overline{X^q}, \overline{X^{q-1}}) \to (Y, \overline{Y^{q-1}})$, there exists a definable map $g : (\overline{X^q}, \overline{X^{q-1}}) \to (Y, \overline{Y^{q-1}})$ such that $f \simeq g$ rel $\overline{X^{q-1}}$ and $g(\overline{X^q}) \subset \overline{Y^q}$.*

Proof. Let e be a q-cell not contained in A. Since $f(\overline{e})$ is definably compact, there exists a finite subcomlex Z of Y with $f(\overline{e}) \subset Z$. Put $C = Z \cap \overline{Y}^{q-1}$. Then $f(e^r) \subset C$, where e^r denotes the boundary of e. Let $\phi : (D^q, S^{q-1}) \to (\overline{e}, e^r)$ be the characteristic map of e. Applying Claim 2.10 to $f \circ \phi : (D^q, S^{q-1}) \to (Z, C)$, there exists a definable map g' such that $f \circ \phi \simeq g'$ rel S^{q-1}, $g'(D^q) \subset \overline{Z}^q$. Then $g = g' \circ \phi$ is the required map. □

Claim 2.12. *For any definable map $f : (X, A) \to (Y, B)$, there exists a definable homotopy $H_q : (X, A) \times [0, 1]_R \to (Y, B)$ such that:*

(1) $H_0(x, t) = f(x)$ for all $x \in X$.

(2) $H_q(x, 0) = H_{q+1}(x, 0)$ for all $x \in X$.

(3) $H_q(x, t) = (x, t)$ for all $(x, t) \in \overline{X}^q \times [0, 1]_R$.

(4) $H_q(\overline{X}^q \times \{1\}) \subset \overline{Y}$.

Proof. Let $H_0(x, t) = f(x)$ for $(x, t) \in X \times [0, 1]_R$. Assume we have H_{q-1}. Since $H_{q-1}(\overline{X}^{q-1} \times \{1\}) \subset \overline{Y}^{q-1}$ and by Claim 2.11, there exists a definable homotopy H'_q rel $\overline{X}^{q-1} : (\overline{X}^q, \overline{X}^{q-1}) \times [0, 1]_R \to (\overline{Y}^q, \overline{Y}^{q-1})$ such that $H'_q|\overline{X}^q \times \{0\} = H_{q-1}|\overline{X}^q \times \{1\}, H'_q(\overline{X}^q \times \{1\}) \subset \overline{Y}^q$. By Lemma 2.8, there exists a definable homotopy $H_q : X \times [0, 1]_R \to Y$

such that $H_q|X\times\{0\} = H_{q-1}|X\times\{1\}, H_q|\overline{X}^q \times [0,1]_R = H'_q$, and H_q satisfies (1)-(4). □

Proof of Theorem 2.7. Let $q = \dim X$. By Claim 2.12, we have a definable homotopy H_q. Then the definable map $g : (X, A) \to (Y, B)$ defined by $g(x) = H_q(x, 1)$ is the required map. □

3 . Proof of Theorem 1.2.

Lemma 3.1. *Let* i *be the inclusion* $X_n \to X_{n+1}$ *and* $x_0 \in X_n$. *Then* $i_* : \pi_1(X_n, x_0) \to \pi_1(X_{n+1}, x_0)$ *is surjective if* $n = 1$ *and an isomorphism* $n > 1$.

Proof. Let $n \geq 1$ and $\alpha : S^1 \to X_{n+1}$ a definable map. By Theorem 2.7, there exists a definable map $\alpha' : S^1 \to X_1 \subset X_n$ such that α is definably homotopic to α'. Since $i_*([\alpha']) = [\alpha]$, i_* is surjective.

Assume $n \geq 2$ and $i_*([\alpha]) = 0$. Then $\alpha : S^1 \to X_{n+1}$ is null homotopic and there exists a definable map $H : S^1 \times [0,1]_R \to X_{n+1}$ such that $H(-,0) = \alpha, H(-,1) = c$, where c denotes a constant map. By Theorem 2.7 and since $S^1 \times [0,1]_R$ is a 2-dimensional definable set, there exists a definable map $H' : S^1 \times [0,1]_R \to X_2$ such that H is definably homotopic to H' relative to $S^1 \times \{0,1\}$. Thus $[\alpha] = 0$ and i_* is injective. □

Lemma 3.2. *If* $k \leq n, n > 1$ *and* $x_0 \in X_n$, *then* $\pi_k(X_{n+1}, X_n, x_0) = 0$.

Proof. Consider an exact sequence $\cdots \to \pi_k(X_n, x_0) \to \pi_k(X_{n+1}, x_0) \to \pi_k(X_{n+1}, X_n, x_0) \to \pi_{k-1}(X_n, x_0) \to \pi_{k-1}(X_{n+1}, x_0) \to \ldots$. We prove that $i_{*k} : \pi_k(X_n, x_0) \to \pi_k(X_{n+1}, x_0)$ is surjecitve and $i_{*k-1} : \pi_{k-1}(X_n, x_0) \to \pi_{k-1}(X_{n+1}, x_0)$ is injective.

Let $\alpha : S^k \to X_{n+1}$ be a definable map. Then by Theorem 2.7, there exists a definable map $\alpha' : S^k \to X_k$ such that α is definably homotpic to α'. Then $i_{*k} : \pi_k(X_n, x_0) \to \pi_k(X_{n+1,x_0})$ is surjecitve.

Assume $i_{*k-1}([\alpha]) = 0$. Then $\alpha : S^{k-1} \to X_{n+1}$ is null homotopic and there exists a definable map $H : S^{k-1} \times [0,1]_R \to X_{n+1}$ such that $H(-,0) = \alpha, H(-,1) = c$. By Theorem 2.7 and since $S^{k-1} \times [0,1]$ is a k-dimensional definable set, there exists a definable map $H' : S^{k-1} \times [0,1]_R \to X_k \subset X_n$ such that H is definably homotopic to H' relative to $S^{k-1} \times \{0,1\}$. Thus $[\alpha] = 0$ and i_{*k-1} is injective.

By the above results and exactness, we have the lemma. □

The following is the o-minimal relative Hurewicz theorem.

Theorem 3.3 (5.4 [1])**.** *Let* (X, A, x_0) *be a definable pointed pair and* $n \geq 2$. *Suppose that* $\pi_r(X, A, x_0) = 0$ *for any* $1 \leq r \leq n-1$. *Then the o-minimal Hurewicz homomorphism* $h_n : \pi_n(X, A, x_0) \to H_n(X, A)$ *is surjective and its kernel is the subgroup generated by* $\{\beta_{[u]}([f])[f]^{-1} | [u] \in \pi_1(A, x_0), [f] \in \pi_n(X, A, x_0)\}$. *In particular,* h_n *is an isomorphism for* $n \geq 3$.

Put $\pi^+_{n+1}(X_{n+1}, X_n) = \pi_{n+1}(X_{n+1}, X_n)/\ker h_n$. Let $g : X_n \to Y$ be a definable map and $\pi : \pi_{n+1}(X_{n+1}, X_n) \to \pi^+_{n+1}(X_{n+1}, X_n)$ denote the projection.

Lemma 3.4. *There exits a factorization* $\overline{g_* \circ \partial} : \pi^+_{n+1}(X_{n+1}, X_n) \to \pi_n(Y)$ *such that* $\pi \circ \overline{g_* \circ \partial} = g_* \circ \partial$.

Proof. If $\alpha \in \pi_1(X_n)$, then $\partial(\alpha x) = a\partial x$. Since Y is n-simple, for any $z \in \pi_n(X_n)$, $g_*(\alpha z) = g_*(\alpha) g_*(z) = g_*(z)$. □

By Lemma 3.4, we can define the composition map $C_{n+1}(X, A) = H_{n+1}(X_{n+1}, X_n) \overset{h^{-1}}{\to} \pi^+_{n+1}(X_{n+1}, X_n) \overset{g_*\circ\partial}{\to} \beta\pi_n(Y)$, where $h : \pi^+_{n+1}(X_{n+1}, X_n) \to H_{n+1}(X_{n+1}, X_n)$ denotes the Hurewicz isomorphism. This composition map defines another cochain in $Hom_{\mathbb{Z}}(C_{n+1}(X, A), \pi_n(Y))$ which we again denote by $\theta^{n+1}(g)$.

Proposition 3.5. *The two definitions of* $\theta^{n+1}(g)$ *coincide.*

Proof. For an $(n+1)$-cell e^{n+1}_i, let $\phi_i : (D^{n+1}, S^n) \to (X_{n+1}, X_n)$ be the characteristic map of e^{n+1}_i. We define a map $(\phi_i \vee u) \circ q : (D^{n+1}, S^n, p) \to (X_{n+1}, X_n, x_0)$ as the composition of a map $q : (D^{n+1}, S^n, p) \to (D^{n+1} \vee I, S^n \vee I, p)$ and a map $D^{n+1} \vee I \stackrel{\phi_i \vee u}{\to} X_{n+1}$, where u is a definable path in X_n to the base point x_0. Then $(\phi_i \vee u) \circ q$ is definably homotopic to the characteristic map ϕ_i. Hence $h((\phi_i \vee u) \circ q)$ is the generator of $H_{n+1}(X_{n+1}, X_n)$ represented by the cell e_i^{n+1} and $(\phi_i \vee u) \circ q$ represents the element $h^{-1}(e_i^{n+1})$ in $\pi^+_{n+1}(X_{n+1}, X_n)$. By definition, $\partial((\phi_i \vee u) \circ q) \in \pi_n(X_n)$ is represented by the composition of the map $\overline{q} : S^n \to S^n \vee I$ obtained by restricting the map q to the boundary and the attaching map $f_i = \phi_i | S^n$ together with a definable path u to x_0: $\partial((\phi_i \vee u) \circ q) = (f_i \vee u) \circ \overline{q} : S^n \to X_n$. By the second definition, $\theta(g)(e_i^{n+1}) = g \circ (f_i \vee u) \circ \overline{q} = (g_i \circ f_i \vee g \circ u) \circ \overline{q}$. Moreover this is equal to $[f_i] \in [S^n, Y] = \pi_n(Y)$, which is the first definition of $\theta(g)(e_i^{n+1})$. □

Theorem 3.6. *The obstruction cohain $\theta^{n+1}(g)$ is a cocycle.*

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc}
\pi_{n+2}(X_{n+2}, X_{n+1}) & \to & H_{n+2}(X_{n+2}, X_{n+1}) \\
\downarrow & & \downarrow \\
\pi_{n+1}(X_{n+1}) & \to & H_{n+1}(X_{n+1}) \\
\downarrow & & \downarrow \\
\pi_{n+1}(X_{n+1}, X_n) & \to & H_{n+1}(X_{n+1}, X_n) \\
\downarrow & & \downarrow \theta(g) \\
\pi_n(X_n) & \stackrel{g_*}{\to} & Hn(Y_n)
\end{array}$$

The unlableled horizontal arrows are the Hurewicz maps and the unlableled vertical arrows are obtained from homotopy or homology exact sequences of the pair (X_{n+2}, X_{n+1}) and (X_{n+1}, X_n).

The composition of the bottom two vertical maps on the left are zero because they occur in the homotopy exact sequence of the pair (X_{n+1}, X_n). Since $\delta\theta(g)$ is the composition of all the right vertical maps, $\delta\theta^{n+1}(g)(x) = \theta^{n+1}(g)(\partial x) = 0$. Thus $\theta^{n+1}(g)$ is a cocycle. □

By a way to similar to the topological category, we have the following proposition.

Proposition 3.7. *If X is a definable CW complex, then $X \times I$ is a definable CW complex.*

Theorem 3.8. *Let (X, A) be a relative definable CW complex, Y a definably connected n-simple definable set and $g : X_n \to Y$ a definable map.*

(1) $\theta^{n+1}(g) = 0$ if and only if there exists a definavble map $\tilde{g} : X_{n+1} \to Y$ extending g.

(2) $[\theta^{n+1}(g)] = 0$ if and only if there exists a definavble map $\tilde{g} : X_{n+1} \to Y$ extending $g|X_{n-1}$.

Lemma 3.9. *Let $f_0, f_1 : X_n \to Y$ be definable maps such that $f_0|X_{n-1}$ is definably homotopic to $f_1|X_{n-1}$. Then there exists a difference cochain $d \in C^n(X, A, \pi_n(Y))$ such that $\delta d = \theta^{n+1}(f_0) - \theta^{n+1}(f_1)$.*

Proof. Let $\hat{X} = X \times I, \hat{A} = A \times I$. Then $(\hat{X}, \hat{A})$ is a relative definable CW complex with $\hat{X}^k = X_k \times \partial I \cup X_{k-1} \times I$. Take a definable homotopy H between f_0 and f_1. Hence a definable map $\hat{X}_n \to Y$ is obtained from $f_0, f_1 : X_n \to Y$ and a definable homotopy $G = H|X_{n-1} \times I : X_{n-1} \times I \to Y$. Thus we have the cocycle $\theta(f_0, G, f_1) \in C^{n+1}(\hat{X}, \hat{A}, \pi_n(Y))$ which obstructs finding an extension of $f_0 \cup G \cup f_1$ to $\hat{X}_{n+1}$. we define the difference cochain $d(f_0, G, f_1) \in C^n(X, A, \pi_n(Y))$ by restricting to cells of the form $e^n \times I$, that is $d(f_0, G, f_1)(e_i^n) = (-1)^{n+1}$ $\theta(f_0, G, f_1)(e_i^n \times I)$ for each n-cell e_i^n of X. Since $\theta(f_0, G, f_1)$ is a cocycle, $0 = (\delta\theta(f_0, G, f_1))(e_i^{n+1} \times I) = \theta(f_0, G, f_1)(\partial((e_i^{n+1} \times I)) = \theta(f_0, G, f_1)(\partial(e_i^{n+1} \times I) + (-1)^{n+1}(\theta(f_0, G, f_1)$ $(e_i^{n+1} \times \{1\}) - \theta(f_0, G, f_1)(e_i^{n+1} \times \{0\})) = (-1)^{n+1}(\delta(d(f_0, G, f_1))(e_i^{n+1}) + \theta(f_1)(e_i^{n+1}) - \theta(f_0)(e_i^{n+1}))$. Thus $\delta d = \theta^{n+1}(f_0) - \theta^{n+1}(f_1)$. □

Proposition 3.10. *Let $f_0 : X_n \to Y$ be a definable map, $G : X_{n-1} \times I \to Y$ a definable homotopy such that $G(-, 0) = f_0|X_{n-1}$ and $d \in C^n(X, A, \pi_n(Y))$. Then there exists a definable map $f_1 : X_n \to Y$ such that $G(-, 1) = f_1|X_{n-1}$ and $d = d(f_0, G, f_1)$.*

To prove Proposition 3.10, we need the following lemma.

Lemma 3.11. *For any definable map* $f: D^n \times \{0\} \cup S^{n-1} \times I \to Y$ *and for any definable homotopy class* $\alpha \in [\partial(D^n \times I), Y]$*, there exists a definable map* $F: \partial(D^n \times I) \to Y$ *such that* $F|D^n \times \{0\} \cup S^{n-1} \times I = f$ *and* $[F] = [\alpha]$.

Proof. Take a definable map $K: \partial(D^n \times I) \to Y$ with $[K] = [\alpha]$. Let $D = D^n \times \{0\} \cup S^{n-1} \times I$. Then D is definably contractible and $K|D$ and f are null homotopic. Thus $K|D$ and f are definably homotopic. Applying Theorem 2.7 to $(\partial(D^n \times I), D)$, there exists an extension $H: \partial(D^n \times I) \times I \to Y$. Hence $F = H(-, 1)$ is the required definable map. □

Proof of Proposition 3.10. Let e_i^n be an n-cell of X_n and $\phi: (D^n, S^{n-1}) \to (X_n, X_{n-1})$ the characteristic map of e_i^n. Applying Lemma 3.11 to $f = f_0 \circ \phi_i \cup G \circ (\phi_i | S^{n-1} \times id_I)$ and $\alpha = d(e_i^n)$, we have a definable map F_i. We define $f_1: X_n \to Y$ on the n-cells by $f_1(\phi_i(x)) = F_i(x, 1)$. Then $d(f_0, G, f_1)(e_i^n) = d(e_i^n)$. □

Proof of Theorem 3.8. We now prove that if $g: X_n \to Y$ and $\theta(g)$ is a coboundary δd, then $g|X_{n-1}$ extneds to X_n. Applying Proposition 3.10 to g, d and the stationary homotopy $((x,t) \mapsto g(x))$ from $g|X_{n-1}$ to itself, there exists a definable map $g': X_n \to Y$ such that $g'|X_{n-1} = g|X_{n-1}$ and $\delta d = \theta(g) - \theta(g')$. Since $\theta(g') = 0$, g' extends to X_{n+1}. □

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