Computing the Conway polynomial of several closures of oriented 3-braids

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Abstract

This paper deals with polynomial invariants of a class of oriented 3-string tangles and the knots (or links) obtained by applying six different closures. In [1], expressions were given to compute the Conway polynomials of four of different closures of the composition of two such 3-string tangles. By using the expressions and results from that reference, and using an algorithm developed on the basis of Giller's calculations for 3-string tangles, we provide new results concerning six closures of 3-braids. Surprisingly, for 3-braids two of the closures turn out to be affine functions of the four previously defined. Among the contributions in this paper one finds computational tools to obtain the Conway polynomial of closures of 3-braids in terms of continuous fractions and their expansions. An interesting feature is that our calculations yield explicit, nonrecursive formulas in the case of 3-braids, thereby considerably lowering the time required to compute them. As a byproduct, explicit expressions are also given to obtain both numerators and denominators of continuous fractions in a nonrecursive way.

Keywords: Conway polynomial, 3-tangle, 3-braid, closure, continued fraction
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1. Introduction

In this article we analyze the relation between the Conway polynomials associated to six different closures of a certain family of 3-tangles and the ones associated to the composition $s_1 \cdot s_2$ of two of them. An invariant $i(S)$ is defined, for any 3-tangle S with an orientation described further below, to be the element in $\mathbb{F}^6$ given by

$$i(S) = (S^{c_1}, S^{c_2}, S^{c_3}, S^{c_4}, S^{c_5}, S^{c_6})^T, \tag{1}$$

where $\mathbb{F}$ is the field of fractions of $\mathbb{Z}[z]$ and S^{c_i} $(i = 1, \ldots, 6)$ represents the Conway polynomial of the knot (or link) obtained by closing S in one of six different ways (shown

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in Figure 8). Moreover, explicit formulas are given to compute the invariant of the 3-tangle $i(S \cdot T)$, obtained by juxtaposition of tangles S and T, as a function of the invariants of $i(S)$ and $i(T)$. One interesting aspect of the results in this paper is that nonrecursive expressions are derived to find the polynomial of a number of different closures of a class of 3-braids. In particular, as shown in [1], given a 3-tangle s a 2×2 matrix $M_\nabla(s)$ which satisfies $M_\nabla(s_1 \cdot s_2) = M_\nabla(s_1)M_\nabla(s_2)$ can be assigned. In [2], similar formulas were given in the case of 2-tangles. Here we apply these results in order to compute $i(\mathcal{T}(2a_1, \ldots, 2a_n))$, where $T(2a_1, \ldots, 2a_n)$ is a 3-braid composed of boxes with an even number of crossings.

The paper is organized as follows. In Section 2 basic notions are recalled, including the definition of the Conway polynomial, tangles, rooms and their skein, and some computations regarding the same polynomial taken from Giller's seminal work [2]. In Section 3, a 3-room R' with a nonstandard orientation is defined and the invariant i is introduced in terms of the Conway polynomial and six different closures. The results are then particularized to 3-braids in the skein of R' in Section 4, where explicit expressions are given to compute their invariant i in terms of continued fractions. Section 5 gives explicit, nonrecursive formulas for continued fractions and, building upon the results in the previous section, for the invariant i of 3-braids. Concluding remarks are presented in Section 6.

2. Preliminary Recalls

2.1. The Conway polynomial

The Conway polynomial of an oriented link is computed by applying the following recursive equations to any of its diagrams:

(i) $\nabla_{\bigcirc}(z) = 1$

(ii) $\nabla_{L_l}(z) = \nabla_{L_r}(z) + z\nabla_{L_s}(z)$,

where (L_l, L_r, L_s) is a ***skein triple***, that is, an ordered triple of oriented links which are identical except at a specific crossing, where they look as illustrated in Figure 1. As is clear from its definition, the recursive nature of the computations required to find the Conway polynomial of a given link entails a significant amount of computational complexity: at each "step," the number of diagrams whose polynomials are to be computed is doubled until no further crossing can be eliminated.

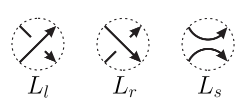


Figure 1: Skein triple: An ordered triple (L_l, L_r, L_s) of oriented links all of which are identical except that they differ at a given crossing, where they look as depicted.

2.2. Tangles

An n-***tangle*** S is a pair (B^3, A), where B^3 is the closed unit ball in $\mathbb{R}^3$ and $A \subset B^3$ is a one-dimensional, embedded submanifold with nonempty boundary, which contains n arcs (i.e., subsets homeomorphic to the closed unit interval $I = [0, 1]$) and satisfies $\partial A = A \cap \partial B^3$ (cf. e.g. [3] or [4, Chap. 3]). An ***oriented*** n-***tangle*** is a tangle (B^3, A) such that each connected component of A is oriented.

2.3. Rooms and their skeins

The material in this subsection is adapted from [2]. Let P be an ordered set of n points in the interior of D^2, the closed unit disk in $\mathbb{R}^2$, and let $S \subset P$. The triple $R = (D^2 \times I, P, S)$ is called an n-***room***. The points in $P \times \{0\}$ are called ***left ports*** and those in $P \times \{1\}$ ***right ports***; the ports in $S \times \{0, 1\}$ are said to be ***positively*** oriented and those in $(P \setminus S) \times \{0, 1\}$ are called ***negatively*** oriented. An illustration of an n-room is shown in Figure 2. An n-room $(D^2 \times I, P, S)$ is said to have the ***standard orientation*** if $S = P$, that is, if its ports are all positively oriented.

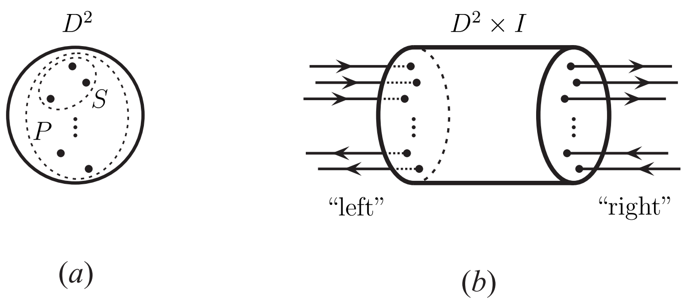


Figure 2: An *n-room* $(D^2 \times I, P, S)$. (a) The disk D^2 with the set P of interior points and the subset S which defines the *positively* oriented ports. (b) The n-room with its *left* and *right* ports and lines tagged by arrows to show their orientations.

An oriented n-tangle $S = (B^3, A)$ is called an ***inhabitant*** of an n-room R if there exists a homeomorphism $\varphi : B^3 \longrightarrow D^2 \times I$, with $\varphi(\partial A) = P \times \{0, 1\}$, such that for each arc $a \subset A$: **(i)** φ maps ∂a into exactly one left and one right port, both with the same orientation; and **(ii)** the orientation of a coincides with the one induced by R in the obvious way. The ***skein*** $\mathcal{S}(R)$ ***of a room*** R is the set of all inhabitants of R. For any inhabitant $S \in \mathcal{S}(R)$, the connections from left to right ports made by the arcs of S determine a unique permutation $\pi(S)$ in the symmetric group S_n; namely, if $a \subset A$ is such an arc and $\varphi(\partial a) = \{(p_i, 0), (p_j, 1)\}$, then $\pi(S)(i) = j$. As with knots and links, it is convenient to work with planar projections of n-rooms and their skein. Accordingly, n-rooms and their inhabitants are represented, as in Figure 3, by diagrams drawn under the same conventions as when passing from a knot or link to one of its diagrams, basically by representing $D^2 \times I$ by a rectangle, by drawing strands in general position, by using arrows to indicate orientations and fractured lines to signify under-passes.

Given a room R and inhabitants $S, T \in \mathcal{S}(R)$, one writes $S = T$ if there exists an ambient isotopy carrying S into T, keeping $\partial(D^2 \times I)$ fixed. The set $\mathcal{S}(R)$ is endowed with a binary operation called ***juxtaposition*** (or ***concatenation***) which maps inhabitants S and T of R to the inhabitant $S \cdot T$ defined as in Figure 4.

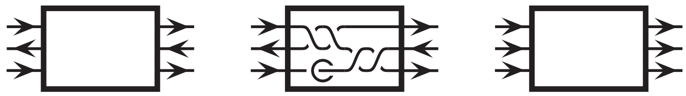


Figure 3: Planar representation of 3-rooms. (a) A 3-room with two positively and one negatively oriented ports, denoted R'. (b) The same room shown with an inhabitant in its skein. (c) The 3-room, denoted R_{std}, with the so called *standard* orientation, i.e., all ports positively oriented.

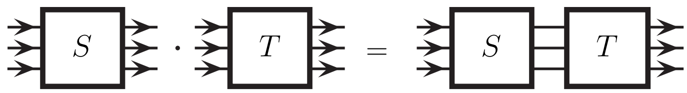


Figure 4: Inhabitants S and T of a room R and their juxtaposition $S \cdot T$, again an inhabitant of R.

2.4. The skein vector space of the standard 3-room

Let R be any 3-room and let $\mathbb{F}$ represent the fraction field of $\mathbb{Z}[z]$. One writes $V(R)$ for the free vector space over $\mathbb{F}$ generated by $\mathcal{S}(R)$, and $N(R)$ for the subspace generated by the elements of the form $S_l - S_r - zS_s$, where $(S_l, S_r, S_s) \in \mathcal{S}(R)^3$ is a skein triple. The quotient space $V(R)/N(R)$ is denoted by $L(R)$. As per a customary abuse of notation, one uses the same symbol S to denote an inhabitant $S \in \mathcal{S}(R)$, the corresponding vector $S \in V(R)$ and the class of S in $L(S)$. A left $\mathbb{F}$-algebra structure is defined on $L(R)$ by extending the juxtaposition "$\cdot$" to all of $L(R)$, that is, by setting $S \cdot (T + \alpha U) = S \cdot T + \alpha(S \cdot U)$ for $S, T, U \in L(R)$ and $\alpha \in \mathbb{F}$. It is worth recalling that, for $n \in \mathbb{N}$, $\mathbb{F}^n$ admits a structure of vector space over $\mathbb{F}$ in a natural way.

Specializing the present discussion to the standard 3-room, let $B = \{\Sigma_1, \ldots, \Sigma_6\}$ be the set of inhabitants of R_{std} depicted in Figure 5. The group of permutations $\{\pi(\Sigma_1), \ldots, \pi(\Sigma_6)\}$ associated with the elements of B clearly equals S_3. Moreover, B was shown in [2] to be a basis of $L(R_{\text{std}})$, so every element in $L(R_{\text{std}})$ admits a unique expression as a linear combination of elements in B.

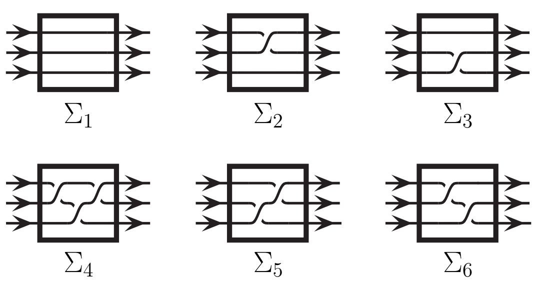


Figure 5: Six elements in $S(R_{\text{std}})$ whose set of associated permutations equals S_n, and whose images in $L(R_{\text{std}})$ form a basis.

2.5. The c_6 closure of inhabitants of R_{std}

Inhabitants of R_{std} give rise to oriented knots or links by an operation, referred to as *closure*, which consists in connecting left ports with right ports, in a way that is consistent with the orientations, via arcs external to the room R_{std}. Many such closures exist, of course, but one that is especially useful and particularly simple is the one in which the ith left port is connected to the ith right one without introducing additional crossings, as illustrated in Figure 6. The designation "c_6," the significance of which shall become evident below, obeys the fact that five other closures are defined in order to obtain invariants for inhabitants of a 3-room with a nonstandard orientation.

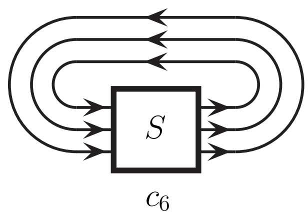


Figure 6: The c_6 closure of an inhabitant of R_{std}.

Given $S \in \mathcal{S}(R_{\mathrm{std}})$, let $c_6(S)$ be the oriented knot (or link) obtained by closing S via c_6. To shorten the notation, the Conway polynomial of $c_6(S)$ shall be denoted by S^{c_6}, i.e., $S^{c_6} = \nabla(c_6(S))$. Every element in $L(R_{\mathrm{std}})$ determines an element in the dual space $L(R_{\mathrm{std}})^*$ as follows. Regard $B = \{\Sigma_1, \ldots, \Sigma_6\}$ as a subset of $L(R_{\mathrm{std}})$, and for each $S, X \in B$, set $S^*(X) = (X \cdot S)^{c_6} = \nabla(c_6(X \cdot S))$. Since B is a basis of $L(R_{\mathrm{std}})$, S^* extends by linearity to a mapping $S^* : L(R_{\mathrm{std}}) \longrightarrow \mathbb{Z}[z] \subset \mathbb{F}$, which belongs to $L(R_{\mathrm{std}})^*$. Let M be the 6×6 matrix whose entries are defined by $M_{ij} = \Sigma_i^*(\Sigma_j)$, $i, j \in \{1, \ldots, 6\}$. It was shown in [2] that $\det M = -(z^2+4) \neq 0$, so M is nonsingular, and that its entries are given by:

$$M = \begin{pmatrix} 0 & 0 & 0 & z & 1 & 1 \\ 0 & 0 & 1 & 1+z^2 & z & z \\ 0 & 1 & 0 & 1+z^2 & z & z \\ z & 1+z^2 & 1+z^2 & 3z^2+z^4 & 2z+z^3 & 2z+z^3 \\ 1 & z & z & 2z+z^3 & 1+z^2 & z^2 \\ 1 & z & z & 2z+z^3 & z^2 & 1+z^2 \end{pmatrix}.$$

As a consequence of the invertibility of M, $B^* = \{\Sigma_1^*, \ldots, \Sigma_6^*\}$ is a basis of the dual $L(R_{\mathrm{std}})^*$. Therefore $\alpha = \sum_{i=1}^{6} \Sigma_i^* \otimes \Sigma_i$ may be regarded as an automorphism $\alpha : L(R_{\mathrm{std}}) \longrightarrow L(R_{\mathrm{std}})$ such that $\alpha(S) = \sum_{i=1}^{6} \Sigma_i^*(S)\Sigma_i$. Given that $(S \cdot X)^{c_6} = (X \cdot S)^{c_6}$ for every $S, X \in L(R_{\mathrm{std}})$, it was shown in [2] that $\varphi(S,T) = (S \cdot T)^{c_6}$ actually defines a bilinear φ mapping on $L(R_{\mathrm{std}}) \times L(R_{\mathrm{std}})$. Since frequent use of (multi)linear mappings shall be made below, vectors in $L(R_{\mathrm{std}})$ will be represented, unless otherwise specified, as "column" vectors whose entries are the corresponding components in the basis B. Given a vector x represented as a column vector, x^T denotes its transpose, regarded as a "row" vector. Using these conventions one has $\alpha(S) = (\Sigma_1^*(S), \ldots, \Sigma_6^*(S))^T$ and

$\alpha(T) = (\Sigma_1^*(T), \ldots, \Sigma_6^*(T))^T$ and hence, in terms of these vectors, one of the results in [2] states that φ has matrix M^{-1}, i.e.,

$$(S_1 \cdot S_2)^{c_6} = \varphi(S_1, S_2) = \alpha(S_1)^T M^{-1} \alpha(S_2). \tag{2}$$

The interest of this expression is that the Conway polynomial of the c_6 closure of $S_1 \cdot S_2$ may be expressed as a function of α and M^{-1}.

3. The 3-room R', its associated closures and the invariant i

Another interesting 3-room to study is R', shown in Figure 3(a), for which formulas may be derived to compute its various closures. Notice that the closure c_6 is equally applicable to R' since it obviously respects the orientations. For R' one defines, as in the case of R_{std}, the bilinear form $\psi(S, T) = (S \cdot T)^{c_6}$. To obtain similar formulas for the Conway polynomial of this closure one uses the trick used in [2], namely, reformulate the problem in terms of the c_6 closure of the *standard* room, thus allowing one to still use the matrix M. This is achieved by embedding S, T into R_{std}, with crossings added in order to render the orientations compatible, as shown in Figure 7. One thereby obtains inhabitants σ and τ of R_{std}. It is easy to see that $\psi(S, T) = \varphi(\sigma, \tau)$.

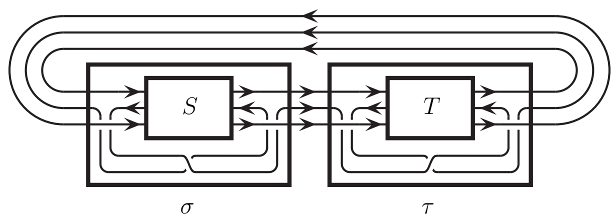


Figure 7: Transformation of inhabitants S, T of R' into inhabitants σ, τ of R_{std} by addition of crossings.

For R' one defines six closures, all of which respect the port orientations, indicated in Figure 8. Note that c_5 is defined in terms of the c_6 closure of $S \cdot E$, where E is the 3-tangle that represents a "180° twist to the front" of the three strands. Thus, $S^{c_5} = (S \cdot E)^{c_6}$. At first glance, c_5 may appear somewhat artificial, all the more that, unlike the other closures, it introduces additional crossings. Nevertheless, it arises naturally when computing $\alpha(\sigma)$ and $\alpha(\tau)$. As before, given $S \in \mathcal{S}(R')$, $c_i(S)$ denotes the c_i closure of S, whereas $S^{c_i} = \nabla(c_i(S))$ $(i = 1, \ldots, 6)$ denotes the Conway polynomial of its closure. For any inhabitant $S \in \mathcal{S}(R')$, one defines $i(S)$ to be the vector in $\mathbb{F}^6$ given by

$$i(S) = (S^{c_1}, S^{c_2}, S^{c_3}, S^{c_4}, S^{c_5}, S^{c_6})^T.$$

Owing to the properties of the Conway polynomial, i is an invariant of S. The previously described embedding actually allows one to use the matrix M^{-1} to find explicit expressions for $i(S)$. To see how this is achieved, define mappings $f, g, h : \mathbb{F}^6 \longrightarrow \mathbb{F}^6$ to

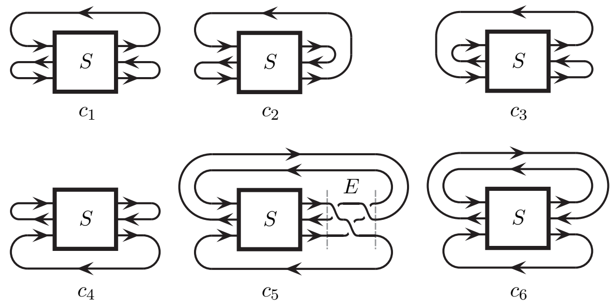


Figure 8: The six closures $c_1, \ldots, c_6$ for the 3-room R'. The additional crossings required to define c_5 equal a 3-braid referred to in this paper as E.

be given by

$$\begin{aligned} f(x) &= (x_2, x_1, x_4, x_3, x_6 + zx_4 - zx_5 + zx_1, x_5)^T \\ g(x) &= (x_6,\ x_4,\ x_1,\ x_5 - zx_3 + zx_6 + zx_2,\ x_2, \\ &\qquad x_3 + zx_1 + zx_4 - zx_5 + z^2x_3 - z^2x_6)^T \\ h(x) &= \left(x_6,\ x_4 + zx_6,\ x_1 + zx_6,\ zx_2 + \left(-z - z^3\right)x_3 + \left(1 + z^2\right)x_5 \right. \\ &\qquad \left. + \left(z + z^3\right)x_6, x_2 + zx_5 - z^2x_3 + z^2x_6,\ x_3 + zx_4 + zx_1 + z^2x_6\right)^T. \end{aligned}$$

It is easily checked that these three mappings are isomorphisms. Note that for $S \in \mathcal{S}(R')$ and the associated $\sigma \in \mathcal{S}(R_{\text{std}})$ (cf. Figure 7), one has $\alpha(\sigma) = g(i(S))$. The definitions of f and h are motivated in a similar way. In turn, consider the bilinear forms $\psi_N, \psi_E : \mathbb{F}^6 \times \mathbb{F}^6 \longrightarrow \mathbb{F}$ defined by

$$\psi_N(x, y) = g(x)^T M^{-1} h(y) \quad \text{and} \quad \psi_E(x, y) = \psi_N(x, f(y)). \tag{3}$$

In a nutshell, given tangles S and T, and recalling that E is the tangle occurring in the definition of c_5 (cf. Figure 8), the relevance of the functions just defined is that f maps $i(S)$ to the invariant $i(S \cdot E)$ of the juxtaposition $S \cdot E$, whereas ψ_N determines the Conway polynomial of the c_6 closure of $S \cdot T$. As an immediate byproduct, the c_5 closure of $S \cdot T$ is given by ψ_E, which is defined in terms of g and ψ_N. These observations are formalized in the following result.

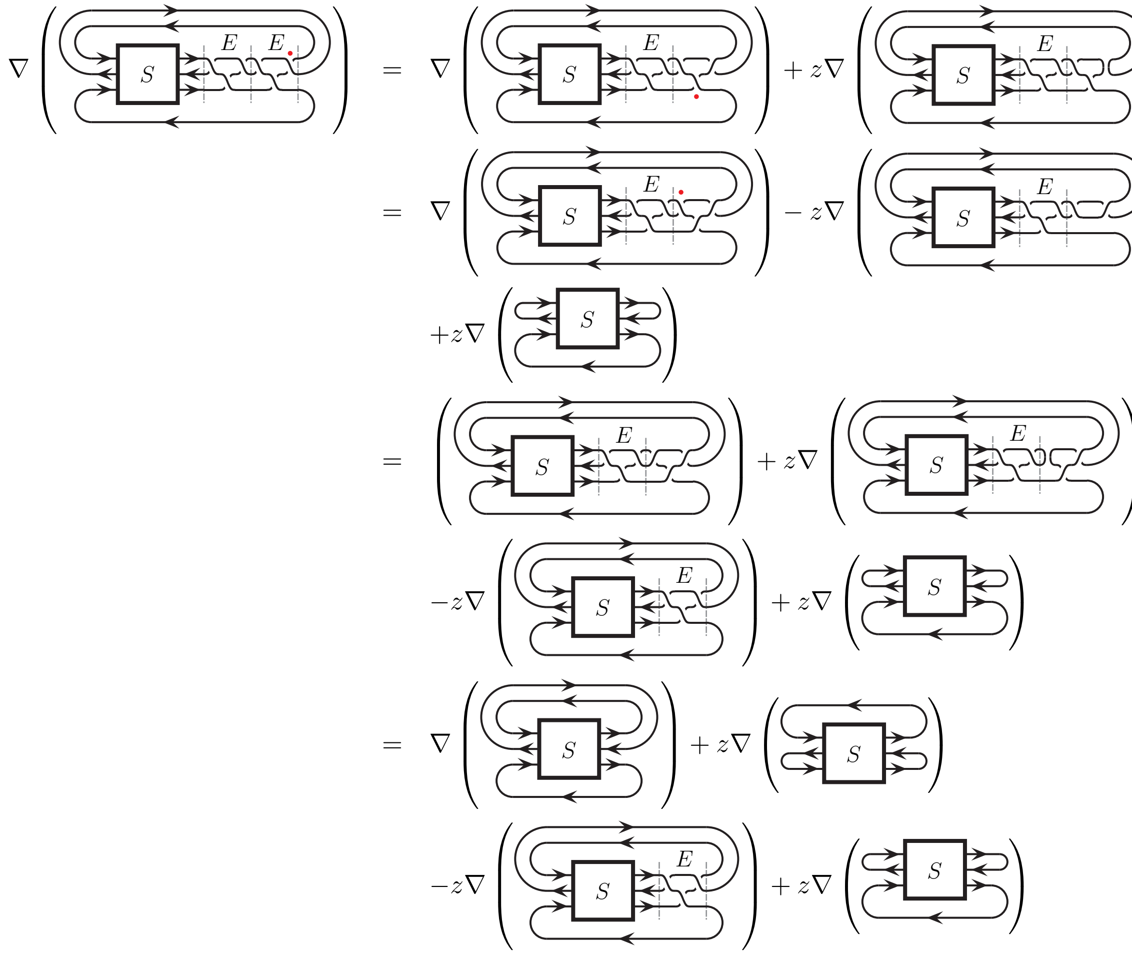
Proposition 1. *Given inhabitants $S, T \in \mathcal{S}(R')$, one has:*

(i) $i(S \cdot E) = f \circ i(S)$;

(ii) $(S \cdot T)^{c_6} = \psi_N(i(S), i(T))$; *and*

(iii) $(S \cdot T)^{c_5} = \psi_E(i(S), i(T)) = \psi_N(i(S), f(i(T)))$.

Proof of (i). As may be checked by inspecting diagrams of closures of $S \cdot E$, the claim is trivial except for the fifth component of $f \circ i(S)$, which corresponds to the c_5 closure of $S \cdot E$. For the latter case one has



$$= \quad S^{c_6} + zS^{c_1} - zS^{c_5} + zS^{c_4},$$

as required.

(Sketch of) Proof of (ii). Let σ, τ the inhabitants of $\mathcal{S}(R_{\mathrm{std}})$ related to S, T as in Figure 7. Since $(\sigma\cdot\tau)^{c_6} = \psi_N(i(\sigma), i(\tau))$, the proof boils down to showing that $i(\sigma) = g(i(S))$ and $i(\tau) = h(i(T))$, where g and h are the linear mappings defined above. This can be achieved by computing the Conway polynomials of the six closures for σ and τ, then expressing each polynomial as a linear combination of $\sigma^{c_1}, \ldots, \sigma^{c_6}$ and $\tau^{c_1}, \ldots, \tau^{c_6}$, respectively, and finally checking that the coefficients of the linear combination correspond with the definitions of g and h. The required calculations, which follow a pattern similar to the one in the proof of **(i)**, are straightforward yet lengthy, and are therefore omitted.

Proof of (iii). One has, by definition, $(S\cdot T)^{c_5} = (S\cdot(T\cdot E))^{c_6}$ whereas, by **(i)**, $i(T\cdot E) = f \circ i(T)$. Using **(ii)**, it follows that $(S \cdot T)^{c_5} = \psi_N(i(S), i(T \cdot E)) = \psi_N(i(S), f \circ i(T)) = \psi_E(S, T)$. ∎

3.1. Computation of i for juxtaposed 3-tangles

Certainly, a desirable property of an invariant such as i would be its computability in practical terms. In particular, in an attempt to use a "divide and conquer" strategy, one would wish to be able to compute $i(S \cdot T)$ as a function of $i(S)$ and $i(T)$. As it turns out, this is the case for i. But, as shown in [1], even more is true for its first four components arranged in matrix form. To see this in more detail consider, for a given $S \in \mathcal{S}(R')$, the 2×2 matrix with entries taken from the components of $i(S)$ arranged as follows

$$M_{\nabla}(S) = \begin{pmatrix} S^{c_3} & S^{c_4} \\ S^{c_1} & S^{c_2} \end{pmatrix}. \tag{4}$$

One has

Proposition 2. [1] *For any $S, T \in \mathcal{S}(R')$, $M_{\nabla}(S \cdot T) = M_{\nabla}(S)M_{\nabla}(T)$.*

Combining Propositions 1 and 2 one obtains the following immediate corollary. It allows one to compute $i(S \cdot T)$ in terms of $i(S)$ and $i(T)$, but simple inspection of the definitions of ψ_N and ψ_E indicate that the situation for i is less fortunate than for M_{∇}. Indeed, while M_{∇} is actually a group homomorphism from $(\mathcal{S}(R'), `\cdot')$ to the multiplicative group $M_{2\times 2}(\mathbb{F})$, there is no evident way to endow $\mathbb{F}^6$ with a multiplicative structure $\star$ so as to have "$i(S \cdot T) = i(S) \star i(T)$." As discussed below, however, explicit expressions shall be obtained for the subclass of $\mathcal{S}(R')$ consisting of 3-braids.

Corollary 3. *For $S, T \in \mathcal{S}(R')$ one has $i(S \cdot T) = \eta(i(S), i(T))$, where $\eta : \mathbb{F}^6 \times \mathbb{F}^6 \longrightarrow \mathbb{F}^6$ is the bilinear mapping defined by*

$$\begin{aligned} \eta(x, y) \quad = \quad & (x_1y_3 + x_2y_1,\ x_1y_4 + x_2y_2,\ x_3y_3 + x_4y_1,\ x_3y_4 + x_4y_2, \\ & \psi_E(x, y),\ \psi_N(x, y))^T . \end{aligned}$$

4. Computations for 3-braids

4.1. Definition of braids and their diagrams

An n-***braid*** is a set of n oriented strands traversing a box steadily from the left to the right (cf. [5]; and some parts of this section are adapted from [6]). For example, the six inhabitants $\Sigma_1, \ldots, \Sigma_6$ of R_{std} shown in Figure 5 are all 3-braids. Every 3-braid may be regarded as a 3-tangle, but the converse need not hold; for instance, the 3-tangle in Figure 3 is not a 3-braid. A braid diagram, viewed as a planar representation of a braid, is uniquely determined by a finite sequence of integers $a_1, \ldots, a_n$, in which case it is denoted by $\mathcal{T}(a_1, \ldots, a_n)$. Pictorially, $\mathcal{T}(a_1, \ldots, a_n)$ represents one of the two diagrams of Figure 9, according to whether n is odd or even. In Figure 9, each box contains an integer a_i representing $|a_i|$ crossings; the sign convention is taken so that alternating diagrams, for which no consecutive under- or over-passes are found as each of its strands is traversed from left to right, correspond to integers a_i with semi-definite sign. In other words, for an alternating braid diagram $\mathcal{T}(a_1, \ldots, a_n)$, either $a_i \geq 0$ for $i = 1, \ldots, n$, or $a_i \leq 0$ for $i = 1, \ldots, n$. As an example, the alternating braid $\mathcal{T}(3, 2, 1, 2, 2)$ is depicted in Figure 10(a).

Given two braids A and B, $A \cdot B$ is, as before, the braid obtained by juxtaposition of A and B. With this operation, n-braids form a noncommutative group with identity

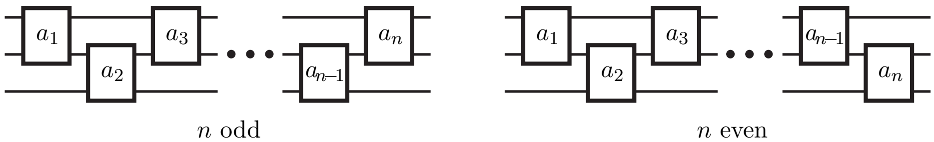


Figure 9: The 3-braid represented by a diagram of the form $\mathcal{T}(a_1, \ldots, a_n)$ according to whether n is odd or even. Note that a_1 always represents crossings involving the uppermost two strands.

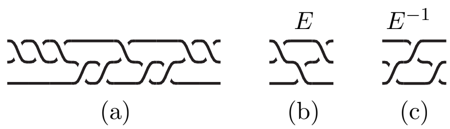


Figure 10: Examples of diagrams of 3-braids: (a) An alternating diagram given by $\mathcal{T}(3, 2, 1, 2, 2)$. (b) and (c): Examples of non-alternating diagrams representing the 3-braids $E = \mathcal{T}(1, -1, 1)$ and $E^{-1} = \mathcal{T}(0, 1, -1, 1)$, respectively.

A diagram for the inverse of an n-braid with diagram A will be denoted by A^{-1}, hence the meaning of A^k is clear for $k \in \mathbb{Z}$. Two important non-alternating diagrams in the representation of braids are E (which was introduced above in the definition of c_5) and its inverse E^{-1}, depicted in Figures 10(b) and (c) respectively, and represented by the following diagrams

$$E = \mathcal{T}(1, -1, 1) \quad \text{and} \quad E^{-1} = \mathcal{T}(0, 1, -1, 1).$$

It was shown in [7] that every braid B admits a ***standard diagram***, that is, a diagram of the form $B = AD \cdot E^k$, where AD is alternating and $k \in \mathbb{Z}$.

4.2. Which 3-braids are inhabitants of R'?

It is easy to see that not all 3-braids are inhabitants of the room R'. Such is the case, for instance, of $T = \mathcal{T}(2, 2, 3)$; to see why, it suffices to inspect the permutation $\pi(T)$ and observe that $\pi(T)(2) \neq 2$, i.e., the strand of T that is attached to the *left* middle port of R' does not end at the middle *right* port, and hence the orientations are not respected. On the other hand, all braids with diagrams $\mathcal{T}(1, k, 3)$ ($k \in \mathbb{Z}$) are in $\mathcal{S}(R')$, and so is E^k for every $k \in \mathbb{Z}$. The essential property for a 3-braid to belong to $\mathcal{S}(R')$ is that the associated permutation leaves 2 invariant, i.e., for a 3-braid T, $\pi(T)(2) = 2$ if and only if $T \in \mathcal{S}(R')$. To that effect, a necessary and sufficient condition is that the braid admits a diagram of the form $\mathcal{T}(2a_1, \ldots, 2a_n) \cdot E^k$, as stated in the following theorem. Before presenting the theorem, however, let us mention that braid diagrams satisfy an interesting property whereby a number of boxes may be "raised" or "lowered" without changing the braid, as long as each of those moves is compensated by including an appropriate power of E. More precisely, let $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$. Then, for k

odd:

$$\begin{aligned}\mathcal{T}(a_1,\dots,a_k,a_{k+1},\dots,a_n) &= \mathcal{T}(a_1,\dots,a_k+1,-1,1-a_{k+1},-a_{k+2},\dots,-a_n)\cdot E^{-1}\\ &= \mathcal{T}(a_1,\dots,a_k-1,1,-1-a_{k+1},-a_{k+2},\dots,-a_m)\cdot E,\end{aligned} \tag{5}$$

whereas, for k even:

$$\begin{aligned}\mathcal{T}(a_1,\dots,a_k,a_{k+1},\dots,a_m) &= \mathcal{T}(a_1,\dots,a_k+1,-1,1-a_{k+1},-a_{k+2},\dots,-a_m)\cdot E\\ &= \mathcal{T}(a_1,\dots,a_k-1,1,-1-a_{k+1},-a_{k+2},\dots,-a_m)\cdot E^{-1}.\end{aligned} \tag{6}$$

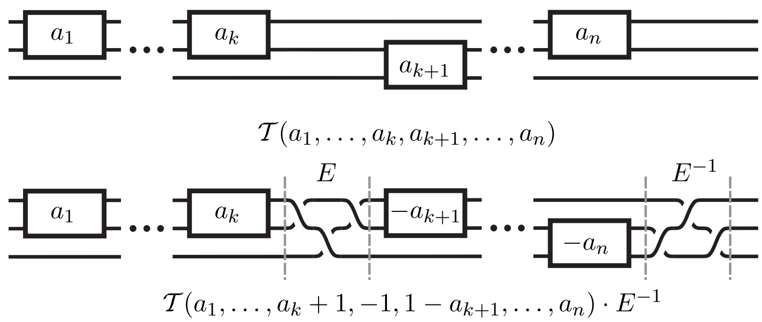


Figure 11: Twisting the portion of the braid consisting of all boxes from the $(k+1)$th to the nth one results in these boxes "swapping" their position, i.e., lower boxes are raised and upper boxes are lowered. The net effect of the swap is that factors E and E^{-1} are inserted after the kth and the nth boxes, respectively. Obviously the diagram changes but the 3-braid remains unaltered. Only the case when k and n are odd is shown; analogous conclusions hold for the other cases.

The proof of these equalities is easily derived by inspection of Figure 11, where only the case when k and n are odd is shown. Notice that the sign change in the boxes affected by the twisting is due to the sign conventions adopted for 3-braids. Another obvious property of 3-braids, to be used below in the proof of Theorem 4, is that $\mathcal{T}(a_1,a_2,\dots,a_n)\in\mathcal{S}(R')$ if and only if $\mathcal{T}(0,-a_1,-a_2,\dots,-a_n)\in\mathcal{S}(R')$.

Theorem 4. *A 3-braid T belongs to $\mathcal{S}(R')$ if and only if there exist integers $a_1,\dots,a_n$ and k such that $T=\mathcal{T}(2a_1,2a_2,\dots,2a_n)\cdot E^k$.*

Sketch of Proof. If $T=\mathcal{T}(2a_1,\dots,2a_n)\cdot E^k$ for some integers $a_1,\dots,a_n$ and k, then it is clear that $T\in S(R')$. Now suppose that $T\in\mathcal{S}(R')$. By the result from [7] cited above, any 3-braid T admits a diagram of the form $T=\mathcal{T}(b_1,\dots,b_m)\cdot E^r$, where $b_ib_{i+1}\geq 0$ for $i=1,2,...,n-1$, $b_jb_{j+1}>0$ for $i=2,...,n-1$, and $r\in\mathbb{Z}$. Since both $T=\mathcal{T}(b_1,b_2,\dots,b_m)\cdot E^r$ and E^{-r} belong to $S(R')$, it suffices to prove that $\mathcal{T}(b_1,b_2,\dots,b_m)=\mathcal{T}(2a_1,2a_2,\dots,2a_n)\cdot E^k$ for some integers $a_1,\dots,a_n$ and k. Let us assume that $b_i\geq 0$ (the case $b_i\leq 0$ is analogous). A necessary condition to have $\mathcal{T}(b_1,b_2,\dots,b_m)\in S(R')$ is that $\pi(\mathcal{T}(b_1,b_2,\dots,b_m))(2)=2$, i.e., the second strand connects the left and right middle ports of R'. Let us proceed by induction on the number m of boxes in the diagram. For $m = 1$ the claim is clear since if b_1 is odd then $\pi(\mathcal{T}(b_1))(2) \neq 2$, a contradiction. Now let $m \in \mathbb{N}$ assume that for any $k \leq m$, the claim is true. Let $b_1, \ldots, b_{m+1} \in \mathbb{Z}$ and let $T = \mathcal{T}(b_1, \ldots, b_{m+1})$. If $b_1, \ldots, b_{m+1}$ are all even, the proof is finished. Otherwise, let q the least integer such that b_q is odd. Since the permutation of the braid consisting of the boxes $b_q, \ldots, b_m$ must leave 2 invariant, there exists a smallest integer $r \in \{q+1, \ldots, m\}$ such that $r \equiv q \mod 2$ and b_r is odd. Assume that $q > 1$ or $r < m$. If $q = 1$ then $\mathcal{T}(b_1, \ldots, b_r)$ has fewer than m boxes, so by the induction assumption it admits a diagram all of whose boxes are even. If $q > 2$ then one applies the induction assumption to $\mathcal{T}(b_q, \ldots, b_{m+1})$ or to $\mathcal{T}(0, b_q, \ldots, b_{m+1})$ according to whether q is odd or even, respectively. If $q = 2$ then apply the induction assumption to $T' = \mathcal{T}(-b_2, -b_3, \ldots, -b_{m+1})$ to write $T' = \mathcal{T}(2a_1, \ldots, 2a_l) \cdot E^p$, which implies that $T = \mathcal{T}(0, -2a_1, \ldots, -2a_l) \cdot E^p$. Now consider the case when $q = 1$ and $r = m+1$. In this case a procedure will be applied to pass from a diagram $\mathcal{T}(b_1, \ldots, b_{m+1})$, with m even, to an equivalent diagram $\mathcal{T}(2a_1, \ldots, 2a_l) \cdot E^p$ for some integers $a_1, \ldots, a_l$ and p. This procedure shall be illustrated only for the case $\mathcal{T}(b_1, b_2, b_3)$; the general case is analogous. The essence of this procedure consists in repeatedly applying either (5) or (6) until the required conclusion is reached. One has

$$\begin{aligned}
\mathcal{T}(b_1, b_2, b_3) &= \mathcal{T}(b_1+1, -1, 1-b_2, -b_3) \cdot E^{-1} \\
&= \mathcal{T}(b_1+1, -1-1, 1, -2+b_2, b_3) \cdot E^{-2} \\
&= \mathcal{T}(b_1+1, -2, 1+1, -1, 3-b_2, -b_3) \cdot E^{-3} \\
&\vdots \\
&= \mathcal{T}(b_1+1, -2, 2, \ldots, (-1)^{b_2-1}2, (-1)^{b_2}(b_3+1)) \cdot E^{-b_2}.
\end{aligned}$$

∎

Having established that every 3-braid $T \in \mathcal{S}(R')$ may be written as $\mathcal{T}(2a_1, \ldots, 2a_n) \cdot E^k$, the next step is to find explicit expressions for its invariant $i(T)$. To this purpose, an instrumental role is played by the so-called continued fractions. Recall that, given $a_1, \ldots, a_n$ in a field $\mathbb{F}$, the ***continued fraction*** associated to those elements, denoted $[a_1, \ldots, a_n]$ is defined by

$$[a_1, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

For a continued fraction $[a_1, \ldots, a_n]$, one defines elements $N([a_1, \ldots, a_n])$ and $D([a_1, \ldots, a_n])$ in $\mathbb{F}$, referred to as the ***numerator*** and the ***denominator*** of $[a_1, \ldots, a_n]$, respectively, in an inductive way:

$$N([a_1]) = a_1, \quad D([a_1]) = 1, \quad N([a_1, a_2]) = 1 + a_1a_2, \quad D([a_1, a_2]) = a_2,$$

and, for $n \geq 3$:

$$\begin{aligned}
N([a_1, \ldots, a_n]) &= a_n N([a_1, \ldots, a_{n-1}]) + N([a_1, \ldots, a_{n-2}]) \\
D([a_1, \ldots, a_n]) &= a_n D([a_1, \ldots, a_{n-1}]) + D([a_1, \ldots, a_{n-2}]).
\end{aligned}$$

Pursuing with the computation of the invariant $i(T)$ for a 3-braid T, and considering that its first four components are the entries of $M_\nabla(T)$, it is useful to obtain expressions for the matrix associated to the "building blocks" of all 3-braids, namely those of the form $\mathcal{T}(2a)$, $\mathcal{T}(0,2a)$ and E^k, with $a,k\in\mathbb{Z}$. Using simple link diagrams one shows that

$$M_\nabla(\mathcal{T}(2n)) = \begin{pmatrix}1 & 0\\ nz & 1\end{pmatrix},\quad M_\nabla(\mathcal{T}(0,2n)) = \begin{pmatrix}1 & -nz\\ 0 & 1\end{pmatrix}$$
$$M(E) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad M(E^2) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Using these equations, along with the fact that

$$\mathcal{T}(a_1,\dots,a_n) = \begin{cases} \mathcal{T}(a_1)\cdot\mathcal{T}(0,a_2)\cdot\mathcal{T}(a_3)\cdots\mathcal{T}(a_n), & n \text{ odd;}\\ \mathcal{T}(a_1)\cdot\mathcal{T}(0,a_2)\cdot\mathcal{T}(a_3)\cdots\mathcal{T}(0,a_n), & n \text{ even,}\end{cases}$$

and Proposition 2, the following result is easily established.

Theorem 5. *For integers $a_1,\dots,a_n$ and a braid $T=\mathcal{T}(2a_1,\dots,2a_n)$ $(n\geq 2)$ one has:*

$$M_\nabla(T) = \begin{cases} \begin{pmatrix} D[a_1z,-a_2z,a_3z,\dots,a_nz] & D[a_1z,-a_2z,a_3z,\dots,-a_{n-1}z]\\ N[a_1z,-a_2z,a_3z,\dots,a_nz] & N[a_1z,-a_2z,a_3z,\dots,-a_{n-1}z]\end{pmatrix}, & n \textit{ odd;}\\ \begin{pmatrix} D[a_1z,-a_2z,a_3z,\dots,a_{n-1}z] & D[a_1z,-a_2z,a_3z,\dots,-a_nz]\\ N[a_1z,-a_2z,a_3z,\dots,a_{n-1}z] & N[a_1z,-a_2z,a_3z,\dots,-a_nz]\end{pmatrix}, & n \textit{ even.}\end{cases} \tag{7}$$

The following theorem determines the fifth and sixth components of $i(T)$ as affine functions of its first four components.

Theorem 6. *Given integers $a_1,\dots,a_n$, let $T=\mathcal{T}(2a_1,\dots,2a_n)$. Then the following relationships hold amongst components of $i(T)$:*

$$T^{c_5} = T^{c_1}+T^{c_4}+z \quad \textit{and} \quad T^{c_6} = T^{c_2}+T^{c_3}-2. \tag{8}$$

Proof. Let us first note, with the aid of Figure 12, that for $a\in\mathbb{Z}$ the six components of $i(\mathcal{T}(2a))$ are easily computed to be $i(\mathcal{T}(2a)) = (az,1,1,0,z(a+1),0)^T$. Similarly one deduces that $i(\mathcal{T}(0,2a)) = (0,1,1,-az,z(1-a),0)^T$.

Let $a_1\in\mathbb{Z}$ and set $T=\mathcal{T}(2a_1)$. Then $i(T)=(a_1z,1,1,0,z(a_1+1),0)^T$ and hence, in particular,

$$T^{c_5} = a_1z+z = T^{c_1}+T^{c_4}+z \quad\text{and}\quad T^{c_6}=0=T^{c_2}+T^{c_3}-2.$$

Now let $a_1,a_2\in\mathbb{Z}$ and consider $T=\mathcal{T}(2a_1,2a_2)=\mathcal{T}(2a_1)\cdot\mathcal{T}(0,2a_2)$. One has $x = i(\mathcal{T}(2a_1)) = (a_1z,1,1,0,z(a_1+1),0)$ and $y=i(\mathcal{T}(0,2a_2)) = (0,1,1,-a_2z,z(1-a_2),0)$; computing $\eta(x,y)$ as in Corollary 3, and considering the expressions for ψ_E and ψ_N given in (3), one gets $i(T)=(a_1z,1-a_1a_2z^2,1,-a_2z,(a_1-a_2+1)z,-a_1a_2z^2)^T$. Consequently,

$$T^{c_5} = a_1z-a_2z+z = T^{c_1}+T^{c_4}+z \quad\text{and}\quad T^{c_6} = -a_1a_2z^2 = T^{c_2}+T^{c_3}-2.$$

Therefore (8) is valid for $n\in\{1,2\}$.

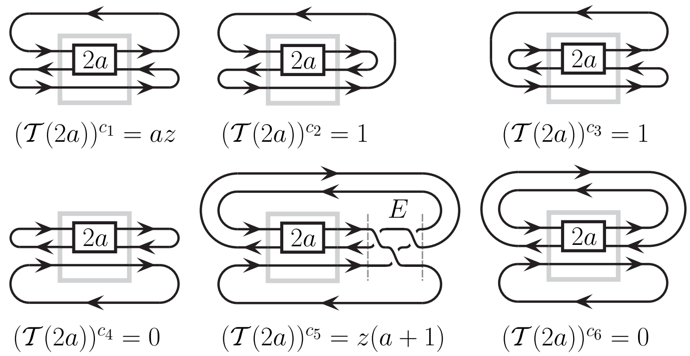


Figure 12: The six components of the invariant i for a 3-braid $\mathcal{T}(2a)$, which consists of $2a$ crossings involving only the upper two strands.

Now let $n \in \mathbb{N}$, $n \geq 2$, and assume that for any $k \leq n$, (8) holds true. Consider integers $a_i \in \mathbb{Z}$ $(i = 1, \ldots, n+1)$ and let $T = \mathcal{T}(2a_1, \ldots, 2a_{n+1})$. Set $x = i(\mathcal{T}(2a_1, \ldots, 2a_n))$. Assume that n is odd. In this case $T = \mathcal{T}(2a_1, \ldots, 2a_n) \cdot \mathcal{T}(0, 2a_{n+1})$ and hence, by virtue of Corollary 3, in order to compute $i(T)$ it suffices to know the values of x and $i(\mathcal{T}(0, 2a_{n+1}))$. The definition (4) of M_∇, along with the expressions in (7) for n odd yield

$$x_1 = N([a_1 z, -a_2 z, a_3 z, \ldots, a_n z]), \quad x_2 = N([a_1 z, -a_2 z, a_3 z, \ldots, -a_{n-1} z]),$$
$$x_3 = D([a_1 z, -a_2 z, a_3 z, \ldots, a_n z]), \quad x_4 = D([a_1 z, -a_2 z, a_3 z, \ldots, -a_{n-1} z]),$$

whereas, by the induction assumption,

$$\begin{aligned} x_5 &= x_1 + x_4 + z \\ &= N([a_1 z, -a_2 z, a_3 z, \ldots, a_n z]) + D([a_1 z, -a_2 z, a_3 z, \ldots, -a_{n-1} z]) + z, \\ x_6 &= x_2 + x_3 - 2 \\ &= N([a_1 z, -a_2 z, a_3 z, \ldots, -a_{n-1} z]) + D([a_1 z, -a_2 z, a_3 z, \ldots, a_n z]) - 2. \end{aligned}$$

Moreover, $i(\mathcal{T}(0, 2a_{n+1})) = (0, 1, 1, -a_{n+1} z, z(1 - a_{n+1}), 0)^T$ and thus, carrying out the computations required for ψ_N and simplifying the resulting expressions yields

$$\begin{aligned} \psi_N(x, i(\mathcal{T}(0, 2a_{n+1}))) &= -a_{n+1} z x_1 + x_6 \\ &= -a_{n+1} z N([a_1 z, -a_2 z, a_3 z, \ldots, a_n z]) \\ &\quad + N([a_1 z, -a_2 z, a_3 z, \ldots, -a_{n-1} z]) \\ &\quad + D([a_1 z, -a_2 z, a_3 z, \ldots, a_n z]) - 2 \\ &= N([a_1 z, -a_2 z, a_3 z, \ldots, -a_{n+1} z]) + \\ &\quad D([a_1 z, -a_2 z, a_3 z, \ldots, a_n z]) - 2 \\ &= (\mathcal{T}(2a_1, \ldots, 2a_{n+1}))^{c_2} + (\mathcal{T}(2a_1, \ldots, 2a_{n+1}))^{c_3} - 2, \end{aligned}$$

which proves that the second equation in (8) holds for $n+1$ even. Similarly:

$$\begin{aligned}\psi_E(x, i(\mathcal{T}(0, 2a_{n+1}))) &= -a_{n+1}zx_3 + x_5 \\ &= -a_{n+1}zD([a_1z, -a_2z, a_3z, \ldots, a_nz]) \\ &\quad +N([a_1z, -a_2z, a_3z, \ldots, a_nz]) \\ &\quad +D([a_1z, -a_2z, a_3z, \ldots, -a_{n-1}z]) + z \\ &= D([a_1z, -a_2z, a_3z, \ldots, -a_{n+1}z]) \\ &\quad +N([a_1z, -a_2z, a_3z, \ldots, a_nz]) + z \\ &= (\mathcal{T}(2a_1, \ldots, 2a_{n+1}))^{c_4} + (\mathcal{T}(2a_1, \ldots, 2a_{n+1}))^{c_1} + z,\end{aligned}$$

which in turn proves the first expression in (8) for $n+1$ even. Now assume that n is even. Thus $T = \mathcal{T}(2a_1, \ldots, 2a_n) \cdot \mathcal{T}(2a_{n+1})$ and, again by Corollary 3, it suffices to know the values of x and $i(\mathcal{T}(2a_{n+1}))$ to get $i(T)$. Using again (4) and (7) for n even

$$\begin{aligned}x_1 &= N([a_1z, -a_2z, a_3z, \ldots, a_{n-1}z]), \quad x_2 = N([a_1z, -a_2z, a_3z, \ldots, -a_nz]), \\ x_3 &= D([a_1z, -a_2z, a_3z, \ldots, a_{n-1}z]), \quad x_4 = D([a_1z, -a_2z, a_3z, \ldots, -a_nz]),\end{aligned}$$

and hence, by the induction assumption,

$$\begin{aligned}x_5 &= x_1 + x_4 + z \\ &= N([a_1z, -a_2z, a_3z, \ldots, a_{n-1}z]) + D([a_1z, -a_2z, a_3z, \ldots, -a_nz]) + z, \\ x_6 &= x_2 + x_3 - 2 \\ &= N([a_1z, -a_2z, a_3z, \ldots, -a_nz]) + D([a_1z, -a_2z, a_3z, \ldots, a_{n-1}z]) - 2.\end{aligned}$$

Since $i(\mathcal{T}(2a_{n+1})) = (a_{n+1}z, 1, 1, 0, z\,(a_{n+1}+1)\,, 0)^T$, it follows from the expression for ψ_N that

$$\begin{aligned}\psi_N(x, i(\mathcal{T}(2a_{n+1}))) &= a_{n+1}zx_4 + x_6 \\ &= a_{n+1}zD([a_1z, -a_2z, a_3z, \ldots, -a_nz]) + \\ &\quad +N([a_1z, -a_2z, a_3z, \ldots, -a_nz]) \\ &\quad +D([a_1z, -a_2z, a_3z, \ldots, a_{n-1}z]) - 2 \\ &= D([a_1z, , -a_2z, a_3z, \ldots, a_{n+1}z]) \\ &\quad +N([a_1z, -a_2z, a_3z, \ldots, -a_nz]) - 2 \\ &= (\mathcal{T}(2a_1, \ldots, 2a_{n+1}))^{c_3} + (\mathcal{T}(2a_1, \ldots, 2a_{n+1}))^{c_2} - 2,\end{aligned}$$

which establishes the second expression in (8) for $n+1$ odd. Analogously:

$$\begin{aligned}\psi_E(x, i(\mathcal{T}(2a_{n+1}))) &= a_{n+1}zx_2 + x_5 \\ &= a_{n+1}zN([a_1z, -a_2z, a_3z, \ldots, -a_nz]) \\ &\quad +N([a_1z, -a_2z, a_3z, \ldots, a_{n-1}z]) \\ &\quad +D([a_1z, -a_2z, a_3z, \ldots, -a_nz]) + z \\ &= N([a_1z, -a_2z, a_3z, \ldots, a_{n+1}z]) \\ &\quad +D([a_1z, -a_2z, a_3z, \ldots, -a_nz]) + z \\ &= (\mathcal{T}(2a_1, \ldots, 2a_{n+1}))^{c_1} + (\mathcal{T}(2a_1, \ldots, 2a_{n+1}))^{c_4} + z,\end{aligned}$$

which proves that the first equation in (8) holds for $n+1$ odd. This finishes the proof. ∎

5. Explicit formulas for continued fractions and the invariant i for 3-braids

In this section a number of explicit, *nonrecursive* formulas are given to express the numerator and denominator of continued fractions. On account of Theorems 5 and 6, as a derived result these expressions provide an easy way to compute the Conway polynomials required to obtain the invariant $i(T)$ for 3-braids in $\mathcal{S}(R')$.

In the expressions below, a specific multi-index notation will be used as follows. For $k \in \mathbb{N}$, a k ***multi-index*** is an ordered k-tuple of integers $I = (i_1, \ldots, i_k)$. A k multi-index I is said to have ***alternating parity*** (or more simply, to be ***alternating***) if either $k = 1$ or, for each $j \in \{1, \ldots, k-1\}$, i_j is odd if and only if i_{j+1} is even. Given strictly positive integers n and k, let

$$Q(n,k) = \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \;\mid\; i_1 < i_2 < \cdots < i_n \text{ and } I \text{ is alternating}\right\}.$$

In other words, $Q(n,k)$ is the set of alternating, strictly increasing k multi-indices with entries in $\{1, \ldots, n\}$. Clearly, $Q(n,k) = \emptyset$ whenever $k > n$. Each set $Q(n,k)$ is further partitioned, according to the parity of the first entry of each of its elements, into the following two subsets:

$$Q^s(n,k) = \{(i_1, \ldots, i_k) \in Q(n,k) \;\mid\; i_1 \equiv s \mod 2\}, \quad s = 0, 1.$$

Hence $Q^0(n,k)$ and $Q^1(n,k)$ consist of all multi-indices in $Q(n,k)$ which have even and odd leading entries, respectively. An obvious property of the sets just defined is that if $k, n, m \in \mathbb{N}$ satisfy $n < m$, then $Q(n,k) \subset Q(m,k)$ and $Q^s(n,k) \subset Q^s(m,k)$, $s = 0, 1$. The main use that will be made of these indexing sets is to represent certain multinomials in a compact way. Thus, if $a_1, \ldots, a_n$ are indeterminates (or, for that matter, elements in any ring) and $I = (i_1, \ldots, i_k)$ is a k multi-index with entries in $\{1, \ldots, n\}$, one sets $a_I = a_{i_1} a_{i_2} \cdots a_{i_k}$. The notation $\sum_{I \in \mathcal{I}} a_I$ is therefore clear for any finite set $\mathcal{I}$ of multi-indices with entries in $\{1, \ldots, n\}$.

Example 1. With $n = 4$ and $k = 4$ one has

$$Q(4,1) = \{1,2,3,4\}, \quad Q(4,2) = \{(1,2),(1,4),(2,3),(3,4)\}$$
$$Q(4,3) = \{(1,2,3),(2,3,4)\} \quad \text{and} \quad Q(4,4) = \{(1,2,3,4)\},$$

whereas

$$Q^0(4,1) = \{2,4\}, \quad Q^0(4,2) = \{(2,3)\}, \quad Q^0(4,3) = \{(2,3,4)\}, \quad Q^0(4,4) = \emptyset,$$

and

$$Q^1(4,1) = \{1,3\}, \quad Q^1(4,2) = \{(1,2),(1,4),(3,4)\},$$
$$Q^1(4,3) = \{(1,2,3)\}, \quad Q^1(4,4) = \{(1,2,3,4)\}.$$

And if $a_1, \ldots, a_4$ belong to a ring, then, for instance,

$$\sum_{k=1}^{4} \sum_{I \in Q^1(4,k)} a_I = a_1 + a_3 + a_1a_2 + a_1a_4 + a_3a_4 + a_1a_2a_3 + a_1a_2a_3a_4.$$

The following technical lemmas, which single out basic properties of the sets defined above, will be useful to prove the ensuing results.

Lemma 7. *Let $k, n \in \mathbb{N}$, with $k \le n$. Then $Q^1(n,k) = Q^1(n+1,k)$ whenever $k \equiv n \mod 2$, and $Q^0(n,k) = Q^0(n+1,k)$ whenever $k \not\equiv n \mod 2$.*

Proof. Let $k, n \in \mathbb{N}$ and assume that $k \le n$. Obviously, $Q^s(n,k) \subset Q^s(n+1,k)$, $s = 0, 1$, hence only the reverse inclusions need to be proven. Suppose that $k \equiv n \mod 2$ and let $I = (i_1, \dots, i_k) \in Q^1(n+1,k)$ so that $i_1 < \cdots < i_k \le n+1$. Since the parities of k and n coincide, $i_k \not\equiv n+1 \mod 2$, so $i_1 < \cdots < i_k \le n$. But $k \le n$, hence $I \in Q^1(n,k)$. Now suppose that $k \not\equiv n \mod 2$ and let $I = (i_1, \dots, i_k) \in Q^0(n+1,k)$. Since the parities of k and n are different, one again has $i_k \not\equiv n+1 \mod 2$, so $i_1 < \cdots < i_k \le n$. But $k \le n$, hence $I \in Q^0(n,k)$. ∎

Lemma 8. *Let $k, n \in \mathbb{N}$, with $n > 2$, and for any set S of k multi-indices let*

$$A_n(S) = \{(i_1, \dots, i_k) \in S \mid i_k = n\}.$$

Then $Q^1(n,k) = A_n(Q^1(n,k)) \cup Q^1(n-2,k)$ whenever $k \equiv n \mod 2$, and $Q^0(n,k) = A_n(Q^0(n,k)) \cup Q^0(n-2,k)$ whenever $k \not\equiv n \mod 2$.

Proof. Let $k, n \in \mathbb{N}$. If $k > n-2$ then $Q(n-2,k) = \emptyset$ and one immediately checks that the claim holds, so let us assume that $k \le n-2$. Suppose that $k \equiv n \mod 2$ and let $I \in Q^1(n,k) \setminus A_n(Q^1(n,k))$. Then $i_k \ne n$, so $i_1 < \cdots < i_k < n$ and, since k and n are both even or both odd, it follows that $i_k \equiv n \mod 2$. But $i_k < n$, thus $i_1 < \cdots < i_k \le n-2$ and $I \in Q^1(n-2,k)$. Since $Q^1(n-2,k) \subset Q^1(n,k)$, and given that none of the multi-indices in $Q^1(n-2,k)$ have last entry equal to n, one has $Q^1(n,k) \setminus A_n(Q^1(n,k)) = Q^1(n-2,k)$, as required. Now suppose that $k \not\equiv n \mod 2$ and let $I \in Q^0(n,k) \setminus A_n(Q^0(n,k))$. Then $i_k \ne n$, so $i_1 < \cdots < i_k < n$ and, since the parity of k differs from that of n, it again follows that $i_k \equiv n \mod 2$. But $i_k < n$, thus $i_1 < \cdots < i_k \le n-2$ and $I \in Q^0(n-2,k)$. Since $Q^0(n-2,k) \subset Q^0(n,k)$, and given that none of the multi-indices in $Q^0(n-2,k)$ have last entry equal to n, one has $Q^0(n,k) \setminus A_n(Q^0(n,k)) = Q^0(n-2,k)$, as was to be shown. ∎

The sets and notations described above provide explicit nonrecursive formulas for the numerator and denominator of continued fractions, as stated in the following result.

Theorem 9. *Given elements $a_1, \dots, a_n$ in $\mathbb{F}$, one has*

$$N([a_1, \dots, a_n]) = \begin{cases} \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{I \in Q^1(n,2k-1)} a_I, & \text{if } n \text{ is odd;} \\ 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{I \in Q^1(n,2k)} a_I, & \text{otherwise;} \end{cases} \tag{9}$$

and

$$D([a_1, \dots, a_n]) = \begin{cases} 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{I \in Q^0(n,2k)} a_I, & \text{if } n \text{ is odd;} \\ \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{I \in Q^0(n,2k-1)} a_I, & \text{otherwise.} \end{cases} \tag{10}$$

Proof. First note that

$$Q(1,1) = \{1\}, \quad Q^0(1,1) = \emptyset, \quad Q^1(1,1) = \{1\},$$
$$Q(2,1) = \{1,2\}, \quad Q^0(2,1) = \{2\}, \quad Q^1(2,1) = \{1\},$$
$$Q(2,2) = \{(1,2)\}, \quad Q^0(2,2) = \emptyset, \quad Q^1(2,2) = \{(1,2)\}.$$

Clearly,

$$N([a_1]) = a_1 \sum_{k=1}^{1} \sum_{I\in Q^1(1,2k-1)} a_I,$$

$$D([a_1]) = 1 = 1 + \sum_{k=1}^{0} \sum_{I\in Q^0(1,2k)} a_I$$

$$N([a_1,a_2]) = 1 + a_1a_2 = 1 + \sum_{k=1}^{1} \sum_{I\in Q^1(2,2k)} a_I$$

$$D([a_1,a_2]) = a_2 = \sum_{k=1}^{1} \sum_{I\in Q^0(2,2k-1)} a_I,$$

so (9) and (10) hold for $n \in \{1,2\}$. Now suppose that these formulas hold for some $n \geq 2$ and every family $a_1,\ldots,a_k$ of elements in $\mathbb{F}$, with $k \leq n$. Let $a_1,\ldots,a_{n+1}$ be in $\mathbb{F}$. By definition,

$$\begin{aligned} N([a_1,\ldots,a_{n+1}]) &= a_{n+1}N([a_1,\ldots,a_n]) + N([a_1,\ldots,a_{n-1}]) \\ D([a_1,\ldots,a_{n+1}]) &= a_{n+1}D([a_1,\ldots,a_n]) + D([a_1,\ldots,a_{n-1}]). \end{aligned}$$

Suppose that n is odd, so that $\lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2}$ and $\lfloor \frac{n+1}{2} \rfloor = \frac{n+1}{2} = \frac{n-1}{2} + 1$. Then

$$\begin{aligned} N([a_1,\ldots,a_{n+1}]) &= a_{n+1}N([a_1,\ldots,a_n]) + N([a_1,\ldots,a_{n-1}]) \\ &= a_{n+1} \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{I\in Q^1(n,2k-1)} a_I + \\ &\qquad 1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{I\in Q^1(n-1,2k)} a_I. \end{aligned} \tag{11}$$

Using the fact that $Q^1(n,n) = \{(1,\ldots,n)\}$, incorporating the common factor a_{n+1} into the first summation in the right-hand member of (11), and appealing to the notation of Lemma 8, the same sum may rewritten as

$$a_{n+1} \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{I\in Q^1(n,2k-1)} a_I \quad = \quad a_1\cdots a_{n+1} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{I\in A_{n+1}(Q^1(n+1,2k))} a_I.$$

But in view of Lemma 8, $Q^1(n+1,2k) = A_{n+1}(Q^1(n+1,2k)) \cup Q^1(n-1,2k)$, hence

$$\begin{aligned} N([a_1,\ldots,a_{n+1}]) &= 1 + a_1\cdots a_{n+1} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{I\in Q^1(n+1,2k)} a_I \\ &= 1 + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{I\in Q^1(n+1,2k)} a_I, \end{aligned}$$

as required. For $D([a_1,\dots,a_{n+1}])$ one has

$$\begin{aligned} D([a_1,\dots,a_{n+1}]) &= a_{n+1}D([a_1,\dots,a_n]) + D([a_1,\dots,a_{n-1}]) \\ &= a_{n+1}\left(1+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{I\in Q^0(n,2k)} a_I\right) \\ &\quad +\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{I\in Q^0(n-1,2k-1)} a_I. \end{aligned} \tag{12}$$

Shifting the limits of summation index by one, compensating accordingly, and noting that $Q^0(n+1,1)=\{(n+1)\}$, the first term on the right-hand member of (12) may be rewritten as

$$\begin{aligned} a_{n+1}\left(1+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{I\in Q^0(n,2k)} a_I\right) &= a_{n+1}+\sum_{k=2}^{\lfloor\frac{n}{2}\rfloor+1} a_{n+1}\sum_{I\in Q^0(n,2k-2)} a_I \\ &= a_{n+1}+\sum_{k=2}^{\lfloor\frac{n}{2}\rfloor+1}\sum_{I\in A_{n+1}(Q^0(n+1,2k-1))} a_I \\ &= \sum_{k=1}^{\lfloor\frac{n+2}{2}\rfloor}\sum_{I\in A_{n+1}(Q^0(n+1,2k-1))} a_I. \end{aligned}$$

On the other hand, if $k=\lfloor\frac{n}{2}\rfloor+1=\lfloor\frac{n+2}{2}\rfloor$ then $Q^0(n-1,2k-1)=Q^0(n-1,n)=\emptyset$, thus the upper limit of the second summation in (12) may be replaced by $\lfloor\frac{n+2}{2}\rfloor$ without affecting the sum. Using again Lemma 8 to deduce that $Q^0(n+1,2k-1)=A_{n+1}(Q^0(n+1,2k-1))\cup Q^0(n-1,2k-1)$, it follows that

$$D([a_1,\dots,a_{n+1}]) = \sum_{k=1}^{\lfloor\frac{n+2}{2}\rfloor}\sum_{I\in Q^0(n+1,2k-1)} a_I,$$

as required. The proof in the case when n is even is the same, *mutatis mutandis.* ∎

As a corollary of Theorem 6, the above result leads to explicit, nonrecursive expressions for the components of the polynomial invariant $i(T)$ for T a 3-braid. As remarked earlier, the availability of this type of expressions represents a significant advantage when computing $i(T)$ in practical situations.

Corollary 10. *Given integers $a_1,\dots,a_n$, let $T=\mathcal{T}(2a_1,-2a_2,2a_3\dots,(-1)^{n-1}2a_n)$. Then, if n is odd one has:*

$$T^{c_1}=\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor} z^{2k-1}\sum_{I\in Q^1(n,2k-1)} a_I,\quad T^{c_2}=1+\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} z^{2k}\sum_{I\in Q^1(n-1,2k)} a_I$$

$$T^{c_3}=1+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} z^{2k}\sum_{I\in Q^0(n,2k)} a_I,\quad T^{c_4}=\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor} z^{2k-1}\sum_{I\in Q^0(n-1,2k-1)} a_I,$$

whereas, if n is even:

$$T^{c_1} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} z^{2k-1} \sum_{I \in Q^1(n-1,2k-1)} a_I, \quad T^{c_2} = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} z^{2k} \sum_{I \in Q^1(n,2k)} a_I$$

$$T^{c_3} = 1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} z^{2k} \sum_{I \in Q^0(n-1,2k)} a_I, \quad T^{c_4} = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} z^{2k-1} \sum_{I \in Q^0(n,2k-1)} a_I.$$

The values of T^{c_5} and T^{c_6} are independent of the parity of n and are given by

$$T^{c_5} = z + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} z^{2k-1} \sum_{I \in Q(n,2k-1)} a_I, \quad T^{c_6} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} z^{2k} \sum_{I \in Q(n,2k)} a_I.$$

Remark 1. The definition $T = \mathcal{T}(2a_1, -2a_2, 2a_3 \ldots, (-1)^{n-1}2a_n)$ merely reflects an artifice to simplify the expressions and prevent the need to account for the minus signs in the even entries of the numerators and denominators in (7). Not using this artifice would force one to include additional "sign bookkeeping" factors in the indexed summations for the T^{c_i}s, making the resulting expressions more cumbersome.

Proof of Corollary 10. Owing to the alternating signs used in the definition of T, in this case one has

$$M_\nabla(T) = \begin{cases} \begin{pmatrix} D[a_1z,\ldots,a_nz] & D[a_1z,\ldots,a_{n-1}z] \\ N[a_1z,\ldots,a_nz] & N[a_1z,\ldots,a_{n-1}z] \end{pmatrix}, & n \text{ odd;} \\ \begin{pmatrix} D[a_1z,\ldots,a_{n-1}z] & D[a_1z,\ldots,a_nz] \\ N[a_1z,\ldots,a_{n-1}z] & N[a_1z,\ldots,a_nz] \end{pmatrix}, & n \text{ even.} \end{cases} \tag{13}$$

The proof that the expressions given for $T^{c_1}, \ldots, T^{c_4}$ are correct trivially boils down to computing the numerators and denominators in (13) via the formulas in Theorem 9. The equalities requiring nontrivial proofs are the ones for T^{c_5} and T^{c_6}. Suppose that n is odd. By Theorem 6 one has

$$T^{c_5} = T^{c_1} + T^{c_4} + z \quad \text{and} \quad T^{c_6} = T^{c_2} + T^{c_3} - 2.$$

But

$$T^{c_1} + T^{c_4} = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} z^{2k-1} \sum_{I \in Q^1(n,2k-1)} a_I + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} z^{2k-1} \sum_{I \in Q^0(n-1,2k-1)} a_I. \tag{14}$$

Since n is odd, $\lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$, thus if $k = \lfloor \frac{n+1}{2} \rfloor$ then $2k-1 = n$. Moreover, since $n-1$ is even, Lemma 7 entails that $Q^0(n-1, 2k-1) = Q^0(n, 2k-1)$ and thus $Q(n, 2k-1) = Q^1(n, 2k-1) \cup Q^0(n-1, 2k-1)$. Using these observations in (14) yields

$$\begin{aligned} T^{c_5} &= z + z^n a_1 \cdots a_n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} z^{2k-1} \sum_{I \in Q(n,2k-1)} a_I \\ &= z + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} z^{2k-1} \sum_{I \in Q(n,2k-1)} a_I, \end{aligned}$$

as required. Similarly, from Theorem 6:

$$T^{c_2} + T^{c_3} = 1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} z^{2k} \sum_{I \in Q^1(n-1,2k)} a_I + 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} z^{2k} \sum_{I \in Q^0(n,2k)} a_I. \tag{15}$$

Since n is odd, $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$ and $n-1$ is even. Hence, using Lemma 7, $Q^1(n-1,2k) = Q^1(n,2k)$ and thus $Q(n,2k) = Q^0(n,2k) \cup Q^1(n-1,2k)$. Therefore

$$T^{c_6} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} z^{2k} \sum_{I \in Q(n,2k)} a_I,$$

as was to be shown. The case when n is even is proven similarly. ∎

6. Conclusions

This work extends previous work on closures of 3-tangles viewed as elements in the skein $\mathcal{S}(R')$ of a room R'. The nonstandard orientation of R' allows the definition of five different closures—referred to throughout as $c_1, \ldots, c_4$ and c_6—which do not introduce any additional crossings. A sixth closure, c_5, involving three additional crossings is introduced in order to enable the computation of the Conway polynomial of the c_6 closure of 3-tangles in $\mathcal{S}(R')$. For such tangles, an invariant i is defined with values in the 6-fold product $\mathbb{F}^6$ of the fraction field of $\mathbb{Z}[z]$; the components of i are the Conway polynomials of the six closures $c_1, \ldots, c_6$. For the specific case of 3-braids in the skein of R', this work extends former explorations by providing explicit formulas for all the six closures and, therefore, for the invariant i. One interesting feature of the expressions and formulas exhibited in this paper is that they bypass the recursive nature found in the usual definitions of polynomial invariants and continued fractions. Computationally speaking, such nonrecursive formulas turn out to be more time-efficient and simple to program in computer algebra environments. As a somewhat unexpected byproduct of our findings, similar nonrecursive formulas are obtained for the numerators and denominators of continued fractions in terms of simple indexing sets. This contributes to shed more light on the connections between continued fractions and 3-braids. The envisaged future work includes the exploitation of the tools presented in this paper, including the systematic use of the invariant i and the associated nonrecursive formulas, to pursue the study of knots and links obtained as closures of 3-braids in $\mathcal{S}(R')$.

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