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# Stability conditions for integral delay systems

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## SUMMARY

In this paper we consider a special class of integral delay systems arising in several stability problems of time-delay systems. For these integral systems we derive stability and robust stability conditions in terms of Lyapunov-Krasovskii functionals. More explicitly, after providing the stability conditions we compute quadratic functionals and apply them to derive exponential estimates for solutions, and robust stability conditions for perturbed integral delay systems.

KEY WORDS: integral delay systems; Lyapunov-Krasovskii functionals; stability; robust stability

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## 1. INTRODUCTION

Recently, in References [1, 2] it has been shown that some system transformations, commonly used to obtain delay-dependent stability conditions for time-delay systems, introduce additional dynamics. Additional dynamics are responsible for the lack of equivalence in the stability property of the original and the transformed systems, *i.e.*, the original system may be stable while the transformed one is not. It has been shown in Reference [3] that the additional dynamics are described by a special integral delay system. Stability of the integral system is a necessary condition for the stability of the transformed system, see Reference [3].

Similar integral systems describe the internal dynamics of the controllers used for finite spectrum assignment of time-delay systems, see Reference [4]. It has been demonstrated in References [5, 6, 7] that the internal stability of such controllers is an essential condition for their successful implementation.

Integral delay systems also appear in the stability analysis of the difference operator of some neutral type functional differential equations. Stability of the difference operator is a necessary condition for stability of the neutral type functional differential equation, see Reference [8] for details.

These three different sources of stability problems associated with time-delay systems motivate us to look for stability and robust stability conditions of a special class of integral delay systems. To the best of our knowledge, no attempt has been made to derive such conditions for integral delay systems in terms of Lyapunov-Krasovskii functionals.

We present such stability conditions, and give a statement guaranteeing the existence of Lyapunov-Krasovskii functionals for integral delay systems, which leads to a procedure for finding the functionals. Finally, we show how the functionals can be used to derive exponential

estimates for solutions, and robust stability conditions for perturbed systems.

The paper is organized as follows: In section 2, we shortly describe how additional dynamics appear in time-delay systems. We discuss the internal stability problem of the controllers used for finite spectrum assignment of time-delay systems, and the stability problem of difference operators in neutral functional differential equations. We also introduce the class of integral delay systems that will be studied in the paper. Section 3 presents some preliminary results. The existence, uniqueness and stability of solutions are briefly discussed. The Cauchy formula for solutions of integral delay systems is also presented. In section 4, Lyapunov-Krasovskii stability conditions for integral delay systems are given. A converse result, providing a constructive procedure for computing Lyapunov-Krasovskii functionals of a given exponentially stable integral delay system, is presented in section 5. Functionals are used to derive exponential estimates for the solutions in section 6. In section 7, the functionals are applied for robust stability analysis of perturbed systems. Concluding remarks end the paper.

## 2. MOTIVATION OF THE INTEGRAL SYSTEMS

We first briefly describe how additional dynamics appear in the stability analysis of time-delay systems. Let us consider a time-delay system of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^0 G(\theta)x(t+\theta)d\theta, \quad (1)$$

where  $A_0, A_1$  are real constant  $n \times n$  matrices, delay  $h > 0$ , and  $G(\theta)$  is a continuous real matrix function defined for  $\theta \in [-h, 0]$ .

In order to obtain delay-dependent stability conditions for (1) one usually applies a special transformation to the system, see for instance Reference [9]. The aim of the transformation is

to present the system in a form more suitable for the stability analysis. The transformation replaces in (1) the delay terms  $x(t-h)$  and  $x(t+\theta)$  by the Newton-Leibnitz formula

$$x(t-h) = x(t) - \int_{-h}^0 \dot{x}(t+\xi)d\xi, \quad x(t+\theta) = x(t) - \int_{\theta}^0 \dot{x}(t+\xi)d\xi,$$

and substitutes the derivative under the integral by the right-hand side of (1). As a result, the transformed system can be written in the form, see Reference [3],

$$\begin{cases} \dot{y}(t) = A_0 y(t) + A_1 y(t-h) + \int_{-h}^0 G(\theta) y(t+\theta) d\theta + z(t) \\ z(t) = \int_{-h}^0 \left( A_1 + \int_{-h}^{\theta} G(\xi) d\xi \right) z(t+\theta) d\theta. \end{cases} \quad (2)$$

The second equation of the above system

$$z(t) = \int_{-h}^0 \left( A_1 + \int_{-h}^{\theta} G(\xi) d\xi \right) z(t+\theta) d\theta \quad (3)$$

describes the additional dynamics introduced by the transformation. Stability of the dynamics is a necessary condition for stability of (2).

We now address the problem of finite spectrum assignment for time-delay systems. Consider the linear system with delayed input

$$\dot{x}(t) = Ax(t) + Bu(t-h), \quad (4)$$

where  $h > 0$ ,  $x(t) \in \mathbb{R}^n$  and  $u(t) \in \mathbb{R}^m$  represent the state and control vectors, and  $A, B$  are real constant matrices of appropriate dimensions.

The control law

$$u(t) = C \left[ e^{Ah} x(t) + \int_{-h}^0 e^{-A\theta} Bu(t+\theta) d\theta \right] \quad (5)$$

assigns a finite spectrum to the closed-loop system (4)-(5) which coincides with the spectrum of the matrix  $(A + BC)$ , see Reference [4]. Recently, in References [5, 6, 7] it has been shown that if the integral at the right-hand side of (5) is approximated by a finite sum, then the

closed-loop system may become unstable if the controller (5) is not internally stable. So, the practical implementation of the control law (5) demands its internal stability. The internal dynamics of (5) are described by the integral system

$$z(t) = \int_{-h}^0 C e^{-A\theta} B z(t + \theta) d\theta. \quad (6)$$

Now consider a neutral functional differential equation of the form

$$\frac{d}{dt}[Dx_t] = Lx_t, \quad (7)$$

where

$$Dx_t = x(t) - \int_{-h}^0 M(\theta)x(t + \theta)d\theta, \text{ and } Lx_t = A_0x(t) + A_1x(t - h) + \int_{-h}^0 N(\theta)x(t + \theta)d\theta.$$

Here  $M(\theta)$  and  $N(\theta)$  are continuous real matrix functions defined for  $\theta \in [-h, 0]$ . Stability of the difference operator  $Dx_t$  is a necessary condition for stability of (7), see Reference [8].

Operator  $Dx_t$  is stable if and only if the integral system

$$x(t) = \int_{-h}^0 M(\theta)x(t + \theta)d\theta \quad (8)$$

is stable.

Comparing systems (3), (6) and (8) one can conclude that all of them are of the form

$$z(t) = \int_{-h}^0 F(\theta)z(t + \theta)d\theta, \quad t \geq 0, \quad (9)$$

where  $F(\theta) = A_1 + \int_{-h}^{\theta} G(\xi)d\xi$  for the case of (3),  $F(\theta) = C e^{-A\theta} B$  for the case of (6), while for (8)  $F(\theta) = M(\theta)$ .

Therefore, the stability and robust stability of (9) turns out to be an interesting issue arising in several problems associated with time-delay systems.

We will assume that in (9),  $h$  is a positive constant and  $F(\theta)$  is a continuously differentiable matrix function on  $[-h, 0]$ , where a right-hand side continuous derivative at  $-h$  and a left-hand

side continuous derivative at 0 are assumed to exist. This differentiability assumption holds for the cases of additional dynamics and internal dynamics of control laws with distributed delay, while for the case of difference operator  $Dx_t$ , it imposes certain restrictions on matrix  $M(\theta)$ .

Throughout this paper we will use the Euclidean norm for vectors and the induced norm for matrices, both denoted by  $\|\cdot\|$ . We denote by  $A^T$  the transpose of  $A$ ,  $I$  stands for the identity matrix,  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  denote the smallest and largest eigenvalues of a symmetric matrix  $A$ , respectively.

### 3. PRELIMINARIES

#### 3.1. Solutions

In order to define a particular solution of (9) an initial vector function  $\varphi(\theta)$ ,  $\theta \in [-h, 0)$  should be given. We assume that  $\varphi$  belongs to the space of piecewise continuous bounded functions  $\mathbb{C}^0([-h, 0), \mathbb{R}^n)$ , equipped with the uniform norm  $\|\varphi\|_h = \sup_{\theta \in [-h, 0)} \|\varphi(\theta)\|$ .

For a given initial function  $\varphi \in \mathbb{C}^0([-h, 0), \mathbb{R}^n)$ , let  $z(t, \varphi)$ ,  $t \geq 0$ , be a solution of (9) satisfying  $z(t, \varphi) = \varphi(t)$ ,  $t \in [-h, 0)$ . This solution is continuously differentiable for  $t \in [0, \infty)$ , at  $t = 0$  the right-hand side derivative is assumed, and it suffers a jump discontinuity

$$\Delta z(0, \varphi) = z(0, \varphi) - z(-0, \varphi) = \int_{-h}^0 F(\theta)\varphi(\theta)d\theta - \varphi(-0).$$

Let us consider the time-delay system

$$\dot{y}(t) = F(0)y(t) - F(-h)y(t-h) - \int_{-h}^0 \dot{F}(\theta)y(t+\theta)d\theta, \quad t \geq 0. \quad (10)$$

For a given initial function  $\varphi \in \mathbb{C}^0([-h, 0], \mathbb{R}^n)$ , define the function

$$\tilde{\varphi}(\theta) = \begin{cases} \varphi(\theta), & \theta \in [-h, 0) \\ \int_{-h}^0 F(\xi) \varphi(\xi) d\xi, & \theta = 0 \end{cases}.$$

Denote by  $y(t, \tilde{\varphi}), t \geq 0$ , the solution of (10) with initial function  $\tilde{\varphi}$ . Existence and uniqueness of  $z(t, \varphi)$ , as well as some other properties of the solution, can be easily derived from the following statement.

**Lemma 1.**  $y(t, \tilde{\varphi}) = z(t, \varphi)$ .

**Proof.** Function  $y(t, \tilde{\varphi})$  satisfies

$$\dot{y}(t, \tilde{\varphi}) = F(0)y(t, \tilde{\varphi}) - F(-h)y(t-h, \tilde{\varphi}) - \int_{-h}^0 \dot{F}(\theta)y(t+\theta, \tilde{\varphi})d\theta, \quad t \geq 0.$$

Then, integrating the equality from 0 to  $t$  we obtain

$$\begin{aligned} y(t, \tilde{\varphi}) - \tilde{\varphi}(0) &= F(0) \int_0^t y(\xi, \tilde{\varphi}) d\xi - F(-h) \int_{-h}^{t-h} y(\xi, \tilde{\varphi}) d\xi - \int_{-h}^0 \dot{F}(\theta) \left( \int_{\theta}^{t+\theta} y(\xi, \tilde{\varphi}) d\xi \right) d\theta \\ &= F(0) \int_0^t y(\xi, \tilde{\varphi}) d\xi - F(-h) \int_{-h}^{t-h} y(\xi, \tilde{\varphi}) d\xi - F(0) \int_0^t y(\xi, \tilde{\varphi}) d\xi \\ &\quad + F(-h) \int_{-h}^{t-h} y(\xi, \tilde{\varphi}) d\xi + \int_{-h}^0 F(\theta) [y(t+\theta, \tilde{\varphi}) - y(\theta, \tilde{\varphi})] d\theta \\ &= \int_{-h}^0 F(\theta)y(t+\theta, \tilde{\varphi})d\theta - \int_{-h}^0 F(\theta)\tilde{\varphi}(\theta)d\theta, \end{aligned}$$

which means that  $y(t, \tilde{\varphi})$  satisfies (9).

Assume now that  $z(t, \varphi), t \geq 0$ , satisfies (9). Observe first that for  $t \geq 0$

$$z(t, \varphi) = \int_{-h}^0 F(\theta)z(t+\theta, \varphi)d\theta = \int_{t-h}^t F(\xi-t)z(\xi, \varphi)d\xi.$$

It follows that

$$\begin{aligned} \dot{z}(t, \varphi) &= F(0)z(t, \varphi) - F(-h)z(t-h, \varphi) - \int_{t-h}^t \dot{F}(\xi-t)z(\xi, \varphi)d\xi \\ &= F(0)z(t, \varphi) - F(-h)z(t-h, \varphi) - \int_{-h}^0 \dot{F}(\theta)z(t+\theta, \varphi)d\theta, \end{aligned}$$



which implies that  $z(t, \varphi)$  satisfies (10).

By definition function  $\tilde{\varphi}(\theta)$  coincides with  $\varphi(\theta)$  for  $\theta \in [-h, 0)$ , and

$$z(0, \varphi) = y(0, \tilde{\varphi}) = \tilde{\varphi}(0) = \int_{-h}^0 F(\xi) \varphi(\xi) d\xi.$$

■

**Definition 1.** *System (9) is said to be exponentially stable if there exist  $\alpha > 0$  and  $\mu > 0$  such that every solution of (9) satisfies the inequality*

$$\|z(t, \varphi)\| \leq \mu \|\varphi\|_h e^{-\alpha t}, \quad t \geq 0. \quad (11)$$

**Remark 1.** *System (10) is not exponentially stable. Indeed, any constant vector is a solution of (10).*

### 3.2. Cauchy formula

In this subsection, we present the Cauchy formula for solutions of (9). This formula will play an important role in the construction of quadratic Lyapunov-Krasovskii functionals for (9). Let  $K_1(t)$  be the fundamental matrix of (10). Matrix  $K_1(t)$  is the solution of the matrix equation

$$\dot{K}_1(t) = K_1(t)F(0) - K_1(t-h)F(-h) - \int_{-h}^0 K_1(t+\theta)\dot{F}(\theta)d\theta, \quad t \geq 0,$$

with initial condition  $K_1(t) = 0$ ,  $t \in [-h, 0)$  and  $K_1(0) = I$ , see Reference [10].

Then, the solution  $y(t, \tilde{\varphi})$  of (10) can be written as

$$\begin{aligned} y(t, \tilde{\varphi}) &= K_1(t)\tilde{\varphi}(0) - \int_{-h}^0 K_1(t-h-\theta)F(-h)\tilde{\varphi}(\theta)d\theta \\ &\quad - \int_{-h}^0 \left( \int_{-h}^{\theta} K_1(t+\xi-\theta)\dot{F}(\xi)d\xi \right) \tilde{\varphi}(\theta)d\theta, \quad t \geq 0. \end{aligned} \quad (12)$$

The spectrum of (9) consists of all roots of the characteristic equation

$$\det \left( I - \int_{-h}^0 F(\theta)e^{s\theta}d\theta \right) = 0.$$

**Lemma 2.** *Suppose that the spectrum of (9) does not contain the point  $s = 0$ , then matrix  $K_1(t)$  can be written as*

$$K_1(t) = K(t) + K_0, \quad (13)$$

where

$$K_0 = \left( I - \int_{-h}^0 F(\theta) d\theta \right)^{-1},$$

and  $K(t)$  is the solution of the matrix equation

$$K(t) = \int_{-h}^0 K(t+\theta) F(\theta) d\theta, \quad t \geq 0, \quad (14)$$

with initial condition  $K(t) = -K_0, t \in [-h, 0)$ .

**Proof.** Note first that if  $s = 0$  does not belong to the spectrum of (9) then matrix  $I - \int_{-h}^0 F(\theta) d\theta$  is non-singular, so matrix  $K_0$  is well defined. Observe now that matrix  $K_1(t)$  satisfies the equality

$$K_1(t) = \int_{-h}^0 K_1(t+\theta) F(\theta) d\theta + I, \quad t \geq 0.$$

Introducing (13) into the above we obtain (14). The initial condition for matrix  $K(t)$  follows directly from (13). ■

From (14) it follows that in spite of the fact that matrix  $K_1(t)$  does not admit a strictly decreasing exponential upper bound, matrix  $K(t)$  may do it.

**Lemma 3.** *If  $s = 0$  does not belong to the spectrum of (9), then  $z(t, \varphi)$  can be written as follows*

$$\begin{aligned} z(t, \varphi) = & K(t) \int_{-h}^0 F(\theta) \varphi(\theta) d\theta - \int_{-h}^0 K(t-h-\theta) F(-h) \varphi(\theta) d\theta \\ & - \int_{-h}^0 \left( \int_{-h}^{\theta} K(t+\xi-\theta) \dot{F}(\xi) d\xi \right) \varphi(\theta) d\theta, \quad t \geq 0. \end{aligned} \quad (15)$$

**Proof.** From (12) and (13) we get, for  $t \geq 0$ ,

$$\begin{aligned} z(t, \varphi) &= K(t) \int_{-h}^0 F(\theta) \varphi(\theta) d\theta - \int_{-h}^0 K(t-h-\theta) F(-h) \varphi(\theta) d\theta \\ &- \int_{-h}^0 \left( \int_{-h}^{\theta} K(t+\xi-\theta) \dot{F}(\xi) d\xi \right) \varphi(\theta) d\theta + K_0 \int_{-h}^0 \left[ F(\theta) - F(-h) - \int_{-h}^{\theta} \dot{F}(\xi) d\xi \right] \varphi(\theta) d\theta. \end{aligned}$$

Since the expression in the square brackets above is equal to zero, it follows that the Cauchy formula (15) holds. ■

Matrix  $K(t)$  is known as the fundamental matrix of (9).

#### 4. A LYAPUNOV TYPE THEOREM

In this section we give exponential stability conditions for system (9).

For any  $t \geq 0$  we denote the restriction of the solution  $z(t, \varphi)$  on the interval  $[t-h, t]$  by  $z_t(\varphi) = \{z(t+\theta, \varphi), \theta \in [-h, 0]\}$ . When the initial function  $\varphi$  is irrelevant we simply write  $z(t)$  and  $z_t$  instead of  $z(t, \varphi)$  and  $z_t(\varphi)$ . A simple inspection shows that for  $t \in [0, h)$ ,  $z_t(\varphi)$  belongs to  $\mathbb{C}^0([-h, 0], \mathbb{R}^n)$ , and for  $t \geq h$ ,  $z_t(\varphi)$  belongs to the space of continuous vector functions  $\mathbb{C}([-h, 0], \mathbb{R}^n)$ .

**Theorem 4.** *System (9) is exponentially stable if there exists a continuous functional  $v : \mathbb{C}^0([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$  such that  $t \rightarrow v(z_t(\varphi))$  is differentiable on  $\mathbb{R}_+$  and the following conditions hold:*

1.  $\alpha_1 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta$ , for some  $0 < \alpha_1 \leq \alpha_2$ ;
2.  $\frac{d}{dt} v(z_t(\varphi)) \leq -\beta \int_{-h}^0 \|z(t+\theta, \varphi)\|^2 d\theta$ ,  $t \geq 0$ , for some  $\beta > 0$ .

**Proof.** Given any  $\varphi \in \mathbb{C}^0([-h, 0], \mathbb{R}^n)$  it follows from the theorem conditions that for

$2\alpha = \beta\alpha_2^{-1} > 0$  the following inequality holds:

$$\frac{d}{dt}v(z_t(\varphi)) + 2\alpha v(z_t(\varphi)) \leq 0, \quad t \geq 0.$$

Thus, on one hand it follows that

$$v(z_t(\varphi)) \leq e^{-2\alpha t}v(\varphi) \leq \alpha_2 e^{-2\alpha t} \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \leq h\alpha_2 e^{-2\alpha t} \|\varphi\|_h^2, \quad t \geq 0.$$

On the other hand, one gets

$$\|z(t, \varphi)\|^2 \leq \left( m \int_{-h}^0 \|z(t + \theta, \varphi)\| d\theta \right)^2 \leq m^2 h \int_{-h}^0 \|z(t + \theta, \varphi)\|^2 d\theta, \quad t \geq 0,$$

where

$$m = \max_{\theta \in [-h, 0]} \|F(\theta)\|.$$

We therefore have the following exponential upper bound:

$$\|z(t, \varphi)\| \leq \mu \|\varphi\|_h e^{-\alpha t}, \quad t \geq 0,$$

where

$$\mu = mh \sqrt{\frac{\alpha_2}{\alpha_1}}.$$

■

## 5. A CONVERSE THEOREM

In this section we present a converse statement to theorem 4. More precisely, for an exponentially stable system (9) we construct a quadratic functional satisfying the conditions of the theorem.

Given positive definite matrices  $W_0$  and  $W_1$ , let us define on  $\mathbb{C}^0([-h, 0], \mathbb{R}^n)$  the following functional:

$$w(\varphi) = \varphi^T(-h)W_0\varphi(-h) + \int_{-h}^0 \varphi^T(\theta)W_1\varphi(\theta)d\theta.$$

**Remark 2.** For  $\beta = \lambda_{\min}(W_1)$  functional  $w(\varphi)$  admits the following lower bound:

$$\beta \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \leq w(\varphi). \quad (16)$$

Let system (9) be exponentially stable. We aim at constructing a functional  $v : \mathbb{C}^0([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$ , such that along the solutions of (9), the following equality holds:

$$\frac{d}{dt}v(z_t) = -w(z_t), \quad t \geq 0. \quad (17)$$

To this end, for a positive definite matrix  $W$ , we first define a functional  $v_0(\varphi, W)$  that satisfies

$$\frac{d}{dt}v_0(z_t, W) = -z^T(t)Wz(t), \quad t \geq 0.$$

The functional is of the form

$$v_0(\varphi, W) = \int_0^\infty z^T(t, \varphi)Wz(t, \varphi)dt. \quad (18)$$

Observe that inequality (11) guarantees the existence of the improper integral at the right-hand side of (18).

**Lemma 5.** Let system (9) be exponentially stable. Then the functional  $v(\varphi)$  satisfying (17) can be written as

$$v(\varphi) = v_0(\varphi, W_0 + hW_1) + \int_{-h}^0 \varphi^T(\xi) [W_0 + (\xi + h)W_1] \varphi(\xi) d\xi. \quad (19)$$

**Proof.** Let  $z(t, \varphi)$ ,  $t \geq 0$ , be a solution of (9). For a fixed  $t$  consider the trajectory segment  $z_t(\varphi)$  as a new initial function and denote by  $z(\tau, z_t(\varphi))$ ,  $\tau \geq 0$ , to the corresponding solution.

Then from (18) we have

$$v_0(z_t(\varphi), W_0 + hW_1) = \int_0^\infty z^T(\tau, z_t(\varphi)) [W_0 + hW_1] z(\tau, z_t(\varphi)) d\tau.$$

The uniqueness of solutions of (9) with a given initial function implies that

$$z(\tau, z_t(\varphi)) = z(\tau + t, \varphi), \quad \tau \geq 0,$$

and

$$\begin{aligned} v_0(z_t(\varphi), W_0 + hW_1) &= \int_0^\infty z^T(\tau + t, \varphi) [W_0 + hW_1] z(\tau + t, \varphi) d\tau \\ &= \int_t^\infty z^T(\xi, \varphi) [W_0 + hW_1] z(\xi, \varphi) d\xi. \end{aligned}$$

Hence for such a solution the functional  $v(\varphi)$  takes the form

$$v(z_t(\varphi)) = \int_t^\infty z^T(\xi, \varphi) [W_0 + hW_1] z(\xi, \varphi) d\xi + \int_{-h}^0 z^T(t + \xi, \varphi) [W_0 + (\xi + h)W_1] z(t + \xi, \varphi) d\xi.$$

We then have

$$\begin{aligned} \frac{d}{dt}v(z_t(\varphi)) &= -z^T(t, \varphi) (W_0 + hW_1) z(t, \varphi) + z^T(t, \varphi) (W_0 + hW_1) z(t, \varphi) \\ &\quad - z^T(t - h, \varphi) W_0 z(t - h, \varphi) - \int_{-h}^0 z^T(t + \xi, \varphi) W_1 z(t + \xi, \varphi) d\xi \\ &= -z^T(t - h, \varphi) W_0 z(t - h, \varphi) - \int_{-h}^0 z^T(t + \xi, \varphi) W_1 z(t + \xi, \varphi) d\xi \\ &= -w(z_t(\varphi)). \end{aligned}$$

■

We now construct the functional  $v_0(\varphi, W)$ . Substituting (15) into (18) we obtain

$$\begin{aligned} v_0(\varphi, W) &= \tilde{\varphi}^T(0)U(0)\tilde{\varphi}(0) - 2\tilde{\varphi}^T(0) \int_{-h}^0 U(-h - \theta)F(-h)\varphi(\theta)d\theta \\ &\quad - 2\tilde{\varphi}^T(0) \int_{-h}^0 \left( \int_{-h}^\theta U(\xi - \theta)\dot{F}(\xi)d\xi \right) \varphi(\theta)d\theta \\ &\quad + \int_{-h}^0 \varphi^T(\theta_1)F^T(-h) \left( \int_{-h}^0 U(\theta_1 - \theta_2)F(-h)\varphi(\theta_2)d\theta_2 \right) d\theta_1 \\ &\quad - \int_{-h}^0 \varphi^T(\theta_1)F^T(-h)K_0^T W \left[ \int_{-h}^0 \left( \int_{-h-\theta_2}^{\theta_1-\theta_2} K(\xi)d\xi \right) F(-h)\varphi(\theta_2)d\theta_2 \right] d\theta_1 \\ &\quad + 2 \int_{-h}^0 \varphi^T(\theta_1)F^T(-h) \left[ \int_{-h}^0 \left( \int_{-h}^{\theta_2} U(h + \theta_1 + \xi - \theta_2)\dot{F}(\xi)\varphi(\theta_2)d\xi \right) d\theta_2 \right] d\theta_1 \\ &\quad - 2 \int_{-h}^0 \varphi^T(\theta_1)F^T(-h)K_0^T W \left[ \int_{-h}^0 \left( \int_{-h}^{\theta_2} \left( \int_{\xi-\theta_2}^{h+\theta_1+\xi-\theta_2} K(\eta)d\eta \right) \dot{F}(\xi)\varphi(\theta_2)d\xi \right) d\theta_2 \right] d\theta_1 \end{aligned}$$

$$\begin{aligned}
& + \int_{-h}^0 \varphi^T(\theta_1) \left( \int_{-h}^{\theta_1} \dot{F}^T(\xi_1) \left[ \int_{-h}^0 \left( \int_{-h}^{\theta_2} U(\theta_1 - \xi_1 + \xi_2 - \theta_2) \dot{F}(\xi_2) d\xi_2 \right) \varphi(\theta_2) d\theta_2 \right] d\xi_1 \right) d\theta_1 \\
& - \int_{-h}^0 \varphi^T(\theta_1) \left[ \int_{-h}^{\theta_1} \dot{F}^T(\xi_1) K_0^T W \left( \int_{-h}^0 \left[ \int_{-h}^{\theta_2} \left( \int_{\xi_2 - \theta_2}^{\theta_1 - \xi_1 + \xi_2 - \theta_2} K(\eta) d\eta \right) \dot{F}(\xi_2) d\xi_2 \right] \varphi(\theta_2) d\theta_2 \right) d\xi_1 \right] d\theta_1.
\end{aligned}$$

Here  $\tilde{\varphi}(0) = \int_{-h}^0 F(\theta)\varphi(\theta)d\theta$ , and matrix function

$$U(\tau) = \int_0^\infty K^T(t)W K(t + \tau)dt, \quad \tau \in [-h, h]. \quad (20)$$

The exponential stability of (9) guarantees the existence of the improper integral in (20).

**Lemma 6.** *Let system (9) be exponentially stable. Then there exist constants  $0 < \alpha_1 \leq \alpha_2$ , such that functional (19) satisfies the inequalities*

$$\alpha_1 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta. \quad (21)$$

**Proof.** Let us define  $u_0 = \max_{\tau \in [-h, h]} \|U(\tau)\|$  and  $r = \int_{-h}^0 \|\dot{F}(\theta)\| d\theta$ . Then the following set of inequalities hold:

$$\begin{aligned}
\tilde{\varphi}^T(0)U(0)\tilde{\varphi}(0) & \leq hm^2u_0 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta, \\
-2\tilde{\varphi}^T(0) \int_{-h}^0 U(-h - \theta)F(-h)\varphi(\theta)d\theta & \leq 2hmu_0 \|F(-h)\| \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta, \\
-2\tilde{\varphi}^T(0) \int_{-h}^0 \left( \int_{-h}^{\theta} U(\theta - \xi)\dot{F}(\xi)d\xi \right) \varphi(\theta)d\theta & \leq 2hu_0mr \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta, \\
\int_{-h}^0 \varphi^T(\theta) [W_0 + (\theta + h)W_1] \varphi(\theta)d\theta & \leq \lambda_{\max}(W_0 + hW_1) \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta,
\end{aligned}$$

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$$\begin{aligned}
& \int_{-h}^0 \varphi^T(\theta_1) F^T(-h) \left( \int_{-h}^0 U(\theta_1 - \theta_2) F(-h) \varphi(\theta_2) d\theta_2 \right) d\theta_1 \\
\leq & \quad hu_0 \|F(-h)\|^2 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta, \\
& 2 \int_{-h}^0 \varphi^T(\theta_1) F^T(-h) \left[ \int_{-h}^0 \left( \int_{-h}^{\theta_2} U(h + \theta_1 + \xi - \theta_2) \dot{F}(\xi) \varphi(\theta_2) d\xi \right) d\theta_2 \right] d\theta_1 \\
\leq & \quad 2hu_0 r \|F(-h)\| \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta, \\
& \int_{-h}^0 \varphi^T(\theta_1) \left( \int_{-h}^{\theta_1} \dot{F}^T(\xi_1) \left[ \int_{-h}^0 \left( \int_{-h}^{\theta_2} U(\theta_1 - \xi_1 + \xi_2 - \theta_2) \dot{F}(\xi_2) d\xi_2 \right) \varphi(\theta_2) d\theta_2 \right] d\xi_1 \right) d\theta_1 \\
\leq & \quad hu_0 r^2 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta.
\end{aligned}$$

In order to estimate the terms of (19) involving the fundamental matrix  $K(t)$  we observe that for  $t \in [0, h]$

$$K(t) = -K_0 \int_{t-h}^0 F(\xi - t) d\xi + \int_0^t K(\xi) F(\xi - t) d\xi,$$

so

$$\|K(t)\| \leq \|K_0\| mh + m \int_0^t \|K(\xi)\| d\xi.$$

A direct application of the Bellman's lemma leads to

$$\|K(t)\| \leq \|K_0\| m h e^{mt} \leq \|K_0\| m h e^{mh}, \quad t \in [0, h].$$

Taking into account this inequality we obtain that, for  $\gamma_1 \in [-h, 0]$  and  $\gamma_2 \in [0, h]$ , the following inequality holds:

$$\left\| \int_{\gamma_1}^{\gamma_2 + \gamma_1} K(\theta) d\theta \right\| \leq (1 + m h e^{mh}) \|K_0\| h. \quad (22)$$



By (22) it is easy to verify that

$$\begin{aligned}
& - \int_{-h}^0 \varphi^T(\theta_1) F^T(-h) K_0^T W \left[ \int_{-h}^0 \left( \int_{-h-\theta_2}^{\theta_1-\theta_2} K(\xi) d\xi \right) F(-h) \varphi(\theta_2) d\theta_2 \right] d\theta_1 \\
& \leq h^2 \|F^T(-h) K_0^T W\| \|F(-h)\| (1 + m h e^{mh}) \|K_0\| \int_{-h}^0 \|\varphi(\tau)\|^2 d\tau, \\
& - 2 \int_{-h}^0 \varphi^T(\theta_1) F^T(-h) K_0^T W \left[ \int_{-h}^0 \left( \int_{-h}^{\theta_2} \left( \int_{\xi-\theta_2}^{h+\theta_1+\xi-\theta_2} K(\eta) d\eta \right) \dot{F}(\xi) \varphi(\theta_2) d\xi \right) d\theta_2 \right] d\theta_1 \\
& \leq 2h^2 r \|F^T(-h) K_0^T W\| (1 + m h e^{mh}) \|K_0\| \int_{-h}^0 \|\varphi(\tau)\|^2 d\tau, \\
& \int_{-h}^0 \varphi^T(\theta_1) \left[ \int_{-h}^{\theta_1} \dot{F}^T(\xi_1) K_0^T W \left( \int_{-h}^0 \left[ \int_{-h}^{\theta_2} \left( \int_{\xi_2-\theta_2}^{\theta_1-\xi_1+\xi_2-\theta_2} K(\eta) d\eta \right) \dot{F}(\xi_2) d\xi_2 \right] \varphi(\theta_2) d\theta_2 \right) d\xi_1 \right] d\theta_1 \\
& \leq h^2 r^2 \|K_0^T W\| (1 + m h e^{mh}) \|K_0\| \int_{-h}^0 \|\varphi(\tau)\|^2 d\tau.
\end{aligned}$$

As a consequence, we obtain the following upper bound for  $v(\varphi)$  :

$$v(\varphi) \leq \alpha_2 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta,$$

where

$$\begin{aligned}
\alpha_2 & \geq h u_0 \left( (m + \|F(-h)\|)^2 + r^2 + 2rm + 2\|F(-h)\| r \right) + \lambda_{\max}(W_0 + hW_1) \\
& + (1 + m h e^{mh}) h^2 \|K_0\| \left( \|F^T(-h) K_0^T W\| \|F(-h)\| + 2\|F^T(-h) K_0^T W\| r + \|K_0^T W\| r^2 \right).
\end{aligned}$$

In order to check the left-hand side inequality in (21) we define the functional

$$v_\epsilon(\varphi) = v(\varphi) - \epsilon \int_{-h}^0 \varphi^T(\theta) \varphi(\theta) d\theta.$$

We have

$$\frac{d}{dt} v_\epsilon(z_t(\varphi)) = -w(z_t(\varphi)) - \epsilon [z^T(t, \varphi) z(t, \varphi) - z^T(t-h, \varphi) z(t-h, \varphi)].$$

It follows that

$$\frac{d}{dt} v_\epsilon(z_t(\varphi)) \leq -w_\epsilon(z_t(\varphi)), \quad t \geq 0, \quad (23)$$

where

$$w_\epsilon(\varphi) = \epsilon \|\varphi(0)\|^2 + [\lambda_{\min}(W_0) - \epsilon] \|\varphi(-h)\|^2 + \lambda_{\min}(W_1) \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta.$$

Choosing  $0 < \epsilon \leq \lambda_{\min}(W_0)$  we have  $w_\epsilon(\varphi) \geq 0$ . Integrating inequality (23) from 0 to  $\infty$  we get

$$v_\epsilon(\varphi) \geq \int_0^\infty w_\epsilon(z_t(\varphi)) dt \geq 0,$$

which implies that

$$v(\varphi) \geq \alpha_1 \int_{-h}^0 \|\varphi(\theta)\|^2 d\theta$$

for  $\alpha_1 = \epsilon$ . Finally, observe that

$$\alpha_2 \geq \lambda_{\max}(W_0 + hW_1) \geq \lambda_{\min}(W_0) \geq \alpha_1 > 0,$$

which ends the proof. ■

We thus obtain the following converse Lyapunov-Krasovskii theorem for (9).

**Theorem 7.** *Let system (9) be exponentially stable. Then for any given positive definite matrices  $W_0$  and  $W_1$  there exist positive constants  $\alpha_1, \alpha_2$  and  $\beta$  such that functional (19) satisfies the conditions of theorem 4.*

**Proof.** Given any positive definite matrices  $W_0$  and  $W_1$ , the exponential stability of (9) implies that functional  $v(\varphi)$  is defined by (19) and (20). Then, the proof follows from lemma 6 and remark 2. ■

In order to compute  $v_0(\varphi, W)$  one has to know matrix  $U(\tau)$  for  $\tau \in [-h, h]$ . Expression (20) defines  $U(\tau)$  as an improper integral. This is not convenient from a practical point of view. Hence we derive some important properties of  $U(\tau)$  that will provide an alternative way to compute it.

**Lemma 8.** *Matrix  $U(\tau)$  satisfies*

$$U(\tau) = \int_{-h}^0 U(\tau + \theta)F(\theta)d\theta, \quad \tau \in [0, h]. \quad (24)$$

**Proof.** Substituting (14) into (20) we obtain (24). ■

**Lemma 9.** *Let  $W^T = W$ , then matrix  $U(\tau)$  satisfies the symmetry property*

$$U(\tau) = K_0^T W \int_0^\tau K(\xi)d\xi + U^T(-\tau), \tau \in [0, h]. \quad (25)$$

**Proof.** The statement can be easily verified by direct calculations. ■

**Corollary 10.** *Matrix  $U(0)$  is symmetric,  $U^T(0) = U(0)$ .*

**Lemma 11.** *Matrix  $U(\tau)$  satisfies*

$$\int_{-h}^0 \dot{U}(\theta)F(\theta)d\theta + \left[ \int_{-h}^0 \dot{U}(\theta)F(\theta)d\theta \right]^T = -K^T(0)WK(0). \quad (26)$$

**Proof.** Let us compute

$$\frac{d}{dt} [K^T(t)WK(t)] = \left( \int_{-h}^0 \dot{K}(t + \theta)F(\theta)d\theta \right)^T WK(t) + K^T(t)W \left( \int_{-h}^0 \dot{K}(t + \theta)F(\theta)d\theta \right).$$

Integrating this equality with respect to  $t$  from 0 to  $\infty$  we obtain (26). ■

Lemma 8 defines  $U(\tau)$  as a solution of (24). In order to compute such a solution one needs to know the corresponding initial condition. The initial condition is not given explicitly. On the other hand, the symmetry property (25) along with (26) serve as a replacement of the unknown initial condition. Indeed, a piecewise linear approximation of matrix  $U(\tau)$  for  $\tau \in [-h, h]$  can be computed from equations (24)-(26).

Despite the fact that there are still no efficient algorithms to check the positivity of functionals of the form (19), and that functionals cannot be directly used for the stability analysis of (9), they can be directly applied to derive exponential estimates for solutions and

robust stability conditions for perturbed systems. Such applications will be presented in the next two sections.

## 6. EXPONENTIAL ESTIMATES FOR THE SOLUTIONS

In this section, we show how functionals (19) can be used to obtain exponential estimates for the solutions of a given exponentially stable integral delay system (9).

Given positive definite matrices  $W_0$  and  $W_1$ , based on lemma 6 and remark 2 we compute positive constants  $\alpha_1, \alpha_2$  and  $\beta$ . Then the functional  $v(\varphi)$  defined by (19) and (20) satisfies conditions 1 and 2 of theorem 4. From this theorem

$$\|z(t, \varphi)\| \leq hm \sqrt{\frac{\alpha_2}{\alpha_1}} \|\varphi\|_h e^{-\alpha t}, \quad t \geq 0,$$

where  $\alpha = \frac{\beta}{2\alpha_2}$ , for all solutions  $z(t, \varphi), \varphi \in \mathbb{C}^0([-h, 0], \mathbb{R}^n)$ , of (9).

Now we illustrate the result by an example. Let us consider the following integral delay system:

$$z(t) = G \int_{-h}^0 z(t + \theta) d\theta. \quad (27)$$

System (27) is a particular case of (9), where  $F(\theta) = G$ .

Let  $h = 1$  and

$$G = \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}.$$

As all eigenvalues of  $G$  lie in the open domain  $\Gamma$  whose boundary in the complex plane is described by

$$\partial\Gamma = \left\{ \frac{\omega \sin(\omega)}{2[1 - \cos(\omega)]} + i\frac{\omega}{2} \mid \omega \in (-2\pi, 2\pi) \right\},$$

system (27) is exponentially stable, see Reference [2].

In this case the functional (19) takes the form

$$\begin{aligned} v(\varphi) &= \left( G \int_{-h}^0 \varphi(\theta) d\theta \right)^T U(0) \left( G \int_{-h}^0 \varphi(\theta) d\theta \right) - 2 \left( G \int_{-h}^0 \varphi(\theta) d\theta \right)^T \int_{-h}^0 U(-h-\theta) G \varphi(\theta) d\theta \\ &+ \int_{-h}^0 \varphi^T(\theta_1) G^T \left( \int_{-h}^0 U(\theta_1 - \theta_2) G \varphi(\theta_2) d\theta_2 \right) d\theta_1 + \int_{-h}^0 \varphi^T(\theta) [W_0 + (\theta + h) W_1] \varphi(\theta) d\theta \\ &- \int_{-h}^0 \varphi^T(\theta_1) G^T K_0^T W \left[ \int_{-h}^0 \left( \int_{-h-\theta_2}^{\theta_1-\theta_2} K(\xi) d\xi \right) G \varphi(\theta_2) d\theta_2 \right] d\theta_1, \end{aligned}$$

and equations (24)-(26) for matrix  $U(\tau)$  are

$$\begin{aligned} U(\tau) &= \left( \int_{-h}^0 U(\tau + \theta) d\theta \right) G, \quad \tau \in [0, h], \\ U(\tau) &= K_0^T W \int_0^\tau K(\xi) d\xi + U^T(-\tau), \quad \tau \in [0, h], \\ -K^T(0) W K(0) &= [U(0)G - U(-h)G]^T + [U(0)G - U(-h)G], \end{aligned}$$

where  $K(t)$  is the solution of

$$K(t) = \left( \int_{-h}^0 K(t + \theta) d\theta \right) G, \quad t \in [0, h],$$

with initial condition  $K(t) = -K_0 = -(I - hG)^{-1}$ ,  $t \in [-h, 0)$ .

Let us assume that  $W_0 = W_1 = I$ . A piecewise linear approximation of matrix  $U(\tau)$  is given on Fig. 1. From the computed values of  $U(\tau)$  on  $[-1, 1]$  we get  $u_0 = 0.1752$ .

From remark 2 we have that for  $\beta = 1$ , inequality (16) holds. Simple calculations derived from lemma 6 show that for  $\alpha_2 = 2.7611 \times 10^3$  and  $\alpha_1 = 1$ , inequalities (21) hold.

So, solutions of (27) satisfy the following exponential estimate:

$$\|z(t, \varphi)\| \leq \mu \|\varphi\|_1 e^{-\alpha t}, \quad t \geq 0, \quad (28)$$

where

$$\mu \approx 245.43 \text{ and } \alpha \approx 1.8 \times 10^{-4}.$$

## 7. ROBUST STABILITY CONDITIONS

In this section we address the robust stability analysis of perturbed integral delay systems (9).

Consider the perturbed system

$$y(t) = \int_{-h}^0 [F(\theta) + \Delta(\theta)] y(t + \theta) d\theta, \quad (29)$$

where  $\Delta(\cdot)$  is an unknown continuous matrix function satisfying

$$\|\Delta(\theta)\| \leq \rho, \quad \theta \in [-h, 0]. \quad (30)$$

Let the nominal system (9) be exponentially stable. The main goal of this section is to show how functional (19) computed for the nominal system (9) can be used to find an upper bound on  $\rho$  such that the perturbed system (29) remains exponentially stable for all perturbations  $\Delta(\cdot)$  satisfying (30).

The time derivative of functional (19) along solutions of the perturbed system (29) is

$$\begin{aligned} \frac{dv(y_t)}{dt} = & -w(y_t) - 2\eta^T(y_t) \left[ F^T(-h)K_0^T W K_0 h + \int_{-h}^0 \dot{U}(\theta)F(\theta)d\theta - K_0^T W \right] \left( \int_{-h}^0 F(\tau)y(t + \tau)d\tau \right) \\ & + 2\eta^T(y_t) \int_{-h}^0 \left( \int_{-h}^{\tau} \left( \int_{-h}^0 \dot{F}^T(\xi) \left[ U^T(\tau - \theta + \xi) - \int_0^{\tau - \theta + \xi} K^T(\eta)d\eta W K_0 \right] d\xi \right) \dot{F}(\theta)d\theta \right) y(t + \tau)d\tau \\ & + 2\eta^T(y_t) \int_{-h}^0 \left( \int_{-h}^{\tau} \dot{F}^T(\xi) \left[ U^T(\tau + h + \xi) - \int_0^{\tau + h + \xi} K^T(\eta)d\eta W K_0 \right] d\xi \right) F(-h)y(t + \tau)d\tau \\ & + 2\eta^T(y_t) \int_{-h}^0 \left( \int_{-h}^{\tau} [F^T(-h)U^T(\tau - \theta - h) - F^T(0)U(\theta - \tau)] \dot{F}(\theta)d\theta \right) y(t + \tau)d\tau \\ & - 2\eta^T(y_t) \int_{-h}^0 ([F^T(0)U(-h - \tau) - F^T(-h)U^T(\tau)] F(-h) \\ & - F^T(-h)K_0^T W K_0 \left( \int_{-h}^{\tau} F(\theta)d\theta \right)) y(t + \tau)d\tau + \eta^T(y_t)W\eta(y_t), \end{aligned} \quad (31)$$

where

$$\eta(y_t) = \int_{-h}^0 \Delta(\theta)y(t + \theta)d\theta.$$

The following set of inequalities hold:

$$\begin{aligned}
& 2\eta^T(y_t) \int_{-h}^0 \left( \int_{-h}^{\tau} \left( \int_{-h}^0 \dot{F}^T(\xi) \left[ U^T(\tau - \theta + \xi) - \int_0^{\tau - \theta + \xi} K^T(\eta) d\eta W K_0 \right] d\xi \right) \dot{F}(\theta) d\theta \right) y(t + \tau) d\tau \\
& \leq 2hr^2\rho (2u_0 + h \|K_0^T W K_0\|) \int_{-h}^0 \|y(t + \tau)\|^2 d\tau, \\
& 2\eta^T(y_t) \int_{-h}^0 \left( \int_{-h}^0 \dot{F}^T(\xi) \left[ U^T(\tau + h + \xi) - \int_0^{\tau + h + \xi} K^T(\eta) d\eta W K_0 \right] d\xi \right) F(-h) y(t + \tau) d\tau \\
& \leq 2h (2u_0 + h \|K_0^T W K_0\|) \|F(-h)\| r\rho \int_{-h}^0 \|y(t + \tau)\|^2 d\tau, \\
& 2\eta^T(y_t) \int_{-h}^0 \left( \int_{-h}^{\tau} [F^T(-h)U^T(\tau - \theta - h) - F^T(0)U(\theta - \tau)] \dot{F}(\theta) d\theta \right) y(t + \tau) d\tau \\
& \leq 2ru_0 (\|F^T(-h)\| + \|F^T(0)\|) h\rho \int_{-h}^0 \|y(t + \tau)\|^2 d\tau, \\
& -2\eta^T(y_t) \left[ F^T(-h)K_0^T W K_0 h + \int_{-h}^0 \dot{U}(\theta)F(\theta) d\theta - K_0^T W \right] \left( \int_{-h}^0 F(\tau) y(t + \tau) d\tau \right) \\
& \leq 2m [\|F^T(-h)K_0^T W K_0 h + U(0)F(0) - U(-h)F(-h) - K_0^T W\| + u_0 r] h\rho \int_{-h}^0 \|y(t + \tau)\|^2 d\tau, \\
& -2\eta^T(y_t) \int_{-h}^0 ([F^T(0)U(-h - \tau) - F^T(-h)U^T(\tau)] F(-h) \\
& - F^T(-h)K_0^T W K_0 \left( \int_{-h}^{\tau} F(\theta) d\theta \right)) y(t + \tau) d\tau \\
& \leq 2 (\|F^T(0)\| + \|F^T(-h)\|) u_0 \|F(-h)\| + \|F^T(-h)K_0^T W K_0\| mh) h\rho \int_{-h}^0 \|y(t + \tau)\|^2 d\tau, \\
& \eta^T(y_t) W \eta(y_t) \leq \|W\| \|\eta(y_t)\| \|\eta(y_t)\| \leq \|W\| \rho^2 h \int_{-h}^0 \|y(t + \tau)\|^2 d\tau.
\end{aligned}$$

From (31) and the above inequalities we obtain the following upper bound:

$$\frac{dv(y_t)}{dt} \leq -w(y_t) + (a\rho^2 + b\rho) \int_{-h}^0 \|y(t + \tau)\|^2 d\tau,$$

where  $a = \|W\| h$  and

$$\begin{aligned}
b &= 2h (r + \|F(-h)\|) [r (2u_0 + h \|K_0^T W K_0\|) + u_0 (\|F^T(-h)\| + \|F^T(0)\|)] \\
&+ 2mh [\|F^T(-h)K_0^T W K_0 h + U(0)F(0) - U(-h)F(-h) - K_0^T W\| + u_0 r + \|F^T(-h)K_0^T W K_0\| h].
\end{aligned}$$

**Theorem 12.** *Let system (9) be exponentially stable. Given positive definite matrices  $W_0$  and  $W_1$ , the perturbed system (29) remains exponentially stable for all perturbations  $\Delta(\cdot)$  satisfying (30) if  $\rho$  is such that the following inequality holds:*

$$\lambda_{\min}(W_1) - (a\rho^2 + b\rho) > 0. \quad (32)$$

**Remark 3.** *Matrices  $W_0$  and  $W_1$  can be used to improve the bound for  $\rho$ .*

We now illustrate the results by two examples. We first investigate the robust stability of the exponentially stable integral delay system (27) introduced in section VI. Thus, let us consider the perturbed system

$$y(t) = \int_{-h}^0 [G + \Delta] y(t + \theta) d\theta, \quad (33)$$

where  $\Delta$  is an unknown matrix satisfying

$$\|\Delta\| \leq \rho. \quad (34)$$

As in section VI, let us take  $W_0 = W_1 = I$ . From the computed values of  $U(\tau)$ , see Fig. 1, we get

$$U(-1) = \begin{pmatrix} -0.1451 & -0.0653 \\ -0.0590 & 0.0026 \end{pmatrix}, U(0) = \begin{pmatrix} 0.0752 & -0.0513 \\ -0.0524 & 0.1412 \end{pmatrix} \text{ and } u_0 = 0.1752.$$

Then, direct calculations from (32) show that the perturbed system (33) remains exponentially stable for all perturbations  $\Delta$  satisfying (34) if

$$\rho < 0.0267.$$

Consider now the scalar integral delay system

$$z(t) = \int_{-h}^0 F(\theta) z(t + \theta) d\theta, \quad (35)$$



where  $F(\theta) = -c_1\theta + c_2$ ,  $\theta \in [-h, 0]$ . System (35) describes the internal dynamics of controller (5), where  $C = \begin{pmatrix} c_1 & c_2 \end{pmatrix}$ ,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The corresponding linear system (4) represents a double integrator with delayed input which is very commonly found in mechanical systems. Note that (35) also describes some scalar difference operators (8), where  $M(\theta) = -c_1\theta + c_2$ ,  $\theta \in [-h, 0]$ .

Let us assume that (35) is exponentially stable and consider the perturbed system

$$y(t) = \int_{-h}^0 [F(\theta) + \Delta(\theta)]y(t + \theta)d\theta, \quad (36)$$

where  $\Delta(\theta) = -\Delta c_1\theta + \Delta c_2$  is an unknown continuous function satisfying

$$|\Delta(\theta)| \leq \rho, \quad \theta \in [-h, 0]. \quad (37)$$

Let  $h = 1$ ,  $c_1 = -0.5$  and  $c_2 = -0.25$ . For this choice of parameters

$$\max_{\theta \in [-h, 0]} |F(\theta)| h < 1$$

holds, and thus the nominal system (36) is exponentially stable, see Reference [11]. It is worth mentioning that for these particular values the ideal controller (5) assigns the closed-loop poles of system (4)-(5) at  $\lambda_{1,2} = -0.1250 \pm 0.696i$ .

Let us assume that  $W_0 = W_1 = 1$ . A piecewise linear approximation of scalar function  $U(\tau)$  is given on Fig. 2. From the computed values of  $U(\tau)$  on  $[-1, 1]$  we obtain

$$U(-1) = -0.2063, U(0) = 0.0752 \text{ and } u_0 = 0.2177.$$

Direct calculations from (32) show that the perturbed system (36) remains exponentially stable for all continuous perturbations  $\Delta(\theta)$  satisfying (37) if

$$\rho < 0.2633.$$

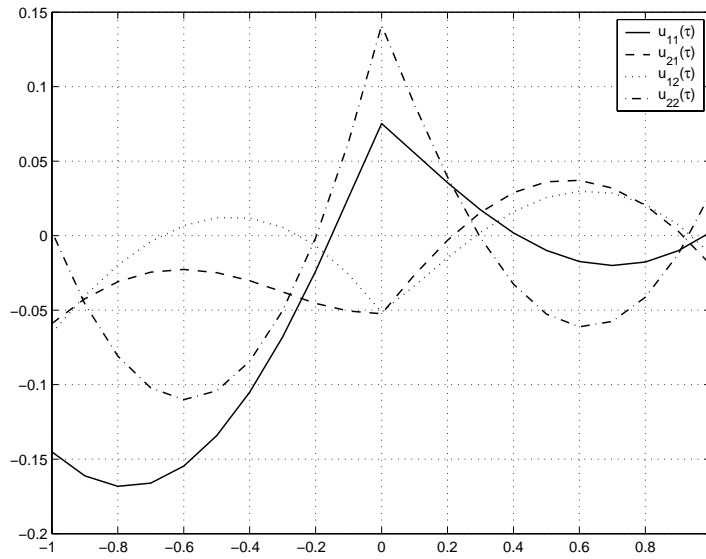


Figure 1. Components of a piecewise linear approximation of matrix  $U(\tau)$

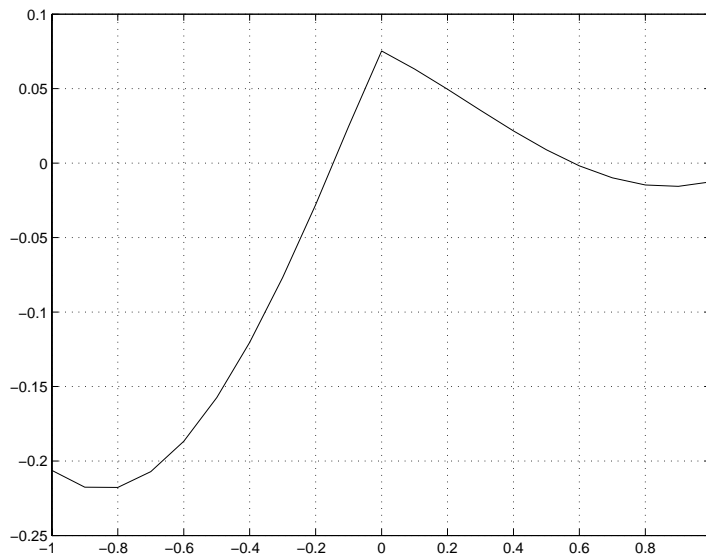


Figure 2. Piecewise linear approximation of scalar function  $U(\tau)$

## 8. CONCLUSIONS

In this paper, we studied the exponential stability of integral delay systems which appear in several stability problems of time-delay systems. Lyapunov type necessary and sufficient conditions for the exponential stability are given. A constructive procedure for computing quadratic Lyapunov-Krasovskii functionals for a given exponentially stable system is provided. It is shown that the functionals depend on matrix function  $U(\tau)$  which satisfies a matrix integral delay equation. Some important properties of this matrix function that allow its computation are explicitly given. The proposed functionals are used to obtain exponential estimates of the system solutions as well as robust stability conditions for perturbed systems.

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## REFERENCES

1. Gu K, Niculescu SI. Additional dynamics introduced in model system transformation. *IEEE Trans. Automat. Contr.* 2000; **45**(3):572-575.
2. Kharitonov VL, Melchor-Aguilar D. On delay dependent stability conditions. *Syst. Control Lett.* 2000; **40**:71-76.
3. Kharitonov VL, Melchor-Aguilar D. Additional dynamics for general class of time delay systems. *IEEE Trans. Automat. Contr.* 2003; **48**(6):1060-1064.
4. Manitus AZ, Olbrot AW. Finite spectrum assignment problem for systems with delay. *IEEE Trans. Automat. Contr.* 1979; **24**(4):541-553.
5. Engelborghs K, Dambrine M, Roose D. Limitations of a class of stabilization methods for delay systems. *IEEE Trans. Automat. Contr.* 2001; **46**(2):336-339.

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6. Mondié S, Dambrine M, Santos O. Approximation of control laws with distributed delays: a necessary condition for stability. *Kybernetika*, 2002; **38**(5):541-551.
7. Van Assche V, Dambrine M, Lafay JF, Richard JP. Some problems arising in the implementation of distributed-delay control law, in *Proc. of 39th IEEE Conf. on Decision and Contr.*, Phoenix, Az., 1999.
8. Hale JK, Verduyn Lunel SM. *Introduction to Functional Differential Equations*. Springer-Verlag, 1993.
9. Niculescu SI. *Delay Effects on Stability. A Robust Control Approach*, vol. 269. Lecture Notes in Control and Information Sciences: Springer-Verlag, 2001.
10. Bellman R, Cooke KL. *Differential-Difference Equations*. Academic Press, 1963.
11. Melchor-Aguilar D, Tristán-Tristán B. On the implementation of control laws for finite spectrum assignment: the multiple delays case, in *Proc. of 4th IEEE Conference on Electrical and Electronics Engineering*, México City, México, 2007.