Further results on exponential stability of linear continuous time difference systems

Daniel Melchor-Aguilar[a,b]

[a] *Division of Applied Mathematics, IPICYT, 78216, San Luis Potosí, SLP, México*
[b] *Mechatronic Section, CINVESTAV-IPN, 07300, México, DF, México*

Abstract

This paper provides further Lyapunov results for the exponential stability of linear continuous time difference system involving discrete and distributed delays. We consider such a class of systems in the case when the discrete and distributed delays are independent thus completing the recent Lyapunov results obtained for the case when the delays are dependent.

Keywords: Continuous time difference systems, exponential stability, Lyapunov-Krasovskii functionals

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1. Introduction and problem formulation

Consider the following class of linear continuous time difference system:

$$x(t) = Ax(t-h) + \int_{-\tau}^{0} G(\theta)x(t+\theta)d\theta, \tag{1}$$

where A is a Schur stable matrix, $G(\theta)$ is a matrix function with piecewise continuous bounded elements defined in the interval $[-\tau, 0]$, while h and τ are positive independent scalars.

Such systems can be found as delay approximations of the partial differential equations for describing the propagation phenomena in excitable media [1], in the stability analysis of additional dynamics introduced by some system transformations [5, 8, 9], in delay-dependent stability analysis of neutral type systems [7], and in the stability analysis of some difference operators in neutral type functional differential equations [2, 6].

Email address: `dmelchor@ipicyt.edu.mx` (Daniel Melchor-Aguilar)

In [11] (see also [10]), motivated from some limitations on the application of existing Lyapunov approaches [3, 4, 12, 13, 14] to the stability analysis of systems of the form in (1), we introduced a new Lyapunov-Krasovskii approach for properly addressing the stability of (1) in the special case when the discrete and distributed delays are equal, i.e., $h = \tau$. For such a case, we derived delay-dependent stability conditions providing less conservative results than the existing ones in the literature based on matrix norms [2, 6].

In the current paper we extend the Lyapunov results in [11] for the more general of case (1) where the discrete delay h and the distributed delay τ are completely independent. We derive exponential stability conditions which are delay-independent w.r.t. discrete delays and delay-dependent w.r.t. distributed delays, a result certainly expected from Schur property of the matrix A, but that, to the best of our knowledge, it has not been derived by means of Lyapunov approaches in the literature. On the other hand, we show that the corresponding exponential estimates for the solutions depend on the particular values of both the discrete and distributed delays.

The remaining part of the paper is organized as follows: Section 2 presents some preliminaries. After revising some facts about solutions, the Lyapunov-Krasovskii theorem in [11] is modified for the case of system (1). The main results are given in section 3. Examples illustrating the results are provided in section 4 and concluding remarks end the paper in section 5.

2. Preliminaries

2.1. Solutions and stability concept

Let $r = \max\{h, \tau\}$. In order to define a particular solution of (1) an initial function $\varphi(\theta), \theta \in [-r, 0)$, should be given. We assume that $\varphi \in \mathcal{C}([-r, 0), \mathcal{R}^n)$, the space of continuous vector functions mapping $[-r, 0)$ to $\mathcal{R}^n$ equipped with the uniform convergence norm $\|\varphi\|_r = \sup_{\theta\in[-r,0)} \|\varphi(\theta)\|$.

For a given initial function $\varphi \in \mathcal{C}([-r, 0), \mathcal{R}^n)$ there exists a unique solution $x(t, \varphi)$ of (1) defined for all $t \geq 0$, see [2]. This solution presents jump discontinuities which distribution on time can be very difficult to describe in the general case. Clearly, at $t = 0$ the jump discontinuity is explicitly given by

$$\Delta x(0) \triangleq x(0) - \varphi(-0) = A\varphi(-h) + \int_{-\tau}^{0} G(\theta)x(t + \theta)d\theta.$$

When $\tau = h$, the jump discontinuities of the solutions occur at time instants multiple of the delay h, see [11] for more details.

In the following we adopt this concept of exponential stability.

Definition 1. *[2] System (1) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu > 0$ such that any solution of (1) satisfies the inequality*

$$\|x(t,\varphi)\| \leq \mu e^{-\alpha t} \|\varphi\|_r, \forall t \geq 0. \tag{2}$$

2.2. Lyapunov-Krasovskii conditions

For a given $t \geq 0$, we define the natural state $x_t(\varphi) = x(t+\theta,\varphi), \theta \in [-r,0)$. When the initial function is irrelevant we simply write $x(t)$ and x_t instead of $x(t,\varphi)$ and $x_t(\varphi)$. Based on the discontinuities of the solutions it results that $x_t(\varphi) \in \mathcal{PC}([-r,0),\mathcal{R}^n)$, the space of piecewise continuous bounded functions mapping the interval $[-r,0)$ to $\mathcal{R}^n$. As a consequence, in a Lyapunov-Krasovskii functional setting, the functionals should be defined on $\mathcal{PC}([-r,0),\mathcal{R}^n)$.

The following result is a modification of the Lyapunov-Krasovskii theorem introduced in [11] to the case of system (1) where the discrete and distributed delays are independent.

Theorem 1. *Consider system (1) and assume that matrix A is Schur stable. System (1) is exponentially stable if there exists a functional $v : \mathcal{PC}([-r,0),\mathcal{R}^n) \to \mathcal{R}$ such that $t \to v(x_t(\varphi))$ is differentiable and the following conditions hold:*

1. $\alpha_1 \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta \leq v(\varphi) \leq \alpha_2 \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta$, *for some* $0 < \alpha_1 \leq \alpha_2$,
2. $\frac{d}{dt} v(x_t(\varphi)) \leq -\beta \int_{-r}^{0} \|x(t,\varphi)\|^2 d\theta$, *for some* $\beta > 0$.

Moreover, for any initial function $\varphi \in \mathcal{C}([-r,0),\mathcal{R}^n)$, the corresponding solution $x(t,\varphi)$ satisfies the exponential upper bound (2) with

$$\mu = \eta\left(1+\gamma+\frac{\gamma}{he\varepsilon}\right) \text{ and } \alpha = \min\left\{\frac{\beta}{2\alpha_2},\nu\right\} - \varepsilon. \tag{3}$$

Here $\varepsilon \in \left(0, \min\left\{\frac{\beta}{2\alpha_2},\nu\right\}\right)$, $\gamma = \left(\sqrt{\frac{\alpha_2}{\alpha_1}} r\tau\right)\left(\sup_{\theta\in[-\tau,0]} \|G(\theta)\|\right)$, *while* $\eta > 0$ *and* $\nu > 0$ *are such that* $\left\|A^k\right\| \leq \eta e^{-\nu(kh)}, k = 0,1,2,\ldots$

3. Main Results

3.1. A general case

Proposition 2. *Let system (1) be given and assume that matrix A is Schur stable. System (1) is exponentially stable if there exist positive definite matrices W_0, W_1 and Q such that*

$$\tau \left(\sup_{\theta\in[-\tau,0]} \|G(\theta)\| \right)^2 < \frac{\lambda_{\min}(W_1)}{\lambda_{\max}(P + PAW_0^{-1}A^TP)}, \tag{4}$$

with P the unique positive definite solution of the Lyapunov matrix equation

$$A^TPA - P = -\left(W_0 + \tau W_1 + Q\right). \tag{5}$$

Furthermore, an exponential estimate for the solutions of (1) is given by (2) where μ and α are as given in (3) and

$$\alpha_1 = \lambda_{\min}(0.5Q), \tag{6}$$
$$\alpha_2 = \lambda_{\max}\left(A^TPA + W_0\right) + \lambda_{\max}(\tau W_1) + \lambda_{\max}(Q), \tag{7}$$
$$\beta = \lambda_{\min}(0.5r^{-1}Q), \tag{8}$$

where $r = \max\{h, \tau\}$.

Proof. Consider the following functional candidate:

$$v(\varphi) = \int_{-h}^{0} \varphi^T(\theta)\left[A^TPA + W_0\right]\varphi(\theta)d\theta + \int_{-\tau}^{0} \varphi^T(\theta)\left(\theta+\tau\right)W_1\varphi(\theta)d\theta$$
$$+0.5\int_{-r}^{0} \varphi^T(\theta)\left[Q + (\theta + r)\,r^{-1}Q\right]\varphi(\theta)d\theta, \tag{9}$$

where W_0, W_1, Q are any positive definite matrices and P is the unique positive definite solution of the Lyapunov matrix equation (5).

The functional satisfies the following inequalities:

$$\alpha_1 \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta \le v(\varphi) \le \alpha_2 \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta,$$

with $0 < \alpha_1 \le \alpha_2$ determined by (6) and (7), respectively.

The time derivative of the functional (9) along solutions of (1) is

$$\begin{aligned}\frac{dv(x_t)}{dt} &= x^T(t)\left[A^TPA + W_0 + \tau W_1 + Q\right]x(t) - 0.5x^T(t-r)Qx(t-r)\\ &\quad -x^T(t-h)\left[A^TPA + W_0\right]x(t-h) - \int_{-\tau}^{0} x^T(t+\theta)W_1x(t+\theta)d\theta\\ &\quad -0.5r^{-1}\int_{-r}^{0} x^T(t+\theta)Qx(t+\theta)d\theta.\end{aligned}$$

By using the Lyapunov matrix equation (5) and substituting the right-hand side of (1) we obtain

$$\begin{aligned}&\frac{d}{dt}v(x_t) = x^T(t-h)A^TPAx(t-h) + 2x^T(t-h)A^TP\int_{-\tau}^{0} G(\theta)x(t+\theta)d\theta\\ &-0.5x^T(t-r)Qx(t-r) - 0.5r^{-1}\int_{-r}^{0} x^T(t+\theta)Qx(t+\theta)d\theta\\ &-x^T(t-h)\left[A^TPA + W_0\right]x(t-h) - \int_{-\tau}^{0} x^T(t+\theta)W_1x(t+\theta)d\theta\\ &+\left(\int_{-\tau}^{0} G(\theta)x(t+\theta)d\theta\right)^T P\left(\int_{-\tau}^{0} G(\theta)x(t+\theta)d\theta\right).\end{aligned}$$

The Jensen integral inequality implies

$$\begin{aligned}&\left(\int_{-\tau}^{0} G(\theta)x(t+\theta)d\theta\right)^T P\left(\int_{-\tau}^{0} G(\theta)x(t+\theta)d\theta\right)\\ \leq\ &\tau\int_{-\tau}^{0} x^T(t+\theta)G^T(\theta)PG(\theta)x(t+\theta)d\theta.\end{aligned}$$

As a consequence we arrive at the following upper bound for the derivative:

$$\begin{aligned}&\frac{d}{dt}v(x_t) \leq -\int_{-\tau}^{0} \begin{bmatrix} x^T(t-h) & x^T(t+\theta)\end{bmatrix}\mathcal{N}(\theta)\begin{bmatrix} x(t-h)\\ x(t+\theta)\end{bmatrix}d\theta\\ &-0.5x^T(t-r)Qx(t-r) - 0.5r^{-1}\int_{-r}^{0} x^T(t+\theta)Qx(t+\theta)d\theta,\end{aligned}$$

where for $\theta \in [-\tau, 0]$

$$\mathcal{N}(\theta) = \begin{bmatrix} \frac{1}{\tau}W_0 & -A^TPG(\theta)\\ -G^T(\theta)PA & W_1 - \tau G^T(\theta)PG(\theta)\end{bmatrix}.$$

If the inequality (4) holds then

$$\lambda_{\min}(W_1) - \tau\lambda_{\max}(P + PAW_0^{-1}A^T P)\left(\sup_{\theta\in[-\tau,0]}\|G(\theta)\|\right)^2 > 0,$$

which in turn implies

$$W_1 - \tau G^T(\theta)\left[P + PAW_0^{-1}A^T P\right]G(\theta) > 0, \forall\theta \in [-\tau, 0].$$

The above inequality is equivalent to $\mathcal{N}(\theta) > 0, \forall\theta \in [-\tau, 0]$, by Schur complement. Thus, if (4) holds then

$$\frac{d}{dt}v(x_t) \leq -\beta\int_{-r}^{0}\|x(t+\theta)\|^2 d\theta,$$

with $\beta > 0$ given by (8), and the exponential stability of (1) follows.

By calculating the positive constants α_1, α_2 and β from the expressions (6),(7) and (8), respectively, the exponential estimate for the solutions directly follows from Theorem 1. ■

Remark 1. *Notice that the Lyapunov functional (9) has an additional term, the one involving matrix Q and delay $r = \max\{h, \tau\}$, to the functional used in [11] for addressing the case when $h = \tau$. This additional integral term allows us to derive the desired* discrete *delay-independent/*distributed *delay-dependent stability conditions. On the other hand, the exponential decay rate α and the μ-factor in the exponential estimate for the solutions depend on both the discrete and distributed delays.*

Remark 2. *When $\tau = h$, the inequality (4) becomes*

$$h\left(\sup_{\theta\in[-h,0]}\|G(\theta)\|\right)^2 < \frac{\lambda_{\min}(W_1)}{\lambda_{\max}(P + PAW_0^{-1}A^T P)}$$

while the Lyapunov matrix equation (5) takes the form

$$A^T PA - P = -(W_0 + hW_1 + Q).$$

Since W_0, W_1 and Q are free positive definite matrices then by choosing $Q = \varepsilon I, \varepsilon > 0$, and letting $\varepsilon \to 0$ we directly get the stability conditions given in [11].

Remark 3. *As a consequence of the above observation, we have that the best possible result that can be obtained for the case of independent discrete and distributed delays by using the new stability conditions (4) and (5) it is the one obtained for the case when the delays are equal.*

3.2. A particular case

Now let us consider the following perturbed system:

$$x(t) = (A + \Delta A)\, x(t-h) + \sum_{j=1}^{m} (G_j + \Delta G_j) \int_{-\tau_j}^{0} x(t+\theta)d\theta, \tag{10}$$

where $0 < h$ is the discrete delay and $0 \leq \tau_1 \leq \tau_2 \leq \cdots \tau_m$ are the distributed delays, $A, G_j \in \mathcal{R}^n, j = 1, 2, \ldots, m,$ are known matrices and $\Delta A, \Delta G_j, j = 1, 2, \ldots, m,$ are unknown constant matrices such that

$$\|\Delta A\| \leq \delta \text{ and } \|\Delta G_j\| \leq \rho_j, j = 1, 2, \ldots, m. \tag{11}$$

System (10) is a particular case of (1), where the kernel $G(\theta)$ is a piecewise constant matrix, see [9] for details in the case of pure integral delay systems.

The particular perturbed case of (10) when there exists only one integral delay term and the discrete and distributed delays are equal was investigated in [11].

Our problem here is to derive conditions for the exponential stability of (10) for all perturbations $\Delta A, \Delta G_j, j = 1, 2, \ldots, m,$ satisfying (11) and without any assumption between the discrete and distributed delays.

Following [11], we assume that matrix $A + \Delta A$ remains Schur stable for all perturbations ΔA satisfying (11).

Proposition 3. *The perturbed system (10) is exponentially stable for all perturbations satisfying (11) if there exist positive definite matrices $P, Q, W_j, j = 0, 1, \ldots, m,$ and a positive constant λ such that the following inequalities hold:*

$$\mathcal{N}_j^n - \lambda \mathcal{N}_j^{p_1} - \lambda \mathcal{N}_j^{p_2} > 0, \tag{12}$$

$$A^T P A + \lambda\delta\left(2\|A\| + \delta\right) I + W_0 + \sum_{j=1}^{m} \tau_j W_j + Q - P < 0, \tag{13}$$

$$\lambda I - P > 0, \tag{14}$$

where for $j = 1, 2, \ldots, m,$

$$\mathcal{N}_j^n = \begin{bmatrix} \frac{1}{m\tau_j} W_0 & -A^T P G_j \\ -G_j^T P A & W_j - m\tau_j G_j^T P G_j \end{bmatrix}, \tag{15}$$

$$\mathcal{N}_j^{p_1} = \left\{ \left(\|A\| + \delta\right) \rho_j + \delta \|G_j\| \right\} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{16}$$

$$\mathcal{N}_j^{p_2} = \left\{ \left(\|A\| + \delta\right) \rho_j + \delta \|G_j\| + m\tau_j \rho_j \left(2\|G_j\| + \rho_j\right) \right\} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{17}$$

Moreover, for any initial function $\varphi \in \mathcal{C}([-r,0), \mathcal{R}^n), r = \max\{h, \tau_m\}$, *the corresponding solution* $x(t,\varphi)$ *of the perturbed system (10) satisfies the exponential upper bound (2) with* μ *and* α *given by (3), while* $\gamma = \left(\sqrt{\frac{\alpha_2}{\alpha_1}} r\tau\right)\left(\sum_{j=1}^{m}(\|G_j\| + \rho_j)\right)$,

$$\begin{aligned} \alpha_1 &= \lambda_{\min}(0.5Q), &(18)\\ \alpha_2 &= \lambda_{\max}(P)(\|A\|+\delta)^2 + \sum_{j=1}^{m} \lambda_{\max}(\tau_j W_j) + \lambda_{\max}(Q), &(19)\\ \beta &= \lambda_{\min}(0.5 r^{-1} Q). &(20) \end{aligned}$$

Proof. Consider the following functional candidate:

$$\begin{aligned} v(\varphi) &= \int_{-h}^{0} \varphi^T(\theta)\left[(A+\Delta A)^T P (A+\Delta A) + W_0\right]\varphi(\theta)d\theta \qquad (21)\\ &+ \sum_{j=1}^{m} \int_{-\tau_j}^{0} \varphi^T(\theta)(\theta+\tau_j) W_j \varphi(\theta) d\theta \\ &+ 0.5 \int_{-r}^{0} \varphi^T(\theta)\left[Q + (\theta + r) r^{-1} Q\right]\varphi(\theta)d\theta, \end{aligned}$$

where $Q, W_j, j = 0, 1, \ldots, m$, are any positive definite matrices and P is the positive definite solution of the Lyapunov inequality

$$(A+\Delta A)^T P (A+\Delta A) - P < -\left(W_0 + \sum_{j=1}^{m} \tau_j W_j + Q\right). \qquad (22)$$

From (21) we get the following inequalities for the functional:

$$\alpha_1 \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta \le v(\varphi) \le \alpha_2 \int_{-r}^{0} \|\varphi(\theta)\|^2 d\theta,$$

where $0 < \alpha_1 \le \alpha_2$ are determined by (18) and (19), respectively.

The time derivative of the functional (21) along solutions of (10) is

$$\frac{dv(x_t)}{dt} = -x^T(t-h)\left[(A+\Delta A)^T P (A+\Delta A) + W_0\right] x(t-h)$$

$$-\sum_{j=1}^{m}\int_{-\tau_j}^{0} x^T(t+\theta)W_j x(t+\theta)d\theta$$

$$+x^T(t)\left[(A+\Delta A)^T P (A+\Delta A) + W_0 + \sum_{j=1}^{m}\tau_j W_j + R + rS\right] x(t)$$

$$-0.5x^T(t-r)Qx(t-r) - 0.5r^{-1}\int_{-r}^{0} x^T(t+\theta)Qx(t+\theta)d\theta.$$

Taking into account the inequality (22) and substituting the right-hand side of (10) we get

$$\frac{dv(x_t)}{dt} \leq -x^T(t-h)W_0 x(t-h) - 0.5r^{-1}\int_{-r}^{0} x^T(t+\theta)Qx(t+\theta)d\theta$$

$$+\left(\sum_{j=1}^{m}(G_j+\Delta G_j)\int_{-\tau_j}^{0} x(t+\theta)d\theta\right)^T P\left(\sum_{j=1}^{m}(G_j+\Delta G_j)\int_{-\tau_j}^{0} x(t+\theta)d\theta\right)$$

$$+2x^T(t-h)(A+\Delta A)^T P\left(\sum_{j=1}^{m}(G_j+\Delta G_j)\int_{-\tau_j}^{0} x(t+\theta)d\theta\right)$$

$$-\sum_{j=1}^{m}\int_{-\tau_j}^{0} x^T(t+\theta)W_j x(t+\theta)d\theta - 0.5x^T(t-r)Qx(t-r).$$

By using the Jensen inequality the following inequality:

$$\left(\sum_{j=1}^{m}(G_j+\Delta G_j)\int_{-\tau_j}^{0} x(t+\theta)d\theta\right)^T P\left(\sum_{j=1}^{m}(G_j+\Delta G_j)\int_{-\tau_j}^{0} x(t+\theta)d\theta\right)$$

$$\leq m\sum_{j=1}^{m}\tau_j\int_{-\tau_j}^{0} x^T(t+\theta)(G_j+\Delta G_j)^T P (G_j+\Delta G_j) x(t+\theta)d\theta$$

holds. As a consequence we get the following upper bound for the derivative:

$$\begin{aligned}\frac{dv(x_t)}{dt} &\le -\sum_{j=1}^{m}\int_{-\tau_j}^{0}\left[\begin{array}{cc} x^T(t-h) & x^T(t+\theta)\end{array}\right]\left\{\mathcal{N}_j^n-\mathcal{N}_j^p\right\}\left[\begin{array}{c} x(t-h)\\ x(t+\theta)\end{array}\right]d\theta\\ &\quad -0.5r^{-1}\int_{-r}^{0}x^T(t+\theta)Qx(t+\theta)d\theta-0.5x^T(t-r)Qx(t-r),\end{aligned}$$

where $\mathcal{N}_j^n, j=1,2,\ldots,m$, are defined by (15) and

$$\mathcal{N}_j^p=\left[\begin{array}{cc} 0 & A^TP\left(\Delta G_j\right)+\left(\Delta A\right)^T PG_j+\left(\Delta A\right)^T P\left(\Delta G_j\right)\\ * & G_j^TP\left(\Delta G_j\right)+\left(\Delta G_j\right)^T PG_j+\left(\Delta G_j\right)^T P\left(\Delta G_j\right)\end{array}\right].$$

Here $*$ denotes the symmetric term of the symmetric matrix.

Using Lemma 7 in the appendix of [11] for bounding the terms involving perturbation in the matrices $\mathcal{N}_j^p$ we obtain, after simple but tedious calculations, the following upper bound for the derivative:

$$\begin{aligned}&\frac{dv(x_t)}{dt}\le -0.5r^{-1}\int_{-r}^{0}x^T(t+\theta)Qx(t+\theta)d\theta-0.5x^T(t-r)Qx(t-r)\\ &-\sum_{j=1}^{m}\int_{-\tau_j}^{0}\left[\begin{array}{cc} x^T(t-h) & x^T(t+\theta)\end{array}\right]\left\{\mathcal{N}_j^n-\lambda\mathcal{N}_j^{p_1}-\lambda\mathcal{N}_j^{p_2}\right\}\left[\begin{array}{c} x(t-h)\\ x(t+\theta)\end{array}\right]d\theta\end{aligned}$$

where $\mathcal{N}_j^{p_1}, \mathcal{N}_j^{p_2}, j=1,2,\ldots,m$, are respectively defined by (16) and (17) and $\lambda>0$ is such that (14) holds. Clearly, if (12) holds then

$$\frac{dv(x_t)}{dt}\le -\beta\int_{-r}^{0}\left\|x(t+\theta)\right\|^2 d\theta,$$

with $\beta>0$ given by (20), and the exponential stability of the perturbed system (10) follows. Noting that the inequality (13) subject to the restriction (21) implies (22) the stability result is concluded.

Finally, after respectively computing the positive constants α_1, α_2 and β from the expressions (18),(19) and (20), the exponential estimate for the solutions of the perturbed system (10) is directly obtained from Theorem 1 by observing that, in this case,

$$\sup_{\theta\in[-\tau,0]}\|G(\theta)\|\le\sum_{j=1}^{m}\left(\|G_j\|+\rho_j\right).$$

■

In the nominal case, when (10) does not have uncertainty

$$x(t) = Ax(t-h) + \sum_{j=1}^{m} G_j \int_{-\tau_j}^{0} x(t+\theta)d\theta, \tag{23}$$

sufficient conditions for the exponential stability can be directly obtained from Proposition 3.

Corollary 4. *System (23) is exponentially stable if there exist positive definite matrices $P, Q, W_j, j = 0, 1, \ldots, m$, such that for $j = 1, 2, \ldots, m$,*

$$\mathcal{N}_j^n = \begin{bmatrix} \frac{1}{m\tau_j} W_0 & -A^T P G_j \\ -G_j^T P A & W_j - m\tau_j G_j^T P G_j \end{bmatrix} > 0,$$

where P is the unique positive definite solution of the matrix Lyapunov inequality

$$A^T P A + W_0 + \sum_{j=1}^{m} \tau_j W_j + Q - P < 0. \tag{24}$$

Furthermore, for any initial function $\varphi \in \mathcal{C}\left([-r, 0), \mathcal{R}^n\right), r = \max\left\{h, \tau_m\right\}$, the corresponding solution $x(t, \varphi)$ of (23) satisfies the exponential upper bound (2) with μ and α given by (3), while

$$\begin{aligned}
\gamma &= \left(\sqrt{\frac{\alpha_2}{\alpha_1} r\tau}\right)\left(\sum_{j=1}^{m} \|G_j\|\right), \\
\alpha_1 &= \lambda_{\min}(0.5Q), \\
\alpha_2 &= \lambda_{\max}(P)\|A\|^2 + \sum_{j=1}^{m} \lambda_{\max}\left(\tau_j W_j\right) + \lambda_{\max}\left(Q\right), \\
\beta &= \lambda_{\min}(0.5r^{-1}Q).
\end{aligned}$$

Remark 4. *Analogously to Remarks 1,2 and 3, the stability conditions in Proposition 3 and Corollary 4 are* discrete *delay-independent/*distributed *delay-dependent, the exponential estimate for the solutions depends on both the discrete and distributed delays and the best possible result is obtained for the case when $\tau_m = h$.*

4. Examples

Example 1. *Let us consider the following system:*

$$x(t) = Ax(t-h) + G\int_{-\tau}^{0} x(t+\theta)d\theta, \tag{25}$$

where

$$A = \begin{pmatrix} 0.2 & 1 \\ -0.1 & -0.2 \end{pmatrix}.$$

System (25) has been studied in [11] when the discrete and distributed delays are exactly the same, i.e., $h = \tau$. It was shown there that (25) is exponentially stable for $h = \tau = 1$ and any matrix $G \in \mathcal{R}^{2\times 2}$ such that $\|G\| < 0.2082$, a result that improve the one obtained by combining similarity transformations and the known norm inequality $\|A\| + \tau \|G\| < 1$, see [11] for details.

By using our results in Proposition 2 and Remark 3 we directly conclude that (25) is exponentially stable for $\tau = 1$, any matrix $G \in \mathcal{R}^{2\times 2}$ satisfying $\|G\| < 0.2082$, and any arbitrary discrete delay h. *In particular, this result is obtained from the inequality (4) and matrix equation (5) for the following positive matrices: $W_0 = 2.3I_2, W_1 = 2.7I_2$ and $Q = 0.0018I_2$.*

We now use Proposition 2 for computing exponential estimates for the solutions of (25). To this aim let us select $W_0 = W_1 = Q = I_2$ and $\tau = 1$. For these values we get that (25) is exponentially stable for any arbitrary discrete delay h *and any matrix $G \in \mathcal{R}^{2\times 2}$ satisfying $\|G\| < 0.1426$.*

For $h_1 = 1$ and $h_2 = \sqrt{2}$ the inequalities $\left\|A^k\right\| \le \eta_j e^{-\nu_j(kh_j)}, j = 1, 2, k = 0, 1, 2, \ldots,$ respectively holds with $\eta_1 = \eta_2 = 1.05, \nu_1 = 0.0073$ and $\nu_2 = 0.0051$. Direct calculations derived from Proposition 2 lead to the following exponential decay rates α and $\mu-$factors:

$$\begin{aligned} \mu_{h_1} &= 65.7162 \textit{ and } \alpha_{h_1} = 0.0036, \\ \mu_{h_2} &= 77.9516 \textit{ and } \alpha_{h_2} = 0.0026. \end{aligned}$$

Example 2. *Consider the following perturbed system:*

$$\begin{aligned} y(t) &= (A + \Delta A)\, y(t-h) + (G_1 + \Delta G_1)\int_{-\tau_1}^{0} y(t+\theta)d\theta \\ &\quad + (G_2 + \Delta G_2)\int_{-\tau_2}^{0} y(t+\theta)d\theta, \end{aligned} \tag{26}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0.01 & 0 \end{pmatrix}, G_1 = \begin{pmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{pmatrix}, G_2 = \begin{pmatrix} 0.1 & 0 \\ 2 & 0.1 \end{pmatrix},$$

matrices $\Delta A, \Delta G_1, \Delta G_2$ *are unknown satisfying*

$$\|\Delta A\| \leq 0.1, \|\Delta G_1\| \leq 0.1, \|\Delta G_2\| \leq 0.1 \tag{27}$$

The induced matrix norms are $\|A\| = 1, \|G_1\| = 1.2511$ *and* $\|G_2\| = 2.005$. *Hence, even in the nominal case, the known inequality* $\|A\| + \sum_{j=1}^{2} \tau_j \|G_j\| < 1$ *cannot be applied to conclude stability of (26).*

On the other hand, since matrix A *is Schur stable then we can use our results to get stability conditions for (26) in both nominal and perturbed cases.*

For simplicity of the calculations and clarity of the presentation let us fix $\tau_1 = 0.15$ *and search for* $\tau_2 \geq \tau_1$ *such that (26) is exponentially stable.*

By using Corollary 4 we found that the nominal system (26) is exponentially stable for all constant discrete delay $h > 0$ *and distributed delays* $0 \leq \tau_1 \leq 0.15 \leq \tau_2 \leq 0.48$.

By using Proposition 3 we found that the perturbed system (26) is exponentially stable for all perturbations $\Delta A, \Delta G_1, \Delta G_2$ *satisfying (27),* all positive constant values of the discrete delay h, *and distributed delays* $0 \leq \tau_1 \leq 0.15 \leq \tau_2 \leq 0.37$.

Now let us select $\tau_1 = 0.15$ *and* $\tau_2 = 0.3$, *for which the perturbed system (26) is exponentially stable, and compute exponential estimates corresponding to different values of the discrete delay* h.

For $h_1 = 0.5$ *and* $h_2 = \sqrt{3}$ *the inequalities* $\left\|(A + \Delta A)^k\right\| \leq \eta_j e^{-\nu_j (kh_j)}, j = 1, 2, k = 0, 1, 2, \ldots,$ *respectively hold with* $\eta_1 = \eta_2 = 1.2, \nu_1 = 0.1740$ *and* $\nu_2 = 0.034$. *Direct calculations derived from Proposition 3 lead to the following exponential decay rates* α *and* $\mu-$*factors:*

$$\begin{aligned} \mu_{h_1} &= 4.2197 \times 10^3 \text{ and } \alpha_{h_1} = 0.0034 \\ \mu_{h_2} &= 7.8527 \times 10^3 \text{ and } \alpha_{h_2} = 9.84 \times 10^{-4}. \end{aligned}$$

5. Conclusions

In this paper, additional results on the exponential stability of linear continuous time difference systems including discrete and distributed delay terms are presented. The contribution extends previous results in several ways. Firstly, it presents Lyapunov-Krasovskii conditions for the exponential stability of such class of systems in the more general case when the discrete and distributed delays are independent. Secondly, by using the general Lyapunov-Krasovskii result, several Lyapunov functionals are constructed for providing stability and robust stability conditions which are discrete delay-independent/distributed delay-dependent. Finally, a constructive procedure for deriving exponential estimates for the solutions that depend on both the discrete and distributed delays is given.

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