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Self-adjoint oscillator operator from a modified factorization

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Abstract

By using an alternative factorization, we obtain a self-adjoint oscillator operator of the form $\mathcal{L}_\delta = \frac{d}{dx}\left(p_\delta(x)\frac{d}{dx}\right) - \left(\frac{x^2}{p_\delta(x)} + p_\delta(x) - 1\right)$, where $p_\delta(x) = 1 + \delta e^{-x^2}$, with $\delta \in (-1, \infty)$ an arbitrary real factorization parameter. At positive values of δ, this operator interpolates between the quantum harmonic oscillator Hamiltonian for $\delta = 0$ and a scaled Hermite operator at high values of δ. For the negative values of δ, the eigenfunctions look like deformed quantum mechanical Hermite functions. Possible applications are mentioned.

Keywords: factorization; quantum harmonic oscillator; generalized Hermite polynomials; Ornstein-Uhlenbeck processes

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I. INTRODUCTION

One of the most prominent areas of research in mathematical physics in the last 30 years has been that of supersymmetric quantum mechanics (SUSY QM), based on the work of Witten on supersymmetry breaking in quantum field theory [1], who not only introduced a standard terminology but also boosted up the old factorization method already systematically investigated by Infeld and Hull [2, 3]. It is well known that the factorization method has applications in many areas [4, 5] extending also to nonlinear physics [6]. Another important breakthrough in the SUSY QM context was the work of Mielnik [7, 8], who showed in the case of the harmonic oscillator that the mutually-adjoint factorization operators of non-constant commutator $[b, b^*] = 2\beta_g'$ based on the general Riccati solution $\beta_g(x)$ led to strictly isospectral potentials. Mielnik's procedure has been applied to different quantum problems [9], and further applications in terms of higher-order supersymmetric operators [10], higher order creation/anihilation operators [11], as well as for nonlinear differential equations [12] have been developed. Here, an unexplored application in terms of a pair of non-mutually adjoint factorization operators is dealt with. These operators, denoted by B^- and B^+, are introduced in equation (11) below and their commutator is $[B^-, B^+] = (\alpha^{-1} + \alpha)\beta'$, where the functions $\alpha(x)$ and $\beta(x)$ are determined in this paper.

One way to introduce SUSY QM is through the easiest example which is that of the simple harmonic oscillator (SHO), whose Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 \tag{1}$$

possesses the eigenfunctions and eigenvalues given by

$$\psi_n(x) = c_n \mathrm{H}_n(x)\, e^{-x^2/2} \ , \quad E_n = n + \frac{1}{2} \ , \tag{2}$$

where $\mathrm{H}_n(x)$ are the Hermite polynomials, satisfying Hermite's equation

$$\mathrm{H}_n''(x) - 2x\,\mathrm{H}_n'(x) + 2n\,\mathrm{H}_n(x) = 0 \ . \tag{3}$$

Hermite's equation can be derived from the SHO Schrödinger's equation by acting the Hamiltonian (1) on the functions (2) and leaving the equation for the functions $\mathrm{H}_n(x)$ alone. In the literature, one can find generalizations of equation (3), which define generalized Hermite polynomials $\mathrm{H}_m(u;t)$ [13], $\mathrm{H}_n(x;\gamma)$ [14, 15], and $\mathrm{H}_n^N(x)$ [16, 17], or simply define the Hermite functions as $h_n(x)$=$\mathrm{H}_n(x)\, e^{-x^2/2}$, the eigenfunctions of the quantum oscillator.

The Hamiltonian (1) can be factorized using the annihilation and creation operators

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^* = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right), \tag{4}$$

such that $a\,a^* = H + \frac{1}{2}$, and $a^*\,a = H - \frac{1}{2}$.

Mielnik proposed a different way to factorize the harmonic oscillator Hamiltonian, by introducing new operators [7]

$$b = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \beta(x)\right), \quad b^* = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \beta(x)\right), \tag{5}$$

which, in order to satisfy the factorization $bb^* = H + \frac{1}{2}$, lead to a Riccati equation for the function $\beta(x)$

$$\beta' + \beta^2 = 1 + x^2. \tag{6}$$

The particular solution $\beta_p = x$ leads to the original anihilation/creation operators, while the general solution of the Riccati equation

$$\beta_g(x) = x + \phi_\gamma(x) = x + \frac{\gamma e^{-x^2}}{1 + \gamma\int_0^x e^{-x'^2}dx'} \tag{7}$$

leads to new Hamiltonians (and potentials) defined by $b^*\,b = \tilde{H} - \frac{1}{2}$,

$$\tilde{H} = H - \phi'_\gamma(x)\,, \quad \tilde{V}(x) = \frac{x^2}{2} - \frac{d}{dx}\left[\frac{\gamma e^{-x^2}}{1 + \gamma\int_0^x e^{-x'^2}dx'}\right], \tag{8}$$

isospectral to the SHO potential, whose eigenfunctions are defined by

$$\tilde{\psi}_{n+1} = b^*\psi_n \tag{9}$$

for $n \geq 0$, and with the ground state defined by the equation

$$b\tilde{\psi}_0 = 0\,. \tag{10}$$

The new potentials and eigenfunctions depend on the SUSY parameter $|\gamma| < 2/\sqrt{\pi}$. A multiple-parameter generalization of this construction producing multiple-parameter families of new potentials can be also found in the literature [18].

In this paper, we will show that not all has been said about factorizing the Hamiltonian (1), and that we can still find some hidden information through the factorization procedure. We shall propose an alternative factorization, which includes both Mielnik's factorization and the original factorization, in terms of anihilation/creation operators, as particular cases, and which in its most general form leads to a bi-parametric factorization of the Hamiltonian. However, we shall focus herein only on a simpler one-parameter form of this factorization leading to a general equation for the SHO which includes its Schrödinger's equation and a scaled Hermite's equation as particular cases. We are aware of only one previous paper in which, in the context of q-deformations, a non-local q-deformed interpolation between the quantum harmonic oscillator and the Hermite equation has been investigated [19].

II. ALTERNATIVE FACTORIZATION

To begin with, let us introduce the pair of non-mutually adjoint operators

$$B^{-} = \frac{1}{\sqrt{2}}\left(\alpha^{-1}(x)\frac{d}{dx} + \beta(x)\right), \quad B^{+} = \frac{1}{\sqrt{2}}\left(-\alpha(x)\frac{d}{dx} + \beta(x)\right), \tag{11}$$

and let us require again that they factorize the Hamiltonian as $B^{-}B^{+} = H + \frac{1}{2}$. Then, the functions α and β have to satisfy the coupled equations

$$\alpha' \; + \; \beta\alpha^2 - \beta = 0\,, \tag{12}$$

$$\beta' \; + \; \alpha\beta^2 = (1 + x^2)\,\alpha\,. \tag{13}$$

These equations can be uncoupled by dividing the second one by α, multiplying the first one by β/α^2, and subtracting, to obtain

$$\frac{d}{dx}\left(\frac{\beta}{\alpha}\right) + \left(\frac{\beta}{\alpha}\right)^2 = 1 + x^2\,. \tag{14}$$

Clearly, Mielnik's solution (7) is the general solution for this equation, introducing one parameter, γ. A second parameter, δ, will appear when we insert this solution into one of the equations (12) or (13) to find the complete biparametric solution. When these parameters acquire the values that make $\alpha \equiv 1$ we shall recover Mielnik's factorization for the SHO, just as when $\gamma \to 0$ in Mielnik's factorization we recover the original factorization by means of (4).

Using Mielnik's solution (7), it is easy to calculate the general solution to equations (12,13) and to find the operators that factorize the SHO hamiltonian in terms of the product $B^{-}B^{+}$. Then, we may consider carrying out the inverse product $B^{+}B^{-}$ to see where we are led to. Obviously, we shall not obtain a new Hamiltonian, due to the factors α^{-1} and α in (11), which may be the reason why in the past nobody paid attention to this factorization. In addition, it is obvious that this operator product will be lengthy and without providing any new insight to the problem. Therefore, we shall consider here only the most simple solution to equations (12,13) to show that it leads to a new general SHO equation that till now has been lost in the trends of the SUSY factorization schemes.

III. PARAMETRIC HERMITE-LIKE DIFFERENTIAL OPERATOR

Let us now consider the simplest solution to equation (14), which stems from the linear relationship $\beta = \alpha\, x$. Introducing this ansatz into equation (12), we get a Bernoulli equation for $\alpha(x)$

$$\alpha' + x\alpha^3 - x\alpha = 0\,. \tag{15}$$

This equation is easily integrated, giving α and β as

$$\alpha(x) = \frac{1}{\sqrt{1+\delta e^{-x^2}}}\ , \qquad \beta(x) = \frac{x}{\sqrt{1+\delta e^{-x^2}}}\,. \tag{16}$$

To avoid singularities, we simply require that $-1 < \delta < \infty$. Note that the appearance here of the product $\left(\frac{1}{\alpha}\frac{d}{dx}\right)\left(\alpha\frac{d}{dx}\right)$ in B^-B^+ resembles the factor ordering operation of Hartle and Hawking [20], a scheme which has been applied in quantum SUSY cosmology [21].

As we said before, the inverse operator product B^+B^- will not give us a new Hamiltonian. However, we can still introduce a second order operator defined by $\tilde{\mathcal{L}}_\delta = B^+B^- + 1/2$

$$\tilde{\mathcal{L}}_\delta = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\delta x e^{-x^2}}{1+\delta e^{-x^2}}\frac{d}{dx} + \frac{1}{2}\left[\frac{x^2}{\left(1+\delta e^{-x^2}\right)^2} - \frac{1}{1+\delta e^{-x^2}} + 1\right]. \tag{17}$$

In addition, defining the functions $\mathrm{H}_n^\delta(x)$ as follows

$$\mathrm{H}_{n+1}^\delta(x) = B^+\psi_n(x) \tag{18}$$

we have that

$$\tilde{\mathcal{L}}_\delta\, \mathrm{H}_{n+1}^\delta = \left(B^+B^- + \frac{1}{2}\right)\left(B^+\psi_n\right) = B^+\left(B^-B^+ + \frac{1}{2}\right)\psi_n = (E_n+1)\,\mathrm{H}_{n+1}^\delta\,. \tag{19}$$

Now, as in SUSY QM, we can define the missing function $\mathrm{H}_0^\delta(x)$ by requiring that $\tilde{\mathcal{L}}_\delta \mathrm{H}_0^\delta = E_0\mathrm{H}_0^\delta$, leading to the equation

$$B^-\,\mathrm{H}_0^\delta = \frac{1}{\sqrt{2}}\left(\frac{1}{\alpha}\frac{d}{dx} + \alpha x\right)\mathrm{H}_0^\delta = 0$$

whose solution is

$$\mathrm{H}_0^\delta = \alpha\psi_0\,. \tag{20}$$

In fact, since $\beta = \alpha x$, equations (4) and (11) tell us that $B^+ = \alpha a^*$, and hence all unnormalized functions H_n^δ become $\mathrm{H}_n^\delta(x) = \alpha(x)\,\psi_n(x)$, $n \geq 0$.

According to the Sturm-Liouville theory [22], any second order differential equation $Pu''+Qu'+Ru+\lambda u = 0$ can be taken to the self-adjoint form $\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu+\lambda w(x)u = 0$ simply by multiplying by the factor $\frac{1}{P}\exp\left(\int^x (Q/P)\,dx\right)$. In our case, multiplying Eq. (19) by the factor $-2\left(1+\delta e^{-x^2}\right)$ shows that the functions $\mathrm{H}_n^\delta(x)$ are orthogonal by construction, and we get the eigenvalue equation

$$\mathcal{L}_\delta\,\mathrm{H}_n^\delta(x) + E_n\,\omega_\delta(x)\,\mathrm{H}_n^\delta(x) = 0\,, \tag{21}$$

where

$$\mathcal{L}_\delta = \left(1+\delta e^{-x^2}\right)\frac{d^2}{dx^2} - 2\delta x e^{-x^2}\frac{d}{dx} - \left[\frac{x^2}{1+\delta e^{-x^2}} + \delta e^{-x^2}\right] \tag{22}$$

is a new self-adjoint operator for the SHO with

$$\omega_\delta(x) = 2\left(1+\delta e^{-x^2}\right) \tag{23}$$

the appropriate weight function, which according to the general theory of Sturm-Liouville self-adjointness should be strictly positive unless possibly at isolated points at which $\omega_\delta(x) = 0$ [23]. The normalized eigenfunctions of the operator $\mathcal{L}_\delta$ are

$$\mathrm{H}_n^\delta(x) = \left(\frac{1}{2^{n+1}n!\sqrt{\pi}}\right)^{\frac{1}{2}}\left(e^{x^2}+\delta\right)^{-\frac{1}{2}}\mathrm{H}_n(x) \tag{24}$$

Note that the eigenfunctions (24) are not just the product of the Hermite polynomials and *any* other function, but a direct consequence of the factorization based on the operators B^- and B^+ (11).

The operator $\mathcal{L}_\delta$, the weight factor $\omega_\delta(x)$, and the eigenfunctions $\mathrm{H}_n^\delta(x)$, all depend on the factorization parameter δ. Two limiting values of this parameter become important here: In the case $\delta = 0$, we have that $\alpha(x) \equiv 1$, and B^- and B^+ become the original anihilation/creation operators (4). Also, the eigenvalue equation (21) becomes Schrödinger's equation for the SHO, and $\mathrm{H}_n^\delta(x)$ become the quantum eigenfunctions.

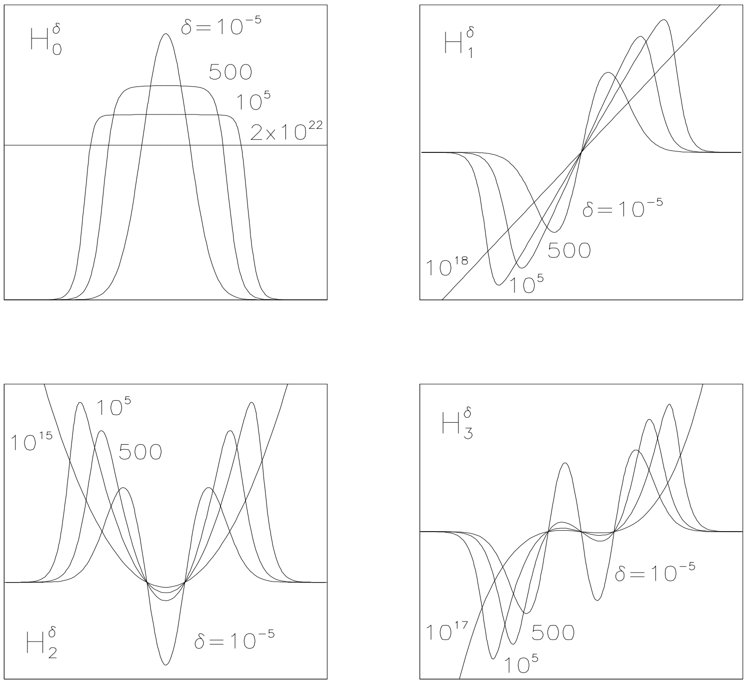

FIG. 1: The first four generalized Hermite functions $\mathrm{H}_n^\delta(x)$ (arbitrary scale) for the following values of the parameter δ: 0.00001, 500, 10^5, and the final huge numbers $2\cdot10^{22}$, 10^{18}, 10^{15}, 10^{17}, respectively. For the first value of the parameter, we are very close to the quantum eigenfunctions $\psi_n(x)$, $n=0,1,2,3$. At increasing δ, the eigenfunctions H_n^δ, $n=0,1,2,3$ begin to acquire, on a larger and larger part of their plots, a form close in shape to the Hermite polynomials, $\mathrm{H}_n(x)$, $n=0,1,2,3$, strongly damped at $x\to\pm\infty$ by the vanishing Gaussian tails $e^{-x^2/2}$.

On the other hand, when δ is a big but still a finite number, we are led to the following approximations

$$\begin{aligned}\mathcal{L}_\delta \mathrm{H}_n^\delta(x) &\to \delta e^{-x^2}\left[\frac{d^2}{dx^2}-2x\frac{d}{dx}-1\right]\delta^{-1/2}c_n\mathrm{H}_n(x)\,,\\ E_n\omega_\delta(x)\mathrm{H}_n^\delta(x) &\to 2\delta e^{-x^2}E_n\delta^{-1/2}c_n\mathrm{H}_n(x)\;=\delta e^{-x^2}(2n+1)\delta^{-1/2}c_n\mathrm{H}_n(x)\;,\end{aligned}$$

and from their summation, we find that the eigenvalue equation (21) becomes Hermite's differential equation. Therefore, we can consider Eq. (21) as a new *generalized Hermite equation for the SHO*, which includes Schrödinger's and Hermite's equations as particular cases for limiting values of the parameter δ, linked to each other by a continuous change of

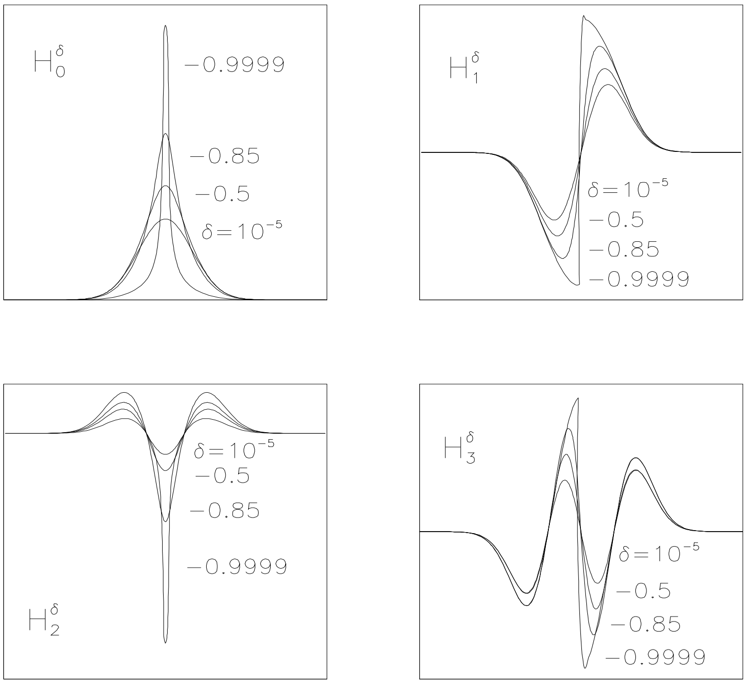

FIG. 2: The first four generalized Hermite functions $\mathrm{H}_n^\delta(x)$ (arbitrary scale) for the following negative values of the parameter δ: -0.5, -0.85, -0.9999 as compared with the positive case $\delta = 10^{-5}$, which is very close to the quantum Hermite eigenfunctions $\psi_n(x)$, $n = 0, 1, 2, 3$.

this parameter. This also allows us to think of the eigenfunctions $\mathrm{H}_n^\delta(x)$ as a new type of *generalized Hermite functions*. Arbitrarily scaled plots of the first four of them are shown in Fig. (1) for a few positive values of the parameter δ. As δ becomes bigger and bigger at constant polynomial order, a continuous shift from the true quantum mechanical eigenfunctions to eigenfunctions sharing a larger Hermite polynomial region can be noticed. On the other hand, in Fig. (2), we plot some negative δ cases as compared with the positive $\delta = 10^{-5}$ one. In the latter cases, one merely notices a deformation effect of the quantum oscillator eigenfunctions.

IV. RAISING AND LOWERING OPERATORS

In Mielnik's SUSY construction for the SHO, the raising and lowering operators for the new functions $\tilde{\psi_n}(x)$ become third order operators. In our case, we can see that since $a^* = (1/\alpha)B^+$ and using eq. (19), the raising operator, for example, would be defined by the triple product operation over $\mathrm{H}_n^\delta(x)$,

$$\begin{aligned} & B^+ a^* B^- \mathrm{H}_n^\delta(x) = B^+ a^* B^- B^+ \psi_n(x) = \textit{const.} \times B^+ a^* \psi_n(x) \\ & = \textit{const.} \times B^+ \frac{1}{\alpha} B^+ \psi_n = \textit{const.} \times \alpha\, a^* \, \frac{1}{\alpha}\, \mathrm{H}_n^\delta(x) \,. \end{aligned}$$

This operation can also be easily deduced from the action of a^* on $\psi_n(x)$. Therefore, we can define the first order raising and lowering operators for the functions $\mathrm{H}_n^\delta(x)$ as the non-mutually adjoint operators

$$c^+ = \alpha\, a^* \frac{1}{\alpha} = \frac{1}{\sqrt{2}}\left[-\frac{d}{dx} + x\left(1 + \frac{\delta e^{-x^2}}{1+\delta e^{-x^2}}\right)\right] \tag{25}$$

$$c^- = \alpha\, a\, \frac{1}{\alpha} = \frac{1}{\sqrt{2}}\left[\frac{d}{dx} + x\left(1 - \frac{\delta e^{-x^2}}{1+\delta e^{-x^2}}\right)\right] \tag{26}$$

They satisfy the relations

$$c^+ \mathrm{H}_n^\delta = \sqrt{n+1}\, \mathrm{H}_{n+1}^\delta \,, \qquad c^- \, \mathrm{H}_n^\delta = \sqrt{n}\, \mathrm{H}_{n-1}^\delta \,, \tag{27}$$

which are the quantum creation/annihilation operations when $\delta = 0$, and become the raising/lowering operations for the Hermite polynomials in the big δ limit.

They also satisfy

$$\begin{aligned} c^- c^+ \, \mathrm{H}_n^\delta &= \alpha \left(H + \frac{1}{2}\right) \frac{1}{\alpha}\, \mathrm{H}_n^\delta = (n+1)\, \mathrm{H}_n^\delta \,, \\ c^+ c^- \, \mathrm{H}_n^\delta &= \alpha \left(H - \frac{1}{2}\right) \frac{1}{\alpha}\, \mathrm{H}_n^\delta = n\, \mathrm{H}_n^\delta \,, \end{aligned}$$

and, therefore,

$$[c^-, c^+] = 1 \,. \tag{28}$$

an operation that lead to the operator algebra in the quantum mechanics case.

V. CONCLUSION

In this paper, we have used an alternative factorization for the SHO Hamiltonian that leads to the parametric self adjoint differential operator (21) that becomes the SHO Schrödinger's equation in the limit $\delta \to 0$ and is a scaled Hermite's equation at high values of δ. The eigenfunctions of this interpolation operator have similar features to the quantum mechanical wavefunctions as far as we refer to the asymptotic Gaussian tails. Nevertheless, the portion where they look as Hermite polynomials can be extended at will by tuning the factorization parameter δ to ever bigger values. Such a property could have applications in the manipulation of the spatial structure of Hermite-Gaussian optical beams [24, 25]. Other applications could be in the context of exactly solvable Fokker-Planck equation for the Ornstein-Uhlenbeck process, which is known to be directly connected to the oscillator potential in quantum mechanics since the drift coefficient is linear in x and the diffusion coefficient is constant [26]. All the mathematical structure of this paper can be kept on for the Fokker-Planck representation of these processes by slightly redefining the initial factorization operators in (4) as follows [27]: $A_1 = \frac{d}{dx} - \eta f'$ and $A_2 = \frac{d}{dx} + \eta f'$, where $f' = \frac{df}{dx}$ is the drift force up to a sign ($f' \sim x$ for the Ornstein-Uhlenbeck stochastic processes). The expression of the free parameter γ depends on the particular problem under consideration that can be as varied as neuronal responses, laser noise, mathematical finance, dynamics of interest rates, and volatilities of asset prices. Finally, we discussed here only the particular single parameter case $\frac{\beta}{\alpha} = \beta_p$ but the full biparametric case is easy to develop by writing $\frac{\beta}{\alpha} = \beta_g$, i.e., $\beta = (x + \phi_\gamma)\alpha$.

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