On friction effects as a mechanism to induce complex dynamical behavior in earthquakes

Valentina Castellanos-Rodríguez,[1, a)] Eric Campos-Cantón,[2, b)] and Ricardo Femat[2, c)]

[1)] *División de Geociencias Aplicadas, IPICyT, Camino a la Presa San José 2055 col. Lomas 4a Sección, 78216, San Luis Potosí, SLP, México*

[2)] *División de Matemáticas Aplicadas, IPICyT, Camino a la Presa San José 2055 col. Lomas 4a Sección, 78216, San Luis Potosí, SLP, México*

In this paper we analyze a nonlinear dynamical system that describes the kinetic mechanism between tectonic plates on the crust's earth undergoing stick slip movement. The analysis includes friction effects and an empirical friction law of granite rocks. The phenomena involved in the analyzed model are Stribeck's effect; Dieterich-Ruina's law; and properties of media as a presence of fluids and deformation. Outcomes arise from analysis of the system, which is conceived by a single slider block of one degree of freedom over a roughness and lubricated surface and formulated by space-state model through a differential equation system. We describe the oscillatory behavior for both continuous and switched conditions in terms of the mathematical solutions. Periodic and aperiodic orbits exist under a driven force and even more complex behavior. A relationship is given between the stability of the switched system and the parameter related with the oscillation frequency associated to characteristic longitude of displacement of slider. A necessary condition for stability in an unstable regime is deduced, under certain conditions in terms of frictional and seismic parameters of the analyzed model. Thus, we show the stationary and aperiodic solutions that describe the friction mechanism inducing earthquakes with a complex and nonlinear behavior.

PACS numbers: 05.45.Ac, 05.45.Pq

Keywords: Earthquake mechanism; Stribeck's effect; Dieterich-Ruina's law; dynamical systems theory; stick slip movement.

[a)] Electronic mail: valentina.castellanos@ipicyt.edu.mx

[b]Electronic mail: eric.campos@ipicyt.edu.mx
[c]Correspondence author: rfemat@ipicyt.edu.mx

Many systems in nature can depict very complex dynamical phenomena. One of them is relative to the earthquakes mechanism, which can be mathematically modeled by spring blocks in order to capture the behavior stick slip. It should be noted that the friction is a resistance force that introduces nonlinear dynamics into natural systems. Thus, the effect of a friction force (namely Dieterich-Ruina's law) related to displacement between existing geological faults suggests that nonlinear behavior can be depicted. We propose a model describing this phenomenon. The model departs from classical 2nd order differential equation but the Dieterich-Ruina's laws induces an additional dimension to the model. That is, the proposed model is a 3rd order dynamical system with interesting dynamical properties. It is our believe these results can open transdisciplinary studies about earthquakes dynamics.

I. INTRODUCTION

Study of earthquakes and seaquakes is of great scientific interest due to complex characteristics and behaviors that are observed in nature[2,4,17,33,47–49]. Complexity of earthquake's mechanism arises from the amount of variables and processes involved at the displacement of tectonic plates[18]; its dynamics is considered a nonlinear oscillatory process,[9,11,13–17] however, the nature of such phenomena suggests the friction is related to the nonlinear complex behavior. Typically, earthquakes occur in the upper ten kilometers of the earth's crust and they arise as a consequence of frictional instabilities that cause stress, which is accumulated by large-scale plate motions over periods of hundreds of years, therefore sudden stick-slip events are generating showing recurrence with irregular slip during large events and pronounced asperities in the slip distribution[3,4]. Since seminal contribution by Brace and Byerlee (1966)[5], the stick-slip is proposed as a possible earthquakes mechanism and a great amount of spring-block models have arisen, as the simplest analogy to represent the earthquakes mechanism on a single fault or a collection faults.[1,3,7–10,12,16,19,22–24] These models consist of an elastically coupled chain of blocks in contact with a moving rough surface, in which the friction laws are introduced and obey Newton's laws of motion. Since the spring-block model by Burridge and Knopoff (1967)[7](Figure 1), others have been proposed; e.g., Carlson *et al.* (1994)[3] and Pelletier (2000)[24]. A classical approach at most such models

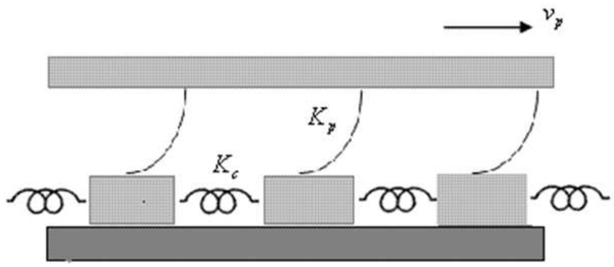

Figure 1. Burridge and Knopoff model[7] it consists of N identical blocks of mass M moving over a rough surface and coupled through coil springs of constant K_p to a driving plate with constant velocity v_p, representing the other side of the fault, and K_c is the spring coefficient between blocks.

of the friction laws is to lump it in a single term with static features, more recently Dragoni and Santini (2010)[13]and Amendola and Dragoni (2013)[11] introduce a static or dynamic law of friction for purely elastic and viscoelastic cases, respectively. Another is related to the mathematical framework, which is related to dynamical systems with stochastic boundary conditions.

An additional fact is the nature of the frictional instabilities and the conditions under which they occur[25,26] have been determined experimental work. Classical experimental approaches have allowed to model the friction terms without dynamical properties, other approaches model the friction as dynamical phenomenon[27–29,39], actually, some of friction laws are obtained directly from laboratory experiments, explain fault instabilities associated to friction with dynamical features[25,30,31]. In regard to the dynamical system with stochastic initial or boundary conditions; Brown *et al.* (1991)[6] and Nakanishi (1991)[21] used an approach based on automata cellular with deterministic dynamics and included initial randomness in the block position. Otsuka (1992)[32] assigned values to the spring constants and frictional parameters with random stochastic fluctuations. Bak and Tang (1989)[33] presented an automata cellular model bases on stochastic dynamics, Barriere and Turcotte (1991)[34], and Ito and Matsizaki (1990)[35] introduced randomness incrementing the stress randomly time until some uniform threshold have been reached, in those models a site is chosen randomly during each time step and a unit of stress is added. When a site has four units of stress accumulated on it, the site becomes unstable and redistributes stress to its nearest neighbors.[24]

We propose a system for the generation and analysis of a model nonlinear deterministic. We introduce heterogeneities in the medium (i.e., matrix of the rock surrounding the fault surface) in a way which allows us to study (i) the model dependence on heterogeneity; and (ii) the presence of fluid between surfaces in contact with stick slip friction and shear stress. Our main contribution is the analysis of a dynamical model, that introduce friction terms associated to phenomena described by Dieterich-Ruina[25,26] and Stribeck[36], such analysis involve that the vector field representing the system is hybrid. The model takes into account three issues (i) the evolution of frictional stress depending of grade of asperities contact as a consequence of the layer of fluid between the surfaces, (ii) a state variable (i.e., the renew of contact with asperities, from Dieterich-Ruina's law of friction), and (iii) a switching in the system. Moreover, our proposal analysis to considers the presence of an external force driving the system attributable to vibrations of neighboring faults. These features make more realistic the description of movement between the tectonic plates.

The ideas in manuscript are presented in the following order. The Section 2 comprises an explanation of the model including the motion equation, friction effects and the dimensionless system of differential equations. The analysis of the system is in Section 3, where for existence, uniqueness, equilibrium points and stability are discussed for both cases: the homogeneous and the perturbed scenarios so also for continuous and switched cases. Finally, the section 4 has the manuscript with concluding remarks.

II. SPACE-STATE MODEL

In order to model the relative movement of tectonic blocks, with normal stress constant along time, we consider a viscoelastic system with single degree of freedom undergoing frictional slip. This system is represented by a slider block of mass M, which slips over a rough lubricated surface with speed v and is connected by a stiffness spring k to a plate point where motion is enforced at constant speed v_0 (Figure 2). It should be noted that the connected plate point represents the other side of the plate.

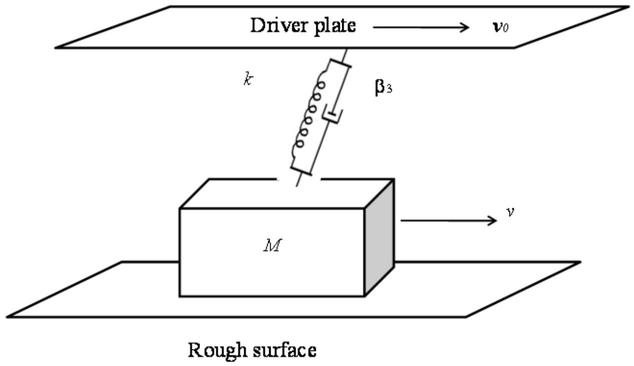

Figure 2. Single degree of freedom slider block model[14,24] is a slider in a viscoelastic medium with dynamic viscosity coefficient β_3, coupled by spring with coefficient k to a driving plate with constant velocity v_0 that represents the other side of plate. The slider of mass M is moving with relative velocity v over a rough and lubricated surface under a combined friction law: Dieterich-Ruina, Stribeck, Coulomb and viscous friction[36].

The motion equation of the system is given by the following equation

$$M\ddot{u} + F(\dot{u}, \theta) + ku = \tau(t), \tag{1}$$

where $u = x - v_0 t$ is the relative horizontal displacement of the block respect to the tectonic plate which has constant velocity v_0; and $\dot{u}$ is the relative velocity of the displacement between block and plate where $\dot{x} = v$, $\ddot{u}$ stand for the block acceleration and θ represents sliding history effects[50].

The first term of Equation (1) contains the inertial forces, the second, $F(\dot{u}, \theta) = F(v, \theta)$, includes all friction effects that will be discussed in the next section, and third is the force due to the deformation of the spring with stiffness k. The term $\tau(t)$ is an external force that disturbs the system.

The relative position u is available from measurements using Global Positional System (GPS) data[37] and nominal values of the parameters k and M can be derived from geophysical and geological observations. Friction forces $F(v, \theta)$ in plates are unknown during a seismic event, although have been studied in laboratory experiments[5,25,38,39].

A. On the friction effects

Up today, we know that the friction effects at moderated and low velocities have a strong influence on the dynamical behavior of mechanical systems[36,40]. A model based on slider block of mass M is essentially a mechanical representation. Henceforth, our analysis includes, among others, the friction components in mechanical systems, which is a function for increasing bounded velocity by an upper limit equal to the static force F_{max} and a lower limit equal to the Coulomb force F_c. Such friction effects are induced by displacement of granite blocks in the presence of fluids. At velocity close to zero, a friction F_c predominates between two dry surfaces but, complementarily, at high velocity a hydrodynamic friction F_v becomes dominant. The minimal friction is reached after intermediate velocity depending on the pressure of the fluid between the solid surfaces in presence of asperities, thus the static friction has been overcome, the friction decreases exponentially with respect to velocity and is called Stribeck friction (denoted with F_s). When the motion direction changes the frictions F_c and F_s change of sign and are defined for all velocities except to zero. These friction forces are represented as follows:

$$F_c(\dot{u}) = \beta_1 sign(\dot{u}) \qquad F_s(\dot{u}) = \beta_2 e^{-\mu/\dot{u}/} sign(\dot{u}) \qquad F_v(\dot{u}) = \beta_3 \dot{u}. \tag{2}$$

where real positive constants β_1, β_2, β_3, and μ are determined as follows: β_1 is equal to the product of normal force and dynamic frictional coefficient of granite; $\beta_2 = F_{max} - \beta_1$; β_3 is a scalar representing the dynamical viscosity coefficient of fluid due to the energy dissipation of fluid between the surfaces ; and μ denotes the slip constant in the Stribeck friction F_s. Because of the slider block velocity v is relative to velocity of plate v_0, we consider negative direction when velocity of the block is lower than the moving plate and it is positive on the contrary. Thus, the function $sign(\dot{u}) = sign(v - v_0) : \mathbf{R} \rightarrow \{-1, 0, 1\}$ at equation (2) is defined by

$$sign(v) = \begin{cases} -1, & \text{if } v < v_0, \\ 0, & \text{if } v = v_0, \\ 1, & \text{if } v > v_0. \end{cases} \tag{3}$$

Friction forces F_c and F_v combined case (Figure 3(a)) show linear dependence on the velocity, whereas when being combined with the F_s, the behavior of friction is nonlinear with respect to velocity (Figure 3(b)).

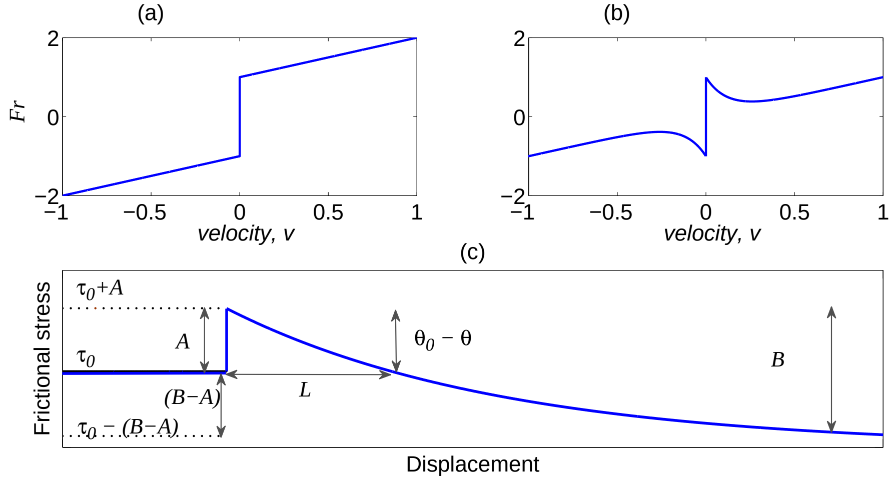

Figure 3. Velocity versus Friction forces in mechanical systems: (a) Coulomb friction plus viscous and (b) combination of Coulomb, viscous and Stribeck friction, from static level, nonlinear behavior is observed. (c) Frictional response versus displacement with the friction law of Dieterich-Ruina[14]. When the sliding velocity v_0 increase instantaneously $\triangle v$, the frictional stress, initially in τ_0, increases to $\tau_0 + A$, following by a constant sliding velocity in $v_0 + \triangle v$ where the frictional stress decreases exponentially B until $\tau_0 - (A - B)$, L is the characteristic distance required to renew the contact population (from the variable θ to a new steady-state θ_0).

Extensive investigations on the stick-slip phenomenon in rocks have been performed due to the relevance into the mechanism of crustal earthquakes. Such investigations have disclosed that friction depends on sliding historical effects. This characteristic is exhibited by rocks and denoted by θ in geosciences argot and θ is interpreted as a measure of the average age of load-supporting contacts between sliding surfaces[42,50]. In order to capture such historical effects on friction terms, we include the dependence on rate and state of the frictional constitutive relations, named $F_{dr}(v, \theta)$, for the single slider block (Figure 2). $F_{dr}(v, \theta)$ describes accurately the experimental results from rocks mechanics[27,28] obtained over a range of slip speeds v. It should be noted that the force associated to $F_{dr}(v, \theta)$ is opposite to the relative displacement of the granite plates and considers the renovation of asperities between them; such opposition is not included in above friction terms. Although $F_{dr}(v, \theta)$ has been isolately used to model earthquake dynamics (cf. with recent reports[14,15]), the combination of $F_{dr}(v, \theta)$, $F_c(v)$, $F_s(v)$ and $F_v(v)$ offers an alternative model. Taking into account all friction forces, the friction becomes

$$F(v,\theta) = F_{dr}(v,\theta) + F_c(v) + F_s(v) + F_v(v). \tag{4}$$

Now, $F_{dr}(v,\theta)$ is shown to derive the proposed model (more details on $F_{dr}(v,\theta)$ are in Figure 3(c), in the appendix; Marone (1998)[25]and Daub and Carlson (2008)[26]). The friction stress depends additively on a term $A\ln(v)$ and the state variable θ. The friction stress is characterized by a change from $A\ln(v)$ to the term $-B\ln(v)$, where the scalars A and B depend on the material properties and determine the sign of the velocity dependence. Instabilities attributable to stick-slip arise only in friction laws with steady-state velocity weakening, i.e., steady-state friction decreases when slip velocity increases[43]. Thus, we assumed $B > A > 0$ is satisfied (Figure 3(c)). L is the characteristic sliding displacement required to stabilize the friction to satisfy F after a change of sliding conditions[42]. $F_{dr}(v,\theta)$ is given by two equations[28]:

$$\left.\begin{array}{l} F_{dr}(v,\theta) = \theta + A\ln(v/v_0) \\ \dot{\theta} = -(v/L)[\theta + B\ln(v/v_0)] \end{array}\right\} \tag{5}$$

which are coupled to the second order Equation (1) to construct the single slider block model.

B. A single slider block model for complex dynamical behavior

By combining the Equations (1), (4), the dynamical model for single slider block can be formulated as the following system of first order differential equations:

$$\left.\begin{array}{l} \dot{\theta} = -(v/L)[\theta + B\ln(v/v_0)] \\ \dot{u} = v - v_0 \\ \dot{v} = -(1/M)[ku + F(v,\theta)] + \tau(t) \end{array}\right\} \tag{6}$$

where $F(v,\theta)$ is given by Equation (4). The dimensionless version of system (6) can be derived by defining new variables $\hat{\theta}$, $\hat{u}$, $\hat{v}$ and $\hat{t}$ as is suggested by Erickson *et al.* (2008)[14]; or equivalently $\theta = A\hat{\theta}$, $v = v_0\hat{v}$, $u = L\hat{u}$, and $t = (L/v_0)\hat{t}$:

$$\left.\begin{array}{l} \dot{\hat{\theta}} = -\hat{v}[\hat{\theta} + (1+\varepsilon)\ln\hat{v}] \\ \dot{\hat{u}} = \hat{v} - 1 \\ \dot{\hat{v}} = -\gamma^2[\hat{u} + (1/\xi)(\hat{\theta} + \ln\hat{v})] + \alpha F_0(\hat{v}) + \tau(\hat{t}) \end{array}\right\} \tag{7}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (1/M)(\beta_1, \beta_2, \beta_3)$, $F_0(\hat{v}) = (sign(\hat{v}-1), e^{-\mu|\hat{v}-1|}sign(\hat{v}-1), \hat{v}-1)^T$. The parameters $\Pi = (\varepsilon, \xi, \gamma) \subset \mathbf{R}^p$ are directly related to earthquakes dynamics as follows: $\varepsilon = (A - B)/A \in \mathbf{R}$ is a measure related to the sensitivity of the velocity relaxation and is associated with the stress drop during displacement; $\xi = kL/A \in \mathbf{R}$ stands for the dimensionless spring constant; and $\gamma = \sqrt{(k/M)}(L/v_0) \in \mathbf{R}$ regards the dimensionless oscillation frequency of the single slider block. In what follows, the dynamical system given by (7) is analyzed in two parts as study cases towards the discussion about the complexity of friction effects on earthquakes.

III. ANALYSIS OF THE DYNAMICAL SYSTEM

The theory of nonlinear dynamical systems plays an important role in almost all the areas of science due to the phenomena of the real world are in most cases nonlinear. The theory of dynamical deterministic systems is particularly useful in the study of complex behaviors like the earthquake mechanism. Generally, the analysis of the behavior of nonlinear systems is, at the beginning, local dynamic analysis in sense of the space state of system (7). Such a local analysis allows us to focus on the discussion, as a first step, to simplify the system while its qualitative properties are preserved. Next, we analyze the system behavior for two study cases. Each of both cases is firstly analyzed unforced ($\tau(\hat{t})$=0 for all t) and after that forcing is added ($\tau(\hat{t})$=sin($\omega\hat{t}$) for all t) to show the complex behavior:

Case 1: Smoothness. We assume that the vector field in the model (7) is continuously differentiable to explorer its behavior with $\alpha F_0(\hat{v}) = \alpha_1 + \alpha_2 e^{-\mu\hat{v}} + \alpha_3\hat{v}$. It should be noted that friction terms are included concerning to Coulomb, Stribeck and Viscous. That is, we firstly assume friction effects except to change of sign, attributable to change in movement sense, but including the grade of contact of asperities as a consequence of the layer of fluid between surfaces.

Case 2: Discontinuity. By other hand, since Dieterich-Ruina's friction shows that the rock behavior obeys non-smooth friction terms, the smoothness assumption in Case 1 is relaxed to suppose the vector field captures a behavior related to the non-continuous differentiability. Such a discontinuity can appear of natural way in physical systems that involve friction forces. In particular, in the analyzed model, given by system (7), a discontinuity exists at $\hat{v} = 1$ for Equation (3). Note that $\hat{v} = 1$ means that the single slider block and plate are moving in same sense with equal velocity. In order to capture non-continuous differentiability of vector field in system (7), the function of variable structure is considered to be $\alpha F_0(\hat{\upsilon}) = \alpha_1 sign(\hat{\upsilon}-1) + \alpha_2 e^{-\mu/\hat{\upsilon}-1/} sign(\hat{\upsilon}-1) + \alpha_3(\hat{\upsilon}-1)$ and the system is referred like a switched system.

A. Case 1: Smoothness

1. *Unforced,* $\tau(\hat{t})$= 0 *for all* t

Let us re-write the Equation (7) as $\dot{x} = f(x)$ where $x = (\hat{\theta}, \hat{u}, \hat{\upsilon})^T$ denotes the state vector. The vector field $f : U \to R^3$ is a smooth function (C^1) in a set $U \subset R^3$, and becomes

$$f(x) = \begin{pmatrix} -\hat{\upsilon}[\hat{\theta} + (1+\varepsilon)\ln\hat{\upsilon}] \\ \hat{\upsilon} - 1 \\ -\gamma^2[\hat{u} + (1/\xi)(\hat{\theta} + \ln\hat{\upsilon})] + \alpha F_0(\hat{\upsilon}) \end{pmatrix} \tag{8}$$

where $\Pi = (\varepsilon, \xi, \gamma)$ stands for the parameter vector, the initial condition $x(0) = x_0 \in U$ and $\Pi \subset R^p$. The vectorial field f generates a flow $\Phi_t : U \to R^n$, where $\Phi_t = \Phi(x,t)$ is a smooth defined function for all $x \in U$ and $t \in I = (a,b) \subseteq R$. Φ_t is solution of $\dot{x} = f(x)$ in sense that it satisfies the Equation (7). Then, the following results holds for the system $\dot{x} = f(x)$.

Theorem III.1. [44] *Let $U \subset R^n$ be an open set of the euclidian real space, $f : U \to R^n$ a differentiable function (C^1), and a point $x_0 \in U$. Then, there exist a real scalar $c > 0$ and a unique solution $\phi : (-c, c) \to U$ such that ϕ satisfies the differential equation $\dot{x} = f(x)$ with initial condition $x(0) = x_0$ and for all $t \in (-c, c)$.*

Actually, only it is required that f is locally Lipchitz for existence and uniqueness of solution $x(t)$, i e., $\| f(y) - f(x) \| \leq K \| x - y \|$ for some real constant $K < \infty$. For the existence and uniqueness of equilibrium point $x^\star$, belonging to domain of vector field (8), is needed for the analysis of stable behavior of the single slider block. A point $x = x^\star$ in the space state is defined to be an equilibrium point of a vector field if $f(x^\star) = 0$ for all t. Thus, given the parameter value Π, we found that the vector field (8) has a unique equilibrium point whose components are at $x^\star = (\hat{\theta}^\star, \hat{u}^\star, \hat{\upsilon}^\star) = (0, \eta, 1)$, where second component depends on parameter as follows: $\eta = (\alpha_1 + \alpha_2 e^{-\mu} + \alpha_3)/\gamma^2$. Note that η is a bijective function of γ implying there is not multiplicity. The second component, $\hat{u}^* = \eta$, corresponds to relative position of the single slider block. This means that $x^\star \to (0, 0, 1)$ as $\gamma \to \infty$. The stability of $x^\star$ is analyzed with the indirect method of Lyapunov stated in the following theorem:

Theorem III.2. [45] *Let $x^\star$ be an equilibrium point of the nonlinear system $\dot{x} = f(x)$, where $f : D \to R^n$ with $D \subset R^n$ is continuously differentiable and D is a surroundings of the $x^\star$. Let us denote the Jacobian matrix as $D_f^\star = (\partial f_i(x)/\partial x_j) \mid_{x^\star}$, with $i, j = 1, 2, 3$, and let λ_i be the eigenvalues of $D_f^\star$. It follows that the equilibrium point $x^\star$ is locally asymptotically stable if the real part of all the eigenvalues is negative, i.e. $Re(\lambda_i) < 0$, and, contrary, it is unstable if $Re(\lambda_i) \geq 0$ for one or more eigenvalues of $D_f^\star$.*

Now, defining a change of variable $y = x - x^\star$, the system (7) is approached by

$$\dot{y} = D_f(x^\star) y \tag{9}$$

at the neighborhood of $x^\star$. For the specific form of vector field, given by Equation (9), the Jacobian matrix becomes

$$D_f = \begin{pmatrix} -\hat{v} & 0 & -\hat{\theta} - (1+\varepsilon)(1 + \ln \hat{v}) \\ 0 & 0 & 1 \\ -\frac{\gamma^2}{\xi} & -\gamma^2 & -\frac{\gamma^2}{\xi}\frac{1}{\hat{v}} + \alpha \frac{\partial F_0(\hat{v})}{\partial \hat{v}} \end{pmatrix} \tag{10}$$

where $\alpha \partial F_0(\hat{v})/\partial \hat{v} = -\alpha_2 \mu e^{-\mu \hat{v}} + \alpha_3$; and $\alpha \partial F_0(\hat{v})/\partial \hat{v} \mid_{x^\star} = -\phi$; with $\phi = \alpha_2 \mu e^{-\mu \hat{v}} - \alpha_3$. Evaluating the Jacobian matrix at the equilibrium point $x^\star = (\hat{\theta}^\star, \hat{u}^\star, \hat{v}^\star) = (0, \eta, 1)$, D_f takes the form

$$D_f^\star = \begin{pmatrix} -1 & 0 & -(1+\varepsilon) \\ 0 & 0 & 1 \\ -\frac{\gamma^2}{\xi} & -\gamma^2 & -\frac{\gamma^2}{\xi} - \phi \end{pmatrix} \tag{11}$$

whose polynomial characteristic is

$$P(\lambda) = a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 \tag{12}$$

where

$$a_0 = 1, \qquad a_1 = 1 + \gamma^2/\xi + \phi, \qquad a_2 = \gamma^2(1 - \varepsilon/\xi) + \phi, \qquad a_3 = \gamma^2. \tag{13}$$

Note that a_1 =-trace($D_f^\star$), a_3 =-det($D_f^\star$). The dynamics of the linear system (9) is characterized by the set of eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\} \in \mathbf{C}$. We assume that the system (9) is naturally dissipative, i.e., trace($D_f^\star$) $= \sum \lambda_I < 0$. Next, by definition, $\xi = kL/A > 0$ and $\gamma^2 = (k/M)(L/v_0) > 0$, then $1 + \gamma^2/\xi = (Mv_0 + A)/Mv_0 > 0$; in addition, $\phi = \alpha_2 \mu e^{-\mu} - \alpha_3 = (\beta_2/)\mu e^{-\mu} - \beta_3/M$. Hence, $1 + \gamma^2/\xi + \phi = (Mv_0 + A)/Mv_0 + (\beta_2/M)\mu e^{-\mu} - \beta_3/M \Rightarrow$trace($D_f^\star$) $= -(1 + \gamma^2/\xi + \phi) < 0$ if $(Mv_0 + A)/v_0 + \beta_2 \mu e^{-\mu} > \beta_3$; as a consequence, the system is dissipative at the neighborhood of $x^\star$ in sense that sum of all eigenvalues is strictly real negative as this condition is satisfied.
Complementarily, det($D_f^\star$) $= -\gamma^2 = -(k/M)(L/v_0) < 0$. As a consequence of facts trace(D_f^*) < 0 and det(D_f^*) < 0, $x^\star$ is dissipative and hyperbolic the equilibrium point if $(Mv_0 + A)/v_0 + \beta_2 \mu e^{-\mu} > \beta_3$. The physical interpretation is as follows: Since (i) the value of the decay parameter μ for mixed lubricated surfaces is within the interval $1 < \mu < 5$ (see references from[36,40]), (ii) α_2, α_3 are constants whose dimensionless magnitude is of order 10^{-2}; from where $|\phi| < 1$ implying that $\phi < 1 + \gamma^2/\xi$. Moreover, it should be noted that the analyzed model allows one to determine mathematical properties in terms of friction coefficients as the involved at Viscous and Stribeck friction terms.

Now, we analyze the condition for stability/instability of $x^\star$. Note that $x^\star$ is unstable if at least one of eigenvalues of (11) has real positive part. In order to determine how many eigenvalues of $D_f^\star$, given by the roots of polynomial (12), are real, we use the Descartes rule of signs. The coefficient signs of the polynomial (12) are $(+, +, sign(a_2), +)$ when $(Mv_0 + A)/v_0 + \beta_2 \mu e^{-\mu} > \beta_3$ is satisfied. Now, if $sign(a_2) > 0$, there are two possibilities on the three eigenvalues of $D_f^\star$: The first possibility involves all eigenvalues are negative real. The second involves one eigenvalue is negative real and the other two are conjugated complex with positive real part; which corresponding to the oscillatory behavior. Note that the $sign(a_2) > 0$ if $\gamma^2(1 - \varepsilon/\xi) + \phi > 0$ or, equivalently, $\varepsilon < \xi\psi$ with $\psi = 1 + \phi/\gamma^2$ We are interested in analysing the stability for the case $\varepsilon > 0$, which means that the stress drop is positive $B - A > 0$ involving relative displacement between plates. Since $\xi > 0$ the stability conditions for $x^\star$ are such that $\psi > 0 \Rightarrow \gamma^2 > -\phi$. That is, it is necessary that $kL/v_0 + \beta - 2\mu e^{-\mu} > \beta_3$. As summary, a necessary condition to $x^\star$ is stable if $\varepsilon < \xi\psi$ and is unstable if $\varepsilon > \xi\psi$. Next, if $sign(a_2) < 0$, the unique possibility implies there are two positive real roots and one negative real root. This is possible only if the condition $\varepsilon < \xi\psi$

is not satisfied.

The Figures 4(a) and (b) show the locus of the $\{\lambda_1, \lambda_2, \lambda_3\} \in \mathbf{C}$ for fixed $\gamma = 0.8, 10$

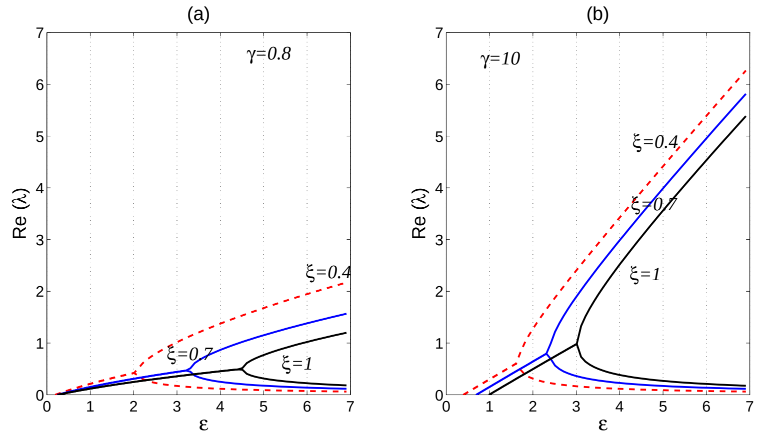

Figure 4. Real Part of eigenvalues for fixed γ. (a) For $\gamma = 0.8$ the real part of eigenvalues for the oscillatory behavior is observed approximately for values of ε between (0.2 to 2), (0.24 to 3.2) and (0.26 to 4.4), for $\xi = 0.4, 0.7, 1$, respectively. (b) For fixed $\gamma = 10$ the variation of ε goes from (0.4 to 1.6), (0.7 to 2.3) and (1 to 3) for $\xi = 0.4, 0.7, 1$, respectively. The oscillatory behavior corresponds to the conjugated complex eigenvalues. After the branch point there are two real positive eigenvalues.

respectively; the oscillatory behavior increases when the value of ξ increases but the range of variation for ε in (a) is greater than (b); with the increasing of γ, one of the real eigenvalues tends to infinity more quickly as is illustrated in Figure 5(a) and (b). The oscillatory behavior can be displayed to the complex eigenvalues located before the branching. After the branching there are two positive real eigenvalues, one tends to zero and the other tends to infinity. After the branch point the system ceases to oscillate. Note that $x^\star$ is stable for $\varepsilon < \xi\psi$ with $\gamma^2 > -\phi$ (Figure 6).

Some comments on above mathematical rationale are in order. Local stability of equilibrium $x^\star = (0, \eta, 1)$, in the Lyapunov sense, can be interpreted in the sense of the frictional stability in earthquakes[4,50] because of the frictional stability is related with a critical value $\overline{\sigma}_c = kL/(A - B)$ which depends on the properties of the rocks surrounding, the earthquake nucleation point, and the frictional parameters $(A - B, L, k)$. The critical value $\overline{\sigma}_c$ is derived for a single degree of freedom oscillator coupled from Dieterich-Ruina friction term $F_{dr}(v, \theta)$ and corresponds to the normal effective stress (or normal stress less pore pressure). As

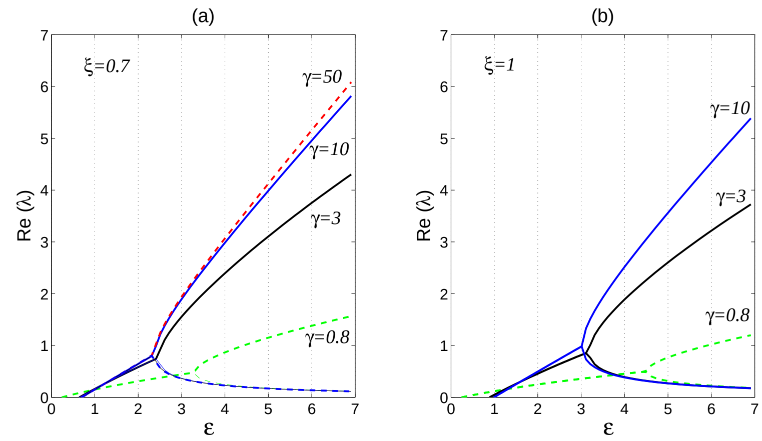

Figure 5. Real Part of eigenvalues for fixed ξ. (a) For $\xi = 0.7$ the real part of eigenvalues for the oscillatory behavior is observed approximately for values of ε between (0.6 to 2.4), for $\gamma > 0.8$, and for $\gamma = 0.8$ this range is greater than first(0.24 to 3.2). (b) For fixed $\xi = 1$ the variation of ε goes from (0.9 to 3.1) for $\gamma > 0.8$ and (0.7 to 2.3) for $\gamma = 0.8$. The oscillatory behavior corresponds to the conjugated complex eigenvalues, before the branch.

any normal stress $\overline{\sigma}$ satisfies $\overline{\sigma} \geq \overline{\sigma}_c$, then changes in the frictional properties occur such that they cause the earthquakes; this phenomenon is named frictional instability. The necessary condition $\varepsilon < \xi\psi$ and the critical value $\overline{\sigma}_c$ are related as follows. By dividing $\xi/\varepsilon = -kL/(A-B) = \overline{\sigma}_c$. Then, we get $(\xi/\varepsilon)\psi = -kL/(A-B)\psi = \overline{\sigma}_c\psi$. As $\varepsilon < \xi\psi$ is satisfied, we get that $1 < \overline{\sigma}_c\psi$ is necessary for stability. In this manner, we derive a corrected critical value $\overline{\sigma}_c^\star = \overline{\sigma}_c\psi$, which combines the frictional parameters from Dieterich-Ruina law and Stribeck effect. Complementarily, $1 > \overline{\sigma}_c\psi$ implies instability and the critical criterion for stability becomes $\overline{\sigma}_c^\star = \overline{\sigma}_c\psi = 1$.

Figure 7(a) shows the evolution of state variables for a set of parameters $\Pi = (0.8, 1, 10)$ where the equilibrium point is asymptotically stable, that means the slider oscillates and later becomes stabilized in $x^\star = (0, \eta, 1)$, with $\eta = (\alpha_1 + \alpha_2 e^{-\mu} + \alpha_3)/\gamma^2$ the relative position of the block is η, its velocity is the velocity of the driver plate but there are not asperities; the phase portrait (Figure 7(b)) shows that the vector field (9) converges to a point making an inward spiral; by the other hand, for the set $\Pi = (0.25, 0.8, 0.8)$, the Jacobian matrix $D_f^\star$ has two of its eigenvalues with null real part, so the equilibrium point is unstable, and would become stable or not, that means it is sensible to disturbs of medium; the variables oscillate around the $x^\star$ (Figures 7(c) and (d))

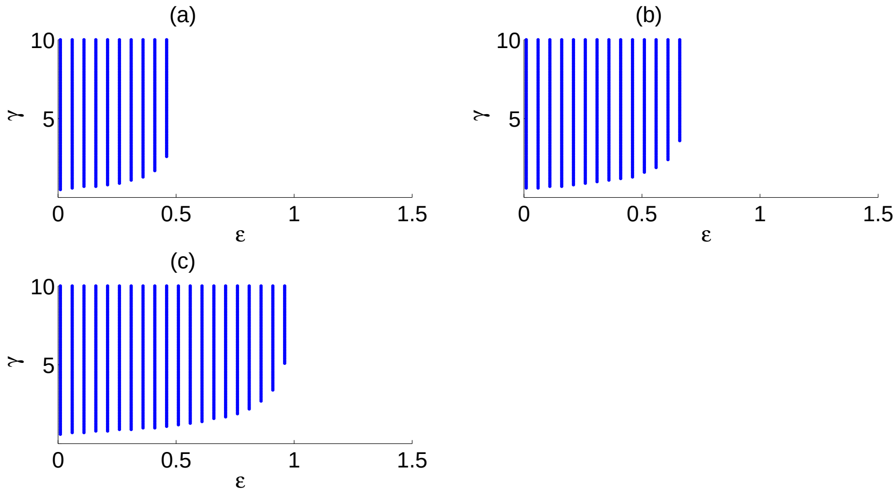

Figure 6. Stability of stationary solutions for different values of Π . We fixed ξ and varied ε and γ. A necessary condition to stability is $\varepsilon < \xi\psi$, here $\xi = 0.5, 0.7, 1.0$, for (a) to (c), respectively; the stability region is bounded by a parabolic relation between ε and γ.

2. ***Forced,*** $\tau(\hat{t}) = \sin(\omega\hat{t})$ ***for all*** t

Now, we investigate the effects of an external, deterministic and periodic force acting onto the system (7). This is motivated because of geological faults are in a complex system interacting with others faults. As a consequence, the motion of a fault might be affected by neighboring faults represented as an external force that disturbs the system (7). From dynamical systems theory this involve that the system (7) with $\tau(\hat{t}) \neq 0$ can display complex behavior. The complex behavior is illustrated varying the angular frequency ω within the interval $(0, 0.3)$ for the parameter values of $\Pi_1 = (0.25, 0.8, 0.8)$ implying $\overline{\sigma}_c^\star < 1$ (i.e., $x^* = (0, \eta, 1)$ is unstable). Specifically, we show the phase portrait for $\omega = 0.1$(Figure 8(a)). For any value of ω at $(0.8, 1.3)$ a periodic orbit is found. The phase portrait of system (7) is depicted for $\omega = 1.2$, note that the periodic orbit has period 2 (Figure 8(b)). For frequencies larger than 1.3, the system trajectories converge to limit cycle with period 1 (Figure 8(c)). The Figure 9 shows the effect of viscosity on the behavior of the system for $\omega = 0.1$; the sub-damped system behavior (Figure 9(a)) is more complex than the cases damped and over-damped (Figures 9(b) and (c))and the range of values for v and u increases as the value of α_3

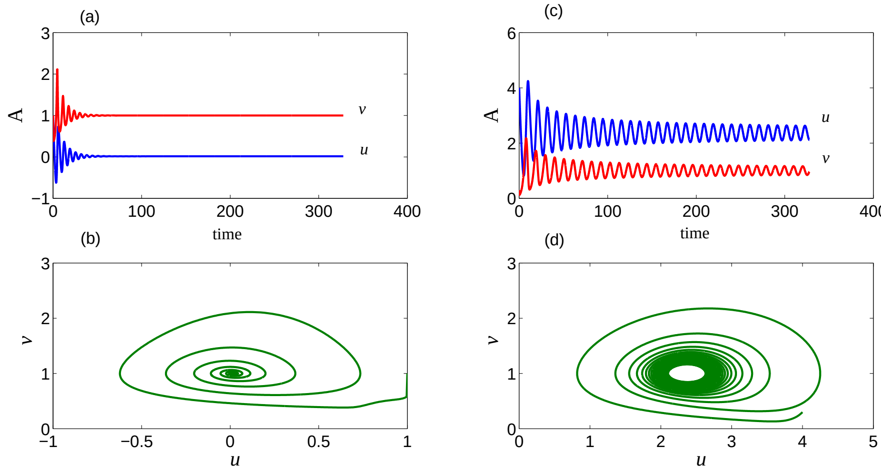

Figure 7. Stationary solution of system (7). (a) Here $\Pi = (.8, 1, 10)$, and $\overline{\sigma}_c^\star = 1.2491$. After a transient region in which the slider oscillate , its velocity stays at a constant rate $\hat{v} = 1$ as it moves with the driver plate, its relative position u is $\eta = 0.0151$ and asperity contact $\hat{\theta}$ is zero. The equilibrium point is local and asymptotically stable. (b) The phase portrait $(\hat{u}, \hat{v})$ shows an inward spiral, and the convergency to a point $x^\star = (0, \eta, 1)$. Periodic solution of system are shown in (c) where $\Pi = (.25, 0.8, 0.8)$, after the slider goes by a transient region in which the amplitude of signal varies, its velocity stays oscillating around the rate $\hat{v} = 1$, its relative position u around $\eta = 2.3593$ and asperity contact $\hat{\theta}$ around zero; (c) the phase portrait shows the convergency to a periodic orbit and $\overline{\sigma}_c^\star = 2.84$

B. Case 2: Discontinuity

Analysis in previous paragraphs allows us to provide relevant information about dynamical behavior. However, seeking completeness, we have to relax the assumptions on continuity to recognize that the structure of the vector field in system (7) includes discontinuous terms induced by switching. Specifically, the term $\alpha F_0(\hat{v}) = \alpha_1 sign(\hat{v} - 1) + \alpha_2 e^{-\mu/\hat{v}-1/} sign(\hat{v} - 1) + \alpha_3(\hat{v} - 1)$ depends on function $sign(\cdot)$, which is related to the Coulomb friction F_c and Stribeck terms F_s. The presence of such discontinuous terms might induce oscillations that are characteristic in nonlinear switched systems like chaotic behavior. In this section we explore computationally the behavior of the system with discontinuities.

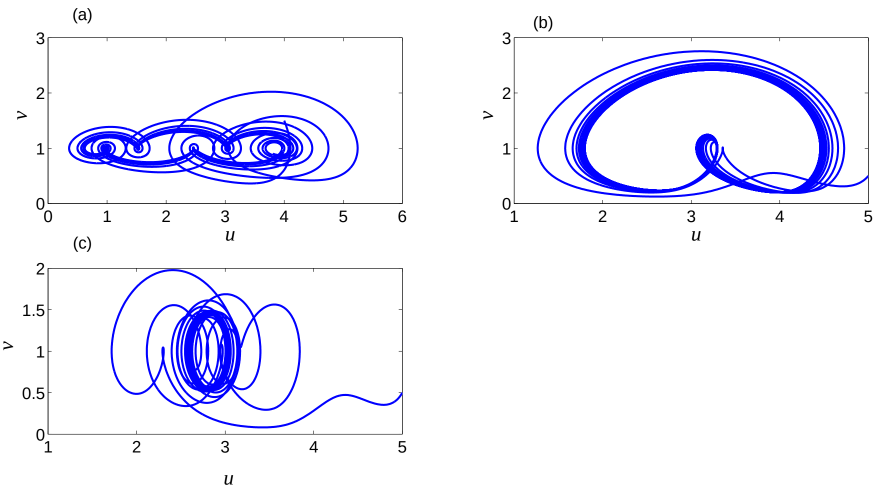

Figure 8. Solutions for perturbed and sub-damped system for ($\alpha_1 = 1.6$,$\alpha_2 = 0.2$,$\alpha_3 = 0.1$,$\mu = 3$). Phase portrait ($\hat{u}$, $\hat{v}$) for $\Pi = (.25, 0.8, 0.8)$ with variable ω. With $\omega = 0.1$ (a) shows the regions in conflict caused by the flow generated by the friction effects, (b) with $\omega = 1.2$ the trajectory makes a closed orbit with a single perfect curl, (c) for $\omega = 2$ the region in conflict is weak and the trajectory quickly forms a closed orbit.

1. Unforced, $\tau(\hat{t}) = 0$ *for all* t

The existence and uniqueness hold as the vector field f be piecewise continuous in t. The concept of fundamental solution can be exploited in order to prove existence and uniqueness, namely, a continuous function $x(\cdot)$ that satisfies the corresponding integral equation: $x(t) = x_0 + \hat{w}\int_\tau f(\tau, x(t))d\tau$. The stability can be analyzed via Lyapunov stability theory as follows[46]: Given a positive definite continuously differentiable (C^1) function $V : R^n \to R$, we will say that V is a Lyapunov function common for the family of systems $\dot{x} = f_p(x)$, $p \in \mathcal{P}$ (where $\mathcal{P}$ is some index set, typically a vector lineal field) if there exists a positive definite continuous function $W : R^n \to R$ such that

$$\frac{\partial V}{\partial x} f_p(x) - W \qquad \forall x, \qquad \forall p \in \mathcal{P}. \tag{14}$$

Next, we analyze the equilibrium points and stability. The equilibrium point is $x^\star = (0, 0, 1) = (\hat{\theta}^\star, \hat{u}^\star, \hat{v}^\star)$. The condition (3)imposes the existence of $\hat{\delta}_i$, $i = 1, 2, 3$ such that it has the family of functions generated by the vector field $\dot{x} = f_i(x) = \hat{\delta}(x) + \hat{\delta}_i(x)$, for all $x = (\hat{\theta}, \hat{u}, \hat{v})$, where $\hat{\delta}_1(x) = \alpha_1 + \alpha_2 e^{-\mu(\hat{v}-1)} + \alpha_3(\hat{v} - 1)$, $\hat{\delta}_2(x) = 0$, and $\hat{\delta}_3(x) = -\alpha_1 - \alpha_2 e^{\mu(\hat{v}-1)} + \alpha_3(\hat{v} - 1)$; this implies to find $V_i = x^T P x > 0$ where $P = P^T$, $P > 0$, and

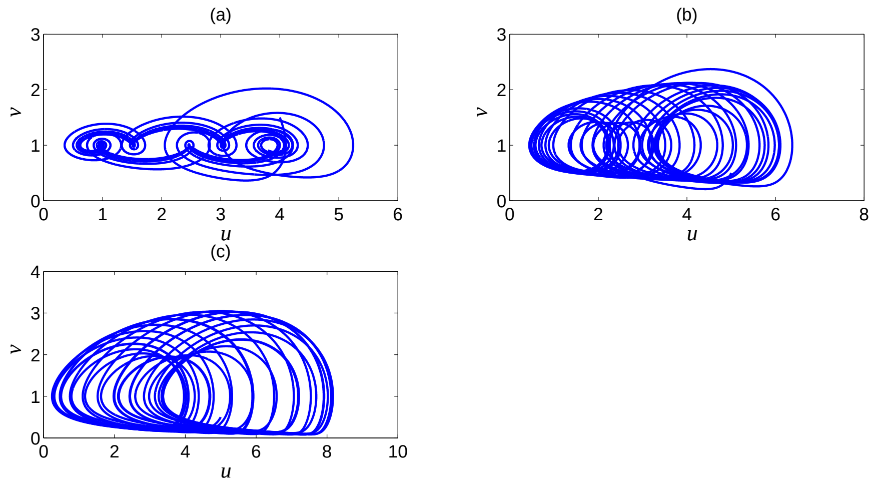

Figure 9. Solutions for perturbed and damped system for ($\alpha_1 = 1.6$,$\alpha_2 = 0.2$,$\omega = 0.1$,$\alpha_3 = 0.1, 0.2, 0.4$,$\mu = 3$). Phase portrait ($\hat{u}$, $\hat{v}$) for $\Pi = (.25, 0.8, 0.8)$ with variable α_3. With $\alpha_3 = 0.1$ (a) shows sub-damped system behavior, (b) for $\alpha_3 = 0.2$ a damped system, and (c) with $\alpha_3 = 0.4$ for an overdamped system.

we find conditions under which $\partial V_i/\partial t = \dot{x}^T P x + x^T P \dot{x} < 0$ such that $\dot{x} = f_i(x)$, $i = 1, 2, 3$ have separately stables equilibrium points. With $P = I$, a candidate Lyapunov function takes the form $V = \hat{\theta}^2 + \hat{u}^2 + \hat{v}^2 > 0$ for all $x = (\hat{\theta}, \hat{u}, \hat{v})^T$. This function V could be the common Lyapunov function for the family $\dot{x} = f_i(x)$, if $\partial V/\partial t < 0$ holds for $i = 1, 2, 3$; subsequently $\partial V/\partial t$ takes the following form for each member of the family $\dot{x} = f_i(x)$, $\partial V_i/\partial t = \hat{\theta}[-\hat{v}(\hat{\theta} + (1 + \varepsilon)\ln v] + \hat{u}(\hat{v} - 1) + \hat{v}[-\gamma^2 \hat{u} - \gamma^2/\xi(\hat{\theta} + \ln \hat{v})] + \hat{v}\delta_i(x)$. For the values of $\Pi = (\varepsilon, \xi, \gamma) > 0$, $\hat{\theta}\,\hat{u} \in R$, and $\hat{v} > 0$ we found that the parameters ε and ξ are not directly related to the release of energy of the system, the inequality $\partial V/\partial t \leq 0$, for all $x = (\hat{\theta}, \hat{u}, \hat{v})^T$, is determined by the values of γ.
Taking $W = -\partial V/\partial t > 0$ and $W = \partial V/\partial t$ the condition (14) is satisfied. Some numerical solutions show the vector field for $\Pi = (0.8, 1, 10)$ as show the Figures 10(a) and (b).

2. ***Forced,*** $\tau(\hat{t}) = \sin(\omega\hat{t})$

Now, for the forced case, following the solution of system (7) is numerically explored with $\Pi = (0.25, 0.8, 3)$ and $\omega = 1.3$. The Figures 10(c), and (d) show the complex dynamical behavior, possibly chaotic. The Figure 10(c) depicts the time series of the forced and switched

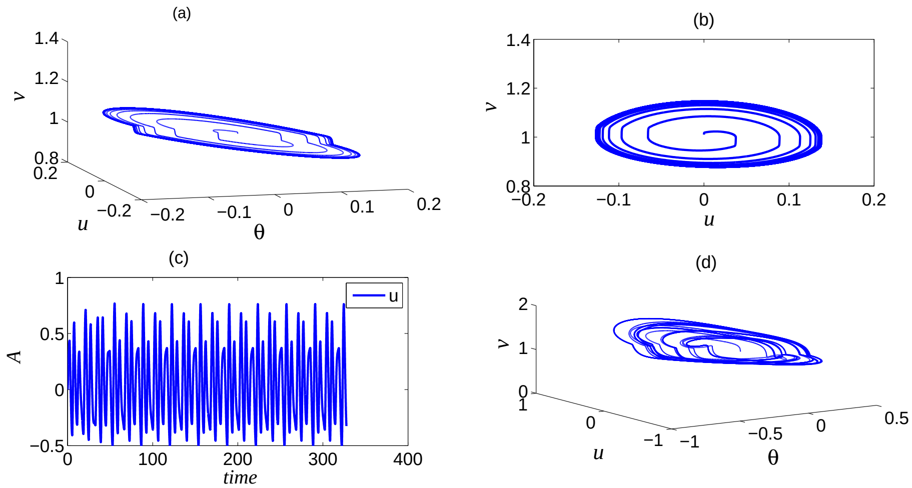

Figure 10. Homogeneous and forced switched system. At $\hat{v} = 1$ exists a no-differentiability in the vector field. (a) and (b) the space phases and the phase portrait, respectively, show an outward spiral for the homogeneous case with $\Pi = (.8, 1, 10)$; (c) the time series show the non-smoothness of the vector field for the forced system; and (d) the space phases for the switched and perturbed system with Π=$(0.25, .8, 3, \omega = 1.3)$ shows a more complex behavior than the homogeneous case.

system (7), note the signal is quite rich in dynamical components. Figure 10(d) shows the phase portrait. Note that a single multiscroll is displayed around the equilibrium point. An interesting challenge is to identify the geophysical properties with the complex time series displayed by such a complex behavior, this is beyond the goals of present manuscript and could be reported elsewhere.

IV. CONCLUSIONS

The analysis of the slider block model (in a viscoelastic medium and over a rough and lubricated surface) with friction laws, including Dieterich-Ruina and the Stribeck's effect, let us to analyze mathematical properties in terms of seismic parameters and friction coefficients that describe a type of behavior observed in the nature of the earthquakes. In stationary state for smoothness and homogeneous case, the stability or instability can appear depending on the parameters $(\varepsilon, \xi, \gamma)$ whose interpretation related to seismic parameters. A necessary condition for stability of equilibrium point is that the parameter related to drop stress, ε, is lower than the parameter related to medium elasticity and friction coefficients, $\xi\psi$, under sub-damped conditions. In this sense, we found a necessary condition for frictional stability within an unstable regime (in the Scholz's (1998)[50] sense) that depends on parameters of Stribecks effect when $\gamma < 10$, approximately. That is, as $\gamma > 10$, the necessary condition only depends on the parameter ξ. Other result is that the position coordinate of the slider (relative displacement) depends on η as a function of α, γ, and μ. Particularly, for $\gamma > 10$, the position $\eta \approx 0$, which means that the relative position of the plate and the block does not depends on the Stribeck parameters. We notice that ψ and η are functions the oscillation frequency γ, which is function of that the characteristic distance to renew the contact population (see Fig. 3), L; *i.e.*, $\psi = \psi(\gamma(L))$ and $\eta = \eta(\gamma(L))$.

Additionally, our analysis shows if the system equilibrium is stable, the slider can be in coupled motion with the driver plate and, as a consequence, the frictional instability (earthquake) is not expected. Since the variable $\hat{\theta}$ is related with the asperities of frictional surface, it converges to zero; nevertheless, if the equilibrium system is unstable, the system stays around it and can be more sensible to disturbances by the medium, which can induce resonance or other phenomena.

As summary, we found periodic and aperiodic trajectories and complex behavior like in chaotic systems. For the smoothness case the trajectories tend to a closed orbit (Figure 8(b)) with a curl for low frequencies with small values for parameters Π and converge to limit cycle with period 1 for frequencies $\omega > 1.3$ (Figure 8(c)); we found a special complex behavior for some values of parameters (Figure 8(a)) strongly related with the parameter of viscosity α_3 (Figure 9). The phase space shows an inward spiral for smoothness case; contrary to the switched case where an outward spiral is shown (Figures 10(c) and (d)). It should be noted that when a discontinuity is introduced (case 2) the released energy of the system depends on the values of γ and the system can display a complex behavior for the forced case even for small values of Π.

V. APPENDIX

A. Law of friction of Dieterich-Ruina

Daub and Carlson (2008)[26], and Scholz (1998)[50] made a summary of the Dieterich-Ruina's Law of friction F_{dr}, that it is a phenomenological friction law introduced to capture experimental observations of steady state and transient friction. The F_{dr} is a rate and state friction law that assumes dependence on a single dynamic state variable. Shear stress τ is a function of rate (slip velocity v) and the state variable θ. The dependence is logarithmic[41]:

$$\tau = \sigma[f_0 + A\ln(v/v_0) + B\ln(\theta v_0/L)] \qquad \frac{d\theta}{dt} = 1 - \frac{\theta v}{L}. \tag{15}$$

Here σ is the normal stress, constants A and B are material properties determining the rate and state dependence, L is the critical slip distance or length scale and reference and is often interpreted as the sliding distance required to renew the contact population, friction coefficient f_0 is the steady-state friction at $v = v_0$ and θ evolves according to second equation of (15). The variable θ has dimensions of time and is often interpreted as the lifetime of surface asperity contacts. When the slider moves at constant velocity v_{ss} (steady-state), $\theta_{ss} = L/v_{ss}$ and the shear stress is $\tau = \sigma[f_0 + (A - B)\ln(v_{ss}/v_0)]$.

According to Rice (1983)[43] $A = \partial\tau/\partial\ln v = v(\partial\tau/\partial v)$ is a measure of the direct velocity dependence, while $(A - B) = \partial\tau_{ss}/\partial\ln v_{ss} = v\partial\tau_{ss}/\partial v$ is a measure of steady-state velocity dependence. A and B determine the velocity dependence of friction, and there is a fixed L for transient effects. Equation (15) are the Dieterich's law, in last equation state (and friction) evolves even for truly stationary contact at $v = 0$ which has been referred to as aging[25]. Ruina (1983)[28] proposed a different evolution law in which velocity and slip, rather than time, were of primary importance (Ruina's law):

$$\tau = \sigma[f_0 + A\ln(v/v_0) + B\ln(\theta v_0/L)] \qquad \frac{d\theta}{dt} = -\frac{\theta v}{L}\ln(v\theta/L). \tag{16}$$

While Dieterich's model (15) casts friction primarily in terms of time dependence and static friction, Ruina's model (16) says that any change in friction requires slip.

REFERENCES

[1]J. B. Rundle, D. L. Turcotte, R. Shcherbakov, W. Klein, and C. Sammis, Rev. of Geophys., **41**, 1-30 (2003).

[2]J. D. Gran, J. B. Rundle and D. L. Turcotte, Geophys. J. Int., **191**, 459 (2012).

[3]J. M. Carlson, J. S. Langer and B. E. Shaw, Rev. Mod. Phys., **66**, 654 (1994).

[4]C. H. Scholz, *The Mechanics of Earthquakes and Faulting*, 2nd edn. (Cambridge University press, 2002).

[5]W. F. Brace and J. D. Byerlee, Science, **153**, 990 (1966).

[6]S. R. Brown, C. H. Scholz, and J. B. Rundle, Geophys. Res. Lett. **18**, 215 (1991).

[7]R. Burridge and L. Knopoff, Seis. Soc. Am. Bull., **57**, 341 (1967).

[8]J. M. Carlson, J. S. Langer and B. E. Shaw, Phys. Rev. A , **40**, 6470 (1989).

[9]M. De Sousa Vieira, Phys. Rev. A **46**,6288 (1992).

[10]M. De Sousa Vieira, Phys. Rev. E **49**, 4534 (1994).

[11]A. Amendola and M. Dragoni, Non. Process. Geophys., **20**, 1(2013).

[12]J. H. Dieterich, J. Geophys. Res., **77**, 360 (1972).

[13]M. Dragoni and S. Santini, Non. Process. Geophys., **17**, 777 (2010).

[14]B. Erickson, B. Birnir and D. Lavallée, Non. Process. Geophys., **15**, 1 (2008).

[15]B. Erickson, B. Birnir and D. Lavalle, Geophys. J. Int., **187**, 178 (2011).

[16]J. Gu, J. R. Rice, A. Ruina and S. Tse, J. Mech. Phys. Solids, **32**, 167 (1984).

[17]T. Matcharashvili, T. Chelidze, and Z. Javakhishvili, Non. Process. Geophys.,**7**, 9 (2000).

[18]J. R. Rice and Y. Ben-Zion. Proc. Natl Acad. Sci. USA; **93**, 3811(1996).

[19]J. Huang, G. Narkounskaia, and D. L. Turcotte, Geophys. J. Int., **111**, 259 (1992).

[20]H. Nakanishi, Phys. Rev. A, **41**, 7086 (1990).

[21]H. Nakanishi, Phys. Rev. A, **43**, 6613 (1991).

[22]J. Naussbaum and A. Ruina, Pageoph., **125**, 631 (1987).

[23]He Changrong, Science in China series D, **46**, 67 (2003).

[24]J. D. Pelletier, Geophys. Monogr., **120**, 27 (2000).

[25]C. Marone, Annu. Rev. Earth Planet Sci., **26**, 643 (1998).

[26]E. G. Daub and J. M. Carlson, J. Geophys. Res., 113(B12309):1-20, 2008. Doi:10.1029/2007JB005377

[27]J. H. Dieterich, Geophys. Monogr., **24**, 103 (1981).

[28]A. Ruina, J. Geophys. Res., **88**, 10359 (1983).

[29]C. H. Scholz and T. Engelder, Int. J. Rock Mech. Min. Sci., **13**, 149 (1976).

[30]S. T. Tse and J. R. Rice, J. Geophys. Res., **91**, 9452 (1986).

[31]J. Szkutnik, B. Kawecka-Magiera, and K. Kulakowski, Tribology series, **43**, 529 (2003).

[32]M. Otsuka, J. Phys. Earth, **20**, 35 (1992).

[33]P. Bak and C. Tang, J. Geophys. Res. **94**, 15635 (1989).

[34]B. Barriere and D. L. Turcotte, Geophys. Res. Lett., **18**, 2011 (1991).

[35]K. Ito and M. Matsuzaki, J. Geophys. Res., **95**, 6853 (1990).

[36]J. Alvarez-Ramírez, R. Garrido and R. Femat, Phys. Rev. E, **51**, 6235 (1995).

[37]Y. Bock and L. Prawirodirdjo, Geophys. Res. Lett. doi:10.1029/2003GL019150.

[38]J. D. Byerlee, Int. J. Rock Mech. Min. Sci., **7**, 577 (1970).

[39]J. D. Byerlee, Pageoph., **116**, 615 (1978).

[40]S. Andersson, A. Sø̈derberg and S. Bjørklund, Tribology Int., **40**, 580 (2007).

[41]J. H. Dieterich, J. Geophys. Res., **84**, 2161 (1979).

[42]J. H. Dieterich and B. Kilgore, Pageoph. **143**, 1/2/3 (1994).

[43]J. R. Rice and A. Ruina, J. Appl. Mech., **50**, 343 (1983).

[44]J. Guckenheimer and P. Holmes, *Nonlinear oscillations, Dynamical Systems, and Bifurcation of Vector Fields*. Springer, 1983.

[45]H. Khalil, *Nonlinear Systems*. 2nd edn. Prentice-Hall, 1996.

[46]D. Liberzon, *Switching in Systems and Control*. Birkhuser, Boston, c/o Springer-Verlag New York, USA, 1973.

[47]C. E. Maloney and M. O. Robbins, Chaos, **17**, 041105 (2007).

[48]B. W. levin, Chaos, **6**, 405 (1996).

[49]N. V. Sarlis and S. R. G. Christopoulos, Chaos, **22**, 023123 (2012).

[50]C. H. Scholz, Nature, **391**, 37 (1998).