Ermakov-Ray-Reid Systems with Additive Noise

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Abstract

Using the methods developed by us in Physica A 401, 141 (2014) for multiplicative noises, we present results on the effects of the additive noise on the Ermakov-Lewis invariant. This case can be implemented in the Euler-Maruyama numerical method if the additive noise is considered as the forcing term of the parametric oscillator and presented as a particular case of the Ermakov-Ray-Reid systems. The results are obtained for the same particular examples as for the multiplicative noise and show a tendency to less robustness of the Ermakov-Lewis invariant to the additive noise as compared to the multiplicative noise.

Key words: Ermakov-Lewis invariant; additive noise; Euler-Maruyama method; forced parametric oscillator; Ermakov-Ray-Reid system.

This work is the second paper in the series concerned with the effects of noises on the stochastic parametric oscillators that we started with [1], where we focused on the effects of multiplicative noises. Here we investigate the robustness of the Ermakov-Lewis (EL) invariant to additive noises by making usage

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as in [1] of the Euler-Maruyama discretization for the numerical calculations. As well known, the standard Ermakov system refers to the motion of a free parametric oscillator

$$\ddot{x} + \Omega^2(t)x = 0 \ , \tag{1}$$

together with the associated Milne-Pinney nonlinear equation

$$\ddot{\rho} + \Omega^2(t)\rho = \frac{k}{\rho^3} \ , \tag{2}$$

where k is an arbitrary real constant. The solutions of (1) and (2) can be related through the following formula [2]

$$x(t) = C\rho(t)\sin(k\Theta_T(t) + \phi) \ , \tag{3}$$

where C and ϕ are arbitrary constants and the total phase $\Theta_T(t)$ is given by

$$\Theta_T(t) = \int^t \frac{1}{\rho^2(t')}dt' \ . \tag{4}$$

Such parametric systems are endowed with the EL invariant given by

$$I_0 = \frac{kx^2}{2\rho^2} + \frac{1}{2}\left(\dot{x}\rho - \dot{\rho}x\right)^2 . \tag{5}$$

Here we show that the effects of the additive noise can be evaluated by extending the Ermakov system to the forced case, as worked out for example in [3] and in the more general context of forced Ermakov-Ray-Reid systems in [4–8]. In particular, we consider the following forced parametric oscillator as presented in [4]

$$\ddot{x} + \Omega^2(t)x = f\left(t\right) + \frac{1}{x^2\rho}g(\rho/x) \ , \tag{6}$$

with $g(\rho/x)$, an arbitrary external force of the Ray-Reid type, for which the EL invariant reads

$$I = I_0 + \dot{\psi}x - \psi\dot{x} + \int^t \psi\left(\tau\right) f\left(\tau\right) d\tau - \rho^2 x f\left(t\right) + \int^{\rho/x} g(\tau)d\tau, \tag{7}$$

where the function $\psi(t)$ is the solution of a second auxiliary equation

$$\ddot{\psi}(t) + \Omega^2(t)\psi(t) = \rho^2\dot{f}(t) + 3\rho\dot{\rho}f\,(t) + \frac{1}{x^3\rho}g\,(\rho/x)\,\psi. \tag{8}$$

It is important to mention that in this case the Milne-Pinney equation (2) keeps its form unchanged [4]. As in [1], we will write the dynamical systems given by the equations (6), (2), and (8) as a matrix version of a stochastic matrix differential equation of the form

$$dY_t = a(t, Y_t)dt + b(t, Y_t)dB_t\ , \tag{9}$$

where B_t is the stochastic variable, for which the Euler-Maruyama numerical method is readily available [9–11]. In the matrix formulation, the corresponding stochastic variables and coefficients are identified in the following explicit forms

$$dX_t = \begin{pmatrix} dx \\ d\dot{x} \end{pmatrix},\quad a\,(t, X_t) = \begin{pmatrix} \dot{x} \\ -\Omega^2 x + \frac{g}{x^2\rho} \end{pmatrix},\quad b\,(t, X_t) = \alpha_\Omega \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{10}$$

$$d\rho_t = \begin{pmatrix} d\rho \\ d\dot{\rho} \end{pmatrix},\quad a\,(t, \rho_t) = \begin{pmatrix} \dot{\rho} \\ -\Omega^2\rho + \frac{1}{\rho^3} \end{pmatrix},\quad b\,(t, \rho_t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{11}$$

$$d\psi_t = \begin{pmatrix} d\psi \\ d\dot{\psi} \end{pmatrix},\quad a\,(t, \psi_t) = \begin{pmatrix} \dot{\psi} \\ \rho^2\dot{f} + \frac{g\psi}{x^3\rho} - \Omega^2\psi \end{pmatrix},\quad b\,(t, X_t) = 3\alpha_\Omega \begin{pmatrix} 0 \\ \rho\dot{\rho} \end{pmatrix}, \tag{12}$$

where α_Ω is the amplitude of the noise [12]. In [1], the simplest case of multiplicative noise, $m = 1$, has been illustrated and a strong robustness of the Ermakov-Lewis invariant has been reported. In the present calculations, the term $\rho^2\dot{f}$ in $a(t, \psi_t)$ has been disregarded because of its negligible effects on the mean values and the standard deviations of the invariant as seen in Table 1. The chosen initial conditions are as in [1], i.e., $x(0) = 1, \dot{x}(0) = 0, \rho(0) = 1, \dot{\rho}(0) = 0$, and additionally, $\psi(0) = 1, \dot{\psi}(0) = 0$.

We consider two particular cases of this type of dynamical system governed by the invariant (7):

(i) Ermakov systems affected only by additive noise by taking the noise as the forcing term $f(t)$, see [13], and setting $g(\rho/x) = 0$. The EL invariant is displayed in Figure 1 for $\Omega(t) = 2$, $\Omega(t) = 2\sin(t)$ and $\Omega(t) = 2t^2$. Three levels of noise have been chosen, $\alpha_\Omega = 0.001$, $\alpha_\Omega = 0.01$, and $\alpha_\Omega = 0.1$, termed weak, intermediate, and strong regimes, respectively.

Table 1
The invariants for the case (ii) with their mean values and standard deviations for the three cases of $\Omega(t)$ considered here, each of them in the presence of weak, intermediate, and strong noise amplitudes.

	I ($\rho^2\dot{f}$ included)	I ($\rho^2\dot{f}$ not included)
$\Omega(t) = 2,\quad \alpha_\Omega = 0$	1.000000 ± 0.000000	1.000000 ± 0.000000
$\alpha_\Omega = 0.001$	1.001280 ± 0.000770	1.001280 ± 0.000770
$\alpha_\Omega = 0.01$	1.012550 ± 0.007800	1.012640 ± 0.007801
$\alpha_\Omega = 0.1$	1.104260 ± 0.089765	1.104990 ± 0.089832
$\Omega(t) = 2\sin t,\ \alpha_\Omega = 0$	1.000000 ± 0.000000	1.000000 ± 0.000000
$\alpha_\Omega = 0.001$	1.000370 ± 0.000805	1.000380 ± 0.000805
$\alpha_\Omega = 0.01$	1.003790 ± 0.008051	1.003920 ± 0.008051
$\alpha_\Omega = 0.1$	1.046310 ± 0.081424	1.047570 ± 0.081271
$\Omega(t) = 2t^2,\quad \alpha_\Omega = 0$	1.000000 ± 0.000000	1.000000 ± 0.000000
$\alpha_\Omega = 0.001$	1.001140 ± 0.001017	1.001140 ± 0.001017
$\alpha_\Omega = 0.01$	1.011460 ± 0.010115	1.011450 ± 0.010116
$\alpha_\Omega = 0.1$	1.116590 ± 0.097451	1.116500 ± 0.097468

(ii) Ermakov-Ray-Reid systems in the presence of the additive noise $f(t)$ and an external force of the type $g(\rho/x) = k'\rho/x$ [4]. In this case, the plots of the invariant are given in Figure 2 for the same parametric oscillators. It is worth noticing that we have taken $k = k' = 1$ and because of the integral over the external force g in (7) the EL invariant in this case is $I = 1$ and not 1/2 in the absence of the noise.

In conclusion, the additive noise in Ermakov systems can be taken into account by considering them as particular cases of forced Ermakov-Ray-Reid systems with the additive noise as the driven terms in the parametric oscillator equation. In general, we conclude that the additive noise has somewhat more pronounced effects on the robustness of the EL invariant as compared to the multiplicative noise as seen in Table 2. On the other hand, the plots provided here show that the average effect of the additive noise on the EL invariant can be considered small when the driving is due only to the noise. However, when there is also driving due to Ray-Reid external forces there is a more pronounced effect as the amplitude of the noise is increased.

Table 2
The invariants for additive (case (i)) and multiplicative noise [1] with their mean values and standard deviations for the three cases of $\Omega(t)$ considered here.

	I (additive noise)	I (multiplicative noise)
$\Omega(t) = 2, \quad \alpha_\Omega = 0$	0.499955 ± 0.000032	0.499955 ± 0.000032
$\alpha_\Omega = 0.001$	0.500077 ± 0.000753	0.499955 ± 0.000032
$\alpha_\Omega = 0.01$	0.501185 ± 0.007480	0.499955 ± 0.000031
$\alpha_\Omega = 0.1$	0.513179 ± 0.076993	0.499961 ± 0.000025
$\Omega(t) = 2\sin t, \quad \alpha_\Omega = 0$	0.500054 ± 0.000041	0.500054 ± 0.000041
$\alpha_\Omega = 0.001$	0.499558 ± 0.000946	0.500054 ± 0.000041
$\alpha_\Omega = 0.01$	0.495245 ± 0.009723	0.500054 ± 0.000041
$\alpha_\Omega = 0.1$	0.466946 ± 0.088735	0.500053 ± 0.000041
$\Omega(t) = 2t^2, \quad \alpha_\Omega = 0$	0.501059 ± 0.001620	0.501059 ± 0.001620
$\alpha_\Omega = 0.001$	0.500883 ± 0.001666	0.501059 ± 0.001620
$\alpha_\Omega = 0.01$	0.499309 ± 0.003966	0.501058 ± 0.001618
$\alpha_\Omega = 0.1$	0.484304 ± 0.035113	0.501046 ± 0.001606

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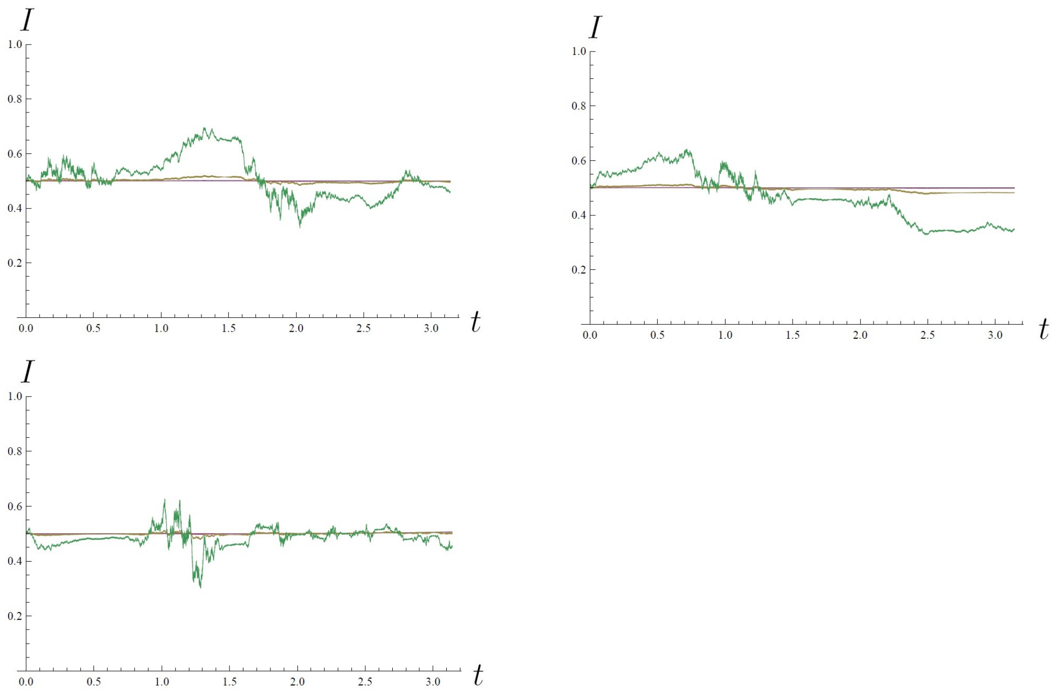

Fig. 1. (color on line) Top left: Plots of the EL invariant I in (7) with $g = 0$ in the interval $t \in [0, \pi]$ for the harmonic oscillator with $\Omega(t) = 2$: (blue color) the case without noise; (magenta color) low-amplitude noise, $\alpha = 0.001$; (olive color) medium-amplitude noise, $\alpha = 0.01$; (green color) high-amplitude noise, $\alpha = 0.1$. Top right: The same but for $\Omega(t) = 2\sin(t)$. Bottom: Same plots for the case $\Omega(t) = 2t^2$.

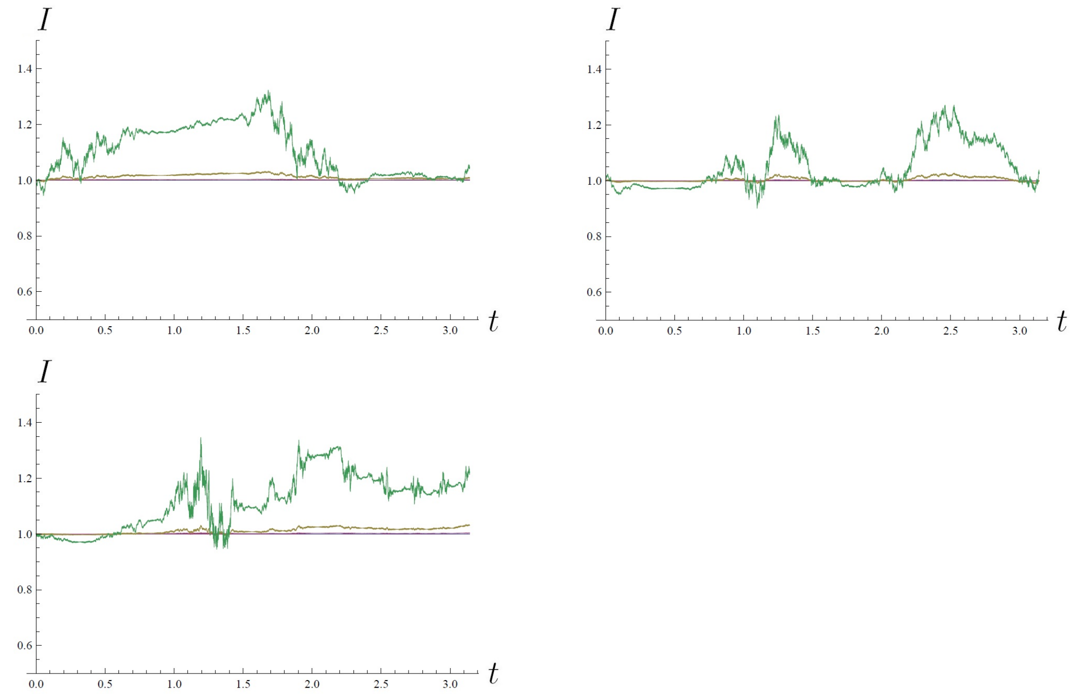

Fig. 2. (color on line) Same plots as in the previous figure but for the EL invariant in (7) with $g = \rho/x$.