SOLUTIONS OF THE PERTURBED KDV EQUATION FOR CONVECTING FLUIDS BY FACTORIZATIONS

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Abstract

In this paper, we obtain some new explicit travelling wave solutions of the perturbed KdV equation through recent factorization techniques that can be performed when the coefficients of the equation fulfill a certain condition. The solutions are obtained by using a two-step factorization procedure through which the perturbed KdV equation is reduced to a nonlinear second order differential equation, and to some Bernoulli and Abel type differential equations whose solutions are expressed in terms of the exponential and Weierstrass functions.

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In a previous paper [1], we have factorized the Korteweg-de Vries-Burgers (KdVB) equation by means of an efficient factorization procedure that we introduced in 2005 [2]. This allowed us to obtain in an easy way some travelling wave solutions of the KdVB equation. Recently, Wang and Li [3] discussed an extended form of our method and applying it to various nonlinear equations. Our goal in the present paper is to use jointly the two methods for yet another nonlinear evolution equation, the so-called perturbed Korteweg de Vries (PKdV) equation, which is one of the most general nonlinear equations with important applications [4]. We shall use the following form of this equation

$$u_t + \lambda(u_{xxx} + 6uu_x) + 5\beta uu_x + (u_{xxx} + 6uu_x)_x = 0 \ . \tag{1}$$

It was used to describe the evolution of long surface waves in a convecting fluid. It has been thoroughly investigated by Cerveró and Zurrón in 1996 [5]. We notice here that the solution of a slightly more complicated equation describing the evolution of a system exhibiting an oscillatory instability with respect to the static state can be obtained from u with an appropriate scaling followed by a constant shift proportional to the excess of the Rayleigh number above its critical value (see page 5 in [5]).

Passing to the travelling variable $z = x - ct$, we convert this equation after integrating it once into the following ODE

$$u_{zzz} + \lambda u_{zz} + 6uu_z + \frac{1}{2}(6\lambda + 5\beta)u^2 - cu + K_1 = 0 \ , \tag{2}$$

where K_1 is the integration constant. Moreover, employing $u = w - \delta$, where the constant δ is equal to $\frac{-c\pm\sqrt{c^2-4\alpha_1 K_1}}{2\alpha_1}$, $\alpha_1 = \frac{1}{2}(6\lambda + 5\beta)$, and denoting $\alpha_2 = 2\delta\alpha_1 + c$, one can get

$$w_{zzz} + \lambda w_{zz} + 6(w - \delta)w_z + w(\alpha_1 w - \alpha_2) = 0 \ . \tag{3}$$

Eq. (3) can be factorized in the form

$$[D_z - \phi_1(w)w_z - \phi_2(w)][D_{zz} - \phi_3(w)D_z - \phi_4(w)]w = 0 \tag{4}$$

by introducing the appropriate ϕ_i functions. It is easily shown by direct calculation that the factorization

$$\left[D_z + \frac{\alpha_1}{3}\right]\left[D_{zz} + (\lambda - \frac{\alpha_1}{3})D_z + \frac{3}{\alpha_1}(\alpha_1 w - \alpha_2)\right] w = 0 \ , \tag{5}$$

is allowed under the restriction

$$\alpha_1^3 - 3\alpha_1^2\lambda - 54\delta\alpha_1 + 27\alpha_2 = 0 \ . \tag{6}$$

Therefore, the travelling wave solutions obtained for Eq. (3) correspond to the case in which Eq. (6) for the coefficients α_1 and α_2 is satisfied. Furthermore, this restriction leads to $c = -\frac{5}{6}\beta(\lambda + \frac{5}{6}\beta)^2$ which represents the velocity of the travelling waves.

Let us consider now the extended factorization scheme [3]. Assuming

$$\left[D_z^2 + (\lambda - \frac{\alpha_1}{3})D_z + \frac{3}{\alpha_1}(\alpha_1 w - \alpha_2)\right] w = \Omega \ , \tag{7}$$

then Eq. (5) can be rewritten as the following system

$$\Omega_z + \frac{\alpha_1}{3}\Omega = 0, \tag{8}$$

$$w_{zz} - \frac{5}{6}\beta w_z + \frac{3}{\alpha_1}(\alpha_1 w - \alpha_2)w = \Omega \ . \tag{9}$$

where $(\lambda - \frac{\alpha_1}{3})$ has been replaced with $-\frac{5}{6}\beta$ in Eq. (9). The first equation implies $\Omega(z) = c_1 e^{-\frac{\alpha_1}{3}z}$, where c_1 is an integration constant. Thus, we can consider the solutions of the second equation of the system, i.e.,

$$w_{zz} - \frac{5}{6}\beta w_z + 3w^2 - 3\frac{\alpha_2}{\alpha_1}w = c_1 e^{-\frac{\alpha_1}{3}z} \tag{10}$$

The transformations

$$w = \lambda(z)W + \mu(z) \ , \quad Z = \Phi(z)$$

where

$$\lambda(z) = (-2)^{1/5}e^{\frac{\beta}{3}z} \ , \quad \mu(z) = \left(\frac{\beta}{6}\right)^2 + \frac{\alpha_2}{2\alpha_1} \ , \quad \Phi(z) = \left(-\frac{1}{2}\right)^{2/5}\frac{6}{\beta}e^{\frac{\beta}{6}z}$$

lead to the following canonical equation [6]

$$\frac{d^2W}{dZ^2} = 6W^2 + S(z) \tag{11}$$

where

$$S(z) = -3\mu^2 + 3\frac{\alpha_2}{\alpha_1}\mu + c_1 e^{-\frac{\alpha_1}{3}z} \ ,$$

which according to Ince's texbook has solutions free of movable singularities other than poles only if $c_1 = 0$. However, this condition leads to $\mu = 0$ and therefore to the Painlevé case

$$\frac{d^2W}{dZ^2} = 6W^2 \ , \tag{12}$$

for which the solutions are

$$W = C^2\left[\frac{-k^2}{1+k^2} + \mathrm{sn}^{-2}(CZ, k)\right] \ . \tag{13}$$

On the other hand, let us consider, for the same case $c_1 = 0$, the second-order nonlinear differential equation coming out from the factorization procedure

$$\frac{d^2w}{dz^2} - \frac{5}{6}\beta\frac{dw}{dz} + 3w\left[w - \alpha_3\right] = 0 \tag{14}$$

where $\alpha_3 = \frac{\alpha_2}{\alpha_1} = \frac{\pm\sqrt{c^2 - 4\alpha_1 K_1}}{3\lambda + \frac{5}{2}\beta}$. We are now able to apply the factorization procedure introduced by Rosu and Cornejo-Pérez [2] as a second-step procedure. Eq. (14) can be factorized in the form

$$\left[D_z - f_2(w)\right]\left[D_z - f_1(w)\right] w = 0 \ , \tag{15}$$

under the conditions

$$\begin{cases} f_1 + f_2 + \frac{df_1}{dw} w = & \frac{5}{6}\beta \\ f_1 f_2 w & = \quad 3w(w-\alpha_3). \end{cases}$$

Let $f_1 = \sqrt{3}a_2(w^{1/2} - \alpha_3^{1/2})$ and $f_2 = \sqrt{3}a_2^{-1}(w^{1/2} + \alpha_3^{1/2})$. From the last factorization conditions one can get after some algebra the following values of the parameters $a_2 = \pm i\sqrt{\frac{2}{3}}$ and $\alpha_3^{1/2} = \pm i\frac{\sqrt{2}}{6}\beta$ for which this type of factorization is possible. Therefore, solutions of

$$[D - f_1]w = 0$$

will be solutions of the factorized equation as well. The latter equation has the explicit form

$$\frac{dw}{dz} \pm i\sqrt{2}w^{3/2} \pm i\sqrt{2}\alpha_3^{1/2} w = 0 \ . \tag{16}$$

The solutions of these two Bernoulli equations can be directly written down. Taking into account that $u = w - \delta$ we immediately get:

$$u_{1,2} = \left[-\frac{3\sqrt{2}}{\beta}i + e^{\frac{\beta}{6}(z-z_0)}\right]^{-2} - \frac{5}{36}\lambda\beta - \frac{\beta^2}{36}\left[\frac{25}{6} \pm 1\right] \tag{17}$$

$$u_{3,4} = \left[\frac{3\sqrt{2}}{\beta}i + e^{-\frac{\beta}{6}(z-z_0)}\right]^{-2} - \frac{5}{36}\lambda\beta - \frac{\beta^2}{36}\left[\frac{25}{6} \pm 1\right] \tag{18}$$

$$\tag{19}$$

Choosing now the factorization functions $f_1 = \sqrt{3}a_2(w^{1/2} + \alpha_3^{1/2})$ and $f_2 = \sqrt{3}a_2^{-1}(w^{1/2} - \alpha_3^{1/2})$ the following two Bernoulli equations are obtained

$$\frac{dw}{dz} \mp i\sqrt{2}w^{3/2} \pm i\sqrt{2}\alpha_3^{1/2} w = 0 \ , \tag{20}$$

whose solutions are

$$u_{5,6} = \left[\frac{3\sqrt{2}}{\beta}i + e^{\frac{\beta}{6}(z-z_0)}\right]^{-2} - \frac{5}{36}\lambda\beta - \frac{\beta^2}{36}\left[\frac{25}{6} \pm 1\right] \tag{21}$$

$$u_{7,8} = \left[-\frac{3\sqrt{2}}{\beta}i + e^{-\frac{\beta}{6}(z-z_0)}\right]^{-2} - \frac{5}{36}\lambda\beta - \frac{\beta^2}{36}\left[\frac{25}{6} \pm 1\right] \ . \tag{22}$$

$$\tag{23}$$

On the other hand, combining the factorization conditions

$$f_1 f_2 w = F(w) \ , \qquad f_2 + \frac{d}{dw}(f_1 w) = \frac{5}{6}$$

and introducing the function $l(w) = f_1(w)w$ one obtains an Abel equation of the form

$$l\frac{dl}{dw} - \frac{5}{6}\beta l = -3w^2 + 3\alpha_3 w \ . \tag{24}$$

The solution of this equation can be written as follows [8]

$$w = \frac{1}{2}\left(\frac{\beta}{3}\right)^2 e^{\frac{\beta}{3}(z-z_0)}\mathcal{P}\left(e^{\frac{\beta}{3}(z-z_0)} + c_2, 0, 1\right) \ , \tag{25}$$

which is expressed in terms of the Weierstrass $\mathcal{P}$ function. Therefore,

$$u(z) = \frac{1}{2}\left(\frac{\beta}{3}\right)^2 e^{\frac{\beta}{3}(z-z_0)}\mathcal{P}\left(e^{\frac{\beta}{3}(z-z_0)} + c_2, 0, 1\right) - \frac{5}{36}\lambda\beta - \left(\frac{\beta}{6}\right)^2\left[\frac{25}{6} \pm 1\right] \ . \tag{26}$$

We notice that the latter solution although similar in form to a solution mentioned by Porubov [7] is different by an additive constant that depends on the coefficients of the PKdV equation and by the variable of the Weierstrass $\mathcal{P}$ function which in the case of Porubov's result is $\exp(y) = \exp(e^{\gamma(z-z_0)})$, $\gamma =$ constant.

The Weierstrass component of the solution (26) can be also written in the following form, see also [9]

$$\frac{1}{2}\left(\frac{\beta}{3}\right)^2 \frac{e^{\frac{\beta}{3}z}}{4\sqrt{3}}k_0^2\left[1+\sqrt{3}\frac{1+\mathrm{cn}\left(k_0(e^{\frac{\beta}{6}z}+c_2e^{\frac{\beta}{6}z_0})|m\right)}{1-\mathrm{cn}\left(k_0(e^{\frac{\beta}{6}z}+c_2e^{\frac{\beta}{6}z_0})|m\right)}\right]\ , \tag{27}$$

where $k_0 = 2H_2^{1/2}e^{-\frac{\beta}{6}z_0}$, $H_2 = \frac{\sqrt{3}}{4^{1/3}}$, $m = \frac{1}{2} - \frac{\sqrt{3}}{4}$. Expanding now the cnoidal function in a Taylor series

$$\mathrm{cn}\left(k_0(e^{\frac{\beta}{6}z}+c_2e^{\frac{\beta}{6}z_0})|m\right) = 1 - \frac{1}{2}k_0^2\left(e^{\frac{\beta}{6}z}+c_2e^{\frac{\beta}{6}z_0}\right)^2 + \dots\ ,$$

one gets in the small k_0 limit:

$$\lim_{k_0\to 0} u(z) = \frac{1}{2}\left(\frac{\beta}{3}\right)^2\left(\frac{e^{\frac{\beta}{6}(z-z_0)}}{c_2+e^{\frac{\beta}{6}(z-z_0)}}\right)^2 + \mathrm{const}\ . = \left(\frac{3\sqrt{2}}{\beta}+c_3e^{\frac{\beta}{6}(z-z_0)}\right)^{-2} - \frac{5}{36}\lambda\beta - \left(\frac{\beta}{6}\right)^2\left[\frac{25}{6}\pm 1\right]\ ,$$

which is a simple particular real PKdV solution.

In conclusion, after performing the travelling variable reduction for the PKdV equation we have jointly used recent factorization methods to obtain some new exact travelling wave solutions of this equation in the particular case when the coefficients fulfill the condition (6). This is equivalent to saying that the factorization of the ODE travelling form of PKdV equation can be performed only for a particular value of the velocity parameter and leads to a second order differential equation that has the Painlevé property, a fact that pinpoints the connection between the technique of factorizations and the Painlevé analysis. The latter connection has been already noticed by Gilson and Pickering for other types of nonlinear third-order partial differential equations [10].

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