# A note on stability of functional difference equations ⋆

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**Abstract**

In this note, we consider perturbed linear functional difference equations with discrete and distributed delays which play a fundamental role in several stability and stabilizability problems of time-delay systems. Linear and nonlinear perturbations are considered. Sufficient conditions for exponential stability of the perturbed solutions are given.

*Key words:* functional difference equations; linear and nonlinear perturbations; exponential stability

## 1 Introduction and problem formulation

Consider the following Functional Difference Equation (FDE):

$$x(t)=\sum_{j=1}^{m}A_jx(t-h_j)+\int_{-\tau}^{0}G(\theta)x(t+\theta)d\theta, t\geq 0, \quad (1)$$

where $A_j\in\mathbb{R}^{n\times n}, 0<h_1<\cdots<h_m, 0<\tau$, and the matrix function $G(\theta)$ has piecewise continuous differentiable elements for $\theta\in[-\tau,0]$.

The stability analysis of equations of the form (1) has recently received a considerable attention Gil' & Cheng (2007); Karafyllis & Krstic (2014); Melchor-Aguilar,Kharitonov & Lozano (2010); Melchor-Aguilar (2010, 2012, 2013a,b); Pepe (2003, 2014); Shaikhet (2004, 2011); Verriest (2001). For specific application problems where systems of the form (1) can be found see the citations therein of these references. In particular, they are essential in the closed-loop stability of predictor-based state feedbacks for stabilization of systems with input delays Krstic (2009).

These recent contributions have shown that new type of results are required in order to properly address the asymptotic behavior of discontinuous solutions of FDEs of the form (1).

For the linear FDE (1) there are still several interesting problems to be addressed. Specifically, the problem of deriving asymptotic stability results for perturbed equations involving general linear and/or nonlinear perturbation terms with discrete and distributed delays has not been sufficiently investigated. As it is well-known, stability results for perturbed linear differential delay equations follows from the variation-of-constants formula which express the solutions of perturbed equations in terms of the fundamental matrix associated to the linear equation. By using such formula to obtain stability results for perturbed linear differential delay equations reduces the discussion to arguments very similar to the ones for ordinary differential equations, see Bellman & Cooke (1963); Hale & Verduyn-Lunel (1993).

By following the definition of a Neutral Functional Differential Equation (NFDE) in the Hale's form, one may be tempted to transform the FDE (1) into the NFDE one

$$\frac{d}{dt}\left[z(t)-\sum_{j=1}^{m}A_jz(t-h_j)\right]=G(0)z(t)$$
$$-G(-\tau)z(t-\tau)-\int_{-\tau}^{0}\dot{G}(\theta)z(t+\theta)d\theta, \quad (2)$$

and then apply the existing results for perturbed linear NFDEs. The problem is that the fundamental matrix $X(t)$, solution of the differential matrix equation

$$\frac{d}{dt}\left[X(t)-\sum_{j=1}^{m}A_jX(t-h_j)\right]=G(0)X(t)$$
$$-G(-\tau)X(t-\tau)-\int_{-\tau}^{0}\dot{G}(\theta)X(t+\theta)d\theta,$$

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with initial condition $X(t) = 0, t < 0$, and $X(0) = I$, cannot admit an exponentially decreasing upper bound. The lack of an exponential decreasing upper bound for $X(t)$ follows from Theorem 3.2 in page 271 of Hale & Verduyn-Lunel (1993) and the fact that $s = 0$ belongs to the spectrum of the NFDE (2). As a consequence, the asymptotic stability problem of a perturbed FDE (1) cannot be solved by using the existing results for NFDEs.

The goal of this note is to present some results for such a problem. We firstly present an appropriate definition of the fundamental matrix that allow the description of piecewise continuous solutions of (1) and could admit an exponentially decreasing upper bound. Then, we obtain the variation-of-constants formula for nonhomogeneous equations and use it for deriving asymptotic stability results of perturbed equations involving general linear and/or nonlinear perturbation terms with discrete and distributed delays.

## 2 Main Results

We recall that the spectrum of (1) consists of all zeros of the characteristic function

$$f(s) = \det\left(I - \sum_{j=1}^{m} A_j e^{-h_j s} - \int_{-\tau}^{0} G(\theta)e^{s\theta}d\theta\right).$$

Let us suppose that the spectrum of (1) does not contain the point $s = 0$. Then the matrix

$$K_0 = \left(I - \sum_{j=1}^{m} A_j - \int_{-\tau}^{0} G(\theta)d\theta\right)^{-1}$$

is well-defined. Let matrix $K(t)$ be the unique solution of the matrix equation

$$K(t) = \sum_{j=1}^{m} A_j K(t-h_j) + \int_{-\tau}^{0} G(\theta)K(t+\theta)d\theta, t \geq 0, \tag{3}$$

with the initial condition $K(t) = -K_0, t \in [-r, 0)$, where $r = \max\{h_m, \tau\}$. The matrix function $K(t)$ is piecewise continuous and of bounded variation.

Next, consider the nonhomogeneous FDE

$$x(t) = \sum_{j=1}^{m} A_j x(t-h_j) + \int_{-\tau}^{0} G(\theta)x(t+\theta)d\theta + w(t), \tag{4}$$
$$x(t) = \varphi(t), t \in [-r, 0),$$

where $w(t)$ is any given piecewise continuous vector function defined for all $t \geq 0$. We assume that the initial function $\varphi$ belongs to the space of piecewise continuous vector functions $\mathcal{PC} = \mathcal{PC}([-r,0), \mathbb{R}^n)$ equipped with the standard uniform norm $\|\varphi\|_r = \sup_{\theta\in[-r,0)} \|\varphi(\theta)\|$.

It is worth of noting that although more general functional spaces than $\mathcal{PC}$ can be considered as, for instance, $\mathcal{L}^p$ spaces Carvalho (1996); Gil' & Cheng (2007); Karafyllis & Krstic (2014), the $\mathcal{PC}$ space suffices the technical proofs of our results and, at the same time, it is sufficiently general for the stability investigation of the kind of problems mentioned in the introduction.

**Lemma 1** *If $s = 0$ does not belong to the spectrum of (1) then for any given initial function $\varphi \in \mathcal{PC}$ the corresponding solution $x(t, \varphi)$ of (4) can be represented as*

$$\begin{aligned} x(t,\varphi) = &\int_{-\tau}^{0}\left(\int_{-\tau}^{\xi}[d_\theta K(t+\theta-\xi)]\,G(\theta)\right)\varphi(\xi)d\xi \\ &- \sum_{j=1}^{m}\int_{-h_j}^{0}[d_\xi K(t-h_j-\xi)]\,A_j\varphi(\xi) \\ &- \int_{0}^{t+0}[d_\xi K(t-\xi)]\,w(\xi),\ t \geq 0, \end{aligned} \tag{5}$$

*where the integrals are Riemann-Stieltjes ones.*

**Proof.** By computing the Laplace image on both sides of (4) one gets

$$\begin{aligned} \hat{x}(s) = &H(s)\sum_{j=1}^{m} A_j \int_{-h_j}^{0}\varphi(\xi)e^{-s(\xi+h_j)}d\xi + H(s)\hat{w}(s) \\ &+H(s)\int_{-\tau}^{0}\left(\int_{-\tau}^{\xi}G(\theta)e^{s\theta}d\theta\right)\varphi(\xi)e^{-s\xi}d\xi, \end{aligned} \tag{6}$$

where $\hat{x}(s)$ and $\hat{w}(s)$ respectively denote the Laplace image of $x(t)$ and $w(t)$, and

$$H(s) = \left(I - \sum_{j=1}^{m} A_j e^{-h_j s} - \int_{-\tau}^{0} G(\theta)e^{s\theta}d\theta\right)^{-1}.$$

From (3) one gets that the Laplace image $\hat{K}(s)$, of the matrix $K(t)$, satisfies

$$H(s) = s\hat{K}(s) + K_0.$$

Taking into account this equation and observing that the Laplace image of $\frac{d}{dt}K(t)$ is equal to $s\hat{K}(s) - K(0) = s\hat{K}(s) - (I - K_0)$, an application of the convolution theorem on the right-hand side of (6) yields

$$x(t,\varphi)=\sum_{j=1}^{m}\int_{-h_j}^{0}\left[\frac{\partial K(t-h_j-\xi)}{\partial t}\right]A_j\varphi(\xi)$$
$$+\sum_{j=1}^{m}A_j\varphi(t-h_j)q(t-h_j)$$
$$+\int_{-\tau}^{0}\left(\int_{-\tau}^{\xi}\left[\frac{\partial K(t+\theta-\xi)}{\partial t}\right]G(\theta)\right)\varphi(\xi)d\xi$$
$$+\int_{t-\tau}^{t}G(\xi-t)\varphi(\xi)q(\xi)d\xi$$
$$+\left[\int_{0}^{t}\frac{\partial K(t-\xi)}{\partial t}\right]w(\xi)d\xi+w(t), \tag{7}$$

where $q(\xi)=1$ for $\xi<0$ and $q(\xi)=0$ for $\xi\geq 0$. Consider the term $A_j\varphi(t-h_j)q(t-h_j)$. If $t\geq h_j$ this term is not present. If $t<h_j$ this term is $A_j\varphi(t-h_j)$, which is precisely the Riemann-Stieltjes integral

$$-\int_{0}^{t-h_j}\left[d_\xi K(t-h_j-\xi)\right]A_j\varphi(\xi).$$

Similarly, if $t\geq\tau$ then the integral term $\int_{t-\tau}^{t}G(\xi-t)\varphi(\xi)q(\xi)d\xi$ is not present while if $t<\tau$ then the term is $\int_{t-\tau}^{0}G(\xi-t)\varphi(\xi)d\theta$, which is precisely the Riemann-Stieltjes integral

$$\int_{t-\tau}^{0}\left(\int_{-\tau}^{\xi-t}\left[d_\theta K(t+\theta-\xi)\right]G(\theta)\right)\varphi(\xi)d\xi.$$

Also, $w(t)$ is the value of the Riemann-Stieltjes integral

$$-\int_{t}^{t+0}d_\xi K(t-\xi)w(\xi).$$

Therefore, if we make use of the Riemann-Stieltjes integral, the expression for $x(t,\varphi)$ given by (7) can be written as (5) and, thus, the Lemma is proved. ■

**Remark 1** *The formula (5) comprises some existing ones for describing piecewise continuous solutions of FDEs of the form (1). For instance, in the particular case of (1) when $m=1, A=A_1, h=h_1$, and $G(\theta)\equiv 0, \tau\in[-\tau,0]$, it is well-known that the variation-of-constants formula for piecewise continuous solutions is given by*

$$x(t,\varphi)=A^{j+1}\varphi(t-(j+1)h)+\sum_{i=0}^{j}A^i w(t-ih),$$

*for $t\in[jh,(j+1)h)$. This expression is directly obtained from (5) by taking into account that, in this particular case, $K(t)$ is piecewise constant and it has jump discontinuities at $jh$ given by $\Delta K(jh)=A^j, j=0,1,2,\ldots$ Also, the formula for the solutions of homogeneous integral delay equations, i.e. when $A_j=0$, derived in Melchor-Aguilar, Kharitonov & Lozano (2010) can be directly obtained from (5).*

**Remark 2** *Suppose that (1) is exponentially stable, i.e., there exist $\mu\geq 1$ and $\alpha>0$ such that*

$$\left\|x_t(\varphi)\right\|_r\leq\mu e^{-\alpha t}\left\|\varphi\right\|_r, \quad t\geq 0.$$

*Here $x_t(\varphi)\in\mathcal{PC}$ is defined by $x_t(\varphi)(\theta)\triangleq x(t+\theta,\varphi), \theta\in[-r,0)$. The exponential stability implies that $s=0$ does not belong to the spectrum of (1) and, therefore, the matrix $K_0$ is well-defined. From the matrix equation (3) the following inequalities:*

$$\left\|K(t)\right\|\leq\mu e^{-\alpha t}\left\|K_0\right\| \;\; \textit{and}\; Var_{[t-r,t]}K\leq\gamma e^{-\alpha t}$$

*hold for all $t\geq 0$, where $\gamma=\max\left\{1,2\mu\left\|K_0\right\|\right\}e^{\alpha r}$.*

*Conversely, suppose that the matrix $K(t)$ satisfies the latter inequalities. Then the formula (5) (with $w(t)=0$) implies that $\left\|x_t(\varphi)\right\|_r\leq\kappa e^{-\alpha t}\left\|\varphi\right\|_r, \forall t\geq 0$, where $\kappa=\max\left\{1,\gamma\left(A_m+G_m\right)\right\}e^{\alpha r}$, with*

$$G_m=\int_{-\tau}^{0}\left(\int_{-\tau}^{\xi}\left\|G(\theta)\right\|d\theta\right)d\xi \;\; \textit{and}\;\; A_m=\sum_{j=1}^{m}h_j\left\|A_j\right\|. \tag{8}$$

The above remark shows that while $X(t)$ cannot admit an exponentially decreasing upper bound, $K(t)$ may do it. Thus, the matrix $K(t)$ is an appropriate fundamental matrix for (1) and the integral representation (5) for piecewise continuous solutions of (4) is the variation-of-constants formula.

We will apply the formula (5) for deriving exponential stability results for the following class of perturbed FDEs:

$$y(t)=\sum_{j=1}^{m}A_j y(t-h_j)+\int_{-\tau}^{0}G(\theta)y(t+\theta)d\theta$$
$$+H(t,y_t)+F(t,y_t), \tag{9}$$
$$y_{t_0}=\varphi, \tag{10}$$

where $H,F:\mathbb{R}\times\mathcal{PC}\to\mathbb{R}^n$ are respectively linear and nonlinear functionals satisfying the following assumptions:

H1) There exists a $h:[0,\infty)\to\mathbb{R}_+$ and constants $\lambda$ and $\nu$ such that

$$\left\|H(t,\phi)\right\|\leq h(t)\left\|\phi\right\|_r,$$
$$\int_{t_0}^{t}h(s)ds\leq\lambda(t-t_0)+\nu, \quad \forall t\geq t_0\geq 0. \tag{11}$$

H2) For every $R>0$ and $\delta\in(0,r)$ there exist constants $L_1=L_1(R), L_2=L_2(R)$ and $M=M(R)$ such that for all $t\geq 0$ and $\phi,\psi\in\mathcal{PC}$ with $\|\phi\|\leq R, \|\psi\|\leq R$, the following inequalities hold:

$$
\begin{aligned}
\|F(t,\phi)-F(t,\psi)\| \leq \delta L_1 &\sup_{\theta\in[-\delta,0)}\|\phi(\theta)-\psi(\theta)\| \\
&+L_2\sup_{\theta\in[-r,-\delta)}\|\phi(\theta)-\psi(\theta)\|, \\
\|F(t,\phi)\| \leq M.&
\end{aligned}
$$

The condition (11) in assumption H1 is similar to the one used in ordinary differential equations for considering constant as well as time-varying perturbations having a bounded integral, see for instance Hale (1980). The assumption H2 is a particular case of the more general conditions for existence and uniquessness of solutions of nonlinear FDEs given in Karafyllis & Krstic (2014).

Additionally, let us assume that the functional $F(t,\phi)$ satisfies

$$
\lim_{\|\phi\|_r\to 0}\frac{\|F(t,\phi)\|}{\|\phi\|_r}=0 \tag{12}
$$

uniformly in $t\geq 0$. The condition (12) assures that the FDE (9) has the trivial solution.

**Definition 1** *The trivial solution of the FDE (9) is said to be exponentially stable if there exit positive constants $\Delta,\mu$ and $\alpha$ such that for any $t_0\geq 0$ and $\varphi\in\mathcal{PC}$, with $\|\varphi\|_r<\Delta$, the following inequality holds:*

$$
\|y_t(t_0,\varphi)\|_r\leq \mu e^{-\alpha(t-t_0)}\|\varphi\|_r, \quad \forall t\geq t_0.
$$

**Theorem 2** *Suppose that the FDE (1) is exponentially stable and the functionals $H(t,\phi)$ and $F(t,\phi)$ respectively satisfy the assumptions H1 and H2. Then there is a $\eta>0$ such that the trivial solution of the perturbed FDE (9) is exponentially stable if $\lambda<\eta$.*

**Proof.** Given $t_0\geq 0$ and an initial function $\varphi\in\mathcal{PC}$, there exists a $\delta\in(0,r)$ such that the initial value problem (9)-(10) admits a unique solution $y(t,t_0,\varphi)$ defined on $[t_0-r,t_0+\delta)$ satisfying $y(t_0+\theta,t_0,\varphi)=\varphi(\theta),\theta\in[-r,0)$, see Theorem 2.1 in Karafyllis & Krstic (2014). The Lemma 1 implies that the solution $y(t,t_0,\varphi)$ can be written as

$$
\begin{aligned}
y(t,t_0,\varphi)=y_0(t,t_0,\varphi)&\\
-\int_{t_0}^{t+0}&[d_\xi K(t-\xi)]\,H(\xi,y_\xi(t_0,\varphi))\\
-\int_{t_0}^{t+0}&[d_\xi K(t-\xi)]\,F(\xi,y_\xi(t_0,\varphi)),
\end{aligned}
$$

where

$$
\begin{aligned}
y_0(t,t_0,\varphi)&=-\sum_{j=1}^{m}\int_{-h_j}^{0}[d_\xi K(t-t_0-h_j-\xi)]\,A_j\varphi(\xi)\\
&+\int_{-\tau}^{0}\left(\int_{-\tau}^{\xi}[d_\theta K(t-t_0+\theta-\xi)]\,G(\theta)\right)\varphi(\xi)d\xi.
\end{aligned}
$$

The above expressions are directly derived from (5) by considering as initial instant $t_0\geq 0$ instead of $t_0=0$.

This solution satisfies $\|y_t(t_0,\varphi)\|_r\leq\rho$, for all $t\in[t_0,t_0+\delta)$, where $\rho=\rho(M(\|\varphi\|_r),A_m,G_m)$ with $A_m,G_m$ given by (8), and it can be arbitrary continued for $t\geq t_0+\delta$ while $\|y_t(t_0,\varphi)\|_r\leq\rho$ holds, see Karafyllis & Krstic (2014).

From the exponential stability of the FDE (1) and Remark 2 follow that there are positive constants $\alpha,\mu$ and $\gamma$ such that $\|y_0(t,t_0,\varphi)\|\leq\mu e^{-\alpha(t-t_0)}\|\varphi\|_r$ and $Var_{[t-r,t]}K\leq\gamma e^{-\alpha(t-t_0)},\forall t\geq t_0$. Consequently, for all $t\geq t_0$,

$$
\begin{aligned}
&\|y(t,t_0,\varphi)\|\leq\mu e^{-\alpha(t-t_0)}\|\varphi\|_r\\
&+\gamma e^{-\alpha(t-t_0)}\int_{t_0}^{t}e^{\alpha(\xi-t_0)}h(\xi)\|y_\xi(t_0,\varphi)\|_r\,d\xi\\
&+\gamma e^{-\alpha(t-t_0)}\int_{t_0}^{t}e^{\alpha(\xi-t_0)}\|F(\xi,y_\xi(t_0,\varphi))\|\,d\xi.
\end{aligned}
$$

Observing that $e^{-\alpha(t-t_0+\theta)}\leq e^{-\alpha(t-t_0-r)}$ and $\int_{t_0}^{t+\theta}(\cdot)\leq\int_{t_0}^{t}(\cdot)$ hold for $\theta\in[-r,0]$, the following inequality holds for $t+\theta\geq t_0$:

$$
\begin{aligned}
&\|y(t+\theta,t_0,\varphi)\|\leq\mu e^{-\alpha(t-t_0-r)}\|\varphi\|_r\\
&+\gamma e^{-\alpha(t-t_0-r)}\int_{t_0}^{t}e^{\alpha(\xi-t_0)}h(\xi)\|y_\xi(t_0,\varphi)\|_r\,d\xi\\
&+\gamma e^{-\alpha(t-t_0-r)}\int_{t_0}^{t}e^{\alpha(\xi-t_0)}\|F(\xi,y_\xi(t_0,\varphi))\|\,d\xi,
\end{aligned} \tag{13}
$$

When $t+\theta<t_0$ we have $\|y(t+\theta,t_0,\varphi)\|\leq\|\varphi\|_r\leq\mu e^{-\alpha(t-t_0-r)}\|\varphi\|_r$. It follows that the inequality (13) holds for all $t\geq t_0$ and $\theta\in[-r,0]$.

From the condition (12) we have that for any given $\varepsilon>0$ there exists $\beta=\beta(\varepsilon)>0$ such that

$$
\|F(t,\phi)\|<\varepsilon\|\phi\|_r \text{ when } \|\phi\|_r<\beta.
$$

Thus, if $\|y_t(t_0,\varphi)\|_r<\beta$ for all $t\geq t_0$ then from (13) we obtain

$$
\begin{aligned}
\|y_t(t_0,\varphi)\|_r\,e^{\alpha(t-t_0)}\leq m\|\varphi\|_r&\\
+\int_{t_0}^{t}&e^{\alpha(\xi-t_0)}n(\xi)\|y_\xi(t_0,\varphi)\|_r\,d\xi,
\end{aligned}
$$

where

$$m = \mu e^{\alpha r} \text{ and } n(\xi) = \gamma e^{\alpha r} (\varepsilon + h(\xi)).$$

A direct application of the Gronwall-Bellman inequality and condition (11) leads to

$$\|y_t(t_0, \varphi)\|_r \leq m\kappa \|\varphi\|_r e^{-(\alpha - \gamma e^{\alpha r}(\varepsilon+\lambda))(t-t_0)}, \forall t \geq t_0,$$

where $\kappa = e^{\gamma e^{\alpha r \nu}}$. Choosing $\varepsilon < \frac{\alpha e^{-\alpha r}}{\gamma}$ and defining $\eta = \frac{\alpha e^{-\alpha r}}{\gamma} - \varepsilon$ follows that if $\lambda < \eta$ then $\|y_t(t_0, \varphi)\|_r$ goes to zero exponentially and

$$\|y_t(t_0, \varphi)\|_r \leq m\kappa \|\varphi\|_r, \forall t \geq t_0.$$

Hence, for an initial function $\varphi \in \mathcal{PC}$ satisfying $\|\varphi\|_r < \Delta = \frac{1}{m\kappa} \min\{\rho, \beta\}$ we have that $\|y_t(t_0, \varphi)\|_r < \beta$ and $\|y_t(t_0, \varphi)\|_r \leq \rho, \forall t \geq t_0$, which in turn implies the solution exists for all $t \geq t_0$ and, thus, the theorem is proved. ■

**Remark 3** *In spite of the fact that the proof of the above Theorem is relatively simple, to the best of our knowledge, the stated results have not been reported in the literature. Indeed, in the case when $H(t, \phi) \equiv 0$, the Theorem 2 generalizes the stability results on first linear approximation given in Gil' & Cheng (2007) (Theorem 4.2) and Pepe (2014) (Theorem 22), where the case of only discrete delays is considered, to the case of FDEs including integral delay terms.*

**Remark 4** *In order to assure the exponential stability of the FDE (1) our previous results reported in Melchor-Aguilar (2012, 2013a,b) can be used. In particular, the given estimations of $\mu \geq 1$ and $\alpha > 0$, characterizing the exponential upper bounds for the solutions of (1), can serve for determining $\eta$ and/or $\varepsilon$ involved in the proof of Theorem 2.*

**Example 1** *A special case of some interest is the following perturbed linear FDE:*

$$y(t) = \sum_{j=1}^{m} [A_j + B_j(t)] \, y(t - h_j) + \int_{-\tau}^{0} [G(\theta) + M(t, \theta)] \, y(t + \theta) d\theta, \tag{14}$$

*Using Theorem 2 one gets that (14) is exponentially stable if the following conditions are satisfied:*

*(1) The FDE (1) is exponentially stable,*

*(2) There are constants $b_j, c_j, f$ and $g$ such that*

$$\int_{t_0}^{t} \|B_j(t)\| \, dt \leq b_j(t - t_0) + c_j,$$

$$\int_{t_0}^{t} \left( \int_{-\tau}^{0} \|M(t, \theta)\| \, d\theta \right) dt \leq f(t - t_0) + g,$$

*hold for $t \geq t_0$, and*

*(3) $\lambda = \sum_{j=1}^{m} b_j + f < \eta = \frac{\alpha e^{-\alpha r}}{\gamma}$, $\gamma = \max\{1, 2\mu \|K_0\|\} e^{\alpha r}$, where $\mu \geq 1$ and $\alpha > 0$ characterizes the decreasing exponential upper bound for the solutions of (1).*

*This is a Dini-Hukuhara type result for the perturbed FDE (14) similar to the one for NFDEs in Bellman & Cooke (1963). However, note that no differentiability of the perturbation functions $B_j(t)$ is needed as in the case of NFDEs.*

*In order to illustrate this result let us consider the FDE*

$$y(t) = \int_{-r}^{0} \left( d(t)q(\theta) + e^{-a(t+\theta)} \right) y(t + \theta) d\theta,$$

*where $a > 0, d(t) \in [-1, 1]$ and $q : [-r, 0] \to \mathbb{R}$ a continuous function. This equation, but without the additive exponential perturbation term, was considered in example 2.7 of Karafyllis & Krstic (2014).*

*The equation is a particular case of (14) that can be regarded as a perturbed one of the trivial exponential stable equation $x(t) \equiv 0$. Observing that*

$$\int_{t_0}^{t} \left( \int_{-\tau}^{0} \left| d(t)q(\theta) + e^{-a(t+\theta)} \right| d\theta \right) dt \leq d(t - t_0) + e,$$

*where*

$$d = \int_{-r}^{0} |q(\theta)| \, d\theta \text{ and } e = \frac{1}{a},$$

*it follows that the equation is exponentially stable if $\lambda < \eta = \frac{\alpha e^{-\alpha r}}{\gamma}$, where $\gamma = \max\{1, 2\mu\} e^{\alpha r}$.*

*Since for the trivial system $x(t) \equiv 0$ the constants $\mu \geq 1$ and $\alpha > 0$ can be arbitrary chosen then by selecting $\mu = 1$ and $\alpha = \frac{1}{2r}$ the constant $\eta$ is maximized.*

*Thus, we conclude that the equation is exponentially stable if $\lambda < \frac{e^{-1}}{4r}$, a delay-dependent condition that complement the delay-independent one, $\lambda \in (0, 1)$, obtained in Karafyllis & Krstic (2014).*

## 3 Concluding remarks

In this note we addressed the stability problem of perturbed linear FDEs with discrete and distributed delays for which the existing results for perturbed linear

NFDEs cannot be directly applied. After defining an appropriate fundamental matrix for the linear equations, the variation-of-constants formula allowing us to express the piecewise continuous solutions of nonhomogeneous equations in a special integral form is obtained. By using such formula, some exponential stability results for linear and nonlinear perturbed equations are derived. In particular, a stability result on first linear approximation for FDEs with discrete and distributed delays is given.


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