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Vertical Transversality and its Applications to Control of Mechanical Systems

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José Miguel Sosa Zúñiga

Para obtener el grado de

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En la opción de

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Director de la Tesis:

Dr. David Antonio Lizárraga Navarro

San Luis Potosí, S.L.P., Julio de 2005.

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"Vertical Transversality and its Applications to Control of Mechanical Systems"

José Miguel Sosa Zúñiga

Sinodales:

Dr. David Antonio Lizárraga Navarro (Director de Tesis)

Dr. Hugo Cabrera Ibarra

Dr. Fausto Ongay Larios (Sinodal Externo)

San Luis Potosí, S.L.P., México, Julio de 2005

Abstract

The transverse control approach proposed by Morin and Samson is a technique based on the use of transverse functions to practically stabilize controllable driftless systems. This control technique is able to cope with practical stabilization of admissible trajectories, including fixed points, as well as practical stabilization of non-admissible trajectories.

In this thesis we attempt to generalize this technique to the control of second-order systems and, in particular, to the case of mechanical systems described on Lie groups. Within this class one finds mechanical systems subjects to (holonomic and non-holonomic) constraints as well as underactuated mechanical systems. It is important to note that for systems in this class, the drift vector field is required along with the control vector fields to generate the accessibility distribution.

We define *vertical transversality* and we show how transverse functions satisfy vertical transversality, a property that generalizes transversality to second-order systems. By applying the methodology introduced in this thesis to second-order systems one achieves practical stabilization of the configuration variables, namely one ensures that the projection of the state trajectories onto the configuration manifold converge to an arbitrarily small neighborhood, specified in advance, of the desired equilibrium point.

Although the approach outlined in this thesis does not constitute a complete extension of Morin and Samson's approach based on transverse functions, it takes initial steps toward what might constitute an interesting theory for the stabilization of admissible trajectories for second-order systems.

Resumen

La aplicación de la técnica de control por medio de funciones transversas propuesta por Morin y Samson a sistemas controlables sin deriva da como resultado una estabilización práctica de las trayectorias del sistema. Esta técnica trata con estabilización práctica de puntos fijos, trayectorias admisibles e inclusive trayectorias no admisibles.

En esta tesis se generaliza la noción de transversalidad para sistemas de segundo orden y se plantea el desarrollo de un método de control para estabilizar sistemas de segundo orden, en particular para sistemas mecánicos que evolucionan en grupos de Lie. Dentro de esta clase de sistemas se encuentran sistemas mecánicos sujetos a restricciones (holonómicas y no holonómicas) como también sistemas mecánicos subactuados. Es importante notar que en esta clase de sistemas el término de deriva se requiere, junto con los campos vectoriales de control, para generar la distribución de accesibilidad.

En esta disertación se define *transversalidad vertical* y se muestra cómo las funciones transversas definen funciones verticalmente transversas. Se presenta además un esquema de control para estabilizar sistemas de segundo orden en grupos de Lie usando funciones verticalmente transversas. Este método asegura estabilización práctica de las variables de configuración del sistema, es decir, la proyección de las trayectorias del sistema a la variedad de configuración converge a una vecindad arbitrariamente pequeña del punto deseado de equilibrio.

El esquema expuesto, aún cuando no resuelve por completo el problema de estabilización práctica para sistemas de segundo orden, se presenta como el punto de partida de un esquema que podría llegar a constituir una teoría interesante para la estabilización de trayectorias admisibles para sistemas de segundo orden.

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Contents

Chapter 1

Introduction

In control theory, the stabilization of critical systems to fixed points or, more generally, to admissible trajectories, is still an open research problem. A critical system [1] is a controllable system whose linearization is non-controllable.
Several systems are critical, examples of these systems can be found when one is dealing with nonholonomic systems and underactuated mechanical systems.

Systems which do not satisfy Brockett's condition typically are critical. This condition can be formulated as follows. Consider a control system $\dot{x} = f(x, u)$ where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that $f(0, 0) = 0$. Brockett [2] states that a necessary condition to render the origin asymptotically stable for the given system by means of a continuous feedback control function $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, is that the map f be open at zero. There are generalizations of Brockett's condition that involve asymptotic stabilization of equilibria for systems evolving on more general spaces than on Euclidean spaces [3].

In view of this, difficulties arise while attempting to produce a unified method to solve various control problems such as asymptotic stabilization of fixed points and trajectory tracking for critical systems.

Different control approaches have been proposed to stabilize this class of systems. For instance, as no continuous feedback control function exists in order to stabilize critical systems, some approaches make use of discontinuous feedback functions for the same purpose. However, Ryan [18] has shown, under certain assumptions, that if a control system is stabilizable to a given trajectory using a discontinuous feedback law, then there also exists a continuous feedback law which stabilizes the system to the given trajectory.

Samson in [19] showed that it is possible to surmount the restriction embodied in Brockett's condition by using time-varying feedback to asymptotically stabilize the unicycle-type robot to a fixed configuration. By a time-varying feedback one means a control function that depends not only on the system state but also explicitly on time. A more general result in this direction is presented in [4].

By using the homogeneous approach [8], [17], based on time-varying feedback to control critical systems, one is able to obtain time-varying feedback laws which force the trajectories of the system to converge exponentially to the equilibrium point. Nevertheless, as expounded in [11], the feedback control laws resulting from this control strategy typically are non-differentiable at the desired equilibrium point, thus raising difficulties regarding the robustness of the feedback laws in the presence of modeling errors.

In recent work, Morin and Samson [16], [14] have developed a new framework, the so called *transverse function approach*, which allows one to tackle both, the point stabilization and trajectory tracking problems for controllable driftless systems. This control approach is applicable to systems of the form $\dot{x} = D(x,t) + \sum_{i=1}^{m} u^i X_i(x)$ where x is a curve on M, a finite-dimensional manifold, $D : M \times \mathbb{R} \longrightarrow TM$ is a "time-varying" vector field which depends continuously on its second argument (D may be seen as a perturbing term), and $\{X_1, \ldots, X_m\}$ is a set of vector fields defined on M that satisfy local accessibility at the desired point to be stabilized.

By using this control approach, the trajectories of the resulting feedback system converge to a pre-specified, arbitrarily small neighborhood of the reference trajectory (or fixed point.) This sort of convergence is called practical stabilization.

An advantageous property of this approach is that the resulting feedback laws are smooth with respect to the state, and this allows one to deduce certain properties of robustness for the resulting feedback system.

The aim of this thesis is to provide initial steps towards a generalization of the transverse function control approach for the stabilization of second-order systems. Being more precise, the ultimate purpose would be to stabilize every admissible trajectory for systems of the form:

$$\dot{x} = D(x) + \sum_{i=1}^{m} u^i X_i^{\text{lift}}(x) \tag{1.1}$$

where x now represents a curve on TM, the tangent manifold of a finite dimensional manifold M, D is a second-order vector field on TM, and $\{X_1, \ldots, X_m\}$ ($m \leq \dim(M)$) is a set of vector fields defined on M that satisfies local accessibility at a given point in M. Equation (1.1) defines a second-order system on M and it may represent a given critical system.

We assume that the set $\{D, X_1^{\text{lift}}, \ldots, X_m^{\text{lift}}\}$ generates the accessibility distribution for every $t \in \mathbb{R}$, while $\{X_1^{\text{lift}}, \ldots, X_m^{\text{lift}}\}$ only spans a proper subdistribution thereof, thus the transverse function control approach cannot be directly applied to System (1.1).

In this thesis we present the characterization of a new property which we refer to as "vertical transversality", which somewhat generalizes the transversality property to the case of second-order systems. We attempt to give a methodology to control second-order systems of the form (1.1) evolving on Lie groups. We focus on mechanical systems since several mechanical systems are naturally modeled as systems on Lie groups. For instance, mechanical systems which usually arise in physical applications are rigid bodies in space, cart-like vehicles, space and underwater robots, which evolve on Lie groups.

It is important to note that this work serves as a starting point towards a generalization of the transverse function control approach to control second-order systems and that research work remains to be done in this respect.

This thesis is organized as follows; in Chapter 2 we fix the notation used throughout this work, we recall some preliminaries on vector bundles, differential geometry, some background in Lagrangian mechanics and basics on Lie theory. In Chapter 3 we review the "transverse function control approach" proposed by Morin and Samson [14] and illustrate, in detail, its application to the control of the chained form system. Chapter 4 presents the main work of this thesis, some necessary lemmas are stated and proved, as well as the *vertical transversality* condition and we expound a possible application thereof to the stabilization of second-order systems. In Chapter 5 some examples are developed in detail. Finally Chapter 6 presents conclusions of this work along with brief descriptions of some of the problems that remain open for research.

Chapter 2

Mathematical preliminaries

The purpose of this chapter is to recall basic mathematical concepts required in the following chapters. We will first fix the notation used throughout the present thesis, then review basic background on vector bundles, mechanical systems and on Lie theory. The interested reader may find additional details in [9], [10], [12], [20].

2.1. Definitions and conventions

We assume the reader is familiar with basic notions of point-set topology and differential geometry.

Let $I \subseteq \mathbb{R}$ denote a nonempty interval in $\mathbb{R}$ which can be finite or infinite, closed or open at either of its endpoints. If A is a set then $\mathrm{id}_A : A \longrightarrow A$ denotes the identity map on A. Let $a \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$. A mapping $f : A \longrightarrow B$ where A and B are manifolds, is said to be of class C^a (or simply f is C^a) if it is a times continuously differentiable. If $f : A \longrightarrow B$ is of class C^∞ one says that f is smooth. $C^a(A; B)$ denotes the space of mappings of class C^a of A into B while $C^a(A)$ stands for $C^a(A; \mathbb{R})$. $\mathbb{T}^n = \mathbb{T} \times \cdots \times \mathbb{T}$ (n copies of $\mathbb{T}$), denotes the n-torus, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

The Kronecker's delta is denoted by δ^i_j which equals 1 if $i = j$ while equals zero if $i \neq j$. Sometimes we shall use Einstein summation convention for the sake of readability, that is, repeated doubled indices indicate a summation. In particular let $x \in \mathbb{R}^n$ and let $\{e_1, \ldots, e_n\}$ be the canonical basis for $\mathbb{R}^n$ (namely, e_i is the vector which components are all zero except the i-th component which is 1), then $x = x^i e_i$

for $i = 1, \ldots, n$ equals $x = \sum_{i=1}^{n} x^i e_i$.

By a manifold we refer to a finite-dimensional, paracompact, differentiable manifold. Let M be a manifold, T_pM denotes the tangent space of M at $p \in M$ while TM, the tangent bundle of M, is the disjoint union of each T_pM for $p \in M$, ($TM = \coprod_{p \in M} T_pM$) endowed with the differentiable structure inherited from the differentiable structure in M. Let $p \in M$, the bundle projection of TM, denoted by $\pi_M : TM \longrightarrow M$, maps v in T_pM to $\pi_M(v) = p$.
Likewise, T_p^*M denotes the cotangent space of M at $p \in M$, T^*M is the cotangent bundle of M consisting of the disjoint union of the T_p^*M for all $p \in M$, ($T^*M = \coprod_{p \in M} T_p^*M$) equipped with the differentiable structure inherited from the differentiable structure in M. $\pi_M^* : T^*M \longrightarrow M$ stands for the bundle projection of T^*M which, for any $p \in M$ and every $\varrho \in T_p^*M$ satisfies $\pi_M^*(\varrho) = p$.

Given a mapping $f \in C^1(M; N)$ where M and N are manifolds, we write $T_pf : T_pM \longrightarrow T_{f(p)}N$ for the (linear) tangent map of f at p. Tf denotes the bundle map covering f, that is, Tf maps the tangent space of M at any $p \in M$ into the tangent space of N at $f(p)$. One says that Tf covers f if the diagram below commutes,

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

i.e. $\pi_N \circ Tf = f \circ \pi_M$. Whenever the base point p is clear from the context we will simply write Tf for T_pf.

$\Gamma^a(TM)$ stands for the space of sections of class C^a of the tangent bundle of M. An element X in $\Gamma^a(TM)$ is a mapping $M \longrightarrow TM$ such that $\pi_M \circ X = \mathrm{id}_M$, X is said to be a C^a vector field (defined) on M. On the other hand, $\Gamma^a(T^*M)$ is the space of C^a sections of the cotangent bundle of M, $\Upsilon \in \Gamma^a(T^*M)$ is a mapping $M \longrightarrow T^*M$ satisfying $\pi_M^* \circ \Upsilon = \mathrm{id}_M$. Υ is said to be a C^a 1-form on M. The commutative diagrams below will help to make a clearer idea.

$$\begin{tikzcd} M \arrow[rr, "X"] \arrow[dr, "\mathrm{id_M}"'] & & TM \arrow[dl, "\pi_M"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, "\Upsilon"] \arrow[dr, "\mathrm{id_M}"'] & & T^*M \arrow[dl, "\pi_M^*"] \\ & M & \end{tikzcd}$$

In the sequel we shall usually write X_p for $X(p)$ and Υ_p for $\Upsilon(p)$ in order to not overburden some expressions. We shall also assume that manifolds, vector fields and functions are of class C^∞ unless otherwise stated, (i.e. M, N manifolds, $X \in \Gamma(TM)$ and $f \in C(M, N)$ are smooth).

One endows $\Gamma(TM)$ with a $C^\infty(M)$-module structure by defining the sum of vector fields as $(X + Y)_p = X_p + Y_p$ and the product of functions in $C^\infty(M)$ and vector fields as $(fX)_p = f(p)X_p$ for all $X, Y \in \Gamma(TM)$, $f \in C^\infty(M)$ and $p \in M$. We can also provide $\Gamma(TM)$ with an $\mathbb{R}$-algebra structure by defining, in addition to the module structure, a mapping $[\cdot,\cdot] : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ such that $[X, Y](f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$. The operation $[\cdot,\cdot]$ defined this way is called the Lie bracket product and it can be shown to satisfy, for every X, Y, Z in $\Gamma(TM)$, the following properties:

1. $[\cdot,\cdot]$ is an $\mathbb{R}$-bilinear mapping,
2. $[X, Y] = -[Y, X]$, i.e. it is anti-commutative,
3. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$, i.e. satisfies Jacobi's identity.

An algebra with product operation $[\cdot,\cdot]$ satisfying the three latter properties is said to be a Lie algebra.

Suppose $\{X_1, \ldots, X_n\} \subset \Gamma(TM)$, then $\mathrm{Lie}(\{X_1, \ldots, X_n\})$ denotes the Lie algebra generated by $\{X_1, \ldots, X_n\}$, i.e. it denotes the intersection of every Lie subalgebra of $\Gamma(TM)$ which contains $\{X_1, \ldots, X_n\}$.

Let X be a vector field on a manifold M, we shall write $\exp(tX)(p)$ for the solution (whenever it exits), of the differential equation $\dot{x} = X(x)$ at time t and with initial condition $x(0) = p$. If for every $p \in M$, $\dot{x} = X(x)$ has a solution with initial value $x(0) = p$ defined for all $t \in \mathbb{R}$, then X is said to be complete.

Let M be a smooth manifold, we denote the Lie derivative of $f \in C^\infty(M)$, with respect to $X \in \Gamma(TM)$ by $L_X f$.

Let E denote a vector space and E^* its associated dual space, then $T_r^s(E) \triangleq (\bigotimes_{i=1}^r E^*) \otimes (\bigotimes_{i=1}^s E)$ is the tensor space of type (s, r) over E. If $t \in T_r^s(E)$, t is said to be a tensor of type (s, r) over E.

Let M be a smooth manifold, then $T_r^s(T_pM) = \left(\bigotimes_{i=1}^r T_p^*M\right) \otimes (\bigotimes_{i=1}^s T_pM)$ is the tensor space of type (s, r) over the tangent space of M at $p \in M$. The tensor space of type (s,r) over TM is denoted by $T_r^s(TM)$ and it is the disjoint union of all the $T_r^s(T_pM)$ for $p \in M$ ($T_r^s(TM) = \coprod_{p\in M} T_r^s(T_pM)$), endowed with the differential structure inherited from the differential structure on M. $T_r^s(TM)$ possesses, indeed, a vector bundle structure (cf. the following section). $\Gamma(T_r^s(TM))$ denotes the set smooth sections of the tensor space of type (s,r) over TM. If $T \in \Gamma(T_r^s(TM))$ then T is said to be a tensor field of type (s,r) over M. It is easy to check that a vector field on M is a tensor field of type $(1,0)$ while a 1-form is a tensor field of type $(0,1)$.

2.2. Vector bundle geometry

In this section we recall notions on vector bundles since these arise as state manifolds for systems considered in this work. For instance, in the case of simple mechanical systems the configuration manifold represents all possible positions or orientations of the system while the space of velocities plus positions and orientations can be seen as the tangent bundle of the configuration manifold. Let us recall the definition of a smooth vector bundle.

Definition 1 *A **vector bundle** is a 4-tuple (E, B, F, π_B) having the following properties:*

1. *E, B are smooth manifolds and F is an $\mathbb{R}$-vector space.*
2. *$\pi_B : E \longrightarrow B$ is a surjective and smooth map.*
3. *For every x in B, $E_x \triangleq \pi_B^{-1}(\{x\})$ is diffeomorphic to F.*
4. *For every open set $U \subset B$, there exist a diffeomorphism $\psi : \pi_B^{-1}(U) \longrightarrow U \times F$ which maps E_x linearly to $\{x\} \times F$, so that the following diagram commutes*

$$\begin{tikzcd} \pi_B^{-1}(U) \subset E \arrow[r, "\psi"] \arrow[d, "\pi_B"'] & U \times F \arrow[dl, "p_1"] \\ U \subset B & \end{tikzcd}$$

where p_1 is the canonical projection onto the first factor.

Given a vector bundle (E, B, F, π_B) one calls E the ***total space***, B the ***base space*** (or simply the base), F the ***standard fiber***, π_B the ***bundle projection*** (or the projection), E_x, for $x \in B$, the ***fiber over*** x, and $(\pi_B^{-1}(U), \psi)$ a ***local trivialization***. Sometimes we write $\pi_B : E \longrightarrow B$ instead of (E, B, F, π_B) to denote a vector bundle.

A ***section*** of a vector bundle $\pi_B : E \longrightarrow B$ is a map $S : B \longrightarrow E$ such that $\pi_B \circ S = \mathrm{id}_B$. The ***zero-section*** of a vector bundle is the section that maps every $x \in B$ to the zero vector in the fiber over x. A vector bundle $\pi_B : E \longrightarrow B$ is said to be a ***trivial vector bundle*** if there exists a local trivialization (E, ψ), i.e. if there exists a diffemorphism $\psi : E \longrightarrow B \times F$ such that the following diagram commutes

$$\begin{tikzcd}
{\pi_B}^{-1}(B) = E \arrow[r, "\psi"] \arrow[d, "\pi_B"'] & B \times F \arrow[dl, "p_1"] \\
B &
\end{tikzcd}$$

Let M be an n-dimensional manifold then TM, the tangent bundle of M, admits the vector bundle structure $(TM, M, \mathbb{R}^n, \pi_M)$ where for every $p \in M$ the fiber over p is T_pM.

Let M and N be differential manifolds and f a mapping in $C^1(M; N)$. Let X and Y be vector fields defined on M and N respectively. One says that X is ***f-related*** to Y iff $Tf \circ X = Y \circ f$, i.e. iff the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "X"'] & N \arrow[d, "Y"] \\
TM \arrow[r, "Tf"] & TN
\end{tikzcd}$$

Let $\widehat{A}$ and $\widehat{B}$ be vector fields defined on M and A, B vector fields defined on N such that $\widehat{A}$ is f-related to A and $\widehat{B}$ is f-related to B, then one easily checks that $[\widehat{A}, \widehat{B}]$ is f-related to $[A, B]$.

If $(x^1, \ldots, x^n)$ are local coordinates for M, an n-dimensional manifold, then the natural coordinates for TM are denoted $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ (where $v^i = v(x^i)$, $i = 1, \ldots, n$). We shall often write x for the local coordinates on M and (x, v) for the natural coordinates on TM associated with x. Using this notation, $((x, v), (w_{\mathrm{L}}, w_{\mathrm{H}}))$ are local coordinates on TTM, where the first pair (x, v) represents coordinates for

TM and the pair $(w_{\mathrm{L}}, w_{\mathrm{H}})$ denotes coordinates for the fiber above (x, v).

Let $\rho : I \longrightarrow M$ be a curve on a differentiable manifold M then we set $\dot{\rho} = T\rho \circ \frac{\partial}{\partial r}$. One easily checks that $\dot{\rho}$, so defined, is a curve $\dot{\rho} : I \longrightarrow TM$ satisfying $\pi_M(\dot{\rho}(t)) = \rho(t)$ for every $t \in I$.

A ***vector field*** X ***along a curve*** $\sigma : I \longrightarrow M$ is a map $X : \sigma(I) \longrightarrow TM$ such that $X_{\sigma(t)} \in T_{\sigma(t)}M$ for every $t \in I$.

Given v in TM, the ***vertical space over*** v, denoted by $(T_vTM)^{\mathrm{vert}}$, is defined as the kernel of the tangent map associated to the projection in M, i.e. $(T_vTM)^{\mathrm{vert}} = \mathrm{kernel}(T_v\pi_M) \subset T_vTM$. Having defined the vertical space over a vector in TM, we define a subbundle of the tangent tangent bundle called the ***vertical subbundle*** given by $TTM^{\mathrm{vert}} = \coprod_{v \in TM}(T_vTM)^{\mathrm{vert}}$. An element w in TTM^{vert} is said to be a ***vertical vector***. Let us outline how this is represented in coordinates. For every $v \in TM$ one has $T_v\pi_M : T_vTM \longrightarrow T_{\pi(v)}M$, since $\pi_M : TM \longrightarrow M$. Let $v \in T_pM$, if $v = (\overline{p}, \overline{v})$ in a given coordinate chart $\pi_M(\overline{p}, \overline{v}) = \overline{p}$. It straightforward to verify that for any $w \in TTM$ represented in coordinates by $((\overline{p}, \overline{v}), (\overline{w}_{\mathrm{L}}, \overline{w}_{\mathrm{H}}))$ one has $T_v\pi_M(((\overline{p}, \overline{v}), (\overline{w}_{\mathrm{L}}, \overline{w}_{\mathrm{H}}))) = (\overline{p}, \overline{w}_{\mathrm{L}})$, therefore an element w in the vertical subbundle TTM^{vert} is expressed, in coordinates, as $((\overline{p}, \overline{v}), (0, \overline{w}_{\mathrm{H}}))$.

Given $v, w \in TM$ such that $\pi_M(v) = \pi_M(w)$, one defines the ***vertical lift*** of w by v, denoted $\mathrm{lift}(v, w)$, to be the vector in T_vTM given by

$$\mathrm{lift}(v, w) = T_0\gamma_{v,w}\left(\left.\frac{\partial}{\partial r}\right|_0\right),$$

where $\gamma_{v,w} : \mathbb{R} \longrightarrow TM$ is the curve determined by $\gamma_{v,w}(t) = v + tw$.
Given a vector field $X \in \Gamma(TM)$, the ***vertical lift of*** X is the vector field $X^{\mathrm{lift}} \in \Gamma(TTM)$ defined by $X^{\mathrm{lift}}_v = \mathrm{lift}(v, X_{\pi_M(v)})$.

Assume that $v, w \in T_pM$ are respectively represented in coordinates by $(\overline{p}, \overline{v})$ and $(\overline{p}, \overline{w})$. Note that, since $\gamma_{v,w} : \mathbb{R} \longrightarrow TM$, one has $T_t\gamma_{v,w} : T_t\mathbb{R} \longrightarrow T_{\gamma_{v,w}(t)}TM$. Now, for every $t \in \mathbb{R}$ one has $\gamma_{v,w}(t) = (\overline{p}, \overline{v} + t\overline{w})$, so it is easy to check that $T_t\gamma_{v,w}(\left.\frac{\partial}{\partial r}\right|_t) = ((\overline{p}, \overline{v} + t\overline{w}), (0, \overline{w}))$. Therefore, since $\mathrm{lift}(v, w) = T_0\gamma_{v,w}(\left.\frac{\partial}{\partial r}\right|_0)$, one has that, in coordinates, $\mathrm{lift}(v, w) = ((\overline{p}, \overline{v}), (0, \overline{w}))$. Consider $v \in TM$ represented in coordinates by $(\overline{p}, \overline{v})$, suppose that X is a vector field on M which is represented in coordinates by $X_x = \left(\overline{x}, \overline{X}_x\right)$ for any $x \in M$, thus $X^{\mathrm{lift}}_v = \mathrm{lift}\left(v, X_p\right)$, so in coordinates

$X_v^{\text{lift}} = \big((\overline{p}, \overline{v}), (0, \overline{X}_p)\big)$.

A vector field $X \in \Gamma(TTM)$ is said to be a ***vertical vector field*** if $T\pi_M \circ X = 0$. Here it is important to note that a vertically lifted vector field is vertical, whereas a vertical vector field is not, in general, the result of vertically lifting some vector field.

A vector field $X \in \Gamma(TTM)$ is said to be a ***second-order vector field*** (one also says that X defines a second-order equation on M) if $T\pi_M \circ X = \mathrm{id}_{TM}$. This definition can be extended naturally to vector fields along curves in TM, namely, if $\gamma : I \longrightarrow TM$ is a curve and X is defined along γ then X is said to be second-order along γ if for every $t \in I$, $T\pi_M(X_{\gamma(t)}) = \gamma(t)$.

Consider $v \in TM$, represented in coordinates by $(\overline{p}, \overline{v})$, and assume that X is a vector field on TM, which is represented in coordinates by $X_v = \big((\overline{p}, \overline{v}), (\overline{X}_{\mathrm{L}}(v), \overline{X}_{\mathrm{H}}(v))\big)$. Thus $T\pi_M \circ X(v) = (\overline{p}, \overline{X}_{\mathrm{L}}(v))$, therefore X is vertical if $\overline{X}_L(v) = 0$ while X is second-order if $\overline{X}_L(v) = v$.

A system of the form $\dot{x} = X(x)$, with $x : I \longrightarrow TM$ and $X \in \Gamma(TTM)$ is said to be a ***second-order system*** iff X is second-order.

The ***Liouville vector field*** denoted by C is a vector field on the tangent space at a manifold M ($C \in \Gamma(TTM)$) which is defined by $C(v) = \mathrm{lift}(v, v)$ for any $v \in TM$. In coordinates, if $v \in TM$ is represented by $(\overline{p}, \overline{v})$, then C_v is represented by $((\overline{p}, \overline{v}), (0, \overline{v}))$.

The ***canonical almost tangent structure*** on M, denoted by J_M, is a tensor field of type $(1, 1)$ over TM (i.e. $J_M \in \Gamma\left(T_1^1(TTM)\right)$) defined by

$$(J_M(X))_v = \mathrm{lift}(v, T\pi_M \circ X(v))$$

where X is a vector field on TM and $v \in TM$. The tensor field J_M, so defined, can be shown to satisfy these two properties [5]

$$[J_M, J_M] = 0 \quad \text{and} \quad [C, J_M] = -J_M.$$

In coordinates, if X_v is represented by $\big((\overline{p}, \overline{v}), (\overline{X}_{\mathrm{L}}(v), \overline{X}_{\mathrm{H}}(v))\big)$ then $(J_M(X))_v = \mathrm{lift}((\overline{p}, \overline{v}), (\overline{p}, \overline{X}_{\mathrm{L}}(v))) = \big((\overline{p}, \overline{v}), (0, \overline{X}_{\mathrm{L}}(v))\big)$.

The conditions that define a vector field to be vertical or second-order can also be formulated using the canonical almost tangent structure together with the Liouville vector field [5]. A vector field X on TM is said to be vertical iff $J_M(X) = 0$ or, alternatively, if $[C, X] = -X$. X is said to be second-order if $J_M(X) = C$. One also defines, using the almost tangent structure, the notion of semispray. Let X be a vector field defined on TM, the tangent bundle of a smooth manifold; X is said to be a ***semispray*** (or alternatively an almost spray) iff

$$J_M(X) = C \qquad \text{and} \qquad [C, X] = X.$$

One notices, by observing the properties above, that a semispray is also a second-order vector field.

The vertical space at $v \in TM$, being a subspace of the tangent space at v, is canonically given by the kernel of the tangent map associated with the bundle projection as stated before. However the choice of a horizontal space at v is not canonically determined, but it can be defined, instead, by means of a connection on M.

Definition 2 *Let M be a smooth manifold. A smooth affine* ***connection*** ∇ *on M, (or simply a connection on M), is a linear map*

$$\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$$

such that, for each $f \in C^\infty(M)$ and every $X \in \Gamma(TM)$,

$$\nabla(fX) = df \otimes X + f\nabla(X).$$

We regard a connection as a map $\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ such that $\nabla : (X, Y) \mapsto \nabla_X Y$. Given vector fields X and Y in $\Gamma(TM)$, the ***covariant derivative of*** Y ***along*** X is defined to be the vector field $\nabla_X Y \in \Gamma(TM)$. It is easy to show that the conditions already mentioned in Definition (2) are equivalent to requiring, from ∇, the following conditions, for every X, Y in $\Gamma(TM)$:

1. $(X, Y) \mapsto \nabla_X Y$ is $\mathbb{R}$-bilinear
2. $\nabla_{fX} Y = f(\nabla_X Y), \quad \forall f \in C^\infty(M)$

3. $\nabla_X(fY) = (L_X f)Y + f(\nabla_X Y), \quad \forall f \in C^\infty(M)$.

Let (U,ϕ) be a coordinate chart for M with coordinates $(x^1,\ldots,x^n)$. Given that the covariant derivative of the basis vector field $\frac{\partial}{\partial x^j}$ with respect to the basis vector field $\frac{\partial}{\partial x^i}$ is a vector field on M, there exist functions on M, Γ^k_{ij}, $i=1,\ldots,n$, such that

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k} \qquad\qquad i,j=1,\ldots,n.$$

The terms Γ^k_{ij}, $\;i,j,k=1,\ldots,n$, referred to as the ***Christoffel symbols***, uniquely define a connection. If, in a given coordinate chart, the vector fields X and Y are given by $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^i\frac{\partial}{\partial x^i}$, $i=1,\ldots,n$, then we have

$$\begin{aligned}
\nabla_X Y &= \left(\nabla\left(Y^i\frac{\partial}{\partial x^i}\right)\right)(X)\\
&= \left(dY^i\otimes\frac{\partial}{\partial x^i} + Y^i\,\nabla\frac{\partial}{\partial x^i}\right)(X)\\
&= dY^i\left(X^k\frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^i} + Y^i\,\nabla_{\left(X^k\frac{\partial}{\partial x^k}\right)}\left(\frac{\partial}{\partial x^i}\right)\\
&= \frac{\partial Y^i}{\partial x^l}dx^l\left(X^k\frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^i} + X^kY^i\,\nabla_{\left(\frac{\partial}{\partial x^k}\right)}\left(\frac{\partial}{\partial x^i}\right)\\
&= \frac{\partial Y^i}{\partial x^l}X^k\delta^l_k\frac{\partial}{\partial x^i} + X^kY^i\,\nabla_{\left(\frac{\partial}{\partial x^k}\right)}\left(\frac{\partial}{\partial x^i}\right)\\
&= \left(\frac{\partial Y^l}{\partial x^k}X^k + \Gamma^l_{ki}\,X^kY^i\right)\frac{\partial}{\partial x^l},
\end{aligned}$$

therefore, rearranging the indices, the expression for the covariant derivative of Y along X, is

$$\nabla_X Y = \left(\frac{\partial Y^k}{\partial x^i}X^i + \Gamma^k_{ij}X^iY^j\right)\frac{\partial}{\partial x^k}.$$

A curve $c: I \longrightarrow M$ is said to be a ***geodesic*** for an affine connection ∇ if $\nabla_{\dot c(t)}\dot c(t) = 0$ for every t in I. Let X be a vector field defined on M and suppose that $c: I \longrightarrow M$ is an integral curve of X, thus $X_{c(t)} = \dot c(t)$ for every $t\in I$, or equivalently, $X\circ c = \dot c$. Recall that $\dot c: I \longrightarrow TM$ is defined by $\dot c(t) = T_t c\left(\frac{\partial}{\partial r}\big|_t\right)$.

One has $\ddot{c}(t) = T_t\dot{c}\left(\frac{\partial}{\partial r}\big|_t\right)$, consequently $\ddot{c}(t) = T_t(X\circ c)\left(\frac{\partial}{\partial r}\big|_t\right)$. By applying the chain rule one gets $\ddot{c}(t) = T_{c(t)}X\circ T_tc\left(\frac{\partial}{\partial r}\big|_t\right)$ and, by using the definition of $\dot{c}$ one gets $\ddot{c}(t) = T_{c(t)}X\circ\dot{c}(t)$, or equivalently, $\ddot{c}(t) = T_{c(t)}X\circ X(c(t))$. In coordinates one has $\ddot{c}^i(t) = \frac{\partial X^i}{\partial x^j}X^j(c(t))$. Then the expression, in coordinates, of $\nabla_{\dot{c}(t)}\,\dot{c}(t) = (\nabla_X X)_{c(t)}$ is

$$\begin{aligned}\nabla_{\dot{c}(t)}\,\dot{c}(t) &= \left(\frac{\partial X^i_{c(t)}}{\partial x^j}X^j_{c(t)} + \Gamma^i_{jk}(c(t))X^j_{c(t)}\,X^k_{c(t)}\right)\frac{\partial}{\partial x^i}\\ &= \left(\ddot{c}^i(t) + \Gamma^i_{jk}(c(t))\,\dot{c}^j(t)\,\dot{c}^k(t)\right)\frac{\partial}{\partial x^i}.\end{aligned}\tag{2.1}$$

Now suppose that the covariant derivative of X along X equals zero. Hence the integral curve c of X is geodesic iff

$$\ddot{c}^i(t) + \Gamma^i_{jk}(c(t))\dot{c}^j(t)\,\dot{c}^k(t) = 0, \qquad \text{for } i = 1,\ldots,n;\ \forall\, t\in I.$$

The latter is a second-order differential equation on M. If $x^i = c^i$ and $v^i = \dot{c}^i$ for $i = 1,\ldots,n$, so the curve $\dot{c}$ in natural coordinates for TM is expressed by $(x^1,\ldots,x^n,v^1,\ldots,v^n)$, then the corresponding first-order differential equation on TM is

$$\left.\begin{aligned}\dot{x}^i &= v^i\\ \dot{v}^i &= -\Gamma^i_{jk}(x)\,v^jv^k\end{aligned}\right.\qquad \text{for } i = 1,\ldots,n$$

These first-order equations define a vector field S on TM given in coordinates by $S_{(x,v)} = v^i\frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x)v^jv^k\frac{\partial}{\partial v^i}$. This vector field is known as the ***geodesic spray*** associated with the connection ∇. The integral curves of S are curves in TM whose projection onto M yield geodesics for the connection ∇.

Given a connection ∇, we define its associated ***torsion tensor field*** T by

$$T : (X,Y)\mapsto \nabla_XY - \nabla_YX - [X,Y].$$

T is indeed a tensor field of type $(1,2)$, so one may regard it as a function $T : \Gamma(TM)\times\Gamma(TM)\longrightarrow\Gamma(TM)$ which is bilinear with respect to function multiplication. Indeed given X,Y, vector fields defined on M, and functions f,g in $C^\infty(M)$ T satisfies

$$
\begin{aligned}
T(fX,gY) &= (\nabla(gY))\,(fX) - (\nabla(fX))\,(gY) - [fX,gY] \\
&= (dg\otimes Y + g\nabla Y)(fX) - (df\otimes X + f\nabla X)(gY) - [fX,gY] \\
&= dg(fX)Y + g\nabla_{(fX)}Y - df(gY)X - f\nabla_{(gY)}X - fg\,[X,Y] \\
&\quad -fX(g)Y + gY(f)X \\
&= fg\,\nabla_X Y - fg\,\nabla_Y X + f\,dg(X)Y - g\,df(Y)X - fg\,[X,Y] \\
&\quad -f\,dg(X)Y + g\,df(Y)X \\
&= fg\,\nabla_X Y - fg\,\nabla_Y X - fg\,[X,Y],
\end{aligned}
$$

therefore $T(fX,gY) = fg\,(\nabla_X Y - \nabla_Y X - [X,Y])$ i.e.

$$
T(fX,gY) = fg\,T(X,Y).
$$

One easily checks that T also satisfies $T(X+Y,Z) = T(X,Y)+T(Y,Z)$ and $T(X,Y+Z) = T(X,Y)+T(X,Z)$ for X,Y,Z in $\Gamma(TM)$.
Suppose that X,Y are given in coordinates by $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^i\frac{\partial}{\partial x^i}$, $i = 1,\ldots,n$. In this case

$$
\begin{aligned}
T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] \\
&= \left(\frac{\partial Y^k}{\partial x^i}X^i + \Gamma^k_{ij}X^iY^j\right)\frac{\partial}{\partial x^k} - \left(\frac{\partial X^k}{\partial x^j}Y^j + \Gamma^k_{ji}X^iY^j\right)\frac{\partial}{\partial x^k} \\
&\quad - \left(\frac{\partial Y^k}{\partial x^i}X^i - \frac{\partial X^k}{\partial x^j}Y^j\right)\frac{\partial}{\partial x^k} \\
&= \left(\Gamma^k_{ij}X^iY^j - \Gamma^k_{ji}X^iY^j\right)\frac{\partial}{\partial x^k},
\end{aligned}
$$

therefore the expression for the torsion tensor field evaluated at (X,Y) is

$$
T(X,Y) = \left(\Gamma^k_{ij}X^iY^j - \Gamma^k_{ji}X^iY^j\right)\frac{\partial}{\partial x^k},
$$

hence, the components of the torsion tensor field in coordinates are

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k, \qquad i, j, k = 1, \ldots, n.$$

2.3. Simple Mechanical Systems

In this section we review the basics on modeling mechanical systems in the Lagrangian formulation. This class of systems plays a fundamental role in this thesis for the ultimate purpose consists in devising a control strategy to cope with stabilization and tracking for such systems.

Definition 3 *Let M be a smooth manifold and $\mathcal{G} \in \Gamma(T_2^0(TM))$ a tensor field of type $(0,2)$ over M such that, for all $p \in M$ and for every $w \in T_pM$, $\mathcal{G}$ satisfies:*

1. *$\mathcal{G}_p(v,w) = \mathcal{G}_p(w,v)$ for every $v \in T_pM$ "symmetry"*
2. *If $\mathcal{G}_p(v,w) = 0$ then $v = 0$ "nondegeneracy".*

*The couple $(M, \mathcal{G})$ is said to be a **pseudo-Riemannian manifold** and $\mathcal{G}$ a **pseudo-Riemannian metric**.*
*Moreover, if $\mathcal{G}$ is positive definite, i.e. $\mathcal{G}_p(v,v) > 0$ for every $p \in M$ and every $v \in T_pM \backslash \{0\}$, one says that $\mathcal{G}$ is a **Riemannian metric** on M and that the couple $(M, \mathcal{G})$ is a **Riemannian manifold**.*

Let M be a finite-dimensional, smooth manifold and $\mathcal{G}$ a Riemannian metric on M. Then one defines the bundle map $\mathcal{G}^\flat$ on TM, such that $\mathcal{G}_p^\flat : T_pM \longrightarrow T_p^*M$, by $\mathcal{G}_p^\flat(v) = \mathcal{G}_p(v, \cdot)$, $\forall p \in M$ and $\forall v \in T_pM$. Likewise consider the map $\mathcal{G}^\sharp$ so that $\mathcal{G}_p^\sharp : T_p^*M \longrightarrow T_pM$ given by $\mathcal{G}_p^\sharp = (\mathcal{G}_p^\flat)^{-1}$, $\forall p \in M$. Since $\mathcal{G}$ is nondegenerate these two maps $\mathcal{G}^\flat$ and $\mathcal{G}^\sharp$ define canonical maps between TM and T^*M and between T^*M and TM respectively. In coordinates, the components of the metric $\mathcal{G}$ are given by $\mathcal{G}_{ij} = \mathcal{G}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. As a consequence, the maps $\mathcal{G}^\flat$ and $\mathcal{G}^\sharp$ satisfy $\mathcal{G}^\flat\left(\frac{\partial}{\partial x^i}\right) = \mathcal{G}_{ij}\, dx^j$ and $\mathcal{G}^\sharp\left(dx^i\right) = \mathcal{G}^{ij} \frac{\partial}{\partial x^i}$, where $\mathcal{G}^{ij}$, $(i, j = 1, \ldots, n)$ is defined by $\mathcal{G}_{ij}\mathcal{G}^{jk} = \delta_i^k$, that is, the matrix whose entries are $\mathcal{G}^{ij}$ is the inverse of the matrix with entries $\mathcal{G}_{ij}$. Hence, for a vector field X on M and for a 1-form Υ, $\mathcal{G}$ satisfies

$$\mathcal{G}^\flat\left(X^i\frac{\partial}{\partial x^i}\right) = \mathcal{G}_{ij}X^i\,dx^j \qquad \mathcal{G}^\sharp\left(\Upsilon_i\,dx^i\right) = \mathcal{G}^{ij}\Upsilon_i\,\frac{\partial}{\partial x^j} \tag{2.2}$$

If a manifold is provided with a Riemannian metric then there is a canonical connection which is torsion free and compatible with the metric. More precisely, if $\mathcal{G}$ is a Riemannian metric on a smooth manifold M, then there exists a unique affine connection $\overset{\mathcal{G}}{\nabla}$ on M which satisfies $\forall\, X, Y \in \Gamma(TM)$:

1. $\overset{\mathcal{G}}{\nabla}_X\,\mathcal{G} = 0$, (metric compatible),
2. $T(X, Y) = 0$, (torsion free),

where T is the torsion tensor field associated with $\mathcal{G}$. $\overset{\mathcal{G}}{\nabla}$ is called the ***Levi-Cività connection*** associated with $\mathcal{G}$. Given a coordinate chart, the latter two conditions imply that, [10], the Christoffel symbol, $\overset{\mathcal{G}}{\Gamma}{}^i_{jk}$, of the Levi-Cività connection is given by:

$$\overset{\mathcal{G}}{\Gamma}{}^i_{jk} = \frac{1}{2}\,\mathcal{G}^{il}\left(\frac{\partial\mathcal{G}_{jl}}{\partial x^k} + \frac{\partial\mathcal{G}_{kl}}{\partial x^j} - \frac{\partial\mathcal{G}_{jk}}{\partial x^l}\right) \tag{2.3}$$

where $\mathcal{G}^{ij}$, $i, j = 1, \ldots, n$, are defined by $\mathcal{G}_{ij}\mathcal{G}^{jk} = \delta_i^k$.

Let Q be an n-dimensional smooth manifold. A ***simple mechanical control system*** defined on Q is a 4-tuple $(Q, \mathcal{G}, V, \mathcal{F})$, where Q is the configuration manifold of the system, $\mathcal{G}$ is a Riemannian metric on Q, $V \in C^\infty(Q)$ is called the potential energy function and $\mathcal{F} = \{F^1, \ldots, F^m\}$ is a set of 1-forms on Q that physically correspond to forces or torques.

By the forced Euler-Lagrange equations one represents the dynamics of $(Q, \mathcal{G}, V, \mathcal{F})$ by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \sum_{i=j}^{m} u^j F_i^j, \tag{2.4}$$

where $L : TQ \longrightarrow \mathbb{R}$, defined by $L(v) = \frac{1}{2}\mathcal{G}_{\pi(v)}(v) - V \circ \pi(v)$ for all $v \in TM$, is the Lagrangian of the system. If, in coordinates, $v = (q, \dot{q})$, then $L(q, \dot{q}) = \frac{1}{2}\mathcal{G}_{\pi(v)}(\dot{q}, \dot{q}) - V(q) = \frac{1}{2}\mathcal{G}_{q,ij}\,\dot{q}^i\dot{q}^j - V(q)$.

One can express (2.4) using the Levi-Cività connection associated with $\mathcal{G}$ [10] as:

$$\overset{\mathcal{G}}{\nabla}_{\dot{q}}\, \dot{q} = -\mathcal{G}^{\sharp} \circ dV(q) + \sum_{i=1}^{m} u^i \mathcal{G}^{\sharp} \circ F^i(q) \tag{2.5}$$

Notice that this is a coordinate-free representation. Let us now express the equation in coordinates. By (2.1) and (2.2) one has

$$\begin{aligned} \left(\ddot{q}^i + \overset{\mathcal{G}}{\Gamma}{}^{i}_{jk}(q)\, \dot{q}^j\, \dot{q}^k \right) \frac{\partial}{\partial q^i} &= -\mathcal{G}^{\sharp} \left(\frac{\partial V}{\partial q^i}(q)\, dq^i \right) + \sum_{h=1}^{m} u^h \mathcal{G}^{\sharp} (F^h_i(q)\, dq^i) \\ &= -\mathcal{G}^{li} \frac{\partial V}{\partial q^l}(q)\, \frac{\partial}{\partial q^i} + \sum_{h=1}^{m} u^h \mathcal{G}^{ri} F^h_r(q)\, \frac{\partial}{\partial q^i} \end{aligned}$$

Hence a simple mechanical control system can be represented by the following set of n second-order differential equations

$$\ddot{q}^i = -\overset{\mathcal{G}}{\Gamma}{}^{i}_{jk}(q)\, \dot{q}^j\, \dot{q}^k - \mathcal{G}^{li} \frac{\partial V}{\partial x^l}(q) + \sum_{h=1}^{m} u^h \mathcal{G}^{ri} F^h_r(q) \qquad i = 1, \ldots, n.$$

Alternatively, it can be represented by a set of $2n$ first-order differential equations $(i = 1, \ldots, n)$

$$\begin{aligned} \dot{x}^i &= v^i \\ \dot{v}^i &= -\overset{\mathcal{G}}{\Gamma}{}^{i}_{jk}(x)\, v^j\, v^k - \mathcal{G}^{li} \frac{\partial V}{\partial x^l}(x) + \sum_{h=1}^{m} u^h \mathcal{G}^{ri} F^h_r(x) \end{aligned}$$

with $x^i = q^i$ and $v^i = \dot{q}^i$. From the latter expression one realizes that (2.5) can be written as a system evolving on TQ, namely

$$\dot{v} = S^{\mathcal{G}}(v) - (\mathcal{G}^{\sharp} \circ\, dV)^{\mathrm{lift}}(v) + \sum_{i=1}^{m} u^i (\mathcal{G}^{\sharp} \circ F^i)^{\mathrm{lift}}(v), \tag{2.6}$$

where $S^{\mathcal{G}}$ is the geodesic spray associated with the Levi-Cività connection $\overset{\mathcal{G}}{\nabla}$.

A ***constrained mechanical control system*** [10], represented by a 5-tuple $(Q, \mathcal{G}, V, \mathcal{F}, \mathcal{D})$, is a simple mechanical control system $(Q, \mathcal{G}, V, \mathcal{F})$ subject to constraints represented by D, an $(n-l)$-dimensional distribution on Q given by the annihilator of $\mathcal{D} = \{\omega_1, \ldots, \omega_l\}$, l linearly independent 1-forms on Q.

The dynamics of a given system $(Q, \mathcal{G}, V, \mathcal{F}, \mathcal{D})$ can be obtained by the application of the Lagrange-d'Alembert principle, which yields the following equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \sum_{i=j}^{m} u^j F^j + \sum_{k=1}^{l} \lambda^k \omega_k,$$

where λ^k $(k = 1, \ldots, l)$ are the Lagrange multipliers for the system. Analogously, one can express this system using the Levi-Cività connection by

$$\overset{\mathcal{G}}{\nabla}_{\dot{q}}\, \dot{q} = \lambda - \mathcal{G}^\sharp \circ dV + \sum_{i=1}^{m} u^i \mathcal{G}^\sharp \circ F^i(q)$$

where λ (related to the Lagrange multipliers) is a section of $D^\perp$, the $\mathcal{G}$-orthogonal complement to $\mathcal{D}$, along the curve q. According to [10], if $P : TQ \longrightarrow TQ$ denote the complementary $\mathcal{G}$-orthogonal projection on D and $P' : TQ \longrightarrow TQ$ the $\mathcal{G}$-orthogonal projection onto $D^\perp$ one has

$$\overline{\nabla}_{\dot{q}}\, \dot{q} = -P \circ \mathcal{G}^\sharp \circ dV + \sum_{i=1}^{m} u^i P \circ \mathcal{G}^\sharp \circ F^i(q), \tag{2.7}$$

where ∇ is defined by $\overline{\nabla}_X Y = \overset{\mathcal{G}}{\nabla}_X Y + A^{-1}((\overset{\mathcal{G}}{\nabla}_X (AP'))(Y))$ where A is any invertible tensor field of type $(1,1)$ on Q . Note that (2.7) has the same form as (2.5), therefore it can be expressed as a system evolving on TQ by a formula analogous to (2.6).

2.4. Lie Theory Preliminaries

Several mechanical systems have as configuration manifold a Lie group that is a differentiable manifold having a group structure. This class of systems can be described as evolving on the tangent bundle of the Lie group.

Definition 4 (Group) *A* ***group*** *is a set G together with a map $\mu : G \times G \longrightarrow G$*

(called the group composition or law of composition) which satisfies:

1. *$\mu(x,y) \in G, \;\; \forall\, x, y \in G.$*
2. *$\mu(x,\mu(y,z)) = \mu(\mu(x,y),z), \;\; \forall\, x,y,z \in G.$*
3. *There exists $e \in G$ such that for every $x \in G$, $\mu(x,e) = \mu(e,x) = x$.*
4. *For all $x \in G$ there exists an element $x^{-1} \in G$ such that $\mu(x,x^{-1}) = \mu(x^{-1},x) = e$.*

We refer to e as the ***identity element*** *(or simply as the identity) in G and to x^{-1} as the* ***inverse element of x***.

For convenience we shall sometimes write $g \cdot h$ or gh instead of $\mu(g,h)$. Let G be a group and X a set. By a ***left action*** of G on X we mean a map $l : G \times X \longrightarrow X$ which satisfies, for all $x \in X$:

1. $l(e,x) = x$
2. $l(g, l(h,x)) = l(gh, x) \quad \forall\, g, h \in G$

Similarly, a ***right action*** is a map $r : X \times G \longrightarrow X$ that satisfies, for all $x \in X$:

1. $r(x,e) = x$
2. $r(r(x,g),h) = r(x,gh) \quad \forall\, g,h \in G$

One readily verifies that left and right actions of a group G on itself are naturally defined by setting $l(g,\cdot) = \mu(g,\cdot)$ and $r(\cdot,g) = \mu(\cdot,g)$. These actions are called ***left translation*** and ***right translation***, respectively.

Definition 5 (Lie Group) *A* ***Lie group*** *is a smooth manifold G that has a group structure compatible with its smooth manifold structure in the sense that the group multiplication*

$$\mu : G \times G \longrightarrow G \qquad \mu : (g,h) \mapsto gh$$

is a smooth map.

It is straightforward to show that if G is a Lie group then the inverse map $g \mapsto g^{-1}$ is also smooth.

Whenever we refer to Lie groups $\exp(tX)$ will denote $\exp(tX)(e)$, i.e. if G is a Lie group and X is a vector field defined on G, $\exp(tX)$ stands for the solution of the differential equation $\dot{g} = X(g)$ at time t with initial condition e (the identity element in G).

Let us denote by L_g and R_g the left and right translations on the Lie group G by an element g, thus $L_g(h) = gh$ and $R_g(h) = hg$ for all $h \in G$. The following diagram, involving left and right translations, commutes

$$\begin{tikzcd} G \arrow[r, "L_g"] \arrow[d, "R_h"'] & G \arrow[d, "R_h"] \\ G \arrow[r, "L_g"'] & G \end{tikzcd}$$

i.e. for every g, h in G, $L_g \circ R_h = R_h \circ L_g$ since for a given $p \in G$, $L_g \circ R_h(p) = gph = R_h \circ L_g(p)$.
Since $L_{g^{-1}}(L_g(h)) = g^{-1}(gh) = h$ and $R_{g^{-1}}(R_g(h)) = (hg)g^{-1} = h$, the inverse of L_g is $L_{g^{-1}}$ and likewise the inverse of R_g is $R_{g^{-1}}$.

The tangent space at any point in G can be canonically identified with T_eG using the tangent map associated with the left translation L or with the right translation R. Indeed given a vector $v \in T_gG$, the related vector $\xi \in T_eG$ is given by

$$\xi = T_g L_{g^{-1}}(v) \quad \text{or by} \quad \xi = T_g R_{g^{-1}}(v).$$

In the following we will conventionally use the association specified by the left translation. Given a vector $\zeta \in T_eG$ one defines a vector field X_ζ on G by

$$X_\zeta(g) = T_e L_g(\zeta),$$

(an analogous definition holds using right translations). The vector field defined in this way is said to be left-invariant since $X_\zeta(L_g(h)) = T_e L_{gh}(\zeta) = T_e(L_g \circ L_h)(\zeta) = T_h L_g \circ T_e L_h(\zeta) = T_h L_g(X_\zeta(h))$. One extends this notion as in the following definition.

Definition 6 (Left-invariant vector fields) *Let G denote a Lie group and let X*

be a vector field on G. X is said to be ***left-invariant*** *with respect to the group operation if*

$$X_{gh} = T_h L_g(X_h) \quad \forall\, g, h \in G.$$

It is straightforward to verify that the Lie bracket product of left-invariant vector fields is a left-invariant vector field, i.e. if X and Y are left-invariant vector fields defined on a Lie group then $[X, Y]_{gh} = T_h L_g([X, Y]_h)$.

The ***Lie algebra*** of left-invariant vector fields associated with G, a Lie group, is denoted by $\mathfrak{g}$ and is isomorphic to the tangent space of G at e ($\mathfrak{g} \simeq T_e G$). It is possible to define a Lie bracket product for elements in $T_e G$ due to the association of a vector in $T_e G$ with a vector field defined in $\mathfrak{g}$, namely $[\xi, \zeta] = [X_\xi, X_\zeta](e)$.

Proposition 1 (The tangent Lie group) *Let G be a Lie group, $\widehat{\mu}$ its group operation and $\widehat{e}$ its identity. Then TG can be endowed with a Lie group structure by defining the group operation $\mu : TG \times TG \longrightarrow TG$ by*

$$\mu : (u, v) \mapsto T_{\pi_G(u)} \widehat{R}_{\pi_G(v)}(u) + T_{\pi_G(v)} \widehat{L}_{\pi_G(u)}(v), \tag{2.8}$$

where $\widehat{L}$ and $\widehat{R}$ are the left and right translations in G. The identity element $e \in TG$ is $0_{\widehat{e}}$, that is, the zero vector in the fiber over the identity on G, and the inverse of $u \in TG$ is

$$u^{-1} = -T_{\widehat{e}} \widehat{L}_{\pi_G(u)^{-1}} \circ T_{\pi_G(u)} \widehat{R}_{\pi_G(u)^{-1}}(u) \tag{2.9}$$

Proof: First let prove that the group composition of two elements of TG is in TG. Since $\widehat{R}_{\pi_G(v)} : G \longrightarrow G$ and $\widehat{L}_{\pi_G(u)} : G \longrightarrow G$ one gets that $T\widehat{R}_{\pi_G(v)} : TG \longrightarrow TG$ and $T\widehat{L}_{\pi_G(u)} : TG \longrightarrow TG$. As $T\widehat{R}_{\pi_G(v)}(u) \in T_{\pi_G(u)\pi_G(v)}G$ and $T\widehat{L}_{\pi_G(u)}(v) \in T_{\pi_G(u)\pi_G(v)}G$ for all $u, v \in TG$, the sum in (2.8) makes sense and $\mu(u, v) \in T_{\pi_G(u)\pi_G(v)}G \subset TG$.
Now assume that $u \in T_g G$, $v \in T_h G$, $w \in T_i G$, then by (2.8) one obtains that $\mu(v, w) = T_h \widehat{R}_i(v) + T_i \widehat{L}_h(w)$. By virtue of the linearity of tangent maps one gets

$$\mu(u, \mu(v, w)) \;\; = \;\; T_g \widehat{R}_{hi}(u) + T_{hi} \widehat{L}_g(T_h \widehat{R}_i(v) + T_i \widehat{L}_h(w))$$

$$
\begin{aligned}
&= T_g\widehat{R}_{hi}(u) + T_{hi}\widehat{L}_g \circ T_h\widehat{R}_i(v) + T_{hi}\widehat{L}_g \circ T_i\widehat{L}_h(w) \\
&= T_g\widehat{R}_{hi}(u) + T_{hi}\widehat{L}_g \circ T_h\widehat{R}_i(v) + T_i\widehat{L}_{gh}(w).
\end{aligned}
$$

Using the fact that $T\widehat{L}_g \circ T\widehat{R}_i = T\widehat{R}_i \circ T\widehat{L}_g$ for each g, h in G, (since left and right translations commute) one obtains

$$
\begin{aligned}
\mu(u, \mu(v, w)) &= T_g\widehat{R}_{hi}(u) + T_{gh}\widehat{R}_i \circ T_h\widehat{L}_g(v) + T_i\widehat{L}_{gh}(w) \\
&= T_{gh}\widehat{R}_i \circ T_g\widehat{R}_h(u) + T_{gh}\widehat{R}_i \circ T_h\widehat{L}_g(v) + T_i\widehat{L}_{gh}(w) \\
&= T_{gh}\widehat{R}_i(T_g\widehat{R}_h(u) + T_h\widehat{L}_g(v)) + T_i\widehat{L}_{gh}(w)
\end{aligned}
$$

Therefore $\mu(u, \mu(v, w)) = \mu(\mu(u, v), w)$ since $\mu(u, v) = T_g\widehat{R}_h(u) + T_h\widehat{L}_g(v)$, and this proves the associativity of the product defined in (2.8).
Next, let us verify that $0 \in T_{\hat{e}}G$ is the identity element in TG with μ as group composition.

$$
\begin{aligned}
\mu(u, 0) &= T_{\pi_G(u)}\widehat{R}_{\pi_G(0)}(u) + T_{\pi_G(0)}\widehat{L}_{\pi_G(u)}(0) \\
&= T_{\pi_G(u)}\widehat{R}_{\hat{e}}(u) \\
&= u.
\end{aligned}
$$

Analogously

$$
\begin{aligned}
\mu(0, u) &= T_{\pi_G(0)}\widehat{R}_{\pi_G(u)}(0) + T_{\pi_G(u)}\widehat{L}_{\pi_G(0)}(u) \\
&= T_{\pi_G(u)}\widehat{L}_{\hat{e}}(u) \\
&= u,
\end{aligned}
$$

hence 0 satisfies the properties to be the identity element in TG.
Since $\pi_G(u^{-1}) = \pi_G(-T_{\hat{e}}\widehat{L}_{\pi_G(u)^{-1}} \circ T_{\pi_G(u)}\widehat{R}_{\pi_G(u)^{-1}}(u))$, $\pi_G(u^{-1})$ equals $(\pi_G(u))^{-1}$, thus $u^{-1} \in T_{\pi_G(u)^{-1}}G$ then $\mu(u, u^{-1}) \in T_{\pi_G(u)\pi_G(u)^{-1}}G = T_{\hat{e}}G$ and $\mu(u^{-1}, u) \in T_{\pi_G(u)^{-1}\pi_G(u)}G = T_{\hat{e}}G$. One also has

$$\begin{aligned}
\mu(u,u^{-1}) &= T\widehat{R}_{\pi_G(u)^{-1}}(u) + T\widehat{L}_{\pi_G(u)}\left(-T\widehat{L}_{\pi_G(u)^{-1}}\circ T\widehat{R}_{\pi_G(u)^{-1}}(u)\right)\\
&= T\widehat{R}_{\pi_G(u)^{-1}}(u) - T\widehat{L}_{\pi_G(u)}\circ T\widehat{L}_{\pi_G(u)^{-1}}\circ T\widehat{R}_{\pi_G(u)^{-1}}(u)\\
&= T\widehat{R}_{\pi_G(u)^{-1}}(u) - T\widehat{L}_{\widehat{e}}\circ T\widehat{R}_{\pi_G(u)^{-1}}(u)\\
&= T\widehat{R}_{\pi_G(u)^{-1}}(u) - T\widehat{R}_{\pi_G(u)^{-1}}(u)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\mu(u^{-1},u) &= T\widehat{R}_{\pi_G(u)}\left(-T\widehat{L}_{\pi_G(u)^{-1}}\circ T\widehat{R}_{\pi_G(u)^{-1}}(u)\right) + T\widehat{L}_{\pi_G(u)^{-1}}(u)\\
&= -T\widehat{R}_{\pi_G(u)}\circ T\widehat{L}_{\pi_G(u)^{-1}}\circ T\widehat{R}_{\pi_G(u)^{-1}}(u) + T\widehat{L}_{\pi_G(u)^{-1}}(u)\\
&= -T\widehat{L}_{\pi_G(u)^{-1}}\circ T\widehat{R}_{\pi_G(u)}\circ T\widehat{R}_{\pi_G(u)^{-1}}(u) + T\widehat{L}_{\pi_G(u)^{-1}}(u)\\
&= -T\widehat{L}_{\pi_G(u)^{-1}}(u) + T\widehat{L}_{\pi_G(u)^{-1}}(u)\\
&= 0
\end{aligned}$$

Hence $(\cdot)^{-1} : TG \longrightarrow TG$, defined by (2.9), satisfies the properties of the inverse map. Finally, the map $\widehat{\mu}$, the group law of composition defined in G, is smooth, since $\widehat{L}_g$ and $\widehat{R}_h$ are smooth for all $g,h \in G$ and, consequently, $T\widehat{L}_g$ and $T\widehat{R}_h$ are smooth, therefore so is μ. ■

Note that $\widehat{\mu}$ and μ, being group operations in G and TG respectively, satisfy $\widehat{\mu}(\pi_G(u),\pi_G(v)) = \pi_G(\mu(u,v))$ (and consequently $\pi_G(u)^{-1} = \pi_G(u^{-1})$), i.e. the following diagram commutes

$$\begin{tikzcd}
TG \times TG \arrow[r, "\mu"] \arrow[d, "\pi_G\times\pi_G"'] & TG \arrow[d, "\pi_G"] \\
G \times G \arrow[r, "\widehat{\mu}"] & G.
\end{tikzcd}$$

It is important to remark that the following diagrams also commute

$$
\begin{tikzcd}
G\times G \arrow[r, "p_1\times p_2"] \arrow[d, "\widehat{\mu}"'] & G\times G \arrow[dl, "\widehat{\mu}"] \\
G &
\end{tikzcd}
\qquad
\begin{tikzcd}
T(G\times G) \arrow[r, "Tp_1\times Tp_2"] \arrow[d, "T\widehat{\mu}"'] & TG\times TG \arrow[dl, "\mu"] \\
TG &
\end{tikzcd}
$$

where $p_i : G\times G \longrightarrow G$ is the canonical projection on the i-th factor $(i = 1, 2)$. Thus

$$T\widehat{\mu} = \mu \circ Tp_1 \times Tp_2 \tag{2.10}$$

From now on we shall write $\widehat{L}$, $\widehat{R}$, $\widehat{\mu}$ to denote operations in G, and L, R and μ to denote operations in TG; $\widehat{e}$ shall denote the identity element in G whereas $e = 0_{\widehat{e}} = 0$ the identity element in TG.

Proposition 2 *Let G be a Lie group and let $\sigma, \tau : I \longrightarrow G$ be curves on G. Then*

$$\frac{d}{dt}\left(\widehat{\mu}(\sigma(t), \tau(t))\right) = T_{\sigma(t)}\widehat{R}_{\tau(t)}(X_{\sigma(t)}) + T_{\tau(t)}\widehat{L}_{\sigma(t)}(Y_{\tau(t)}) \tag{2.11}$$

where X and Y are vector fields on G defined along the curves σ and τ such that

$$X_{\sigma(t)} = T_t\sigma\left(\left.\frac{\partial}{\partial r}\right|_t\right), \quad Y_{\tau(t)} = T_t\tau\left(\left.\frac{\partial}{\partial r}\right|_t\right), \quad \forall\, t \in I.$$

Proof: Define $\alpha : I \longrightarrow G\times G$ and $\gamma : I \longrightarrow G$ by $\alpha = (\sigma, \tau)$ and $\gamma = \widehat{\mu}\circ\alpha$. By definition $\dot{\gamma}(t) = T_t\gamma\left(\left.\frac{\partial}{\partial t}\right|_t\right)$, thus $\dot{\gamma}(t) = T_t(\widehat{\mu}\circ\alpha)\left(\left.\frac{\partial}{\partial t}\right|_t\right)$. Using the chain rule one obtains $\dot{\gamma}(t) = T_{\alpha(t)}\widehat{\mu}\circ T_t\alpha\left(\left.\frac{\partial}{\partial t}\right|_t\right)$ but, according to (2.10) one has $T\widehat{\mu} = \mu\circ Tp_1\times Tp_2$, hence

$$\dot{\gamma}(t) = \mu\circ T_{\alpha(t)}p_1 \times T_{\alpha(t)}p_2 \circ T_t\alpha\left(\left.\frac{\partial}{\partial r}\right|_t\right).$$

Using the fact that $\dot{\alpha}(t) = (\dot{\sigma}(t), \dot{\tau}(t)) = (X_{\sigma(t)}, Y_{\tau(t)})$, we obtain $\dot{\gamma}(t) = \mu\circ T_{\alpha(t)}p_1 \times T_{\alpha(t)}p_2(X_{\sigma(t)}, Y_{\tau(t)})$. Thus

$$\frac{d}{dt}\left(\widehat{\mu}(\sigma(t), \tau(t))\right) = \dot{\gamma}(t) = T_{\sigma(t)}\widehat{R}_{\tau(t)}(X_{\sigma(t)}) + T_{\tau(t)}\widehat{L}_{\sigma(t)}(Y_{\tau(t)}),$$

as was to be shown. ∎

Proposition 3 *Let G be a Lie group and $\tau : I \longrightarrow G$ a curve on G. Then the vector field Z along the curve $t \mapsto (\tau(t))^{-1}$ defined by $Z_{\tau(t)^{-1}} = T_t(\tau^{-1})\left(\frac{\partial}{\partial r}\big|_t\right)$ is given by*

$$Z_{\tau(t)^{-1}} = -T_{\widehat{e}}\widehat{L}_{\tau(t)^{-1}} \circ T_{\tau(t)}\widehat{R}_{\tau(t)^{-1}}(Y_{\tau(t)}) \tag{2.12}$$

where Y is the vector field along τ defined by $Y_{\tau(t)} = T_t\tau\left(\frac{\partial}{\partial r}\big|_t\right)$.

Proof: Consider $\widehat{\mu}(\tau, \tau^{-1})$, then by (2.11) we have $\frac{d}{dt}(\widehat{e}) = T_{\tau(t)}\widehat{R}_{\tau(t)^{-1}}(Y_{\tau(t)}) + T_{\tau(t)^{-1}}\widehat{L}_{\tau(t)}(Z_{\tau(t)^{-1}}) = 0$ thus $T_{\tau(t)^{-1}}\widehat{L}_{\tau(t)}(Z_{\tau(t)^{-1}}) = -T_{\tau(t)}\widehat{R}_{\tau(t)^{-1}}(Y_{\tau(t)})$. Using the fact that $(\widehat{L}_\tau)^{-1} = \widehat{L}_{\tau^{-1}}$ we obtain

$$Z_{\tau(t)^{-1}} = -T_{\widehat{e}}\widehat{L}_{\tau(t)^{-1}} \circ T_{\tau(t)}\widehat{R}_{\tau(t)^{-1}}(Y_{\tau(t)}).$$

∎

Chapter 3

Transverse function control approach

In this chapter we review results presented in [13] and [14] about the transverse function control approach applicable to controllable driftless systems. We explain what defines a function to be *transverse* — so called since the condition these functions satisfy bears a resemblance with the transversality condition of differential topology. We also recall a procedure to construct transverse functions for certain cases. We review the methodology followed by Morin and Samson to achieve practical stabilization of points for controllable driftless systems. We also expound how this control approach is applied to control the chained form system, which is feedback equivalent to several other systems, among which the unicycle-type robot.

3.1. Characterization of transverse functions

Let $X_1, \ldots, X_m$ denote smooth, linearly independent vector fields on M, an n-dimensional smooth manifold. Suppose the set $\{X_1, \ldots, X_m\}$ satisfies the Lie Algebra Rank Condition (LARC) for some point $p \in M$, i.e.

$$T_pM = \{X_p : X \in \mathrm{Lie}(\{X_1, \ldots, X_m\})\}.$$

Expressed in more geometric terms, the set $\{X_1, \ldots, X_m\}$ satisfies the LARC at a point p iff the distribution spanned by $\{X_1, \ldots, X_n\}$ is completely nonintegrable around p. It is shown in [13] that the latter condition is equivalent to the fact that, given a neighborhood U of p, there exists an integer $\kappa \geq n - m$ and a function

$f : \mathbb{T}^{\kappa} \longrightarrow M$ such that $f(\mathbb{T}^k) \subset U$ and f satisfies, for every $\theta \in \mathbb{T}^{\kappa}$, the so-called *transversality condition*

$$T_{f(\theta)}M = \mathrm{span}_{\mathbb{R}}(\{X_{1f(\theta)}, \ldots, X_{mf(\theta)}\}) + T_{\theta}f(T_{\theta}\mathbb{T}^{\kappa}). \tag{3.1}$$

Thus the assumption that $\{X_1, \ldots, X_m\}$ satisfies the LARC at $p \in M$ is equivalent to the existence of a function, whose image is contained in an arbitrarily small neighborhood of p, such that the tangent space of M at every q in the image of f equals the sum of the distribution spanned by $\{X_1, \ldots, X_m\}$ at q and the image of the tangent map associated to f at q. Any function f satisfying these conditions is called ***transverse*** for the set $\{X_1, \ldots, X_m\}$ near p. Notice that, in general, the sum in (3.1) is not direct, namely κ may be larger than $n - m$. However, when the manifold M has a Lie group structure, and the vector fields $X_1, \ldots, X_m$ are left-invariant, the sum becomes direct and f turns out to be an immersion. In such a case there exists an explicit method, outlined in the next section, to construct transverse functions.

3.2. Construction of transverse functions for systems on Lie groups

As we have already mentioned, the construction of transverse functions can be readily prescribed when $M = G$ is an n-dimensional Lie group and the elements of $\{X_1, \ldots, X_m\}$ are left-invariant, linearly independent, smooth vector fields defined on G, and the set satisfies the Lie Algebra Rank Condition at e, the identity element in G.

The requirement that the set of controlled vector fields satisfy the LARC at $\widehat{e}$, i.e. $T_{\widehat{e}}G = \{X_{\widehat{e}} : X \in \mathrm{Lie}(\{X_1, \ldots, X_m\})\}$ is equivalent, for any point g in G, to

$$T_gG = \{X_g : X \in \mathrm{Lie}(\{X_1, \ldots, X_m\})\},$$

since by virtue of the left-invariance of the vector fields in $\mathrm{Lie}(\{X_1, \ldots, X_m\})$ one readily transfers the LARC condition at $\widehat{e}$ to an analogous condition at g.

By virtue of the stated assumptions, there exists a function $f : \mathbb{T}^{n-m} \longrightarrow G$ such that

$$T_{f(\theta)}G = \operatorname{span}_{\mathbb{R}}(\{X_{1f(\theta)}, \dots, X_{mf(\theta)}\}) \oplus T_\theta f(T_\theta \mathbb{T}^{n-m}) \tag{3.2}$$

A possible choice for the transverse function f can be described by the following procedure [14]. Let $\xi_i \in \mathfrak{g}$ be the vector associated to X_i $(i = 1, \dots, m)$, i.e. $\xi_i = X_i(\widehat{e})$. Next define a family $\{G_k : k \in \mathbb{N}\}$ of subspaces of $\mathfrak{g}$, by setting

$$G_k \triangleq \operatorname{span}_{\mathbb{R}}(\{[X_{i_1}, [X_{i_2}, [\dots, [X_{i_{j-1}}, X_{i_j}] \cdots]]] : i_1, \dots, i_j \leq m,\ j \leq k\})$$

Consider two mappings $\lambda, \rho : \{m+1, \dots, n\} \longrightarrow \{1, \dots, n\}$ together with an ordered basis $\{\zeta_1, \dots, \zeta_n\}$ of $\mathfrak{g}$, such that:

1. $G_k = \operatorname{span}_{\mathbb{R}}(\{\zeta_1, \dots, \zeta_{\dim(G_k)}\})$, for $k = 1, \dots, \min\{k : G_k = \mathfrak{g}\}$
2. Whenever $k \geq 2$ and $\dim(G_{k-1}) < i \leq \dim(G_k)$, one has $\zeta_i = [\zeta_{\lambda(i)}, \zeta_{\rho(i)}]$, with $\zeta_{\lambda(i)} \in G_a$, $\zeta_{\rho(i)} \in G_b$ and $a + b = k$.

The set $\{\zeta_1, \dots, \zeta_n\}$, together with the mappings λ and ρ constitute what is termed a ***graded basis*** for $\mathfrak{g}$. With this graded basis one can associate a weight vector $(r_1, \dots, r_n)$ such that $r_i = k$ iff $\zeta_i \in G_k \backslash G_{k-1}$.

Given a graded basis of $\mathfrak{g}$, a transverse function f is constructed by selecting strictly positive real numbers $\varepsilon_{m+1}, \dots, \varepsilon_n$ and by defining mappings $f_i : \mathbb{T} \longrightarrow G$, $i = m+1, \dots, n$, given in coordinates by

$$f_i(\theta) = \exp\left(\varepsilon_i^{r_{\lambda(i)}} \sin(\theta) X_{\zeta_{\lambda(i)}} + \varepsilon_i^{r_{\rho(i)}} \cos(\theta) X_{\zeta_{\rho(i)}}\right). \tag{3.3}$$

Once these mappings are defined, the expression in coordinates of a transverse function, $f : \mathbb{T}^{n-m} \longrightarrow G$, is given by

$$f(\theta_{m+1}, \dots, \theta_n) = f_n(\theta_n) f_{n-1}(\theta_{n-1}) \cdots f_{m+1}(\theta_{m+1}) \tag{3.4}$$

where $(\theta_{m+1}, \dots, \theta_n)$ are coordinates on $\mathbb{T}^{n-m}$. Notice that by choosing the positive reals $\varepsilon_{m+1}, \dots, \varepsilon_n$ appropriately one may ensure that the image of f is contained in an arbitrarily small neighborhood of $\widehat{e}$.

3.3. Transverse function control technique for systems on Lie groups

Consider a control system of the form

$$\dot{g} = D(g,t) + \sum_{i=1}^{m} u^i X_i(g) \tag{3.5}$$

where g represents a curve on an n-dimensional Lie group G, $D(g,t)$ is a drift term that depends continuously on t and takes values in the tangent space of G at g (D might be seen as a perturbing term that may be constant with respect its second argument), and $X_1, \dots, X_m$ are linearly independent, left-invariant smooth vector fields on G satisfying the Lie Algebra Rank Condition at $\widehat{e} \in G$. Note that we can choose any $p \in G$ such that (3.5) satisfies the LARC at p, as remarked in the previous section.

Without loss of generality we assume that $\{X_1, \dots, X_m\}$ is linearly independent, otherwise one may apply an input transformation to (3.5) such that the resulting control vector fields become linearly independent.

Given that $\{X_1, \dots, X_m\}$ satisfies the LARC at $\widehat{e}$ there exists a function $f : \mathbb{T}^{n-m} \longrightarrow G$ such that:

$$T_{f(\theta)}G = \mathrm{span}_{\mathbb{R}}(\{X_{1f(\theta)}, \dots, X_{mf(\theta)}\}) \oplus T_\theta f(T_\theta \mathbb{T}^{n-m}). \tag{3.6}$$

One way to take advantage from the transversality condition (3.6) is to adjoin, to system (3.5), an auxiliary system which allows us to control the system also along the image of Tf. Consider this auxiliary system to be

$$\dot{\theta} = \sum_{i=1}^{n-m} w^i \Theta_i(\theta) \tag{3.7}$$

where $\theta : I \longrightarrow \mathbb{T}^{n-m}$ and $\{\Theta_1, \dots, \Theta_{n-m}\}$ is a global frame for $T\mathbb{T}^{n-m}$, that is $T_\theta \mathbb{T}^{n-m} = \mathrm{span}_{\mathbb{R}}(\{\Theta_1(\theta), \dots, \Theta_{n-m}(\theta)\})$ for every $\theta \in \mathbb{T}^{n-m}$. Such a global frame exists because of the triviality of $T\mathbb{T}^{n-m}$ as vector bundle. If $(\theta^1, \dots, \theta^{n-m})$ are local coordinates for $\mathbb{T}^{n-m}$ by using the frame $\{\Theta_1, \dots, \Theta_{n-m}\}$ defined in coordinates by $\Theta_i = \frac{\partial}{\partial \theta^i}$ ($i = 1, \dots, n-m$), (3.7) can be written as $\dot{\theta} = w$, where w is an $\mathbb{R}^{n-m}$-valued control input.

One defines an error signal z, by using the group structure of the manifold where the system evolves, namely

$$z \triangleq g \cdot f(\theta)^{-1}. \tag{3.8}$$

The error quantifies the difference between the state g of System (3.5) and the image by f of the state θ of the auxiliary system. The error equals $\widehat{e}$ iff $g = f(\theta)$. By differentiating the error along the trajectories of the composite system (3.5) and (3.7), we obtain the error dynamics to be

$$\begin{aligned}\dot{z} \;=\; & T_{z\cdot f(\theta)}\widehat{R}_{f(\theta)^{-1}} \circ T_{f(\theta)}\widehat{L}_z \left(\sum_{i=1}^{m} u^i X_{i\,f(\theta)} - \sum_{j=1}^{n-m} w^j\, T_\theta f(\Theta_j(\theta)) \right. \\ & \left. + T_{z\cdot f(\theta)}\widehat{L}_{z^{-1}}\left(D(z\cdot f(\theta),t)\right) \right) \end{aligned} \tag{3.9}$$

From (3.6) one deduces that given any vector field $Z \in \Gamma(TG)$, there exists a smooth feedback function $(u(z,\theta), \dot{\theta}(z,\theta))$ defined by

$$\sum_{i=1}^{m} u^i X_{i\,f(\theta)} - \sum_{j=1}^{n-m} w^j\, T_\theta f(\Theta_j(\theta)) = T_{z\cdot f(\theta)}\widehat{L}_{z^{-1}} \left(T_z\widehat{R}_{f(\theta)}(Z_z) - D(z\cdot f(\theta),t)\right)$$

such that (3.9) takes the form $\dot{z} = Z_z$.
It suffices to choose for Z a vector field having $\widehat{e}$ as an asymptotically stable point (assuming that the projection of $D(\widehat{e},t)$ onto $(\mathrm{span}_{\mathbb{R}}(\{X_{1\widehat{e}},\ldots,X_{m\widehat{e}}\}))^{\perp}$ tends to zero as t tends to infinity) to make the error dynamics $z(t)$ tend to $\widehat{e}$ as $t\to\infty$. The state g of the resulting closed loop system

$$\dot{g} = D(g,t) + \sum_{i=1}^{m} u^i(z,\theta)X_i(g)$$

tends to the image of f, i.e. as $t \to \infty$, $g(t) \to f(\theta(t))$, and since $f(\mathbb{T}^{n-m})$ is a neighborhood of e specified beforehand, one obtains practical stabilization [14, Proposition 1].

3.4. Example: The chained form system

The purpose of this section is to illustrate how the concepts recalled in this chapter may be put to use on a specific example. We show how to apply the transverse function control approach to the problem of stabilization to a fixed point for the chained form system evolving on $\mathbb{R}^3$, which has a Lie group structure that differs from the usual one (derived from its vector space structure) and which is useful for control.
In this example we define a Lie group law of composition, construct a transverse function around $\widehat{e} \in \mathbb{R}^3$, define an auxiliary system and set the error signal as in (3.8), and then we obtain the error dynamics. Then we produce a feedback function that renders the identity element $\widehat{e}$ asymptotically stable for the error dynamics. This ensures practical stabilization of the chained form system towards the identity $\widehat{e}$. We also present a numerical simulation which illustrates the performance of the feedback control law.

Consider the ***chained form*** system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 \end{aligned}$$

which can be written as

$$\dot{x} = u_1 X_1(x) + u_2 X_2(x) \tag{3.10}$$

where X_1, X_2 are vector fields on $\mathbb{R}^3$ with expressions in coordinates $x = (x_1, x_2, x_3)$ given by

$$X_1(x) = \left.\frac{\partial}{\partial x^1}\right|_x + x_2 \left.\frac{\partial}{\partial x^3}\right|_x, \quad X_2(x) = \left.\frac{\partial}{\partial x^2}\right|_x.$$

The configuration manifold $\mathbb{R}^3$ can be endowed with a group multiplication $\widehat{\mu} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that turns it into a Lie group, by setting, for every x, y in $\mathbb{R}^3$,

$$\widehat{\mu}(x,y) = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_2 y_1 \end{pmatrix}.$$

The inverse of an element x in $\mathbb{R}^3$, and the identity element $\widehat{e}$ are given respectively by

$$x^{-1} = \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 + x_1 x_2 \end{pmatrix} \quad \text{and} \quad \widehat{e} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Next, let us verify that the vector fields X_1 and X_2 are left-invariant with respect to the group composition defined above. For any x in $\mathbb{R}^3$, the left translation by x is defined by $\widehat{L}_x(y) = \widehat{\mu}(x,y)$ for all $y \in \mathbb{R}^3$, so the tangent map associated to $\widehat{L}_x$ at y, i.e. $T_y\widehat{L}_x : T_y\mathbb{R}^3 \longrightarrow T_{\widehat{L}_x(y)}\mathbb{R}^3$, is represented by

$$T_y\widehat{L}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix}.$$

Given v in $T_y\mathbb{R}^3$, from its expression in coordinates we shall omit the base coordinates, writing only the fiber coordinates (v_1, v_2, v_3). Thus $T_y\widehat{L}_x(v)$ is given by

$$T_y\widehat{L}_x(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ x_2 v_1 + v_3 \end{pmatrix},$$

so one has

$$X_1(\widehat{L}_x(y)) = \begin{pmatrix} 1 \\ 0 \\ x_2 + y_2 \end{pmatrix}, \quad T_y\widehat{L}_x(X_1(y)) = T_y\widehat{L}_x\left(\begin{pmatrix} 1 \\ 0 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ x_2 + y_2 \end{pmatrix}$$

and

$$X_2(\widehat{L}_x(y)) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T_y\widehat{L}_x(X_2(y)) = T_y\widehat{L}_x\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, for every x, y in $\mathbb{R}^3$, $X_i(\widehat{L}_x(y)) = T_y\widehat{L}_x(X_i(y))$ for $i = 1, 2$, i.e. both X_1 and X_2 are left-invariant with respect the group multiplication $\widehat{\mu}$.

The Lie bracket of X_1 and X_2, $[X_1, X_2]$, is given by

$$[X_1, X_2] = -\frac{\partial}{\partial x^3}$$

therefore the tangent space of $\mathbb{R}^3$ at $\widehat{e}$, $T_{\widehat{e}}\mathbb{R}^3$, is generated by linear combinations of $X_1(\widehat{e})$, $X_2(\widehat{e})$ and $[X_1, X_2](\widehat{e})$. Hence System (3.10) satisfies the LARC at $\widehat{e}$.

The set of linearly independent, left-invariant, control vector fields is $\{X_1, X_2\}$, thus $m = 2$. The vectors ξ_1, ξ_2 and ξ_3 in $T_{\widehat{e}}\mathbb{R}^3$, associated with X_1, X_2 and $[X_1, X_2]$, respectively, are

$$\xi_1 = X_1(\widehat{e}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_2 = X_2(\widehat{e}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \xi_3 = X_3(\widehat{e}) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

The dimension of the Lie algebra spanned by $\{X_1, X_2\}$ is $n = 3$, $\mathfrak{g} = \mathrm{Lie}(\{\xi_1, \xi_2\}) = \mathrm{span}_{\mathbb{R}}(\{\xi_1, \xi_2, \xi_3\})$, since every Lie bracket involving three or more vectors equals zero.

Now let us obtain the subspaces $\{G_k : k \in \mathbb{N}\}$. By construction we have

$$G_1 = \mathrm{span}_{\mathbb{R}}(\{\xi_1, \xi_2\}) \quad G_2 = \mathrm{span}_{\mathbb{R}}(\{\xi_1, \xi_2, [\xi_1, \xi_2]\}),$$

notice that $G_2 = \mathfrak{g}$, hence $K = 2$.
The ordered set $\{\xi_1, \xi_2, \xi_3\}$ ($\xi_3 = [\xi_1, \xi_2]$) with $\xi_i \preccurlyeq \xi_j$ iff $i \leq j$, is an ordered basis for $\mathfrak{g}$. One checks that

$$G_1 = \mathrm{span}_{\mathbb{R}}\{\xi_1, \xi_2\} \qquad G_2 = \mathrm{span}_{\mathbb{R}}\{\xi_1, \xi_2, \xi_3\}$$

Take $k = 2$ and $2 = \dim(G_1) < i \leq \dim(G_k) = 3$, i.e. $i = 3$. Next consider mappings

$\lambda, \rho : \{3\} \longrightarrow \{1,2,3\}$ defined by $\lambda(3) = 1$ and $\rho(3) = 2$. Hence

$$\xi_3 = [\xi_{\lambda(3)}, \xi_{\rho(3)}] = [\xi_1, \xi_2].$$

The set $\{\xi_1, \xi_2, \xi_3\}$, in addition to the mappings λ and ρ, define a graded basis for $\mathfrak{g}$. We may consider the weight vector $(r_1, r_2, r_3) = (1, 1, 2)$, since $\xi_1, \xi_2 \in G_1$ and $\xi_3 \in G_2 \backslash G_1$.

Take (U, θ) to be a coordinate chart for $\mathbb{T}$, for example $U = S^1 \backslash \{(0,1)\}$ and $\theta((p_1, p_2)) = 2 \arctan\left(\frac{p_1}{1-p_2}\right)$. Let $\varepsilon > 0$ and $f : \mathbb{T} \longrightarrow \mathbb{R}^3$ be defined by

$$f(\theta) = \exp\left(\varepsilon^{r_{\lambda(3)}} \sin(\theta) X_{\xi_{\lambda(3)}} + \varepsilon^{r_{\rho(3)}} \cos(\theta) X_{\xi_{\rho(3)}}\right).$$

Thus $f(\theta) = \exp(X_\theta)$, where

$$\begin{aligned} X_\theta(x) &= \varepsilon^{r_{\lambda(3)}} \sin(\theta) X_{\xi_{\lambda(3)}}(x) + \varepsilon^{r_{\rho(3)}} \cos(\theta) X_{\xi_{\rho(3)}}(x) \\ &= \varepsilon \sin(\theta) \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} + \varepsilon \cos(\theta) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

and therefore, the vector field X_θ given in coordinates is

$$X_\theta(x) = \begin{pmatrix} \varepsilon \sin(\theta) \\ \varepsilon \cos(\theta) \\ \varepsilon \sin(\theta)\, x_2 \end{pmatrix}.$$

$f(\theta)$ equals the solution of the differential equation $\dot{x} = X_\theta(x)$ at time $t = 1$ and with initial condition $x_0 = e$. Therefore

$$f(\theta) = \left(\varepsilon \sin(\theta),\, \varepsilon \cos(\theta),\, \frac{1}{4}\, \varepsilon^2 \sin(2\,\theta)\right). \tag{3.11}$$

Figure 3.1 shows the plot of f with $\varepsilon = 1$, together with the vector fields X_1 and X_2 evaluated at $f(\theta)$ for some $\theta \in \mathbb{T}$ and $T_\theta f(\omega)$ for some $\omega \in T_\theta \mathbb{T}$.

We define the error signal z by

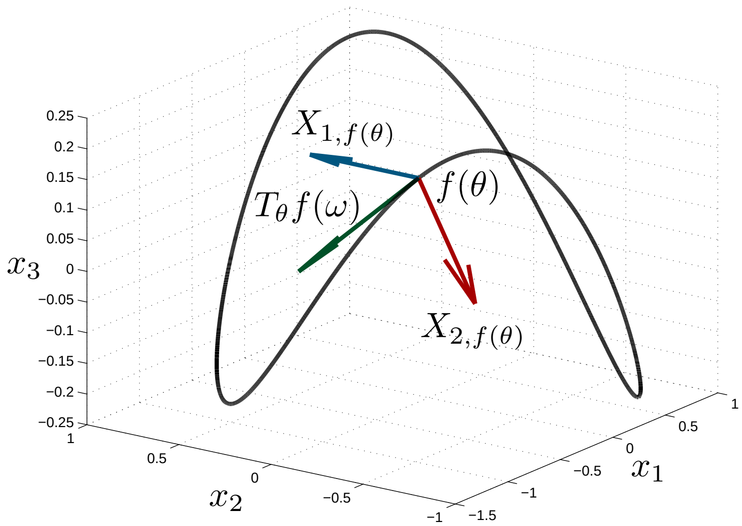

Figure 3.1: Plot of f (Equation 3.11).

$$z = \widehat{\mu}(x, f(\theta)^{-1}) = \begin{pmatrix} x_1 - \varepsilon\,\sin(\theta) \\ x_2 - \varepsilon\,\cos(\theta) \\ x_3 + \frac{1}{4}\,\varepsilon^2\sin(2\,\theta) - x_2\,\varepsilon\,\sin(\theta) \end{pmatrix}$$

where $\theta \in \mathbb{T}$ is the state of the auxiliary system $\dot{\theta} = \alpha$.

According to (3.9), the error dynamics is

$$\dot{z} = \begin{pmatrix} u_1 - \alpha\,\varepsilon\,\cos(\theta) \\ u_2 + \alpha\,\varepsilon\,\sin(\theta) \\ u_1 z_2 + u_1\varepsilon\,\cos(\theta) - \varepsilon\,\sin(\theta) u_2 - \alpha\,\varepsilon\,\cos(\theta) z_2 - \frac{1}{2}\,\alpha\,\varepsilon^2 \end{pmatrix}$$

which can be rewritten, in matrix notation, as

$$\dot{z} = \begin{pmatrix} 1 & 0 & -\varepsilon\,\cos(\theta) \\ 0 & 1 & \varepsilon\,\sin(\theta) \\ z_2 + \varepsilon\,\cos(\theta) & -\varepsilon\,\sin(\theta) & -\varepsilon\,\cos(\theta) z_2 - \frac{1}{2}\,\varepsilon^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \alpha \end{pmatrix}$$

Consider the vector field $Z \in \Gamma(T\mathbb{R}^3)$ defined by $Z(z) = -kz$ for $z \in \mathbb{R}^3$ with k a strictly positive real. Clearly this vector field has $\widehat{e}$ as an exponentially stable point. Let $u = (u_1, u_2, \alpha)$. Then the feedback function $u(z, \theta)$ needed to impose a closed-loop error dynamics of the form $\dot{z} = Z(z)$ is

$$u(z,\theta) = -k \begin{pmatrix} 1 & 0 & -\varepsilon\,\cos(\theta) \\ 0 & 1 & \varepsilon\,\sin(\theta) \\ z_2 + \varepsilon\,\cos(\theta) & -\varepsilon\,\sin(\theta) & -\varepsilon\,\cos(\theta) z_2 - \frac{1}{2}\,\varepsilon^2 \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

or, more explicitly,

$$u(z,\theta) = -2\,\frac{k}{\varepsilon} \begin{pmatrix} -\cos(\theta) z_2 - \frac{1}{2}\,\varepsilon\,\cos(2\,\theta) & \frac{1}{2}\,\varepsilon\,\sin(2\,\theta) & \cos(\theta) \\ \sin(\theta) z_2 + \frac{1}{2}\,\varepsilon\,\sin(2\,\theta) & \frac{1}{2}\,\varepsilon\,\cos(2\,\theta) & -\sin(\theta) \\ -\frac{1}{\varepsilon}\,(z_2 + \varepsilon\,\cos(\theta)) & \sin(\theta) & \varepsilon^{-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{3.12}$$

A numerical simulation of the chained form system with feedback control (3.12) was performed. The initial condition is $x_0 = (-2.5, 0.6, -1.5)$, the controller gain $k = -1.3$, and the value of ε is 0.25. Figure 3.2 presents plots of the trajectories of the chained form system and the trajectories of the error system. Figure 3.3 shows the time-history of the control input defined by (3.12).

One notes from Figure 3.2 that the error trajectories approach zero as t increases and, according to (3.8), the trajectories of the chained form system approach $f(\theta(t))$ as t increases, indeed the state x converges to a given fixed configuration near $f(\theta)$. This behavior can be observed in the plot of the system trajectories. The image of f can be modified by adjusting the value of ε, the smaller the value of ε, the nearer the system trajectories will be to zero.

In this chapter we have recalled the control technique proposed by Morin and Samson to stabilize driftless controllable systems. We applied this technique to control the chained form system equivalent, among others systems, to the unicycle-type robot. The chained form system is a system whose control vector fields satisfy accessibility, and in spite of its simplicity is a system that satisfies Brockett's condition,

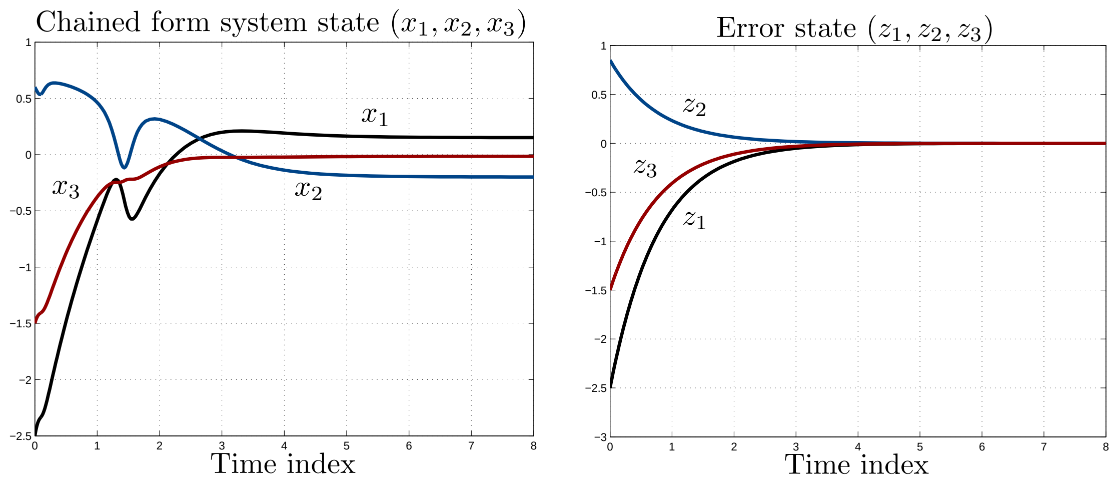

Figure 3.2: Plots of the state of the controlled Chained Form System and of the error function respectively.

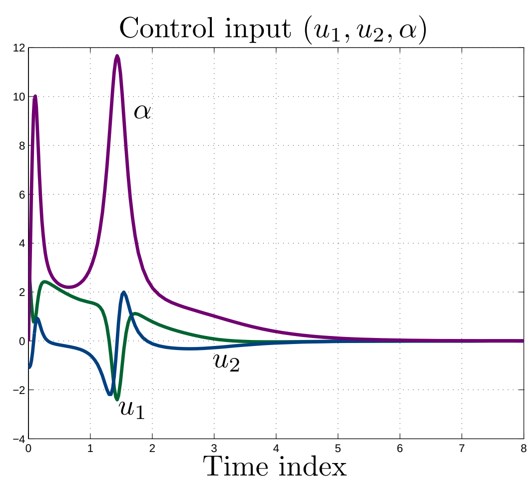

Figure 3.3: Plot of the control input (3.12) applied.

which means that it is a tough stabilization problem. The following chapter deals with second-order systems whose control fields alone do not generate the accessibility distribution. Examples of this systems are vastly found in mechanical systems.

Chapter 4

Vertically transverse functions and their application to control

This chapter contains the main contributions of this thesis, among which the characterization of certain functions we choose to call vertically transverse functions. These functions, based on the transverse functions proposed by Morin and Samson, generalize the property of transversality for second-order systems as we will show later. A possible application of the use of this newly characterized property in the control of second-order systems evolving on Lie groups is outlined.

4.1. Introduction

Now we have enough mathematical background to state the problem in more detail. We have already recalled the transverse function control approach proposed by Morin and Samson to control driftless controllable systems. As remarked earlier, this control technique can deal with two classical control problems, point stabilization and trajectory tracking for systems of the form

$$\dot{x} = f(x,t) + \sum_{i=1}^{m} u^i g_i(x)$$

where $g_1, \ldots, g_m$ represent smooth vector fields on a smooth manifold M and f is a time-varying smooth vector field defined on M (f may represent possibly null additive

disturbances to the system).
In order to use this approach it is necessary that the controlled vector fields $g_1, \ldots, g_m$ ensure local accessibility at the desired stabilization point (or, in the case of trajectory tracking, in a subset of M). As a result, this approach cannot be directly applied to systems in which the drift term f is essential to ensure local accessibility.

Consider now a second-order system

$$\dot{v} = D(v) + \sum_{i=1}^{m} u^i X_i^{\text{lift}}(v) \tag{4.1}$$

where $v : I \longrightarrow TM$ is a curve on the tangent bundle of a smooth manifold M, D is a smooth, second-order vector field on TM, and $X_1, \ldots, X_m$ are smooth vector fields on M (recall that if $X \in \Gamma(TM)$ then $X^{\text{lift}} \in \Gamma(TTM)$).
Assume that the set $\{X_1, \ldots, X_m\}$ satisfies the Lie Algebra Rank Condition at $\pi_M(v)$ for some $v \in TM$. Note that the Lie Algebra spanned by $\{X_1^{\text{lift}}, \ldots, X_m^{\text{lift}}\}$ cannot generate the tangent space of TM at v, therefore the drift term D of system (4.1) is essential to attain local accessibility. However, as we do not have control over the drift vector field it is not clear how to proceed to obtain stabilization to a given trajectory by means of the transverse function control approach.

In particular, one cannot apply the approach to System (4.1) since the control vector fields do not satisfy the LARC at the desired stabilization point. The objective then, is to provide an extension of the transverse function formalism to this class of systems.

Although the class of systems considered has a given structure and this apparently restricts the applicability of the approach proposed in this thesis, it is important to notice that it encompasses a wide class of second-order systems. For example, a subclass of systems of the form (4.1) are simple mechanical control systems,

$$\nabla_{\dot{q}} \dot{q} = -\mathcal{G}^\sharp \circ dV(q) + \sum_{i=1}^{m} u^i \mathcal{G}^\sharp \circ F^i(q) \tag{4.2}$$

where $q : I \longrightarrow Q$ is a curve on the configuration manifold Q (positions and orientations), $\mathcal{G}^\sharp$ is the canonical map $T^*Q \longrightarrow TQ$, associated with the Riemannian metric $\mathcal{G}$, $V \in C^\infty(Q)$ represents a potential energy function, F^i $(i = 1, \ldots, m)$ are smooth

1-forms physically representing forces or torques and finally ∇ is the connection associated to the Riemannian metric $\mathcal{G}$.
Simple mechanical control systems subject to constrains have essentially the same structure as (4.2), except that ∇ no longer represents the Levi-Cività connection, and the vector fields are modified depending on the constrain codistribution [10].

Equation (4.2) can be recast as a system evolving on TM with dynamics

$$\dot{v} = S_v - (\mathcal{G}^\sharp \circ dV)^{\text{lift}}(v) + \sum_{i=1}^{m} u^i (\mathcal{G}^\sharp \circ F^i)^{\text{lift}}(v),$$

where S is the geodesic spray associated with the connection ∇. It is clear that if we take $D = S - (\mathcal{G}^\sharp \circ dV)^{\text{lift}}$ and $X_i = \mathcal{G}^\sharp \circ F^i$ $(i = 1, \ldots, m)$, the class of simple mechanical control systems (both, subject and not subject to constrains) fits into Equation (4.1). The cases $m = \dim(M)$ (fully actuated mechanical system) and $m < \dim(M)$ (underactuated mechanical system), are also included.

4.2. Vertically transverse functions

Having already discussed the transversality property for functions with respect to a set of vector fields in Chapter 3, we show how the tangent mappings associated with transverse functions define vertically transverse functions, which may be regarded as second-order generalizations of transverse functions for second-order systems.

Prior to defining vertical transversality, let us present the following lemma regarding the way the tangent tangent mapping of a differentiable function acts on vertical vectors.

Lemma 1 *Let M and N denote smooth manifolds and let $f : M \longrightarrow N$ be a C^2 mapping. Then:*

1. *TTf maps vertical vectors in TTM into vertical vectors in TTN.*

2. *For every $v, w \in TM$ such that $\pi_M(v) = \pi_M(w)$ we have*

$$TTf(\text{lift}(v, w)) = \text{lift}(Tf(v), Tf(w)).$$

Proof: ***(1.)*** By basic notions regarding tangent bundles and mappings, the following diagram commutes

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

that is, $f \circ \pi_M = \pi_N \circ Tf$. As a consequence, $T(f \circ \pi_M) = T(\pi_N \circ Tf)$ and, by the chain rule, we have $Tf \circ T\pi_M = T\pi_N \circ TTf$, i.e. the following diagram commutes

$$\begin{tikzcd} TTM \arrow[r, "TTf"] \arrow[d, "T\pi_M"'] & TTN \arrow[d, "T\pi_N"] \\ TM \arrow[r, "Tf"] & TN \end{tikzcd}$$

Now let $v \in TM$ and $\xi \in T_vTM^{\text{vert}}$, thus $Tf \circ T\pi_M(\xi) = T\pi_N \circ TTf(\xi)$. Since ξ is vertical it satisfies $T\pi_M(\xi) = 0$ and, by the linearity of Tf we have $T\pi_N \circ TTf(\xi) = 0$, therefore $TTf(\xi) \in T_{Tf(v)}TN^{\text{vert}}$, i.e. $TTf(\xi)$ is vertical, as stated.

(2.) Let $v, w \in TM$ be such that $\pi_M(v) = \pi_M(w)$. Define the curve $\gamma_{v,w} : \mathbb{R} \longrightarrow T_{\pi_M(v)}M$ as $\gamma_{v,w}(t) = v + tw$ for each $t \in \mathbb{R}$. Thus

$$\operatorname{lift}(v, w) = T_0\gamma_{v,w}\left(\left.\frac{\partial}{\partial r}\right|_0\right).$$

By virtue of the linearity of T_pf for every p in M one has

$$Tf \circ \gamma_{v,w}(t) = Tf(v + tw) = Tf(v) + t\,Tf(w) = \gamma_{Tf(v),Tf(w)}(t) \quad \text{for } t \in \mathbb{R}$$

i.e. $Tf \circ \gamma_{v,w} = \gamma_{Tf(v),Tf(w)}$ thus $T(Tf \circ \gamma_{v,w}) = T\gamma_{Tf(v),Tf(w)}$, then $TTf \circ T\gamma_{v,w} = T\gamma_{Tf(v),Tf(w)}$. In particular we have

$$TTf \circ T_0\gamma_{v,w}\left(\left.\frac{\partial}{\partial r}\right|_0\right) = T_0\gamma_{Tf(v),Tf(w)}\left(\left.\frac{\partial}{\partial r}\right|_0\right).$$

Therefore $TTf(\operatorname{lift}(v, w)) = \operatorname{lift}(Tf(v), Tf(w))$ as required. ∎

Lemma 2 *Let M and N be differentiable manifolds and let $f \in C^2(M; N)$, then the Liouville vector field on TM is Tf-related to the Liouville vector field on TN.*

Proof: Let $\widehat{C}$ denote the Liouville vector field on TM and C the Liouville vector field on TN. Recall that $\widehat{C}$ is a vector field on TM defined by $\widehat{C}(v) = \mathrm{lift}(v, v)$ for all $v \in TM$ (an analogous definition holds for C). For $\widehat{C}$ to be Tf-related to C the following diagram must commute

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\widehat{C}"'] & TN \arrow[d, "C"] \\ TTM \arrow[r, "TTf"'] & TTN, \end{tikzcd}$$

i.e. one must have $C \circ Tf = TTf \circ \widehat{C}$. This is equivalent, using the definition of the Liouville vector field, to $\mathrm{lift}(Tf(\cdot), Tf(\cdot)) = TTf(\mathrm{lift}(\cdot, \cdot))$. But, by virtue of Lemma (1), we have $\mathrm{lift}(Tf(v), Tf(w)) = TTf(\mathrm{lift}(v, w))$ for every v, w in TM, so, in particular $\mathrm{lift}(Tf(v), Tf(v)) = TTf(\mathrm{lift}(v, v))$. Therefore $\widehat{C}$ is Tf-related to C. ■

Let us define the vertical transversality condition.

Definition 7 *Let M be a n-dimensional manifold and $\{X_1, \ldots, X_m\}$ a set of vertical vector fields defined on TM. A bundle map $F : T\mathbb{T}^\kappa \longrightarrow TM$ ($\kappa \geqslant n - m$) such that*

$$T_{F(\omega)}TM^{\mathrm{vert}} = T_\omega F((T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}) + \mathrm{span}_{\mathbb{R}}(\{X_1(F(\omega)), \ldots, X_m(F(\omega))\})$$

*is said to be **vertically transverse** for the set $\{X_1, \ldots, X_m\}$.*

Let M be a smooth n-dimensional manifold, and $X_1, \ldots, X_m$ smooth vector fields on M such that the set $\{X_1, \ldots, X_m\}$ satisfies the Lie Algebra Rank Condition (LARC) at a given point $p \in M$, i.e.

$$T_pM = \{X_p : X \in \mathrm{Lie}(\{X_1, \ldots, X_m\})\}$$

If f is a transverse function for $\{X_1, \ldots, X_m\}$ near p then Tf is vertically transverse for the set of the vertically lifted vector fields i.e. the distribution spanned by $\{X_1^{\mathrm{lift}}, \ldots, X_m^{\mathrm{lift}}\}$ along the image of Tf, together with the image by TTf of the vertical subbundle $(TT\mathbb{T}^\kappa)^{\text{vert}}$ ($m \leq \kappa \leq n$), generates the vertical space of TTM along the image of Tf. Let us proceed to formally state this proposition.

Proposition 4 *Let $\{X_1, \ldots, X_m\}$ be a set of smooth vector fields on M satisfying the LARC at $p \in M$. Let $f : \mathbb{T}^\kappa \longrightarrow M$ be a transverse function for $\{X_1, \ldots, X_m\}$ near p so that, for every $\theta \in \mathbb{T}^\kappa$,*

$$T_{f(\theta)}M = T_\theta f(T_\theta \mathbb{T}^\kappa) + \operatorname{span}_{\mathbb{R}}(\{X_1(f(\theta)), \ldots, X_m(f(\theta))\}). \tag{4.3}$$

Then Tf is vertically transverse for $\{X_1^{\text{lift}}, \ldots, X_m^{\text{lift}}\}$, i.e., for every $\omega \in T\mathbb{T}^\kappa$,

$$T_{Tf(\omega)}TM^{\text{vert}} = T_\omega Tf((T_\omega T\mathbb{T}^\kappa)^{\text{vert}}) + \operatorname{span}_{\mathbb{R}}(\{X_1^{\text{lift}}(Tf(\omega)), \ldots, X_m^{\text{lift}}(Tf(\omega))\}). \tag{4.4}$$

Furthermore, if f is such that the sum in (4.3) *is direct ($\kappa = n - m$), then the sum in* (4.4) *is also direct.*

Proof: Let $\omega \in T_\theta \mathbb{T}^\kappa$ with $\theta \in \mathbb{T}^\kappa$, and assume that $v \in T_{Tf(\omega)}TM^{\text{vert}}$. Since v is vertical, there exists $\widetilde{v} \in T_{f(\theta)}M$ such that $\operatorname{lift}(Tf(\omega), \widetilde{v}) = v$. Making use of (4.3) one concludes that there exist $\overline{\omega} \in T\mathbb{T}^\kappa$ and real numbers $a^1, \ldots, a^m$ such that

$$\widetilde{v} = T_\theta f(\overline{\omega}) + \sum_{i=1}^{m} a^i X_i(f(\theta)).$$

Then

$$v = \operatorname{lift}\left(Tf(\omega), T_\theta f(\overline{\omega}) + \sum_{i=1}^{m} a^i X_i(f(\theta))\right).$$

Since $\operatorname{lift}(Tf(\omega), \cdot)$ is linear one has

$$v = \operatorname{lift}(Tf(\omega), T_\theta f(\overline{\omega})) + \sum_{i=1}^{m} a^i \operatorname{lift}\left(Tf(\omega), X_i(f(\theta))\right),$$

by Lemma 1-(*2*) and by the definition of the lift of a vector field one has

$$v = T_\omega Tf(\operatorname{lift}(\omega, \overline{\omega})) + \sum_{i=1}^{m} a^i X_i^{\text{lift}}(Tf(\omega))$$

given that $\mathrm{lift}(\omega, \overline{\omega}) \in (T_\omega T\mathbb{T}^\kappa)^{\mathrm{vert}}$ this concludes the first part of the proof.

Now assume that the sum in (4.3) is direct, i.e.

$$T_{f(\theta)}M = T_\theta f(T_\theta \mathbb{T}^{n-m}) \oplus \mathrm{span}_{\mathbb{R}}(\{X_1(f(\theta)), \ldots, X_m(f(\theta))\}) \tag{4.5}$$

and suppose that there exists $v \in T_{Tf(\omega)}TM^{\mathrm{vert}}$ such that

$$v \in T_\omega Tf((T_\omega T\mathbb{T}^{n-m})^{\mathrm{vert}}) \cap \mathrm{span}_{\mathbb{R}}(\{X_1^{\mathrm{lift}}(Tf(\omega)), \ldots, X_m^{\mathrm{lift}}(Tf(\omega))\}).$$

We shall show that the unique v satisfying this is $v = 0$. Thus there exist $\alpha \in (T_\omega T\mathbb{T}^{n-m})^{\mathrm{vert}}$ and real numbers $a^1, \ldots, a^m$ such that

$$v = TTf(\alpha) = \sum_{i=1}^m a^i X^{\mathrm{lift}}(Tf(\omega))$$

Given that α is vertical, $\alpha = \mathrm{lift}(\omega, \overline{\omega})$ for some $\overline{\omega} \in T_\theta \mathbb{T}^{n-m}$. Then $v = TTf(\alpha) = TTf(\mathrm{lift}(\omega, \overline{\omega}))$ and by Lemma 1-(*2*) one obtains $v = \mathrm{lift}(Tf(\omega), Tf(\overline{\omega}))$. On the other hand

$$\begin{aligned}
\sum_{i=1}^m a^i X^{\mathrm{lift}}(Tf(\omega)) &= \sum_{i=1}^m a^i \,\mathrm{lift}(Tf(\omega), X_i(f(\theta))) \\
&= \mathrm{lift}\left(Tf(\omega), \sum_{i=1}^m a^i X_i(f(\theta))\right),
\end{aligned}$$

thus $\mathrm{lift}(Tf(\omega), Tf(\overline{\omega})) = \mathrm{lift}\left(Tf(\omega), \sum_{i=1}^m a^i X_i(f(\theta))\right)$. The map $\mathrm{lift}(Tf(\omega), \cdot)$ is linear and injective, then

$$\sum_{i=1}^m a^i X_i(f(\theta)) = Tf(\overline{\omega})$$

However this in contradiction with the assumption that f satisfies (4.5), thus $\sum_{i=1}^m a^i X_i(f(\theta)) = Tf(\overline{\omega}) = 0$. From $v = \mathrm{lift}(Tf(\omega), Tf(\overline{\omega}))$ and using linearity of the mapping $\mathrm{lift}(Tf(\omega), \cdot)$ it is easy to deduce that $v = 0$. This concludes the proof. ■

4.3. Application of vertically transverse functions to control

In this section we inspect a possible application of vertically transverse functions to a class of second-order systems evolving on Lie groups. We focus on this class of systems since a large range of second-order and mechanical systems are naturally modeled as systems on Lie groups. For instance, mechanical systems which usually arise in physical applications and evolve on Lie groups are rigid bodies in space, cart-like vehicles, space and underwater robots which, in addition, may present some sort of invariance with respect the Lie group operation. Some examples are the hovercraft, the PPR manipulator, a rigid body in $\mathbb{R}^2$ or $\mathbb{R}^3$, the unicycle-type robot and the snakeboard [10], [7].

Consider a system with dynamics

$$\dot{v} = D(v) + \sum_{i=1}^{m} u^i X_i^{\mathrm{lift}}(v) \tag{4.6}$$

where $v : I \longrightarrow TG$ is a curve on the tangent space of an n-dimensional Lie group G, $D \in \Gamma(TTG)$ is a second-order vector field, and $X_i \in \Gamma(TG)$ $(i = 1, \ldots, m)$ are linearly independent, left-invariant, smooth vector fields on G such that

$$\mathrm{Lie}(\{X_{1,\widehat{e}}, \ldots, X_{m,\widehat{e}}\}) = \mathfrak{g}.$$

Given these conditions we can construct a transverse function $f : \mathbb{T}^{m-n} \longrightarrow G$, following the procedure recalled in Section 3.2 of Chapter 3, such that

$$T_{Tf(\omega)}TG^{\mathrm{vert}} = T_\omega Tf((T_\omega T\mathbb{T}^{n-m})^{\mathrm{vert}}) \oplus \mathrm{span}_{\mathbb{R}}(\{X_1^{\mathrm{lift}}(Tf(\omega)), \ldots, X_m^{\mathrm{lift}}(Tf(\omega))\}) \tag{4.7}$$

The next step is to dynamically extend the system by adding an auxiliary system evolving on $TT\mathbb{T}^{n-m}$. To do this we select a global frame for $(TT\mathbb{T}^{n-m})^{\mathrm{vert}}$, $\{\Omega_1, \ldots, \Omega_m\}$, consisting of vertical vector fields $\Omega_i \in \Gamma(TT\mathbb{T}^{n-m})$ $(i = 1, \ldots, n-m)$, i.e., for every $\omega \in T\mathbb{T}^{n-m}$

$$(T_\omega T\mathbb{T}^{n-m})^{\mathrm{vert}} = \mathrm{span}_{\mathbb{R}}(\{\Omega_1(\omega), \ldots, \Omega_{n-m}(\omega)\}).$$

The existence of such a global frame on $TT\mathbb{T}^{n-m}$ is assured by the triviality of $TT\mathbb{T}^{n-m}$ as a vector bundle, a consequence of the triviality of $T\mathbb{T}^{n-m}$.

The proposed auxiliary system is then

$$\dot{\omega} = \Delta_w + \sum_{i=1}^{n-m} w^i \Omega_{i,\omega} \tag{4.8}$$

where $\Delta \in TT\mathbb{T}^{n-m}$ is an arbitrary, smooth, second-order, vector field. Then we define an error signal z as the group composition of the original system state and the auxiliary system state, namely

$$z = \mu(v, Tf(\omega)^{-1}). \tag{4.9}$$

Roughly speaking, this error function is used to quantify the error difference between the state of the system and the image of the auxiliary system by Tf. Let us remark that one can use alternative error expressions, e.g. $z = Tf(\omega) \cdot v^{-1}$, for which the approach leads to analogous results.

At this stage one may wonder what the dynamics governing the evolution of the error is. Prior to continuing, let us present some results that will be useful later on. The first is a lemma that concerns the induced left-invariance of vertically lifted left-invariant vector fields.

Lemma 3 *Let G be a Lie group and let $X \in \Gamma(TG)$ be a vector field on G. If X is left invariant, then X^{lift} is a left-invariant vector field on the tangent Lie group TG (with respect to the Lie group operation induced by the one on G).*

Proof: We assume that for every $g, h \in G$, $X_{gh} = T_h\widehat{L}_g(X_h)$. We shall prove that $X^{\mathrm{lift}}(vw) = T_w L_v(X^{\mathrm{lift}}(w))$ for all v, w in TG.

Consider the curve $\gamma_{w,X_{\pi_G(w)}}$ on TG defined by $\gamma_{w,X_{\pi_G(w)}} : t \mapsto w + tX_{\pi_G(w)}$ with $t \in \mathbb{R}$, then

$$
\begin{aligned}
T_w L_v(X^{\mathrm{lift}}(w)) &= T_w L_v \circ T_0 \gamma_{w, X_{\pi_G(w)}} \left(\left.\frac{\partial}{\partial r}\right|_0 \right) \\
&= T_0(L_v \circ \gamma_{w, X_{\pi_G(w)}}) \left(\left.\frac{\partial}{\partial r}\right|_0 \right).
\end{aligned}
$$

However,

$$
\begin{aligned}
L_v \circ \gamma_{w, X_{\pi_G(w)}}(t) &= \mu(v, w + tX_{\pi_G(w)}) \\
&= T_{\pi_G(v)} \widehat{R}_{\pi_G(w)}(v) + T_{\pi_G(w)} \widehat{L}_{\pi_G(v)}(w + tX_{\pi_G(w)}) \\
&= T_{\pi_G(v)} \widehat{R}_{\pi_G(w)}(v) + T_{\pi_G(w)} \widehat{L}_{\pi_G(v)}(w) + t\, T_{\pi_G(w)} \widehat{L}_{\pi_G(v)}(X_{\pi_G(w)}) \\
&= \mu(v, w) + t\, T_{\pi_G(w)} \widehat{L}_{\pi_G(v)}(X_{\pi_G(w)})
\end{aligned}
$$

Given that X is left-invariant one has

$$
\begin{aligned}
L_v \circ \gamma_{w, X_{\pi_G(w)}}(t) &= vw + t\, X_{\pi_G(v) \pi_G(w)} \\
&= vw + t\, X_{\pi_G(vw)}
\end{aligned}
$$

Therefore $L_v \circ \gamma_{w, X_{\pi_G(w)}}(t) = \gamma_{vw, X_{\pi_G(vw)}}(t)$ for all $t \in \mathbb{R}$; thus

$$
\begin{aligned}
T_w L_v(X^{\mathrm{lift}}(w)) &= T_0 \gamma_{vw, X_{\pi_G(vw)}} \left(\left.\frac{\partial}{\partial r}\right|_0 \right) \\
&= \mathrm{lift}(vw, X_{\pi_G(vw)}) \\
&= X^{\mathrm{lift}}(vw),
\end{aligned}
$$

which shows that X^{lift} is left-invariant with respect μ. ■

Proposition 5 *Let G be a Lie group and TG its tangent Lie group. Let $X \in \Gamma(TTG)$ be a complete second-order vector field and assume that $Y \in \Gamma(TTG)$ is a second-order vector field defined along the curve $w : I \longrightarrow TG$ by $\dot{w}(t) = Y_{w(t)}$. Then:*

1. *If $v : I \longrightarrow TG$ is an integral curve of X, then the curve $z(t) = v(t) \cdot w^{-1}(t)$*

satisfies, for every $t \in I$,

$$\dot{z}(t) = T_{v(t)}R_{w^{-1}(t)}\left(X_{v(t)} - T_{w(t)}L_{z(t)}(Y_{w(t)})\right). \tag{4.10}$$

2. *The expression* (4.10) *represents a (non-autonomous) second-order differential equation on TG.*

Proof: **(*1.*)** Using Proposition 2 in Chapter 2 one has that if z equals $v(t) \cdot w^{-1}(t)$ then

$$\dot{z}(t) = T_{v(t)}R_{w^{-1}(t)}(X_{v(t)}) + T_{w^{-1}(t)}L_{v(t)}(\widetilde{Y}_{w^{-1}(t)}). \tag{4.11}$$

Now, from Proposition 3 in Chapter 2 one knows that if $\widetilde{Y}$ is the vector field along the curve $t \mapsto w^{-1}(t)$ defined by $\widetilde{Y}_{w^{-1}(t)} = \frac{d}{dt}w^{-1}(t)$, then the expression for $\widetilde{Y}$ is given by

$$\widetilde{Y}_{w^{-1}(t)} = -T_eL_{w^{-1}(t)} \circ T_{w(t)}R_{w^{-1}(t)}(Y_{w(t)}).$$

Substituting this expression in (4.11), using the associativity of left translations, in addition to the commutativity of right and left translations, one gets

$$\begin{aligned}
\dot{z}(t) &= T_{v(t)}R_{w^{-1}(t)}(X_{v(t)}) + T_{w^{-1}(t)}L_{v(t)}(-T_eL_{w^{-1}(t)} \circ T_{w(t)}R_{w^{-1}(t)}(Y_{w(t)})) \\
&= T_{v(t)}R_{w^{-1}(t)}(X_{v(t)}) - T_{w^{-1}(t)}L_{v(t)} \circ T_eL_{w^{-1}(t)} \circ T_{w(t)}R_{w^{-1}(t)}(Y_{w(t)}) \\
&= T_{v(t)}R_{w^{-1}(t)}(X_{v(t)}) - T_eL_{v \cdot w^{-1}(t)} \circ T_{w(t)}R_{w^{-1}(t)}(Y_{w(t)}) \\
&= T_{v(t)}R_{w^{-1}(t)}(X_{v(t)}) - T_{v(t)}R_{w^{-1}(t)} \circ T_{w(t)}L_{v \cdot w^{-1}(t)}(Y_{w(t)}) \\
&= T_{v(t)}R_{w^{-1}(t)}\left(X_{v(t)} - T_{w(t)}L_{v \cdot w^{-1}(t)}(Y_{w(t)})\right).
\end{aligned}$$

Recalling that $z = v \cdot w^{-1}$ this yields

$$\dot{z} = T_{v(t)}R_{w^{-1}(t)}\left(X_{v(t)} - T_{w(t)}L_{z(t)}(Y_{w(t)})\right)$$

(*2.*) Given $\widetilde{w} \in TG$ such that $\widetilde{w} = w(t)$ for some $t \in I$, we define a map $Z_{\widetilde{w}} : TG \longrightarrow TTG$ by

$$Z_{\widetilde{w}}(z) = T_{z \cdot \widetilde{w}}R_{\widetilde{w}^{-1}}\left(X_{z \cdot \widetilde{w}} - T_{\widetilde{w}}L_z(Y_{\widetilde{w}})\right)$$

From the latter expression it is straightforward to show that for every $t \in I$ and every $z \in TG$ one has $\pi_{TG} \circ Z_{\widetilde{\omega}}(z) = z$ therefore $Z_{\widetilde{\omega}}$ is a vector field defined on TG along z.

We need to check that $T\pi_G \circ Z_{\widetilde{w}} = \mathrm{id}_{TG}$ in order to show that $Z_{\widetilde{w}}$ represents a second-order differential equation. Note that, for any $\alpha \in TG$, the following diagrams commute

$$\begin{tikzcd} TG \arrow[r, "L_\alpha"] \arrow[d, "\pi_G"'] & TG \arrow[d, "\pi_G"] \\ G \arrow[r, "\widehat{L}_{\pi_G(\alpha)}"] & G \end{tikzcd} \qquad \begin{tikzcd} TG \arrow[r, "R_\alpha"] \arrow[d, "\pi_G"'] & TG \arrow[d, "\pi_G"] \\ G \arrow[r, "\widehat{R}_{\pi_G(\alpha)}"] & G \end{tikzcd}$$

since for every $\beta \in TG$, one has

$$\pi_G \circ L_\alpha(\beta) = \pi_G(\alpha \cdot \beta) = \pi_G(\alpha) \cdot \pi_G(\beta) = \widehat{L}_{\pi_G(\alpha)}(\pi_G(\beta)) = \widehat{L}_{\pi_G(\alpha)} \circ \pi_G(\beta),$$

and, likewise for the right translation,

$$\pi_G \circ R_\alpha(\beta) = \pi_G(\beta \cdot \alpha) = \pi_G(\beta) \cdot \pi_G(\alpha) = \widehat{R}_{\pi_G(\beta)}(\pi_G(\alpha)) = \widehat{R}_{\pi_G(\beta)} \circ \pi_G(\alpha).$$

Hence $T\pi_G \circ TL_\alpha = T\widehat{L}_{\pi_G(\alpha)} \circ T\pi_G$ and $T\pi_G \circ TR_\alpha = T\widehat{R}_{\pi_G(\alpha)} \circ T\pi_G$.
By using the relations found above, for any $z \in TG$, one has

$$\begin{aligned} T\pi_G \circ Z_{\widetilde{w}}(z) &= T\pi_G \circ T_{z\cdot\widetilde{w}} R_{\widetilde{w}^{-1}} \left(X_{z\cdot\widetilde{w}} - T_{\widetilde{w}} L_z(Y_{\widetilde{w}})\right) \\ &= T_{\pi_G(z\cdot\widetilde{w})} \widehat{R}_{\pi_G(\widetilde{w}^{-1})} \circ T_{z\cdot\widetilde{w}} \pi_G \left(X_{z\cdot\widetilde{w}} - T_{\widetilde{w}} L_z(Y_{\widetilde{w}})\right) \\ &= T_{\pi_G(z\cdot\widetilde{w})} \widehat{R}_{\pi_G(\widetilde{w}^{-1})} \left(T_{z\cdot\widetilde{w}} \pi_G(X_{z\cdot\widetilde{w}}) - T_{z\cdot\widetilde{w}} \pi_G \circ T_{\widetilde{w}} L_z(Y_{\widetilde{w}})\right) \\ &= T_{\pi_G(z\cdot\widetilde{w})} \widehat{R}_{\pi_G(\widetilde{w}^{-1})} \left(T_{z\cdot\widetilde{w}} \pi_G(X_{z\cdot\widetilde{w}}) - T_{\pi_G(\widetilde{w})} \widehat{L}_{\pi_G(z)} \circ T_{\widetilde{w}} \pi_G(Y_{\widetilde{w}})\right) \end{aligned}$$

As a consequence of the second-order property of X and Y as vector fields

$$T_{z\cdot\widetilde{w}} \pi_G(X_{z\cdot\widetilde{w}}) = z \cdot \widetilde{w} \quad \text{and} \quad T_{\widetilde{w}} \pi_G(Y_{\widetilde{w}}) = \widetilde{w}.$$

Moreover, using the definition for the group multiplication in TG (Proposition 1 in

Chapter 2) for $z, \widetilde{w} \in TG$ one has

$$z \cdot \widetilde{w} = T_{\pi_G(z)} \widehat{R}_{\pi_G(\widetilde{w})}(z) + T_{\pi_G(\widetilde{w})} \widehat{L}_{\pi_G(z)}(\widetilde{w}),$$

thus

$$\begin{aligned} T\pi_G \circ Z_{\widetilde{w}}(z) &= T_{\pi_G(z\cdot\widetilde{w})} \widehat{R}_{\pi_G(\widetilde{w}^{-1})} \left(z \cdot \widetilde{w} - T_{\pi_G(\widetilde{w})} \widehat{L}_{\pi_G(z)}(\widetilde{w}) \right) \\ &= T_{\pi_G(z\cdot\widetilde{w})} \widehat{R}_{\pi_G(\widetilde{w}^{-1})} \left(T_{\pi_G(z)} \widehat{R}_{\pi_G(\widetilde{w})}(z) \right). \end{aligned}$$

From $(\widehat{R}_{\pi_G(\widetilde{w})})^{-1} = \widehat{R}_{\pi_G(\widetilde{w}^{-1})}$ (Chapter 2), one deduces that $T\pi_G \circ Z_{\widetilde{w}}(z) = z$, as stated. ■

In order to obtain an expression for the dynamics of (4.9) one applies the result of Proposition 5 to $v \cdot (Tf \circ \omega)^{-1}$ with $v : I \longrightarrow TG$ and $Tf \circ \omega : I \longrightarrow TG$ representing, respectively, the state of the system and the the state of the auxiliary system composed with Tf, the tangent map of a transverse function f.
The vector fields along the curves v and $Tf \circ \omega$ are, respectively,

$$X_v = D_v + \sum_{i=1}^{m} u^i X_{i,v}^{\mathrm{lift}} \qquad \text{and} \qquad Y_{Tf(\omega)} = T_\omega Tf \left(\Delta_\omega + \sum_{i=1}^{n-m} w^i \Omega_{i,\omega} \right).$$

Then the error dynamics of $z = v \cdot (Tf \circ \omega)^{-1}$ is

$$\begin{aligned} \dot{z} &= T_v R_{Tf(\omega)^{-1}} \left(X_v - T_{Tf(\omega)} L_z (Y_{Tf(\omega)}) \right) \\ &= T_v R_{Tf(\omega)^{-1}} \left(D_v + \sum_{i=1}^{m} u^i X_{i,v}^{\mathrm{lift}} - T_{Tf(\omega)} L_z \circ T_\omega Tf \left(\Delta_\omega + \sum_{i=1}^{n-m} w^i \Omega_{i,\omega} \right) \right) \end{aligned}$$

If we group the terms corresponding to drifts, i.e. D and Δ, one gets

$$\dot{z} \;=\; T_v R_{Tf(\omega)^{-1}} \left(D_v - T_{Tf(\omega)} L_z \circ T_\omega Tf(\Delta_\omega) \right)$$

$$+T_vR_{Tf(\omega)^{-1}}\left(\sum_{i=1}^{m} u^i X_{i,v}^{\mathrm{lift}} - T_{Tf(\omega)}L_z \circ T_\omega Tf\left(\sum_{i=1}^{n-m} w^i \Omega_{i,\omega}\right)\right) \tag{4.12}$$

As a consequence of Lemma 3, one has that X_i^{lift} is left-invariant ($i = 1, \ldots, m$). In particular, $X_{i,v}^{\mathrm{lift}} = T_{Tf(\omega)}L_z(X_i^{\mathrm{lift}}(Tf(\omega)))$, since $z = v \cdot Tf(\omega)^{-1}$. Thus

$$\begin{aligned}
\dot{z} &= T_vR_{Tf(\omega)^{-1}}\left(D_v - T_{Tf(\omega)}L_z \circ T_\omega Tf(\Delta_\omega)\right) \\
&\quad + T_vR_{Tf(\omega)^{-1}}\left(\sum_{i=1}^{m} u^i\, T_{Tf(\omega)}L_z\left(X_i^{\mathrm{lift}}(Tf(\omega))\right)\right. \\
&\qquad\qquad \left. - T_{Tf(\omega)}L_z \circ T_\omega Tf\left(\sum_{i=1}^{n-m} w^i \Omega_{i,\omega}\right)\right) \qquad (4.13) \\
&= T_vR_{Tf(\omega)^{-1}}\left(D_v - T_{Tf(\omega)}L_z \circ T_\omega Tf(\Delta_\omega)\right) \\
&\quad + T_vR_{Tf(\omega)^{-1}} \circ T_{Tf(\omega)}L_z\left(\sum_{i=1}^{m} u^i\, X_i^{\mathrm{lift}}(Tf(\omega)) - T_\omega Tf\left(\sum_{i=1}^{n-m} w^i \Omega_{i,\omega}\right)\right) \\
&= T_vR_{Tf(\omega)^{-1}}\left(D_v - T_{Tf(\omega)}L_z \circ T_\omega Tf(\Delta_\omega)\right) \\
&\quad + T_vR_{Tf(\omega)^{-1}} \circ T_{Tf(\omega)}L_z\left(\sum_{i=1}^{m} u^i\, X_i^{\mathrm{lift}}(Tf(\omega)) - \sum_{i=1}^{n-m} w^i\, T_\omega Tf\left(\Omega_{i,\omega}\right)\right)
\end{aligned}$$

By (4.9) one has $v = z \cdot Tf(\omega)$, then the expression one gets for the error dynamics is

$$\begin{aligned}
\dot{z} &= T_{z\cdot Tf(\omega)}R_{Tf(\omega)^{-1}}\left(D_{z\cdot Tf(\omega)} - T_{Tf(\omega)}L_z \circ T_\omega Tf(\Delta_\omega)\right) \\
&\quad + T_{z\cdot Tf(\omega)}R_{Tf(\omega)^{-1}} \circ T_{Tf(\omega)}L_z\left(\sum_{i=1}^{m} u^i\, X_i^{\mathrm{lift}}(Tf(\omega))\right. \qquad (4.14) \\
&\qquad\qquad \left. - \sum_{i=1}^{n-m} w^i\, T_\omega Tf\left(\Omega_{i,\omega}\right)\right)
\end{aligned}$$

The differential equation (4.14) is second-order as a consequence of Proposition 5. This will be the error dynamics we shall use in the sequel.

For second-order systems, the control inputs can only shape the second-order time-derivatives of the base trajectories. When dealing with mechanical systems, this translates into acting only upon accelerations and not upon configurations or velocities. Since the image of TTf, together with the controlled vector fields along the image of Tf span the vertical subbundle, it is possible to use this property to design a feedback law that imposes any desired error dynamics. Let us make this statement precise.

Theorem 1 *Given a second-order vector field $Z_d \in \Gamma(TTG)$, there exists a smooth feedback law $\alpha = (\alpha^1, \ldots, \alpha^n) : TG \times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^n$ such that the error dynamics (4.14) with control inputs $u^i(z,\omega) = \alpha^i(z,\omega)$ $(i = 1, \ldots, m)$ and $w^j(z,\omega) = \alpha^{j+m}(z,\omega)$ $(j = 1, \ldots, n-1)$ writes as $\dot{z} = Z_d(z)$.*

Proof: To find such feedback function α one can proceed by setting the right-hand-side of (4.14) equal to $Z_d(z)$ and solving the resulting equation for $u^1, \ldots, u^m$, $w^1, \ldots, w^{n-m}$ in terms of z and ω.

Define a vector field $\mathcal{D}_\omega \in \Gamma(TTG)$ $(\omega \in T\mathbb{T}^{n-m})$ by

$$\mathcal{D}_\omega : z \mapsto T_{z\cdot Tf(\omega)^{-1}} R_{Tf(\omega)^{-1}} \left(D_{z\cdot Tf(\omega)^{-1}} - T_{Tf(\omega)} L_z \circ T_\omega Tf(\Delta_\omega) \right).$$

Then (4.14) becomes

$$\dot{z} = \mathcal{D}_\omega(z) + T_{z\cdot Tf(\omega)} R_{Tf(\omega)^{-1}} \circ T_{Tf(\omega)} L_z \left(\sum_{i=1}^{m} u^i \, X^{\mathrm{lift}}_{i,Tf(\omega)} - \sum_{i=1}^{n-m} w^i \, T_\omega Tf\,(\Omega_{i,\omega}) \right),$$

by setting this expression equal to $Z_d(z)$ one gets:

$$\begin{aligned} & Z_d(z) - \mathcal{D}_\omega(z) = \\ & T_{z\cdot Tf(\omega)} R_{Tf(\omega)^{-1}} \circ T_{Tf(\omega)} L_z \left(\textstyle\sum_{i=1}^{m} u^i \, X^{\mathrm{lift}}_{i,Tf(\omega)} - \sum_{i=1}^{n-m} w^i \, T_\omega Tf\,(\Omega_{i,\omega}) \right) \end{aligned}$$

$$\begin{aligned} \textstyle\sum_{i=1}^{m} u^i \, X^{\mathrm{lift}}_{i,Tf(\omega)} - \sum_{i=1}^{n-m} w^i T_\omega Tf(\Omega_{i,\omega}) = & \\ & (T_{z\cdot Tf(\omega)} R_{Tf(\omega)^{-1}} \circ T_{Tf(\omega)} L_z)^{-1} (Z_d(z) - \mathcal{D}_\omega(z)) \\ & = T_{zTf(\omega)} L_{z^{-1}} \circ T_z R_{Tf(\omega)} (Z_d(z) - \mathcal{D}_\omega(z)). \end{aligned}$$

From the proof of Proposition 5-(2) one has $T\pi_G \circ TL_\alpha = T\widehat{L}_{\pi_G(\alpha)} \circ T\pi_G$ and $T\pi_G \circ TR_\alpha = T\widehat{R}_{\pi_G(\alpha)} \circ T\pi_G$, from which one easily finds that for every $a, b \in TG$,

1. $T\pi_G \circ TL_b \circ TR_a = T\widehat{L}_{\pi_G(b)} \circ T\widehat{R}_{\pi_G(a)} \circ T\pi_G$
2. $T\pi_G \circ TR_a \circ TL_b = T\widehat{R}_{\pi_G(a)} \circ T\widehat{L}_{\pi_G(b)} \circ T\pi_G$.

Using the equation in 2., together with the fact that X_i^{lift} and Ω_j ($i = 1, \dots, m$; $j = 1, \dots, n-m$) are vertical, one deduces that

$$TR_{Tf(\omega)^{-1}} \circ TL_z(\sum_{i=1}^{m} u^i X^{\text{lift}}_{i,Tf(\omega)} - \sum_{i=1}^{n-m} w^i T_\omega Tf(\Omega_{i,\omega}))$$

is vertical. The latter assertion, coupled to the fact that (4.14) is second-order, implies that $\mathcal{D}_\omega$ is second-order for every $\omega \in T\mathbb{T}^{n-m}$. Now, using 1. one deduces that $TL_{z^{-1}} \circ TR_{Tf(\omega)}(Z_d(z) - \mathcal{D}_\omega(z))$ is vertical for every $z \in TG$.

Making use of Proposition 4 and in particular of

$$T_{Tf(\omega)}TG^{\text{vert}} = T_\omega Tf((T_\omega T\mathbb{T}^{n-m})^{\text{vert}}) \oplus \operatorname{span}_{\mathbb{R}}(\{X_1^{\text{lift}}(Tf(\omega)), \dots, X_m^{\text{lift}}(Tf(\omega))\}),$$

together with the assumption that $\{\Omega_1, \dots, \Omega_m\}$ is a global frame for $(TT\mathbb{T}^{n-m})^{\text{vert}}$, we conclude that there exist a mapping $\alpha : TG \times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^n$ such that, for every $(z, \omega) \in TG \times T\mathbb{T}^{n-m}$,

$$\begin{aligned}\textstyle\sum_{i=1}^{m} \alpha^i(z,\omega)\, X^{\text{lift}}_{i,Tf(\omega)} - \sum_{i=1}^{n-m} \alpha^{i+m}(z,\omega)\, T_\omega Tf(\Omega_{i,\omega}) = \\ T_{zTf(\omega)}L_{z^{-1}} \circ T_z R_{Tf(\omega)}(Z_d(z) - \mathcal{D}_\omega(z)).\end{aligned} \tag{4.15}$$

Since left and right translations and Tf are smooth, their respective tangent maps are also smooth, and as the vector fields Z_d, $\mathcal{D}_\omega$, X_i^{lift} and Ω_i are smooth we conclude that α given by the above equation is also smooth. ■

Given the latter theorem one is able to impose any desired dynamics to the error, in particular one would typically aim at having the error converge to zero, the identity element in TG. To do this one selects for Z_d a vector field which has zero as an asymptotically stable point. This ensures that, in closed-loop, the state v of system (4.6) converges to the image by Tf of $\omega(t)$, the state of the auxiliary system (4.8).

4.4. Zero-dynamics of the closed-loop system

Suppose that we select for Z_d in (4.15) a vector field which has zero as an exponentially stable point. Let us examine the remaining dynamics of the original and auxiliary system when the error z equals zero. Although z may never reach zero, when the initial conditions of z are different from zero, this allows us to study the long-term dynamics of the total feedback system.

Assume that the vector fields D in (4.6), and Δ in (4.8), are semisprays (several simple mechanical systems have a semispray as drift term). By applying the feedback law (4.15) one assures that the error dynamics (4.14) is of the form $\dot{z} = Z_d(z)$. Let $t_0 \in \mathbb{R}$, suppose that $z(t_0) = 0 = e$, then $\dot{z}(t) = Z_d(0) = 0$ for all $t \geqslant t_0$. Thus

$$\begin{aligned} 0 \;=\; & T_{e\cdot Tf(\omega)} R_{Tf(\omega)^{-1}} \left(D_{e\cdot Tf(\omega)} - T_{Tf(\omega)} L_e \circ T_\omega Tf(\Delta_\omega) \right) \\ & + T_{e\cdot Tf(\omega)} R_{Tf(\omega)^{-1}} \circ T_{Tf(\omega)} L_e \left(\sum_{i=1}^{m} u^i \, X^{\text{lift}}_{i,Tf(\omega)} - \sum_{i=1}^{n-m} w^i \, T_\omega Tf \left(\Omega_{i,\omega} \right) \right), \end{aligned}$$

where $u : T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^m$ and $w : T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^{n-m}$. Thus the zero-dynamics is

$$D_{Tf(\omega)} + \sum_{i=1}^{m} u^i \, X^{\text{lift}}_{i,Tf(\omega)} - T_\omega Tf(\Delta_\omega) - \sum_{i=1}^{n-m} w^i \, T_\omega Tf \left(\Omega_{i,\omega} \right) = 0,$$

which can be written as

$$D \circ Tf + \sum_{i=1}^{m} u^i \, X^{\text{lift}}_i \circ Tf = TTf \circ \Delta + \sum_{i=1}^{n-m} w^i \, TTf \circ \Omega_i \tag{4.16}$$

From (4.16) one observes that the zero-dynamics of the auxiliary system entirely governs the zero-dynamics of the target system.

Let ω be in $T\mathbb{T}^{n-m}$, then there exist an open subset V_u of TG, an open neighborhood U_u of ω and map $\widetilde{u} \in C^\infty(V_u; \mathbb{R}^m)$ with $Tf(U_u) \subset V_u$ such that the following diagram commutes [20]

$$\begin{tikzcd} U_u \arrow[rr, "Tf|_{U_u}"] \arrow[dr, "u"'] & & V_u \arrow[dl, "\widetilde{u}"] \\ & \mathbb{R}^m & \end{tikzcd}$$

Therefore (4.16) can be written as

$$D \circ Tf + \left(\widetilde{u}^i \cdot X_i^{\mathrm{lift}}\right) \circ Tf = TTf \circ \Delta + w^j \, TTf \circ \Omega_j, \tag{4.17}$$

with $i = 1, \ldots, m$ and $j = 1, \ldots, n-m$. Define $\widehat{Y}$ to be Tf-related to $D + \widetilde{u}^i X_i^{\mathrm{lift}}$, i.e. $\widehat{Y}$ is a vector field defined on $T\mathbb{T}^{n-m}$ such that the following diagram commutes

$$\begin{tikzcd} T\mathbb{T}^{n-m} \arrow[r, "Tf"] \arrow[d, "\widehat{Y}"'] & TG \arrow[d, "D + \widetilde{u}^i X_i^{\mathrm{lift}}"] \\ TT\mathbb{T}^{n-m} \arrow[r, "TTf"'] & TTG \end{tikzcd}$$

Thus (4.17) is equivalent to

$$\begin{aligned} \left(D + \widetilde{u}^i X_i^{\mathrm{lift}}\right) \circ Tf &= TTf \circ \Delta + w^j \, TTf \circ \Omega_j \\ TTf \circ \widehat{Y} &= TTf \circ \Delta + w^j \, TTf \circ \Omega_j. \end{aligned}$$

Thus, by virtue of the linearity of TTf on fibers, one obtains

$$\widehat{Y} = \Delta + w^j \, \Omega_j. \tag{4.18}$$

By applying $[\widehat{C}, \cdot]$ to both members of (4.18) one gets

$$\begin{aligned} [\widehat{C}, \widehat{Y}] &= [\widehat{C}, \Delta + \omega^j \, \Omega j] \\ &= [\widehat{C}, \Delta] + [\widehat{C}, w^j \, \Omega_j] \\ &= \Delta + w^j \, [\widehat{C}, \Omega_j] + \widehat{C}(w^j)\Omega_j \\ &= \Delta - w^j \, \Omega_j + \widehat{C}(w^j)\Omega_j, \end{aligned} \tag{4.19}$$

since Δ is a semispray (thus $[\widehat{C},\Delta]=\Delta$) and Ω_j is vertical (thus $[\widehat{C},\Omega_j]=-\Omega_j$), $j=1,\ldots,n-m$. By applying TTf to (4.19) one has

$$TTf\circ[\widehat{C},\widehat{Y}]=TTf\circ\Delta-w^j\,TTf\circ\Omega_j+\widehat{C}(w^j)\,TTf\circ\Omega_j.$$

Using that $\widehat{C}$ is Tf-related to C (Lemma 2) in addition to the fact that $[\widehat{C},\widehat{Y}]$ is Tf-related to $[C,D+\widetilde{u}^i\,X_i^{\rm lift}]$ since $\widehat{Y}$ is Tf-related to $D+\widetilde{u}^i\,X_i^{\rm lift}$ (Chapter 2) one obtains

$$[C,D+\widetilde{u}^i\,X_i^{\rm lift}]\circ Tf=TTf\circ\Delta+\left(\widehat{C}(w^j)-w^j\right)TTf\circ\Omega_j.$$

Equivalently

$$\begin{aligned}&[C,D]\circ Tf+[C,\widetilde{u}^i\,X_i^{\rm lift}]\circ Tf=\\&\qquad\qquad TTf\circ\Delta+\left(\widehat{C}(w^j)-w^j\right)TTf\circ\Omega_j,\end{aligned}$$

$$\begin{aligned}&[C,D]\circ Tf+\left(\widetilde{u}^i\cdot[C,X_i^{\rm lift}]\right)\circ Tf+\left(C(\widetilde{u}^i)\cdot X_i^{\rm lift}\right)\circ Tf=\\&\qquad\qquad TTf\circ\Delta+\left(\widehat{C}(w^j)-w^j\right)TTf\circ\Omega_j\end{aligned}$$

Thus, since D is a semispray and $X_i^{\rm lift}$ is vertical ($i=1,\ldots,m$), one obtains

$$\begin{aligned}&D\circ Tf-\left(\widetilde{u}^i\cdot X_i^{\rm lift}\right)\circ Tf+\left(C(\widetilde{u}^i)\cdot X_i^{\rm lift}\right)\circ Tf=\\&\qquad\qquad TTf\circ\Delta+\left(\widehat{C}(w^j)-w^j\right)TTf\circ\Omega_j,\end{aligned}$$

$$\begin{aligned}&D\circ Tf+\left((C(\widetilde{u}^i)-\widetilde{u}^i)\cdot X_i^{\rm lift}\right)\circ Tf=\\&\qquad\qquad TTf\circ\Delta+\left(\widehat{C}(w^j)-w^j\right)TTf\circ\Omega_j\end{aligned}$$

i.e.

$$D\circ Tf-TTf\circ\Delta=\left(\widehat{C}(w^j)-w^j\right)TTf\circ\Omega_j-\left(\left(C(\widetilde{u}^i)-\widetilde{u}^i\right)\cdot X_i^{\rm lift}\right)\circ Tf,$$

but, from (4.17), $D\circ Tf-TTf\circ\Delta=w^j\,TTf\circ\Omega_j-\left(\widetilde{u}^i\cdot X_i^{\rm lift}\right)\circ Tf$, so one has

$$\left(\widehat{C}(w^j)-2w^j\right)TTf\circ\Omega_j-\left(C(\widetilde{u}^i)-2\widetilde{u}^i\right)\,X_i^{\rm lift}\circ Tf=0$$

Given that f satisfies (4.7), this equation implies that $\widehat{C}(w^j) = 2w^j$ and $C(\widetilde{u}^i) = 2\widetilde{u}^i$, i.e. (using (4.19))

$$
\begin{aligned}
[\widehat{C}, \Delta + \omega^j \, \Omega j] &= \Delta - w^j \, \Omega_j + \widehat{C}(w^j)\Omega_j \\
&= \Delta - w^j \, \Omega_j + 2w^j \, \Omega_j \\
&= \Delta + w^j \, \Omega_j .
\end{aligned}
$$

The zero-dynamics of the auxiliary system $\Delta + w^j \, \Omega_j$ is second-order given that Δ is second-order and Ω_j is vertical ($j = 1 \ldots, n - m$). This, in addition to $[\widehat{C}, \Delta + \omega^j \, \Omega j] = \Delta + w^j \, \Omega_j$, shows that the zero-dynamics has a semispray structure under the assumption that D and Δ are semisprays.

Chapter 5

Examples of the application of vertically transverse functions

The main purpose of this chapter is to illustrate how the control technique proposed in this thesis, which makes use of vertically transverse functions, is applied to certain specific systems. The description of each system is given along with a detailed application of the method and a numerical simulation.

5.1. The ENDI system

5.1.1. System description

The ENDI system (ENDI stands for *Extended Nonholonomic Double Integrator*) arises when one includes an integrator in series with each of the inputs of the Brockett's nonholonomic integrator. This latter system does not meet Brockett's necessary condition, i.e. it can not be stabilized to any equilibrium point by means of continuous feedback functions depending only on the state. Brockett's nonholonomic integrator is of the form

$$\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= u_1 y_2 - u_2 y_1. \end{aligned}$$

where $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $u = (u_1, u_2) \in \mathbb{R}^2$. The ENDI system arises when one includes an integrator in series with each of the inputs of the Brockett's nonholonomic integrator.

$$
\begin{aligned}
\ddot{y}_1 &= u_1 \\
\ddot{y}_2 &= u_2 \\
\dot{y}_3 &= \dot{y}_1 y_2 - \dot{y}_2 y_1.
\end{aligned}
\tag{5.1}
$$

Taking the third equation of (5.1) and computing its time-derivative one gets $\ddot{y}_3 = u_1 y_2 - u_2 y_1$. Consider then the following system

$$
\begin{aligned}
\ddot{x}_1 &= u_1 \\
\ddot{x}_2 &= u_2 \\
\ddot{x}_3 &= u_1 x_2 - u_2 x_1,
\end{aligned}
\tag{5.2}
$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

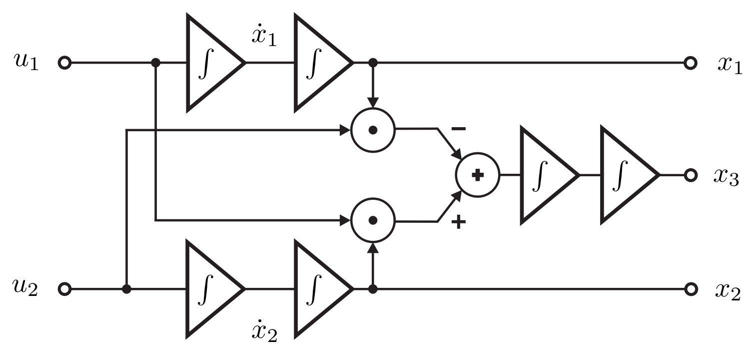

Figure 5.1: The ENDI system.

We refer, in the sequel, to this later system as the ENDI system which can be sketched as in Figure 5.1. The trajectories of system (5.1) are the same than the trajectories of system (5.2) whenever the initial condition of (5.2) is of the form $(x_0, \dot{x}_0) = (y_{10}, y_{20}, y_{30}, \dot{y}_{10}, \dot{y}_{20}, 0)$.

5.1.2. Application of the vertically transverse function approach

System (5.2) can be written as

$$\ddot{x} = u_1 X_1(x) + u_2 X_2(x) \tag{5.3}$$

where x takes values in $\mathbb{R}^3$ and X_1, X_2 are vector fields on $\mathbb{R}^3$ defined by

$$X_1(x) = \left.\frac{\partial}{\partial x_1}\right|_x + x_2 \left.\frac{\partial}{\partial x_3}\right|_x, \quad X_2(x) = \left.\frac{\partial}{\partial x_2}\right|_x - x_1 \left.\frac{\partial}{\partial x_3}\right|_x$$

The Lie bracket of X_1 and X_2 is given by

$$[X_1, X_2] = -2\frac{\partial}{\partial x_3},$$

hence the Lie Algebra generated by $\{X_1, X_2\}$ is $\mathrm{span}_{\mathbb{R}}(\{X_1, X_2, [X_1, X_2]\})$ since Lie brackets involving three or more vector fields are identically zero. As a result $\mathrm{Lie}(\{X_1, X_2\})$ spans $T_x\mathbb{R}^3$ at every point $x \in \mathbb{R}^3$, therefore $\{X_1, X_2\}$ satisfies the Lie Algebra Rank Condition at every $x \in \mathbb{R}^3$.

One can endow $\mathbb{R}^3$ with a Lie group law composition $\widehat{\mu}$ defined by

$$\widehat{\mu}(x, y) = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_2 y_1 - x_1 y_2 \end{pmatrix} \quad \text{for every } x, y \text{ in } \mathbb{R}^3,$$

with inverse group operation defined by $x^{-1} = (-x_1, -x_2, -x_3)$ for $x \in \mathbb{R}^3$, and identity element $\widehat{e} = (0, 0, 0)$.

The equation $\mu(v, w) = T_{\pi_{\mathbb{R}^3}(v)} \widehat{R}_{\pi_{\mathbb{R}^3}(w)}(v) + T_{\pi_{\mathbb{R}^3}(w)} \widehat{L}_{\pi_{\mathbb{R}^3}(v)}(w)$ (Equation (2.8)) allows one to explicitly find the Lie group composition in $T\mathbb{R}^3$ associated with $\widehat{\mu}$ to be

$$\mu(v, w) = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 + v_2 w_1 - v_1 w_2 \\ v_4 + w_4 \\ v_5 + w_5 \\ v_6 + w_6 - w_2 v_4 + w_1 v_5 + v_2 w_4 - v_1 w_5 \end{pmatrix} \quad \forall\, v, w \in T\mathbb{R}^3$$

with inverse $v \mapsto v^{-1} = (-v_1, -v_2, -v_3, -v_4, -v_5, -v_6)$ and identity element $e = 0$.

Given x in $\mathbb{R}^3$, the left translation by x, $\widehat{L}_x$, is defined by

$$\widehat{L}_x(y) = \widehat{\mu}(x, y) = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_2 y_1 - x_1 y_2 \end{pmatrix} \text{ for all } y \in \mathbb{R}^3$$

thus, the tangent map associated to $\widehat{L}_x$ at $y \in \mathbb{R}^3$, i.e. $T_y\widehat{L}_x : T_y\mathbb{R}^3 \longrightarrow T_{\widehat{L}_x(y)}\mathbb{R}^3$ is defined as

$$T_y\widehat{L}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & -x_1 & 1 \end{pmatrix}$$

Let $v = (v_1, v_2, v_3)$ be in $T_y\mathbb{R}^3$. Then $T_y\widehat{L}_x(v)$ is given by

$$T_y\widehat{L}_x(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & -x_1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ x_2 v_1 - x_1 v_2 + v_3 \end{pmatrix}.$$

The vector fields previously defined

$$X_1(y) = \begin{pmatrix} 1 \\ 0 \\ y_2 \end{pmatrix} \quad \text{and} \quad X_2(y) = \begin{pmatrix} 0 \\ 1 \\ -y_1 \end{pmatrix}$$

are left-invariant with respect to the group operation $\widehat{\mu}$ of $\mathbb{R}^3$ since

$$X_1(\widehat{L}_x(y)) = \begin{pmatrix} 1 \\ 0 \\ x_2 + y_2 \end{pmatrix},$$

$$T_y\widehat{L}_x(X_1(y)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & -x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ x_2 + y_2 \end{pmatrix}$$

and

$$X_2(\widehat{L}_x(y)) = \begin{pmatrix} 0 \\ 1 \\ -x_1 - y_1 \end{pmatrix},$$

$$T_y\widehat{L}_x(X_2(y)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & -x_1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -x_1 - y_1 \end{pmatrix}$$

so that, for every x, y in $\mathbb{R}^3$,

$$X_i(\widehat{L}_x(y)) = T_y\widehat{L}_x(X_i(y)) \qquad i = 1, 2.$$

Given that X_1 and X_2 are left-invariant, X_1^{lift} and X_2^{lift} are also left-invariant with respect to μ according to result stated in Lemma 3 of Chapter 4. These vertically lifted vector fields are defined in $T\mathbb{R}^3$ by

$$X_1^{\text{lift}}(v) = \left.\frac{\partial}{\partial \dot{x}_1}\right|_v + x_2 \left.\frac{\partial}{\partial \dot{x}_3}\right|_v, \quad X_2^{\text{lift}}(v) = \left.\frac{\partial}{\partial \dot{x}_2}\right|_v - x_1 \left.\frac{\partial}{\partial \dot{x}_3}\right|_v$$

where we consider coordinates $v = (x, \dot{x})$ on $T\mathbb{R}^3$, naturally associated with the coordinates x.

The system (5.3) can be recast as a system on $T\mathbb{R}^3$ by

$$\dot{v} = \mathcal{S}(v) + u_1 X_1^{\text{lift}}(v) + u_2 X_2^{\text{lift}}(v), \tag{5.4}$$

where $\mathcal{S}$ is the second-order vector field defined by $\mathcal{S}(v) = \sum_{i=1}^{3} \dot{x}_i \left.\frac{\partial}{\partial x_i}\right|_v$.

Next we find a transverse function $f : \mathbb{T} \longrightarrow \mathbb{R}^3$ for X_1 and X_2 near $\widehat{e}$, following the procedure reviewed in Chapter 3, which results in

$$f : \theta \mapsto \begin{pmatrix} \varepsilon \sin(\theta) \\ \varepsilon \cos(\theta) \\ 0 \end{pmatrix}$$

The tangent map associated to f at $\theta \in \mathbb{T}$, $T_\theta f : T_\theta\mathbb{T} \longrightarrow T_{f(\theta)}\mathbb{R}^3$, is defined for each

$\omega = (\theta, \dot{\theta})$ in $T_\theta \mathbb{T}$ by:

$$Tf(\omega) = \begin{pmatrix} \varepsilon \sin(\theta) \\ \varepsilon \cos(\theta) \\ 0 \\ \varepsilon \dot{\theta} \cos(\theta) \\ -\varepsilon \dot{\theta} \sin(\theta) \\ 0 \end{pmatrix}$$

The transversality property of the map f is equivalent to the nonsingularity of the matrix M whose columns are the components of the vector fields X_1 and X_2 evaluated on the image of f, together with $f'(\theta)$, i.e.

$$M(\theta) \triangleq \begin{pmatrix} 1 & 0 & \varepsilon \cos(\theta) \\ 0 & 1 & -\varepsilon \sin(\theta) \\ \varepsilon \cos(\theta) & -\varepsilon \sin(\theta) & 0 \end{pmatrix}$$

One easily proves that $\det(M(\theta))$ equals $-\varepsilon^2$ for all $\theta \in \mathbb{T}$ and, given that ε is a strictly positive real, the determinant of M is nonzero.

Now, we proceed to define the auxiliary second-order system on $T\mathbb{T}$

$$\dot{\omega} = \Delta_\omega + w\, \Omega_\omega$$

as

$$\underbrace{\begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix}}_{\dot{\omega}} = \underbrace{\begin{pmatrix} \dot{\theta} \\ 0 \end{pmatrix}}_{\Delta_\omega} + w \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\Omega_\omega}$$

i.e.

$$\ddot{\theta} = w \tag{5.5}$$

The error is defined as the product in $T\mathbb{R}^3$ of the state $v = (x, \dot{x})$ of System (5.3) and the inverse of the image by Tf of $\omega = (\theta, \dot{\theta})$ (the latter being the state of the auxiliary system (5.5)), i.e.

$$z = \mu(v, Tf(\omega)^{-1}).$$

Carrying out the computations one finds the following expression for the error

$$z(v,\omega) = \begin{pmatrix} v_1 - \varepsilon\,\sin(\theta) \\ v_2 - \varepsilon\,\cos(\theta) \\ v_3 - v_2\varepsilon\,\sin(\theta) + v_1\varepsilon\,\cos(\theta) \\ v_4 - \dot{\theta}\,\varepsilon\,\cos(\theta) \\ v_5 + \dot{\theta}\,\varepsilon\,\sin(\theta) \\ \varepsilon\,\cos(\theta)v_4 - \varepsilon\,\sin(\theta)v_5 + v_6 - v_2\dot{\theta}\,\varepsilon\,\cos(\theta) - v_1\dot{\theta}\,\varepsilon\,\sin(\theta) \end{pmatrix}.$$

The error dynamics, obtained by differentiating the expression for the error z, is given by (4.14)

$$\dot{z} = B(z,\omega) + \sum_{i=1}^{3} u_i H_i(z,\omega) \tag{5.6}$$

with $u_3 = w$,

$$\begin{aligned} H_1(z,\omega) &= (0,0,0,1,0,2\,\varepsilon\,\cos(\theta) + z_2), \\ H_2(z,\omega) &= (0,0,0,0,1,-2\,\varepsilon\,\sin(\theta) - z_1), \\ H_3(z,\omega) &= (0,0,0,-\varepsilon\,\cos(\theta),\varepsilon\,\sin(\theta),-\varepsilon\,\cos(\theta)z_2 - \varepsilon\,\sin(\theta)z_1 - \varepsilon^2) \end{aligned}$$

and

$$B(z,\omega)=\begin{pmatrix} z_4\\ z_5\\ z_6\\ \dot\theta^2\varepsilon\,\sin(\theta)\\ \dot\theta^2\varepsilon\,\cos(\theta)\\ -2\,\dot\theta\,\varepsilon\,\sin(\theta)z_4-2\,\dot\theta\,\varepsilon\,\cos(\theta)z_5+\dot\theta^2\varepsilon\,\sin(\theta)z_2-\dot\theta^2\varepsilon\,\cos(\theta)z_1 \end{pmatrix}.$$

As we showed in Proposition 5 in Chapter 4, the error dynamics represents a second-order differential equation. Consequently we can write it down taking the second-time derivatives of (z_1,z_2,z_3), namely,

$$\begin{aligned}\begin{pmatrix}\ddot z_1\\ \ddot z_2\\ \ddot z_3\end{pmatrix} &= \begin{pmatrix}\dot\theta^2\varepsilon\,\sin(\theta)\\ \dot\theta^2\varepsilon\,\cos(\theta)\\ b_3(z,\theta,\dot\theta)\end{pmatrix}\\ &+\begin{pmatrix}1 & 0 & -\varepsilon\cos(\theta)\\ 0 & 1 & \varepsilon\sin(\theta)\\ 2\,\varepsilon\,\cos(\theta)+z_2 & -2\,\varepsilon\,\sin(\theta)-z_1 & h_{3,3}(z,\theta)\end{pmatrix}\begin{pmatrix}u_1\\ u_2\\ w\end{pmatrix}\end{aligned}\tag{5.7}$$

with $b_3(z,\theta,\dot\theta)=-2\,\dot\theta\,\varepsilon\,\sin(\theta)z_4-2\,\dot\theta\,\varepsilon\,\cos(\theta)z_5+\dot\theta^2\varepsilon\,\sin(\theta)z_2-\dot\theta^2\varepsilon\,\cos(\theta)z_1$ and $h_{3,3}(z,\theta)=-\varepsilon\,\cos(\theta)z_2-\varepsilon\,\sin(\theta)z_1-\varepsilon^2$.

In order to construct a feedback function to make z converge to e, the identity element in $T\mathbb{R}^3$, we take a second-order vector field $S\in\Gamma(TT\mathbb{R}^3)$ which has e as locally asymptotically stable point, for instance,

$$S_z=(z_4,z_5,z_6,-k_1z_1-k_2z_4,-k_1z_2-k_2z_5,-k_1z_3-k_2z_6)$$

where the control gains k_1,k_2 are strictly positive real numbers.

One obtains the feedback function $(u(z,\omega),w(z,\omega))$ by equating the right hand side of (5.6) to S_z and solving for (u_1,u_2,u_3), or equivalently, by equating the right hand side of (5.7) to $S_H(z)$ and solving for (u_1, u_2, w), where

$$S_H(z) = \begin{pmatrix} -k_1 z_1 - k_2 z_4 \\ -k_1 z_2 - k_2 z_5 \\ -k_1 z_3 - k_2 z_6 \end{pmatrix}.$$

For every (z, ω) in $T\mathbb{R}^3 \times T\mathbb{T}$ there exists a solution (u_1, u_2, w) due the nonsingularity of the square matrix in (5.7), since for every $z \in T\mathbb{R}^3$ and $\theta \in \mathbb{T}$ its determinant equals ε^2. After simple manipulations we get

$$\begin{aligned} u_1(z, \theta, \dot\theta) \;\; = \;\; & k_1 \left(\cos(2\,\theta) z_1 - \sin(2\,\theta) z_2 - \tfrac{\cos(\theta)}{\varepsilon} z_3\right) \\ & + k_2 \left(\tfrac{\cos(\theta)}{\varepsilon} \left(z_2 z_4 - z_1 z_5 - z_6\right) + \cos(2\,\theta) z_4 - \sin(2\,\theta) z_5\right) \\ & + \dot\theta \left(\sin(2\,\theta) z_4 + 2\, \left(\cos(\theta)\right)^2 z_5 - \dot\theta\, \varepsilon\, \sin(\theta)\right) \end{aligned}$$

$$\begin{aligned} u_2(z, \theta, \dot\theta) \;\; = \;\; & -k_1 \left(\sin(2\,\theta) z_1 + \cos(2\,\theta) z_2 - \tfrac{\sin(\theta)}{\varepsilon} z_3\right) \\ & - k_2 \left(\tfrac{\sin(\theta)}{\varepsilon} \left(z_2 z_4 - z_1 z_5 - z_6\right) + \sin(2\,\theta) z_4 + \cos(2\,\theta) z_5\right) \\ & - \dot\theta \left(\sin(2\,\theta) z_5 + 2\, \left(\sin(\theta)\right)^2 z_4 + \dot\theta\, \varepsilon\, \cos(\theta)\right) \end{aligned}$$

$$\begin{aligned} w(z, \theta, \dot\theta) = \;\; \tfrac{1}{\varepsilon^2} \;\; & \Big(k_1 \left(2\,\varepsilon\, \cos(\theta) z_1 - 2\,\varepsilon\, \sin(\theta) z_2 - z_3\right) \\ & + k_2 \left(\left(2\,\varepsilon\, \cos(\theta) + z_2\right) z_4 - \left(2\,\varepsilon\, \sin(\theta) + z_1\right) z_5 - z_6\right) \\ & + 2\,\varepsilon\, \dot\theta\, \left(\sin(\theta) z_4 + \cos(\theta) z_5\right)\Big) \end{aligned}$$

Using this feedback function $z(t) = v \cdot (Tf(\omega))^{-1}$ tends to zero as t tends to infinity. Hence $v(t) \longrightarrow Tf(\omega(t))$ as $t \longrightarrow \infty$, therefore $x(t) = \pi_{\mathbb{R}^3} \circ v(t) \longrightarrow f \circ \pi_{\mathbb{T}} \circ \omega(t) = f \circ \theta(t)$, i.e. $x(t)$ converge to the image by f of θ.

One easily verifies that

$$\ddot\theta = 0$$

is the zero-dynamics of the auxiliary system. Thus $\dot\theta$ is bounded with a bound depending on the initial conditions.

The zero-dynamics of the auxiliary system governs the zero-dynamics of the target system in (5.3) (Section 4.4 in Chapter 4), therefore $\dot{x}$ is bounded, and, since $x(t) \longrightarrow f \circ \theta(t)$ as $t \longrightarrow \infty$, $v = (x, \dot{x})$ is bounded for the long-term behavior.

A numerical simulation of the closed-loop system with feedback control $u = (u_1(z, \theta, \dot{\theta}), u_2(z, \theta, \dot{\theta}), w(z, \theta, \dot{\theta}))$ was performed. The initial condition is $v = (x, \dot{x}) = (3.5, -0.3, 0.2, 0.5, -0.1, 0.0)$, the controller gains are $k_1 = k_2 = 1.0$, and the value of ε is 0.5. Figure 5.3 presents plots of the trajectories of the ENDI system and Figure 5.2 presents the evolution in time of the error and the control input applied.

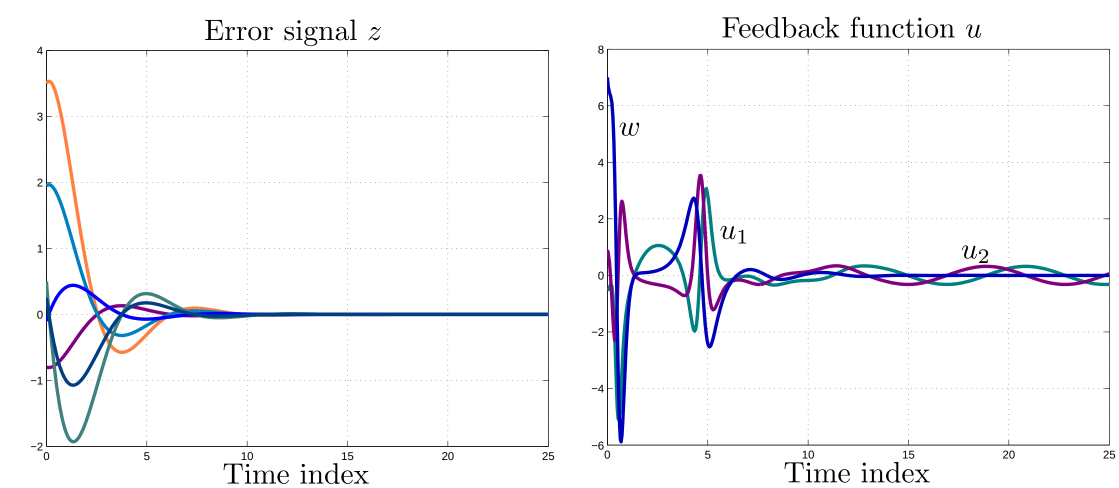

Figure 5.2: Time histories of the error z and the control input $u(z, \theta, \dot{\theta})$ respectively.

The error in Figure 5.2 tends to zero as the time increases. We also note (Figure 5.3) that the configuration variables and velocities of the system, after a transient, seem to converge to a periodic motion. As a matter of fact the configuration variables converge to a neighborhood which can be modified by changing the value of ε.

5.2. PPR manipulator

5.2.1. System description

In this section we deal with the three link planar manipulator PPR, (PPR stands for *Prismatic-Prismatic-Revolute*), in which the two first joints are actuated whereas

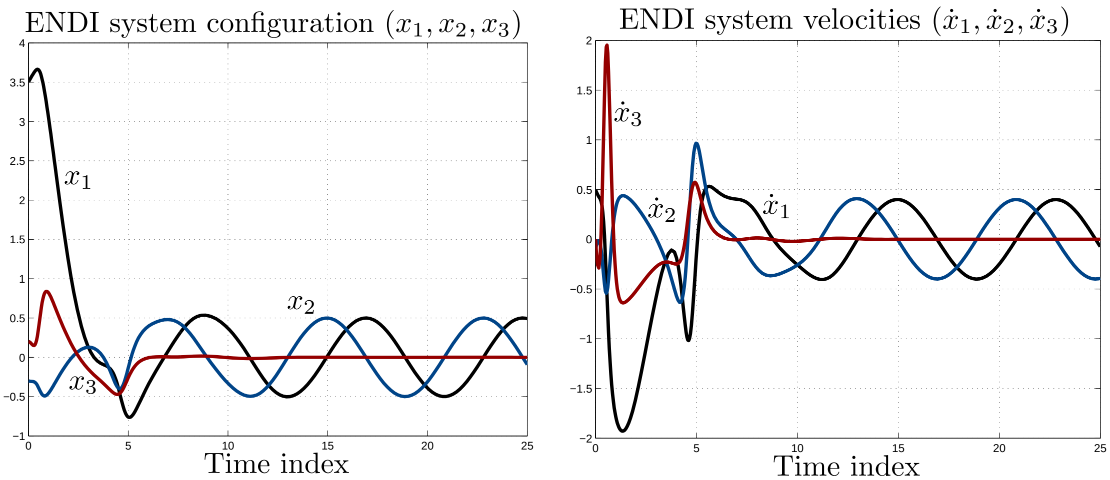

Figure 5.3: Time histories of the configuration x and velocity $\dot{x}$ respectively.

the third revolute joint is passive. The system is schematically represented in Figure 5.4.

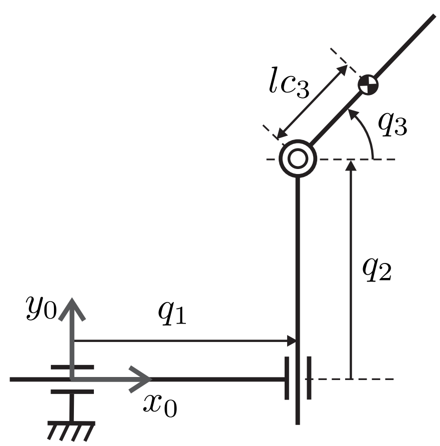

Figure 5.4: Planar *Prismatic-Prismatic-Revolute* manipulator with its third joint unactuated.

The configuration of the system is given by $(q_1, q_2) \in \mathbb{R}^2$ and $q_3 \in \mathbb{S}^1$ so for a given configuration $q = (q_1, q_2, q_3)$ in the configuration manifold $Q = \mathbb{R}^2 \times \mathbb{S}^1 \simeq SE(2)$ of the system $(n = \dim(Q) = 3)$, (q_1, q_2) represents the net displacement in the $\mathbb{R}^2$ plane with respect to a fixed basis while $q_3 \in \mathbb{S}$ corresponds to the orientation of the third link.

5.2.2. Dynamical model

We do not consider friction in the dynamical model and we assume the system moves on a horizontal plane, so that it does not experiment the action of any gravitational force. The mathematical model for the PPR manipulator is derived from the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \tau_i \qquad i = 1, \ldots, 3$$

where $L : TQ \longrightarrow \mathbb{R}$ denotes the Lagrangian of the system, given by $L = K - P \circ \pi_Q$, with $K : TQ \longrightarrow \mathbb{R}$ and $P : Q \longrightarrow \mathbb{R}$ respectively being the kinetic and potential energies of the system. τ_i represents the force applied to the i-th link ($i = 1, \ldots, 3$).

In the sequel we shall write c for cos and s for sin. Let $[p_i]_0$ ($i = 1, \ldots, n$) be the position of the centre of mass of the i-th link with respect to the coordinate frame $\Sigma_0 = (x_0, y_0)$ (see Figure 5.4). One easily checks that

$$[p_1]_0 = \begin{pmatrix} -lc_1 + q_1 \\ 0 \end{pmatrix}, \quad [p_2]_0 = \begin{pmatrix} q_1 \\ -lc_2 + q_2 \end{pmatrix}, \quad [p_3]_0 = \begin{pmatrix} lc_3\,\mathrm{c}(q_3) + q_1 \\ lc_3\,\mathrm{s}(q_3) + q_2 \end{pmatrix}.$$

Differentiating we obtain the velocities of the links to be

$$[v_1]_0 = \begin{pmatrix} \dot{q}_1 \\ 0 \end{pmatrix}, \quad [v_2]_0 = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}, \quad [v_3]_0 = \begin{pmatrix} -lc_3\,\mathrm{s}(q_3)\,\dot{q}_3 + \dot{q}_1 \\ lc_3\,\mathrm{c}(q_3)\,\dot{q}_3 + \dot{q}_2 \end{pmatrix}.$$

Thus the kinetic energy associated with the i-th link, $K_i = \frac{1}{2}\, m_i {[v_i]_0}^2$, is given by

$$\begin{aligned}
K_1(q, \dot{q}) &= \frac{1}{2}\, m_1 \dot{q}_1 \\
K_2(q, \dot{q}) &= \frac{1}{2}\, m_2(\dot{q}_1^2 + \dot{q}_2^2) \\
K_3(q, \dot{q}) &= \frac{1}{2}\, m_3(\dot{q}_1^2 + \dot{q}_2^2 + lc_3^2 \dot{q}_3^2 - 2\, lc_3 \mathrm{s}(q_3)\, \dot{q}_1 \dot{q}_3 + 2\, lc_3 \mathrm{c}(q_3)\, \dot{q}_2 \dot{q}_3).
\end{aligned}$$

Since the potential energy is assumed to be zero, the Lagrangian of the system equals the kinetic energy, i.e. $L(q, \dot{q}) = K(q, \dot{q}) = \sum_{i=1}^{3} K_i(q, \dot{q}) = \frac{1}{2}\mathcal{G}(\dot{q}, \dot{q})$:

$$L(q,\dot q) = \frac{1}{2}\,\dot q^{\mathrm{T}} \begin{pmatrix} m_1+m_2+m_3 & 0 & -m_3lc_3\mathrm{s}(q_3) \\ 0 & m_2+m_3 & m_3lc_3\mathrm{c}(q_3) \\ -m_3lc_3\mathrm{s}(q_3) & m_3lc_3\mathrm{c}(q_3) & m_3{lc_3}^2 \end{pmatrix} \dot q.$$

Let $M_j = \sum_{i=j}^{3} m_i$ $(j = 1,2)$ and $J = m_3{lc_3}^2$. Then

$$L(q,\dot q) = \frac{1}{2}\,\dot q^{\mathrm{T}} \begin{pmatrix} M_1 & 0 & -m_3lc_3\mathrm{s}(q_3) \\ 0 & M_2 & m_3lc_3\mathrm{c}(q_3) \\ -m_3lc_3\mathrm{s}(q_3) & m_3lc_3\mathrm{c}(q_3) & J \end{pmatrix} \dot q$$

From the Euler-Lagrange equations we obtain the dynamics of the PPR manipulator to be

$$\begin{aligned} M_1\ddot q_1 - m_3lc_3\mathrm{s}(q_3)\,\ddot q_3 - m_3lc_3\mathrm{c}(q_3)\,\ddot q_3^2 &= \tau_1 \\ M_2\ddot q_2 + m_3lc_3\mathrm{c}(q_3)\,\ddot q_3 - m_3lc_3\mathrm{s}(q_3)\,\ddot q_3^2 &= \tau_2 \\ J\ddot q_3 - m_3lc_3\mathrm{s}(q_3)\,\ddot q_1 + m_3lc_3\mathrm{c}(q_3)\,\ddot q_2 &= 0 \end{aligned} \tag{5.8}$$

Rewriting the above set of equations into matrix notation we have

$$\begin{pmatrix} M_1 & 0 & -m_3lc_3\mathrm{s}(q_3) \\ 0 & M_2 & m_3lc_3\mathrm{c}(q_3) \\ -m_3lc_3\mathrm{s}(q_3) & m_3lc_3\mathrm{c}(q_3) & J \end{pmatrix} \begin{pmatrix} \ddot q_1 \\ \ddot q_2 \\ \ddot q_3 \end{pmatrix} + \begin{pmatrix} -m_3\,lc_3\mathrm{c}(q_3)\,\dot q_3^2 \\ -m_3\,lc_3\mathrm{s}(q_3)\,\dot q_3^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ 0 \end{pmatrix}$$

According to [6], if we consider the input transformation given by

$$\begin{aligned} \tau_1 &= -m_3lc_3\mathrm{c}(q_3)\,\dot q_3^2 + \left(M_1 - \frac{{m_3}^2{lc_3}^2}{J}\mathrm{s}^2(q_3)\right)\eta_1 + \frac{{m_3}^2{lc_3}^2}{J}\mathrm{s}(q_3)\mathrm{c}(q_3)\,\eta_2 \\ \tau_2 &= -m_3lc_3\mathrm{s}(q_3)\,\dot q_3^2 + \left(M_2 - \frac{{m_3}^2{lc_3}^2}{J}\mathrm{c}^2(q_3)\right)\eta_2 + \frac{{m_3}^2{lc_3}^2}{J}\mathrm{s}(q_3)\mathrm{c}(q_3)\,\eta_1, \end{aligned}$$

the system (5.8) becomes

$$
\begin{aligned}
\ddot{q}_1 &= \eta_1 \\
\ddot{q}_2 &= \eta_2 \\
\ddot{q}_3 &= \frac{m_3 l c_3}{J}\left(\mathrm{s}(q_3)\eta_1 - \mathrm{c}(q_3)\eta_2\right).
\end{aligned}
\tag{5.9}
$$

If we now consider the following state and input transformations

$$
\begin{aligned}
y_1 &= q_1 + \tfrac{J}{m_3 l c_3}\mathrm{c}(q_3) \\
y_2 &= q_2 + \tfrac{J}{m_3 l c_3}\mathrm{s}(q_3) \\
y_3 &= q_3
\end{aligned}
\qquad
\begin{aligned}
\eta_1 &= \left(v_1 + \tfrac{J}{m_3 l c_3}\dot{q}_3^2\right)\mathrm{c}(q_3) + \tfrac{J}{m_3 l c_3}\mathrm{s}(q_3)v_2 \\
\eta_2 &= \left(v_1 + \tfrac{J}{m_3 l c_3}\dot{q}_3^2\right)\mathrm{s}(q_3) - \tfrac{J}{m_3 l c_3}\mathrm{c}(q_3)v_2
\end{aligned},
$$

we obtain the following system

$$
\begin{aligned}
\ddot{y}_1 &= \mathrm{c}(y_3)\, v_1 \\
\ddot{y}_2 &= \mathrm{s}(y_3)\, v_1 \\
\ddot{y}_3 &= v_2.
\end{aligned}
\tag{5.10}
$$

The latter system is locally defined on $\mathbb{R}^2 \times \mathbb{S}^1$. One may verify that its control vector fields $Y_1(y) = \mathrm{c}(y_3)\, \frac{\partial}{\partial y^1}\Big|_y + \mathrm{s}(y_3)\, \frac{\partial}{\partial y^2}\Big|_y$ and $Y_2(y) = \frac{\partial}{\partial y^3}\Big|_y$ satisfy the LARC at every point, moreover, they are left-invariant with respect to the Lie group operation ϕ in $\mathbb{R}^2 \times \mathbb{S}^1$ defined by

$$
\phi(a,b) = \begin{pmatrix} \mathrm{c}(a_3)b_1 - \mathrm{s}(a_3)b_2 + a_1 \\ \mathrm{s}(a_3)b_1 + \mathrm{c}(a_3)b_2 + a_2 \\ a_3 + b_3 \end{pmatrix} \quad \text{for every } a, b \text{ in } \mathbb{R}^2 \times \mathbb{S}^1,
$$

and therefore one is allowed to apply to (5.10) the methodology proposed in this thesis. However, before we proceed, it is worth mentioning that (5.10) is equivalent to

$$\begin{aligned} \ddot{x}_1 &= u_1 \\ \ddot{x}_2 &= u_2 \\ \ddot{x}_3 &= x_2\, u_1, \end{aligned} \tag{5.11}$$

by considering the following input and state transformations

$$\begin{aligned} v_1 &= \sec(y_3)\, u_1 \\ v_2 &= \mathrm{c}(y_3)^2\, u_2 - 2\tan(y_3)\dot{y}_3^2 \end{aligned} \qquad \begin{aligned} x_1 &= y_1 \\ x_2 &= \tan(y_3) \\ x_3 &= y_2. \end{aligned}$$

In this example we work with System (5.11), which also evolves on a Lie group. The control vector fields which define (5.11) are, in addition, left-invariant under an appropriately defined multiplication on $\mathbb{R}^3$ as shown in the next section.

5.2.3. Application of the vertically transverse function approach

The system (5.11) can be recast as

$$\ddot{x} = u_1 X_1(x) + u_2 X_2(x), \tag{5.12}$$

where $x = (x_1, x_2, x_3)$ is a curve on $\mathbb{R}^3$ and X_1, X_2 are vector fields defined by

$$X_1(x) = \left.\frac{\partial}{\partial x^1}\right|_x + x_2 \left.\frac{\partial}{\partial x^3}\right|_x, \qquad X_2(x) = \left.\frac{\partial}{\partial x^2}\right|_x$$

The Lie bracket of X_1 and X_2 is given by

$$[X_1, X_2] = -\frac{\partial}{\partial x^3}.$$

Hence $\mathrm{Lie}(\{X_1, X_2\})$ spans $T_x\mathbb{R}^3$ at every $x \in \mathbb{R}^3$.

Consider the differentiable manifold structure of $\mathbb{R}^3$ together with the group structure defined by

$$\widehat{\mu}(x,y) = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_2 y_1 \end{pmatrix} \quad \text{for every } x, y \text{ in } \mathbb{R}^3,$$

where the inverse of x in $\mathbb{R}^3$ is $x^{-1} = (-x_1, -x_2, -x_3 + x_2 x_1)$ and $\widehat{e} = (0,0,0)$. From the expression $\mu(a,b) = T_{\pi_{\mathbb{R}^3}(a)} \widehat{R}_{\pi_{\mathbb{R}^3}(b)}(a) + T_{\pi_{\mathbb{R}^3}(b)} \widehat{L}_{\pi_{\mathbb{R}^3}(a)}(b)$ (2.8) one finds that the Lie group composition μ in $T\mathbb{R}^3$ associated with $\widehat{\mu}$ is

$$\mu(a,b) = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 + a_2 b_1 \\ a_4 + b_4 \\ a_5 + b_5 \\ a_6 + b_6 + b_1 a_5 + a_2 b_4 \end{pmatrix} \qquad \forall\, a, b \in T\mathbb{R}^3$$

The inverse element of a in $T\mathbb{R}^3$ is then

$$a^{-1} = (-a_1, -a_2, -a_3 + a_2 a_1, -a_4, -a_5, a_1 a_5 - a_6 + a_2 a_4),$$

and the identity $e = 0$. Given x in $\mathbb{R}^3$, the left translation by x, $\widehat{L}_x$, is given by $\widehat{L}_x(y) = \widehat{\mu}(x,y)$ for all y in $\mathbb{R}^3$. Thus, the tangent map associated to $\widehat{L}_x$ at $y \in \mathbb{R}^3$, i.e. $T_y \widehat{L}_x : T_y \mathbb{R}^3 \longrightarrow T_{\widehat{L}_x(y)} \mathbb{R}^3$ is

$$T_y \widehat{L}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix}.$$

Let $v = (v_1, v_2, v_3)$ be in $T_y \mathbb{R}^3$. Then $T_y \widehat{L}_x(v)$ is given by

$$T_y \widehat{L}_x(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ x_2 v_1 + v_3 \end{pmatrix}.$$

Note that

$$X_1(\widehat{L}_x(y)) = \begin{pmatrix} 1 \\ 0 \\ x_2 + y_2 \end{pmatrix},$$

$$T_y\widehat{L}_x(X_1(y)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ x_2 + y_2 \end{pmatrix}$$

and

$$X_2(\widehat{L}_x(y)) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$T_y\widehat{L}_x(X_2(y)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

therefore, for every x, y in $\mathbb{R}^3$, $X_i(\widehat{L}_x(y)) = T_y\widehat{L}_x(X_i(y))$ $i = 1, 2$, i.e. the vector fields X_1 and X_2 are left-invariant and so are X_1^{lift} and X_2^{lift} with respect to μ (Lemma 3 of Chapter 4). X_1^{lift} and X_2^{lift} are vector fields on $T\mathbb{R}^3$ defined by

$$X_1^{\text{lift}}(v) = \left.\frac{\partial}{\partial \dot{x}^1}\right|_v + x_2 \left.\frac{\partial}{\partial \dot{x}^3}\right|_v, \qquad X_2^{\text{lift}}(v) = \left.\frac{\partial}{\partial \dot{x}^2}\right|_v$$

where we consider $(x, \dot{x})$ as coordinates for $T\mathbb{R}^3$.

The system (5.12) can be rewritten as a system on $T\mathbb{R}^3$ by

$$\dot{v} = \mathcal{S}(v) + u_1 X_1^{\text{lift}}(v) + u_2 X_2^{\text{lift}}(v) \tag{5.13}$$

where $\mathcal{S}$ is the second-order vector field on $T\mathbb{R}^3$ defined by $\mathcal{S}(v) = \sum_{i=1}^n \dot{x}^i \left.\frac{\partial}{\partial x^i}\right|_v$.

A transverse function $f : \mathbb{T} \longrightarrow \mathbb{R}^3$ associated with X_1 and X_2 near e can be found by following the procedure recalled in Chapter 3, which yields

$$f(\theta) = \left(\varepsilon\,\mathrm{s}(\theta), \varepsilon\,\mathrm{c}(\theta), \frac{1}{4}\,\varepsilon^2 \mathrm{s}(2\,\theta)\right) \qquad \forall\,\theta \in \mathbb{T}.$$

The tangent map associated with f at $\theta \in \mathbb{T}$, $T_\theta f$ is defined for each $\dot\theta \in T_\theta\mathbb{T}$ by

$$T_\theta f(\dot\theta) = \begin{pmatrix} \varepsilon\, \mathrm{s}(\theta) \\ \varepsilon\, \mathrm{c}(\theta) \\ \frac{1}{4}\,\varepsilon^2 \mathrm{s}(2\,\theta) \\ \varepsilon\,\dot\theta\, \mathrm{c}(\theta) \\ -\varepsilon\,\dot\theta\, \mathrm{s}(\theta) \\ \frac{1}{2}\,\varepsilon^2\dot\theta\, \mathrm{c}(2\,\theta) \end{pmatrix}$$

We proceed to define the auxiliary second-order system on $T\mathbb{T}$ by

$$\ddot\theta = w, \tag{5.14}$$

and the error is given by $z = \mu(v, Tf(\omega)^{-1})$, that is,

$$z(\theta, \dot\theta, v) = \begin{pmatrix} v_1 - \varepsilon\, \mathrm{s}(\theta) \\ v_2 - \varepsilon\, \mathrm{c}(\theta) \\ v_3 - \varepsilon\, \mathrm{s}(\theta)\, v_2 + \frac{1}{4}\,\varepsilon^2 \mathrm{s}(2\,\theta) \\ v_4 - \varepsilon\,\dot\theta\, \mathrm{c}(\theta) \\ v_5 + \varepsilon\,\dot\theta\, \mathrm{s}(\theta) \\ v_6 - \varepsilon\,\dot\theta\, \mathrm{c}(\theta)\, v_2 - \varepsilon\, \mathrm{s}(\theta) v_5 + \frac{1}{2}\,\varepsilon^2\dot\theta\, \mathrm{c}(2\,\theta) \end{pmatrix}.$$

The error dynamics is found by differentiating the expression for the error z:

$$\dot z = \begin{pmatrix} z_4 \\ z_5 \\ z_6 \\ u_1 + \dot\theta^2 \varepsilon\, \mathrm{s}(\theta) - \alpha\,\varepsilon\, \mathrm{c}(\theta) \\ u_2 + \dot\theta^2 \varepsilon\, \mathrm{c}(\theta) + \varepsilon\, \mathrm{s}(\theta)\alpha \\ Z_{6,z}(z, \theta, \dot\theta, u) \end{pmatrix}, \tag{5.15}$$

where

$$Z_{6,z}(z,\theta,\dot\theta,u) \;=\; -2\,\omega\,\varepsilon\,\mathrm{c}(\theta)z_5+\frac{1}{2}\,\omega^2\varepsilon^2\mathrm{s}(2\,\theta)-\varepsilon\,\mathrm{s}(\theta)u_2$$
$$+u_1z_2+u_1\varepsilon\,\mathrm{c}(\theta)+\omega^2\varepsilon\,\mathrm{s}(\theta)z_2-\alpha\,\varepsilon\,\mathrm{c}(\theta)z_2-\frac{1}{2}\,\alpha\,\varepsilon^2$$

From the latter expression, for $\dot z$, is easy to note that the vector field defining the error dynamics is second-order, consequently we can write (5.15) taking the second-time derivatives of (z_1, z_2, z_3).

$$\ddot z = B(z,\theta,\dot\theta) + H(z,\theta)\begin{pmatrix} u_1 \\ u_2 \\ w \end{pmatrix}, \tag{5.16}$$

with

$$B(z,\theta,\dot\theta) = \begin{pmatrix} \dot\theta^2\varepsilon\,\mathrm{s}(\theta) \\ \dot\theta^2\varepsilon\,\mathrm{c}(\theta) \\ -2\,\dot\theta\,\varepsilon\,\mathrm{c}(\theta)z_5+\frac{1}{2}\,\dot\theta^2\varepsilon^2\mathrm{s}(2\,\theta)+z_2\dot\theta^2\varepsilon\,\mathrm{s}(\theta) \end{pmatrix},$$

and

$$H(z,\theta) = \begin{pmatrix} 1 & 0 & -\varepsilon\,\mathrm{c}(\theta) \\ 0 & 1 & \varepsilon\,\mathrm{s}(\theta) \\ z_2+\varepsilon\,\mathrm{c}(\theta) & -\varepsilon\,\mathrm{s}(\theta) & -\varepsilon\,\mathrm{c}(\theta)z_2-\frac{1}{2}\,\varepsilon^2 \end{pmatrix}$$

In order to construct a feedback function such that z converges to zero we take a second-order vector field $S \in \Gamma(TT\mathbb{R}^3)$ which has $e \in T\mathbb{R}^3$ as local exponentially stable point, for instance

$$S_z = (z_4, z_5, z_6, -k_1z_1 - k_2z_4, -k_1z_2 - k_2z_5, -k_1z_3 - k_2z_6)$$

where the control gains k_1, k_2 are strictly positive real numbers. One obtains such a feedback function $u(z,\theta,\dot\theta)$ by solving

$$S_H(z) - B(z,\theta,\dot\theta) = H(z,\theta)\begin{pmatrix} u_1 \\ u_2 \\ w \end{pmatrix}$$

where

$$S_H(z) = \begin{pmatrix} -k_1 z_1 - k_2 z_4 \\ -k_1 z_2 - k_2 z_5 \\ -k_1 z_3 - k_2 z_6 \end{pmatrix},$$

equation which is solvable due the nonsingularity of the matrix H.
The resulting feedback law is $(u_1(z,\theta,\dot\theta), u_2(z,\theta,\dot\theta), u_3(z,\theta,\dot\theta))$, with

$$\begin{aligned} u_1(z,\theta,\dot\theta) \;=\; \tfrac{1}{2\varepsilon}\,(\quad & 4\,\mathrm{c}(\theta) z_2 k_1 z_1 + 4\,\mathrm{c}(\theta) z_2 k_2 z_4 - 3\,\dot\theta^2\varepsilon^2\mathrm{s}(\theta) \\ & -\dot\theta^2\varepsilon^2\mathrm{s}(3\,\theta) + 2\,\varepsilon\,\mathrm{c}(2\,\theta) k_1 z_1 + 2\,\varepsilon\,\mathrm{c}(2\,\theta) k_2 z_4 \\ & -2\,\varepsilon\,\mathrm{s}(2\,\theta) k_1 z_2 - 2\,\varepsilon\,\mathrm{s}(2\,\theta) k_2 z_5 + 4\,\dot\theta\, z_5 \mathrm{c}(2\,\theta)\varepsilon \\ & +4\,\dot\theta\, z_5 \varepsilon - 4\,\mathrm{c}(\theta) k_1 z_3 - 4\,\mathrm{c}(\theta) k_2 z_6 \quad) \end{aligned}$$

$$\begin{aligned} u_2(z,\theta,\dot\theta) \;=\; \tfrac{1}{2\varepsilon}\,(\quad & -4\,\mathrm{s}(\theta) z_2 k_1 z_1 - 4\,\mathrm{s}(\theta) z_2 k_2 z_4 - 2\,\varepsilon\,\mathrm{s}(2\,\theta) k_1 z_1 \\ & -2\,\varepsilon\,\mathrm{s}(2\,\theta) k_2 z_4 - \dot\theta^2\varepsilon^2\mathrm{c}(\theta) - \dot\theta^2\varepsilon^2\mathrm{c}(3\,\theta) \\ & -2\,\mathrm{c}(2\,\theta)\varepsilon\, k_1 z_2 - 2\,\mathrm{c}(2\,\theta)\varepsilon\, k_2 z_5 - 4\,\dot\theta\, z_5 \mathrm{s}(2\,\theta)\varepsilon \\ & +4\,\mathrm{s}(\theta) k_1 z_3 + 4\,\mathrm{s}(\theta) k_2 z_6 \quad) \end{aligned}$$

$$\begin{aligned} u_3(z,\theta,\dot\theta) \;=\; \tfrac{1}{\varepsilon^2}\,(\quad & 2\,z_2 k_1 z_1 + 2\,z_2 k_2 z_4 - \dot\theta^2\varepsilon^2\mathrm{s}(2\,\theta) + 2\,\varepsilon\,\mathrm{c}(\theta) k_1 z_1 \\ & +2\,\varepsilon\,\mathrm{c}(\theta) k_2 z_4 - 2\,\varepsilon\,\mathrm{s}(\theta) k_1 z_2 - 2\,\varepsilon\,\mathrm{s}(\theta) k_2 z_5 \\ & +4\,\dot\theta\,\varepsilon\,\mathrm{c}(\theta) z_5 - 2\,k_1 z_3 - 2\,k_2 z_6 \quad) \end{aligned}$$

The zero-dynamics of the auxiliary system is

$$\ddot\theta = -\sin(2\,\theta)\,\dot\theta^2, \tag{5.17}$$

which can be described as evolving on $T\mathbb{T}$. One may interpret this system as a mechanical system determined by a connection ∇ which unique Christoffel symbol is $\Gamma(\theta) = \sin(\theta)$. Note that ∇ is torsionless, so one may expect the connection be compatible with a Riemannian metric $\mathcal{G}$ defined on $\mathbb{T}$. To obtain this metric we solve for $\mathcal{G}$ in (2.3), which in this case is equivalent to the ordinary differential equation

$$\frac{d\mathcal{G}}{d\theta}(\theta) - 2\sin(2\,\theta)\,\mathcal{G}(\theta) = 0.$$

A family of solutions is given by $\mathcal{G}(\theta) = A\,e^{-2\cos^2(\theta)}$, with $A > 0$. Let us define the Lagrangian of the system $L : T\mathbb{T} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} L(\omega) &= \frac{1}{2}\mathcal{G}_\theta(\dot{\theta}, \dot{\theta}) && (5.18) \\ &= \frac{1}{2}A\,e^{-2\cos^2(\theta)}\,\dot{\theta}^2, && (5.19) \end{aligned}$$

with $\omega = (\theta, \dot{\theta}) \in T\mathbb{T}$. One easily verifies that $\nabla_{\dot{\theta}}\dot{\theta} = 0$ (from (2.1)) is exactly (5.17), therefore the zero-dynamics has the form of a simple mechanical system with zero potential.

It is easy to show that $\frac{d}{dt}\left(L(\omega)\right) = 0$ and so the energy of the system is a conserved quantity. As $\mathcal{G}$ is a continuous function defined on a compact space $\mathbb{T}$, $\mathcal{G}$ is bounded from below. It follows that $\dot{\theta}$ remains bounded for all $t \in [t_0, \infty)$. As a consequence, the state v of the target system (5.13) converges to a bounded neighborhood of zero which depends on the initial conditions.

A numerical simulation of the complete system in closed-loop with $u = (u_1(z, \theta, \dot{\theta}), u_2(z, \theta, \dot{\theta}), u_3(z, \theta, \dot{\theta}))$ was performed.
The initial condition is $v = (x, \dot{x}) = (2.0, 0.5, -1.5, 0.3, -1.0, 0.1)$, the controller gains are $k_1 = k_2 = 1.0$, and the value of ε is 0.6. Figure 5.5 depicts the evolution in time of the error and the control input applied while Figure 5.6 shows plots of the trajectories of the PPR manipulator. By observing the figures one may note that, as in the previous example, the state of the system appears to be ultimately bounded. The configuration variables converge to a neighborhood of zero, while the velocities converge to a bounded set that depends on the initial conditions.

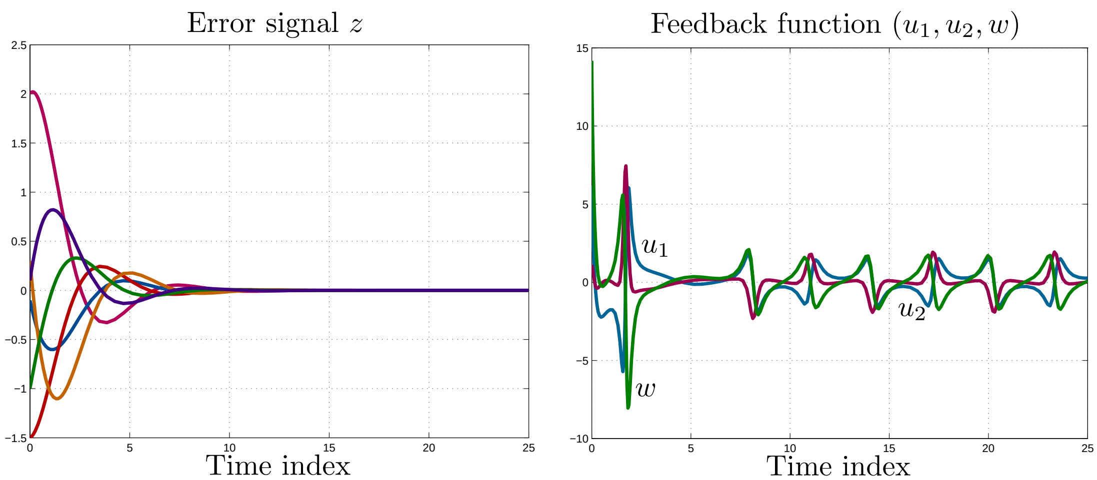

Figure 5.5: Time histories of the error z and the control input $u(z, \theta, \dot{\theta})$ respectively.

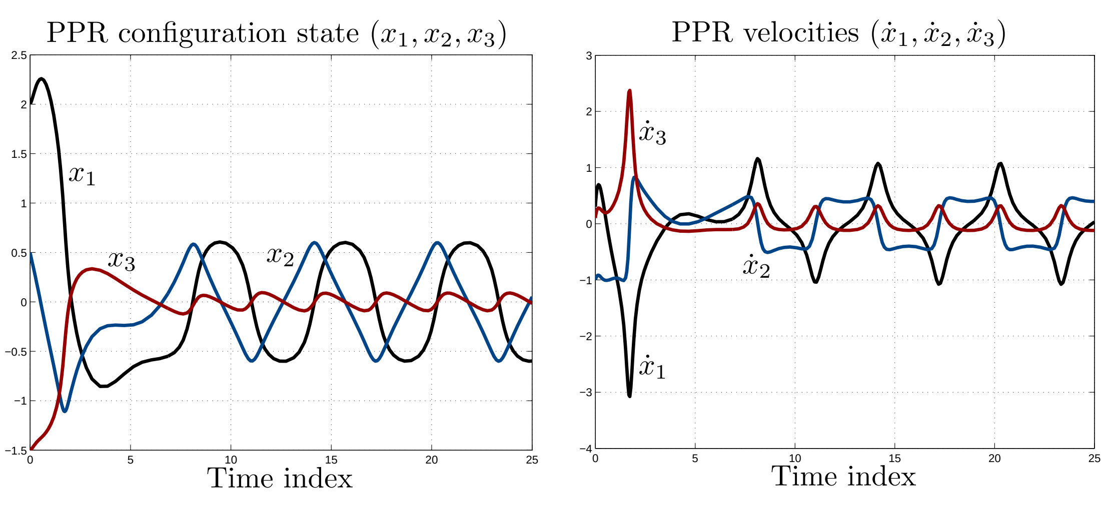

Figure 5.6: Time histories of the configuration x and velocity $\dot{x}$ respectively.

Chapter 6

Conclusions and future work

As shown in this thesis the tangent mappings associated with transverse functions, as defined in the Morin and Samson sense [13], satisfy also vertical transversality (Section 4.2). This property comes as a natural generalization of the transverse condition for functions.

This newly characterized property was used in this work in order to derive a control technique to control second-order systems and, in particular, to tackle stabilization of simple mechanical control systems defined on Lie groups. When this technique is applied to second-order systems one achieves practical stabilization of the configuration variables, namely one ensures that the projection of the state trajectory onto the configuration converge to a previously specified, arbitrarily small neighborhood of the desired equilibrium point.

A mechanical system, as remarked in Section 2.3, can be represented as evolving on the tangent bundle of the configuration manifold. With the proposed approach it is possible to impose any desired dynamics to the error, in particular one desires the identity element to be an asymptotically stable point. Then one assures that configuration trajectories of the system converge to a specified small neighborhood of the bundle projection of desired equilibria. This means that the system configuration will ultimately evolve sufficiently near the target equilibrium point. However, at present, little can be asserted about the evolution of the fiber components of the trajectories (corresponding to velocities of the system in a mechanical system). This is mainly because, even though the error tends to zero as time increases, the closed-loop system evolves according to a nontrivial zero-dynamics entirely characterized by the zero dynamics of the auxiliary system, as shown in Section 4.4.

When the drift term of the target and auxiliary systems are semisprays (which, in the case of simple mechanical systems, it turns out to be the case), we show that the zero-dynamics is also defined by a semispray. One possible way to proceed in order to achieve convergence of the fiber trajectories to a specified neighborhood of the zero section is to examine the drift vector field of the auxiliary system to determine if the imposition of a particular structure ensures that the velocities remain bounded, i.e. it could be possible that, by selecting an appropriate second-order field Δ in Equation (4.8), the target system achieves the desired behavior.

While applying the approach to a particular system, specifically to the one in section 5.2 (PPR manipulator), we discovered that the zero-dynamics of the auxiliary system has an interpretation as a simple mechanical system by finding a Riemannian metric. With this metric we were able to define a smooth energy function for the system and thus to show that the fiber trajectories were bounded, with a bound depending on the initial conditions. This example seems to suggest that, for mechanical systems, the fiber trajectories are bounded, given that in this particular example the spray associated to the zero-dynamics of the auxiliary system defines a torsionless connection which is also compatible with a Riemannian metric. If one can show that the semispray that rules the evolution of the zero-dynamics is the Levi-Cività connection for a Riemannian metric, then the next natural step would be to use the latter to define an energy function for the auxiliary system, which could in turn be useful to guarantee that the velocities do not grow unbounded. However, finding a Riemannian metric associated with the Cristoffel symbols of a connection is, in general, a rather involved task, and hence work remains to be done in this direction.

Another way to achieve stabilization of mechanical systems could be to analyze the possibility of using the so-called *generalized transverse functions*, a more recent result published by Morin and Samson [15]. With these functions, which are also transverse in the original sense, one is able to obtain asymptotic stabilization of trajectories for certain systems in which the drift is not needed to generate the accessibility distribution. Trying to generalize these functions might be instrumental to achieve practical stabilization of mechanical systems.

The implementation of trajectory tracking controllers by using the proposed approach is formally straightforward and is another road to explore in immediate future work. More future work is to study mechanical systems with symmetries, given that symmetries play an important role in the analysis and design of motion control algorithms.

Finally, it is worth mentioning that, although the approach outlined in this thesis does not constitute a complete extension of Morin and Samson's approach based on transverse functions, it takes steps toward what might constitute an interesting theory for the stabilization of admissible trajectories for second-order systems, including fixed points.

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