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Introduction

In the ECRH heating experiment on TFR at Fontenay-aux-Roses, carried out by the group FOM-CEA, 600 kW of microwave power has to be radiated into the plasma. The polarization should be parallel to the main magnetic field used for confining the plasma so that the ordinary mode is excited.

The microwave power is generated by three 60 GHz (5 mm) gyrotrons of 200 kW each (Varian type VGE 8060 B) [1]. It is transported in circular waveguide of radius 13.89 mm. To keep the ohmic losses low, the TE_{01}-mode is used for the transport. Radiation from a truncated waveguide with this mode will not give the polarization needed. Just before launching, the TE_{01}-mode is therefore converted to the TE_{11}-mode which has a field pattern (Fig. 8) very well suited for exciting the ordinary mode. In a later stage of the experiment the HE_{11}-mode will be used. This mode has the ideal field but is not an eigenmode of straight metallic circular waveguide.

In principle, any mode converter can be made by introducing a dielectric rod along or parallel to the axis of the guide while the diameter or other parameter of the rod varies with the position along the axis [2]. For conversion between modes that differ by only one in angular index, as in this case, bending of the guide is also very efficient and better suited for high power applications when the inevitable losses in dielectric material might increase its temperature too much.

To design mode converters of this type the Maxwell equations (ME) are transformed into generalized telegraphist equations (GTE), a set of linear differential equations which is equal to the set of equations describing the currents and voltages in mutually coupled transmission lines [3], [4], or in mutually coupled tuned LC-circuits [5]. Each mode can then be identified with the current in one of the circuits. Power can be transferred from one circuit to another with different resonant frequency by reversing the sign of the coupling at certain points in time. In the microwave analogon this corresponds to a change of curvature along the axis of the guide which leads to a wiggling axis.

In order to keep reflections low, the curvature should not change abruptly. This also simplifies the calculations. For practical reasons both ends of the converter should have the same direction and all "wiggles" should have the same form.

Conversion of Maxwell's equations into generalized telegraphist equations

The ME for fields with an assumed time dependence contained in a separable factor $e^{-i\omega t}$ in the homogeneous isotropic non-conducting medium inside the waveguide are:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \text{.....ME 1}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \qquad \text{.....ME 2}$$

$$\vec{\nabla} \times \vec{E} = i\omega\mu\vec{H} \qquad \text{.....ME 3}$$

$$\vec{\nabla} \times \vec{H} = -i\omega\varepsilon\vec{E} \qquad \text{.....ME 4} \quad .$$

Let the axis of the waveguide be in the w-direction of a general orthogonal curvilinear coordinate system, (u,v,w). Any field can be considered as the sum of a pure transverse magnetic (TM) field ($H_w = 0$) and a pure transverse electric (TE) field ($E_w = 0$). ME 1 and ME 2 are satisfied when we consider $\vec{E}$ and $\vec{H}$ as the rotation of the vectors A', respectively A [6]:

$$\vec{E} = \vec{\nabla} \times \vec{A}' \tag{1}$$

$$\vec{H} = \vec{\nabla} \times \vec{A} \ . \tag{2}$$

For the TE, TM parts of the field $\vec{A}'$, $\vec{A}$ can have only a component in the w-direction π', π which are called stream functions

$$\vec{A}'_{TE} = \pi' \hat{w} \tag{3}$$

$$\vec{A}_{TM} = \pi \hat{w} \quad . \tag{4}$$

This gives for the TE part of the field:

$$\vec{E}_{TE} = \vec{\nabla} \times \pi' \hat{w} = \vec{\nabla}_t \times \pi' \hat{w} \quad . \tag{5}$$

Substitution of this field in ME 3 gives:

$$\vec{\nabla} \times (\vec{\nabla} \times \pi' \hat{w}) = i\omega\mu\vec{H}_{TE}$$

$$\vec{\nabla}(\vec{\nabla}_w \cdot \pi') - \nabla^2 \pi' \hat{w} = i\omega\mu\vec{H}_{TE}$$

$$\vec{\nabla}_t(\nabla_w \pi') - (\nabla^2 - \nabla_w^2)\pi' \hat{w} = i\omega\mu\vec{H}_{TE}$$

$$\vec{\nabla}_t(\nabla_w \pi') - \nabla_t^2 \pi' \hat{w} = i\omega\mu\vec{H}_{TE}$$

$$\vec{H}_{TE} = \frac{1}{i\omega\mu}\left[\vec{\nabla}_t(\nabla_w \pi') - \nabla_t^2 \pi' \hat{w}\right] \quad . \tag{6}$$

Substitution of this field in ME 4 reveals that $\nabla_u \pi'$ and $\nabla_w \pi'$ have to satisfy the wave equation (WE) which is automatically satisfied when π' itself satisfies the WE:

$$(\nabla^2 + k_o^2)\pi' = 0 \; , \tag{7}$$

where k_o is the free space phase constant.
In this way ME 1 through ME 4 are replaced by Eqs. (5), (6) and (7) for the TE part of the field. Although Eq. (6) is derived from ME 3 it also satisfies ME 2 as can be verified by taking the divergence of Eq. (6). When π' does not satisfy the WE, the fields $\vec{E}$ and $\vec{H}$ derived from π' with Eqs. (5) and (6) will still be TE and will still satisfy ME 1, ME 2 and ME 3 but not ME 4.

For the TM part of the field we find in a similar way

$$H_{TM} = \vec{\nabla}_t \times \pi\hat{w} \tag{8}$$

$$E_{TM} = -\frac{1}{i\omega\varepsilon}\left[\vec{\nabla}_t(\nabla_w\pi) - \nabla_t^2\pi\hat{w}\right] \quad , \tag{9}$$

$$(\nabla^2 + k_o^2)\pi = 0 \ . \tag{10}$$

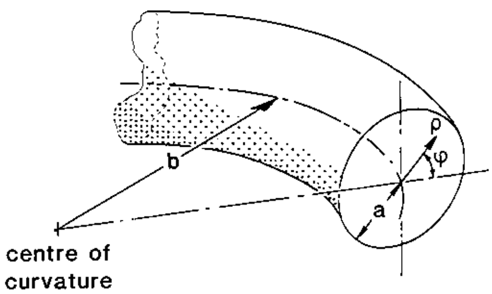
In this case, when π is replaced by an arbitrary function $F(u,v,w)$ the fields derived from F with Eqs. (8) and (9) will still be TM and satisfy the ME except possibly ME 3.

The total field will be the sum of the TE and TM fields

$$\vec{E} = \vec{\nabla}_t \times \pi'\hat{w} - \frac{1}{i\omega\varepsilon}\left[\vec{\nabla}_t(\nabla_w\pi) - \nabla_t^2(\pi\hat{w})\right] \quad , \tag{11}$$

$$\vec{H} = \vec{\nabla}_t \times \pi\hat{w} + \frac{1}{i\omega\mu}\left[\vec{\nabla}_t(\nabla_w\pi') - \nabla_t^2(\pi'\hat{w})\right] \quad . \tag{12}$$

The boundary conditions (B:) on the ideal conducting wall, $\vec{E}(B) = (E_u,0,0)$ and $\vec{H}(B) = (0,H_v,H_w)$, require that the scalars π',π as well as their derivatives ∇_u and ∇_w vanish on the boundary.



$$e_1 = \partial s/\partial\rho = 1$$

$$e_2 = \partial s/\partial\phi = \rho$$

$$e_3 = \partial s/\partial z = 1 + \frac{\rho\cos\phi}{b} \ .$$

Fig. 1. Coordinate system of bent guide.

Figure 1 shows the coordinate system used for the bent waveguide. ρ is the distance from the axis, $0 \leq \rho \leq a$, ϕ is the azimuthal angle $0 \leq \phi \leq 2\pi$, and z is the corresponding distance along the axis. When used as an index ρ, ϕ, and z are generally replaced by the numbers 1, 2 and 3.

Let ψ', ψ be solutions for π', π in straight cylindrical waveguide as obtained by assuming a separable factor as the only z-dependence. These solutions by definition have $\frac{\partial}{\partial\rho}\psi'(a,\phi,z) = 0$, $\psi(a,\phi,z) = 0$ at all z, so that $\nabla_3\psi'(a,\phi,z) = \nabla_3\psi(a,\phi,z) = 0$. From Eqs. (11) and (12) it can then be seen that ψ, ψ' satisfy (B:) for the bent guide. The ψ, ψ' do not necessarily satisfy the WE for the bent guide, but they constitute a complete set in the transverse plane because they are obtained by solving the two dimensional WE $(\nabla_t^2 + k_t^2)\psi, \psi' = 0$, after separation of a phase factor $e^{ik_3 z}$, $e^{ik_3' z}$ and subjected to (B:).

This means that any function in the transverse plane of the bent guide can be expanded in the set of solutions for the straight guide. So the π, π', satisfying the WE for the bent guide, can be written as:

$$\pi(\rho,\phi,z) = \sum_n \alpha_n \psi_n , \tag{13}$$

$$\pi'(\rho,\phi,z) = \sum_n \alpha_n' \psi_n' . \tag{14}$$

As the set ψ, ψ' is complete in the transverse plane it is only required for α, α' to depend on z. The set of coefficients $\alpha_n(z)$, $\alpha_n'(z)$ can be considered as a vector $\vec{\alpha}(z)$, $\vec{\alpha}'(z)$ representing the mode composition along z in terms of TE, TM eigenmodes of the straight guide. With these expansions Eqs. (11) and (12) with the summation convention become:

$$\vec{E} = \vec{\nabla}_t \times (\alpha_n' \psi_n') - \frac{1}{i\omega\varepsilon} \left[\vec{\nabla}_t (\nabla_3 \alpha_n \psi_n) - \nabla_t^2 (\alpha_n \psi_n)\, \hat{w}\right] , \tag{15}$$

$$\vec{H} = \vec{\nabla}_t \times (\alpha_n \psi_n) + \frac{1}{i\omega\mu} \left[\vec{\nabla}_t (\nabla_3 \alpha_n' \psi_n') - \nabla_t^2 (\alpha_n' \psi_n')\, \hat{w}\right] . \tag{16}$$

$\vec{E}$ and $\vec{H}$ from Eqs. (15) and (16) already satisfy ME 1, ME 2 and (B:). To obtain the relation between $\nabla_3(\vec{\alpha},\vec{\alpha}')$ and $(\vec{\alpha},\vec{\alpha}')$ necessary to satisfy all ME, it is sufficient to substitute the fields from Eqs. (15) and (16) into ME 3 and ME 4. Schelkunoff [3], [4] avoided the occurrence of second derivatives $\nabla_3^2(\vec{\alpha},\vec{\alpha}')$ in these last steps by introducing separate expansions for $\alpha_n \psi_n$, $\nabla_3 \alpha_n \psi_n$, $e_3 \nabla_t^2 (\alpha_n \psi_n)$ and the corresponding primed expansions. Equations (15) and (16) are then written as:

$$\vec{E} = (\vec{\nabla}_t V_n' \psi_n' \times \hat{z}) + \vec{\nabla}_t (V_n \psi_n) + \frac{1}{e_3} V_{3n} \psi_n \hat{z} , \tag{17}$$

$$\vec{H} = (\vec{\nabla}_t I_n \psi_n \times \hat{z}) + \vec{\nabla}_t (I_n' \psi_n') + \frac{1}{e_3} I_{3n}' \psi_n' \hat{z} , \tag{18}$$

where the factors $1/i\omega\varepsilon$ and $1/i\omega\mu$ are absorbed into the coefficients V_n, V_n', V_{3n}, I_n, I_n', I_{3n}'. The factor $1/e_3$ is also added for convenience.

Substitution of Eqs. (17) and (18) in ME 3 and ME 4 will give the relations between the $\nabla_3 V_n$, $\nabla_3 V'_n$ and the I_n, I'_n and the relation between the $\nabla_3 I_n$, $\nabla_3 I'_n$ and the V_n, V'_n .

These relations are the generalized telegraphist equations, GTE, a name due to Lord Kelvin who was the first to write down a similar equation for the relations between currents and voltages in a set of mutually coupled telegraph wires. The GTE for the bent waveguide are derived in the Appendix. The result is:

$$\begin{bmatrix} \frac{\partial V_m}{\partial z} \\ \frac{\partial V'_m}{\partial z} \\ \frac{\partial I_m}{\partial z} \\ \frac{\partial I_{m'}}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & Z_{mn} & Z_{mn'} \\ 0 & 0 & Z_{m'n} & Z_{m'n'} \\ Y_{mn} & Y_{mn'} & 0 & 0 \\ Y_{m'n} & Y_{m'n'} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_n \\ V_{n'} \\ I_n \\ I_{n'} \end{bmatrix} \tag{19a}$$

in which

$$Z_{mn} = - i\omega\mu \int e_3 \vec{\nabla}_t \psi_m \cdot \vec{\nabla}_t \psi_n \, dS + Z_{3mn} \tag{19b}$$

$$Z_{mn'} = + i\omega\mu \int e_3 \vec{\nabla}_t \psi_m \cdot (\vec{\nabla}_t \psi'_n \times \hat{z}) \, dS \tag{19c}$$

$$Z_{m'n} = - i\omega\mu \int e_3 (\vec{\nabla}_t \psi'_m \times \hat{z}) \cdot \nabla_t \psi_n \, dS \tag{19d}$$

$$Z_{m'n'} = - i\omega\mu \int e_3 \vec{\nabla}_t \psi'_m \cdot \vec{\nabla}_t \psi'_n \, dS \, , \tag{19e}$$

$$Y_{mn} = - i\omega\varepsilon \int e_3 \vec{\nabla}_t \psi_m \cdot \vec{\nabla}_t \psi_n \, dS \tag{19f}$$

$$Y_{mn'} = - i\omega\varepsilon \int e_3 \vec{\nabla}_t \psi_m \cdot (\vec{\nabla}_t \psi'_n \times \hat{z}) \, dS \tag{19g}$$

$$Y_{m'n} = - i\omega\varepsilon \int e_3 (\vec{\nabla}_t \psi'_m \times \hat{z}) \cdot \vec{\nabla}_t \psi_n \, dS \tag{19h}$$

$$Y_{m'n'} = - i\omega\varepsilon \int e_3 \vec{\nabla}_t \psi'_m \cdot \vec{\nabla}_t \psi'_n \, dS + Y'_{3m'n'} \, , \tag{19i}$$

in which

Z_{3mn} is the mn-th element of the matrix $(\vec{Y}_3)^{-1}$, defined by

$$Y_{3rs} = -\, i\omega\varepsilon \int \frac{\psi_r \psi_s}{e_3}\, dS \ , \tag{19j}$$

and

$Y'_{3m'n'}$ is the mn-th element of the matrix $(\vec{\vec{Z}}{}'_3)^{-1}$, defined by

$$Z'_{3rs} = +\, i\omega\mu \int \frac{\psi'_r \psi'_s}{e_3}\, dS \ . \tag{19k}$$

For straight cylindrical waveguide with circular cross-section the corresponding solutions ψ, ψ' are products of Bessel- and goniometric functions.

$$\Psi_{mn} = N_{mn}\, J_m(k_{tmn}\rho)\, \sin m\phi \ , \tag{20}$$

$$\Psi'_{mn} = N_{mn}\, J_m(k'_{tmn}\rho)\, \cos m\phi \ , \tag{21}$$

with normalization constants:

$$N_{mn} = \frac{1}{k_{tmn}\rho\, J_{m-1}(k_{tmn}\rho)} \sqrt{\varepsilon_m/\pi} \ , \tag{22}$$

$$N'_{mn} = \frac{1}{\sqrt{k^2_{tmn} - m^2}\, J_m(k'_{tmn}\rho)} \sqrt{\varepsilon_m/\pi} \ , \tag{23}$$

in which

$$\begin{aligned} \varepsilon_m &= 1 \ , \quad m=0 \ , \\ &= 2 \ , \quad m\neq 0 \ . \end{aligned} \tag{24}$$

The functions with $\cos(m\phi)$ respectively $\sin(-m\phi)$ have been omitted for simplicity but are required for completeness of the set. The elements with unequal subscript are the coupling coefficients. The elements with equal subscript are the phase constants k_3 in the direction of propagation.

Note that the coupling between a TM-mode and a TE-mode is proportional to the curvature $1/b$, because

$$\int e_3 \vec{\nabla}_t \psi_m \cdot (\hat{z} \times \vec{\nabla}_t) \psi'_n \, dS = \frac{1}{b} \int \rho \cos\phi \; \vec{\nabla}_t \psi_m \cdot (\hat{z} \times \vec{\nabla}_t) \psi'_n \, dS \; . \tag{25}$$

The mutual coupling between TM-modes and the mutual coupling between TE-modes contains a term porportional to the curvature 1/b and a second term, which is only in a first-order approximation proportional to the curvature. A different method to obtain the first-order term of the coupling coefficients is given by [11].

Analogy with mutual coupled LC-circuits [5]

Let I_m be the current through the inductor and V_m be the voltage over the capacitor of tuned circuit No. m of a system of mutually coupled tuned circuits as in Fig. 2.

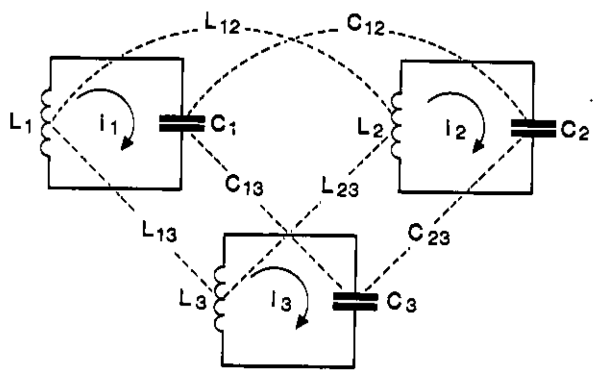


Fig. 2. Tuned circuits with mutual inductance and mutual capacitance.

The rate of change of I_m, V_m is given by: (summation convention assumed)

$$\frac{\partial I_m}{\partial t} = \frac{V_m}{L_m} + \frac{L_{mn}}{L_m L_n} V_n \; , \tag{26}$$

$$\frac{\partial V_m}{\partial t} = \frac{I_m}{C_m} + \frac{C_{mn}}{C_m C_n} I_n \; , \tag{27}$$

in which L_{mn} is the voltage induced in inductor m by a current rate of change of 1 A/V in inductor n (mutual inductance) and C_{mn} is the current induced in capacitor m by a voltage rate of change of 1 V/s in capacitor n (mutual capacitance).

This set of equations differs from the GTE's only therein that it has "t" as independent variable while the GTE's have "z". The analogy is complete. Each tuned circuit corresponds to a mode of propagation, the current in it to magnetic field, the voltage to the electric field, the resonance frequency to the wave number k_3, the mutual reactances to the coupling coefficients and passing of time to distance along the guide.

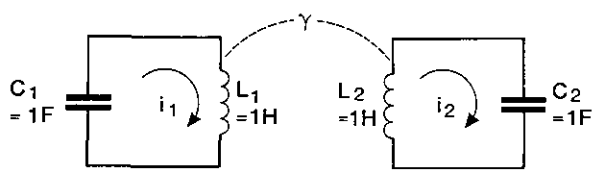
Once the coupling coefficients have been determined from the analysis, the design of mode converters can be carried out completely in terms of coupled LC-circuits.

Two slightly coupled degenerate modes

The TE_{01}-mode and the TM_{11}-mode have the same k_3 and therefore the same phase velocity:

$$V_f = \frac{\omega}{k_3} \left(\frac{\text{rad/s}}{\text{rad/m}} = \frac{m}{s}\right) . \tag{28}$$

A slight bend will weakly couple these two modes. The tuned circuit analagon consists of two coupled tuned circuits with the same resonance frequency ω_o. Normalization to $\omega_o = 1$ and L/C = 1 leads to the circuit of Fig. 3a.



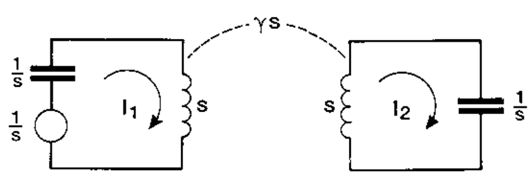


Fig. 3a.
Two equally tuned circuits.

Fig. 3b.
Equivalent circuit in the complex frequency domain.

Let the initial currents and voltages be zero except for $V_{C_1}(t=0) = 1$ representing a pure TE_{01}-mode at the beginning of the bend, z = 0. Laplace transformation of the circuit with initial conditions [7] leads to the circuit of Fig. 3b. Kirchhoff's rules then gives

$$\begin{bmatrix} (s + \frac{1}{s}) & \gamma s \\ \gamma s & (s + \frac{1}{s}) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} , \tag{29}$$

from which the Laplace transform of I_1,

$$I_1(s) = \frac{\frac{1}{s}(s + \frac{1}{s})}{(s + \frac{1}{s})^2 - \gamma^2 s^2} = \frac{1}{2} \left[\frac{\frac{1}{1-\gamma}}{s^2 + \frac{1}{1-\gamma}} + \frac{\frac{1}{1+\gamma}}{s^2 + \frac{1}{1+\gamma}} \right] , \tag{30}$$

which in the time domain represents:

$$i_1(t) = \frac{1}{2} \quad \frac{1}{\sqrt{1-\gamma}} \sin \frac{t}{\sqrt{1-\gamma}} + \frac{1}{\sqrt{1+\gamma}} \sin \frac{t}{\sqrt{1+\gamma}} . \tag{31}$$

$i_1(t)$ is the sum of two oscillators of nearly equal frequency, which gives the well-known beating.

The energy travels back and forth between the two circuits. One complete transfer takes place in the time needed to build up a phase difference of π rad between the two oscillators.

$$\frac{t_{tr}}{\sqrt{1-\gamma}} - \frac{t_{tr}}{\sqrt{1+\gamma}} = \pi . \tag{32}$$

Back to the coupling between the TE_{01}- and the TM_{11}-mode when coupling to other modes, reflections and losses are all neglected.

The GTE, (Eq. (19a)) will be for this case:

$$\frac{\partial V_1}{\partial z} = jk_3V_1 + jcV_2$$

$$\frac{\partial V_2}{\partial z} = jcV_1 + jk_3V_2 \ , \tag{33}$$

where the modes are numbered 1 and 2. k_3 and c can be obtained from Eqs. (19a), (19b) and the fixed ratio between V and I, the wave impedance, for each mode. All coefficients are pure imaginary. Separation of the real and imaginary parts, $\vec{R}$ and $j\vec{I}$ of $\vec{V}$ change (33) into:

$$\frac{\partial R_1}{\partial z} = - k_3I_1 - cI_2 \ ,$$

$$\frac{\partial R_2}{\partial z} = - cI_1 - k_3I_2 \ , \tag{34}$$

$$\frac{\partial I_1}{\partial z} = + k_3R_1 + cR_2 \ ,$$

$$\frac{\partial I_2}{\partial z} = + cR_1 + k_3R_2 \ . \tag{35}$$

Equation (34) is differentiated again. Equation (35) is substituted into it and the result Laplace transformed with respect to z. This gives:

$$\begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \begin{matrix} = \\ \\ = \end{matrix} \begin{bmatrix} s^2 + k_3^2 + c^2 & 2k_3c \\ \\ 2k_3c & s^2 + k_3^2 + c^2 \end{bmatrix} \begin{matrix} = \\ \\ = \end{matrix} \begin{bmatrix} R_1 \\ \\ R_2 \end{bmatrix} \ . \tag{36}$$

The poles are located at

$$k_3\sqrt{1 \pm \frac{2c}{k_3}} \ . \tag{37}$$

As their difference is approximately 2c, complete power transfer will take place, according to (32), at:

$$z_{tr} = \frac{\pi}{2c} . \tag{38}$$

For curved circular waveguide of radius a = 13.9 mm and free space phase constant $k_0 = 200 \times 2\pi$, the value of $c = 3.222\ m^{-1}$ at a radius of curvature of 1 m. With this value $z_{tr} = 0.975$ m, which is at a bending angle of 28°, independent of (small) curvature, as long as coupling is proportional to curvature.

Coupling between two modes with different phase velocity

To analyse this case we detune the circuits of Fig. 3 as follows:

$$L_1 = (1+\delta)H \quad ; \quad L_2 = (1-\delta)H \quad . \tag{39}$$

This will give a current in the frequency domain:

$$I_1(s) = \frac{s^2(1 - \delta) + 1}{\left(s^2(1+w) + 1\right)\left(s^2(1-w) + 1\right)} \quad ; \text{ in which } w = \sqrt{\delta^2 + \gamma^2} \ , \tag{40}$$

or in the time domain:

$$i_1(t) = \frac{1}{2w} \left[\frac{w + \delta)}{\sqrt{1 + w}} \sin \frac{t}{\sqrt{1 + w}} + \frac{w - \delta}{\sqrt{1 - w}} \sin \frac{t}{\sqrt{1 - w}}\right] . \tag{41}$$

As in the previous case $i_1(t)$ is the sum of two terms of different frequency. The difference between the amplitudes, however, is more pronounced. When a and b represent the amplitudes of the two terms, the maximum of $i_1(t)$ varies between a+b and $|a-b|$. The beats will therefore not be complete as in the degenerate case.

Assuming $\delta \ll 1$ and $\gamma \ll 1$ and neglecting all terms of higher order in δ and γ, we obtain:

$$\frac{|a - b|}{a+b} \cong \frac{\delta}{\sqrt{\delta^2 + \gamma^2}} \quad . \tag{42}$$

Vectorial representation

The current I and the voltage V in the coupled circuits can be represented by rotating vectors. Let from both angles be subtracted a fixed value such that vector I_1 no longer rotates. Let us start with I_1 high and $V_2 = 0$ for the two equal tuned circuits.

The induced voltage V_2 will be $\frac{\pi}{2}$ radians out of phase (Fig. 4b). Coupling will be continuous but in order to be able to design a vector diagram we make the coupling discrete, which means that the effect of the coupling within one period is integrated over that period. The next period, the induced voltage will again be $\frac{\pi}{2}$ out of phase with I_1 and thus adds to the voltage still present of the previous induction (no damping), (Fig. 4c). After several periods the amplitude of V_2 has increased considerably and the back-induced I_1, will again lead V_2 in phase by $\frac{\pi}{2}$ and thus be π out of phase with the original. Thus even when V_2 is high and I_1 is low, V_2 will go on increasing and I_1 decreasing until I_1 reaches zero where it changes sign. From there on the process is reversed (Fig. 4d).

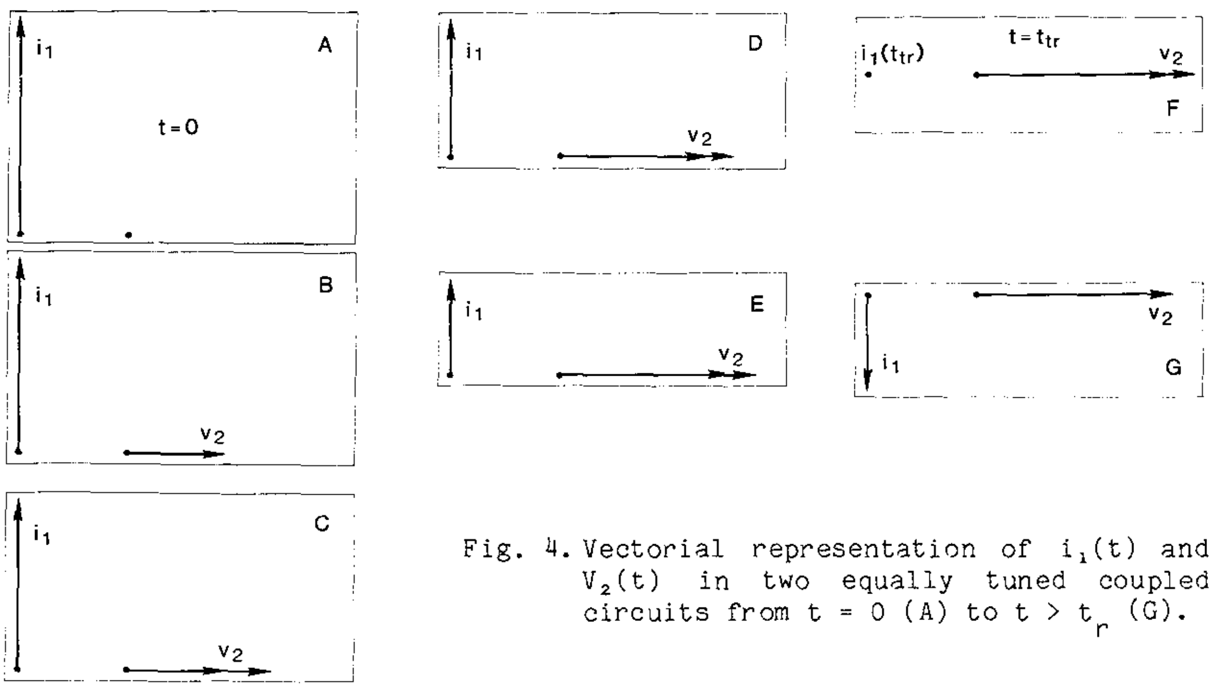


Fig. 4. Vectorial representation of $i_1(t)$ and $V_2(t)$ in two equally tuned coupled circuits from $t = 0$ (A) to $t > t_r$ (G).

For the non-degenerate case, with the detuned circuits, we have in the vectorial representation with discrete coupling (Fig. 5) that, when a new V_2 is induced, the signal still present has rotated so that the signals do not add precisely in phase. When after several periods the voltage V_2 has shifted more than $\frac{\pi}{2}$, the induced voltage will no longer increase but decrease the amplitude of V_2, so that the power flow from one circuit to the other is reversed too early for complete transfer. In this picture the coupling is assumed to be so small that I_1 remains practically unaltered during this part of the process. To obtain complete transfer of power, the sign of the coupling has to be reversed each time the phase difference between I_1 and V_2 passes the value $n\pi$. The effect of coupling reversal is shown in Fig. 6.

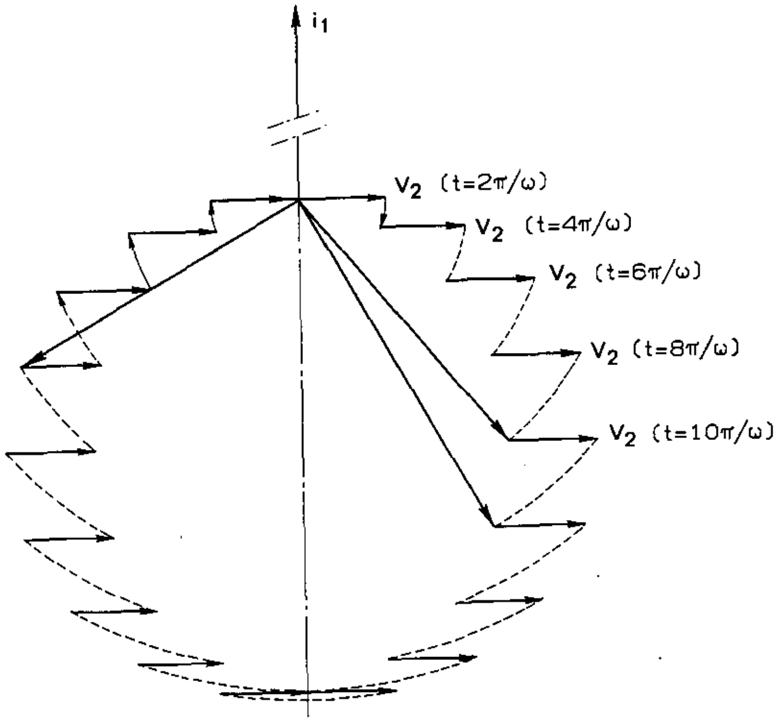


Fig. 5. Vectorial representation of $V_2(t)$ in detuned circuits.

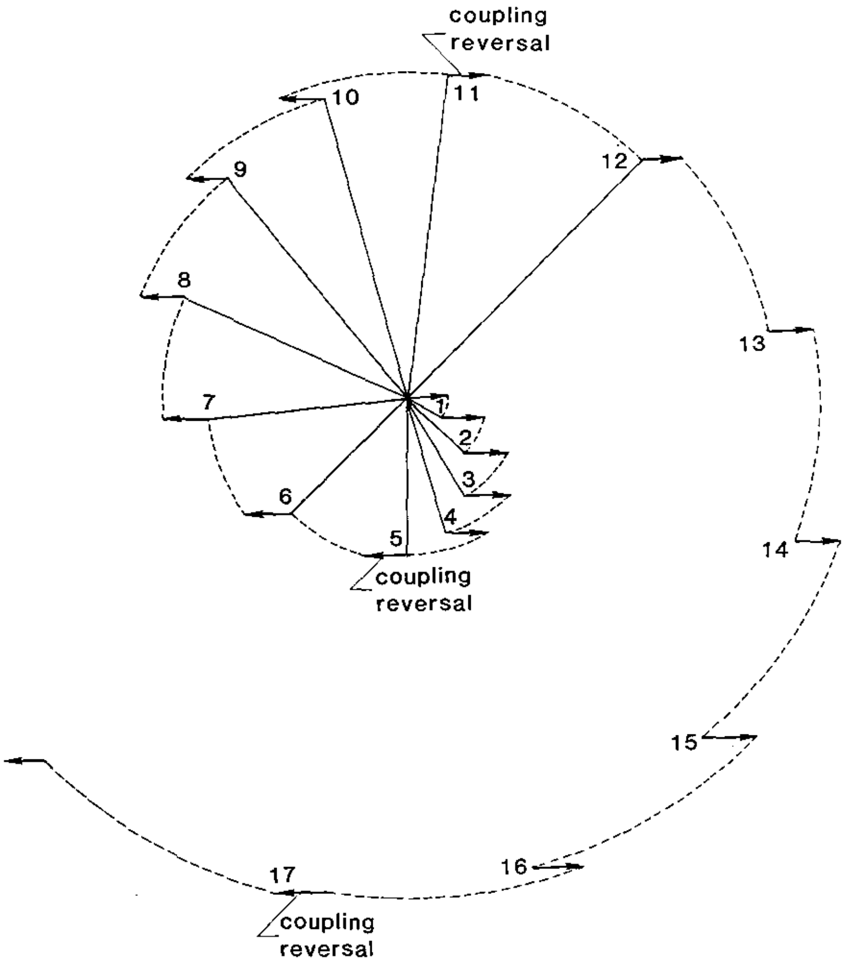


Fig. 6. $V_2(t)$ in detuned circuits with sign reversal of coupling at $\phi = n\pi$.

Around the points $\phi = n\pi$ the coupling has only little influence on the amplitude, so the points, where the coupling is reversed, are not critical. In the qualitative curve of Figs. 5 and 6, the coupling from V_2 to I_1 is not taken into account for simplicity. It is not essential for explaining the principle. The sign of the coupling can be reversed by changing from inductive to capacitive coupling. The sign of the coupling between modes in curved waveguide is reversed by reversing the curvature.
This accounts for the wiggle type of the TE_{01}-TE_{11} converter.

Design procedure TE_{01} - TE_{11}-mode converter

Theory allows radii of curvature equal to the radius of the waveguide. The accuracy is determined by the number of modes that is taken into account. The parameter which determines the importance for a certain mode to be taken into account, is the ratio of its coupling coefficient with other modes that reach high amplitude and the difference in k_3, Δk_3.

For ECRH the mode converter should reflect little power for several reasons, e.g., gyrotron protection, risk of voltage breakdown due to standing waves and because microwave power is expensive. Low reflections can certainly be obtained by using only small values of the curvature and its derivative. In this design we have chosen for a curvature according to

$$cur = c\hat{u}r \sin \Delta k_3 z \quad , \tag{43}$$

where $c\hat{u}r$ is the amplitude of the curvature. The curvature is the inverse of the length of the radius of curvature, b (Fig. 1).

As cur is the second derivative

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{b} = + \, c\hat{u}r \sin \Delta k_3 z \ , \tag{44}$$

the axis of the converter should have the form:

$$y = - \frac{c\hat{u}r}{(\Delta k_3)^2} \sin \Delta k_3 z \ , \tag{45}$$

shown in Fig. 7.

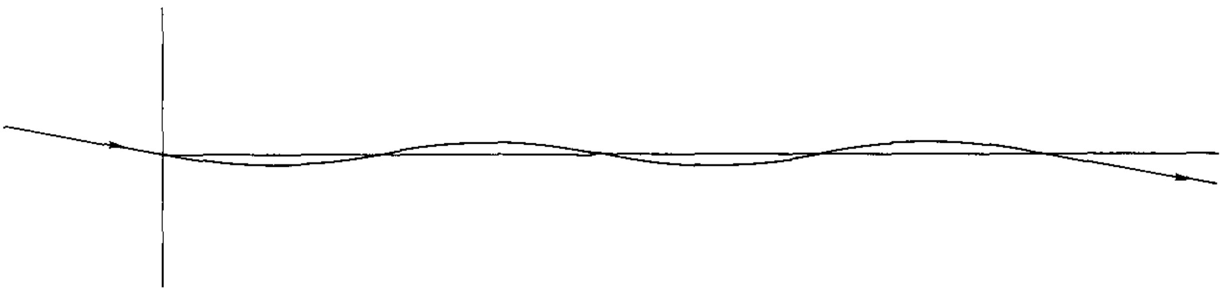


Fig. 7. Axis of guide in wiggle type of converter.

A design with low reflections also has computational advantages. Half the set of equations needs only to be used because for each mode the wave impedance, the ratio of E-field to H-field, is constant. A converter consisting of w-wiggles will have a total length:

$$L_{conv.} = \frac{2\pi w}{\Delta k_3} \quad . \tag{46}$$

Mode conversion will then be as in the degenerate case, except that the gradual curvature causes coupling and deviation from optimal phase to be smeared out proportional to $[\sin(z)]^2$. The effective curvature is thereby reduced by a factor:

$$\frac{\int \sin^2\phi \, d\phi}{\int d\phi} = \frac{1}{2} \, . \tag{47}$$

Thus, according to (38), power is transferred completely after:

$$z_{tr} = 2 \times \frac{\pi}{2 \times (c \times c\hat{u}r)} = \frac{\pi}{c \times c\hat{u}r} \quad , \tag{48}$$

where $(c \times c\hat{u}r)$ is the amplitude of the coupling coefficient.

This length must be equal to the length given by (46). It follows that:

$$c\hat{u}r = \frac{\Delta k_3}{2cw} = \frac{23.62}{2 \cdot 3.207 \, w} = \frac{3.68}{w} \, . \tag{49}$$

The error made in the first-order approximation of the coupling coefficients are of the order of $(a/b)^2$, where a is the waveguide radius and b the radius of curvature. The constraint $(a/b)^2 < 0.01$ sets the number of wiggles at four or more.

Optimizing by computer simulation

The rough design of the converter is a circular waveguide of radius 13.9 mm with a number "w" of wiggles with curvature given by (49). To optimize the design with respect to maximum conversion efficiency, a computer program has been written by C.A.J. Hugenholtz which simulates the converter and adjusts the values of the length and the maximum curvature such as to obtain maximum efficiency in a series of identical wiggles not separated by straight sections as phase shifters.

Simulation of the converter is done by solving the linear differential equation

$$\frac{\partial \vec{V}}{\partial z} = \vec{\vec{C}} \cdot \vec{V} , \tag{50}$$

in which the off-diagonal elements of the matrix $\vec{\vec{C}}$ are the coupling coefficients (imaginary) and the damping (real).

The complex vector $\vec{V}$ represents the amplitudes of the modes normalized with respect to power. Six modes are taken into account, the TE_{11}-, TE_{01}-, TM_{11}-, TE_{21}-, TM_{21}- and TE_{12}-mode. The neglected mode with the highest ratio between coupling coefficient and Δk_3 is the TE_{13}- mode. The calculated coupling coefficients are within 10^{-3} identical to the values given by Morgan [2], see Table 1 and Fig. 8.

TABLE 1

Phase constants and coupling coefficients for a = 0.01389 m, b = 1 m and k_0 = 200×2π rad/m

Mode No.	1 TE_{11}	2 TE_{21}	3 TE_{01}	4 TM_{11}	5 TM_{21}	6 TE_{12}
TE_{11}	1249.6360	5.185	-3.207	0	0	0
TE_{21}	5.185	1237.2765	0	2.201	0	1.743
TE_{01}	-3.207	0	1226.0274	-3.223	0	-5.245
TM_{11}	0	2.201	-3.223	1226.0274	5.081	0
TM_{21}	0	0	0	5.081	1201.0919	0
TE_{12}	0	1.743	-5.245	0	0	1196.6673

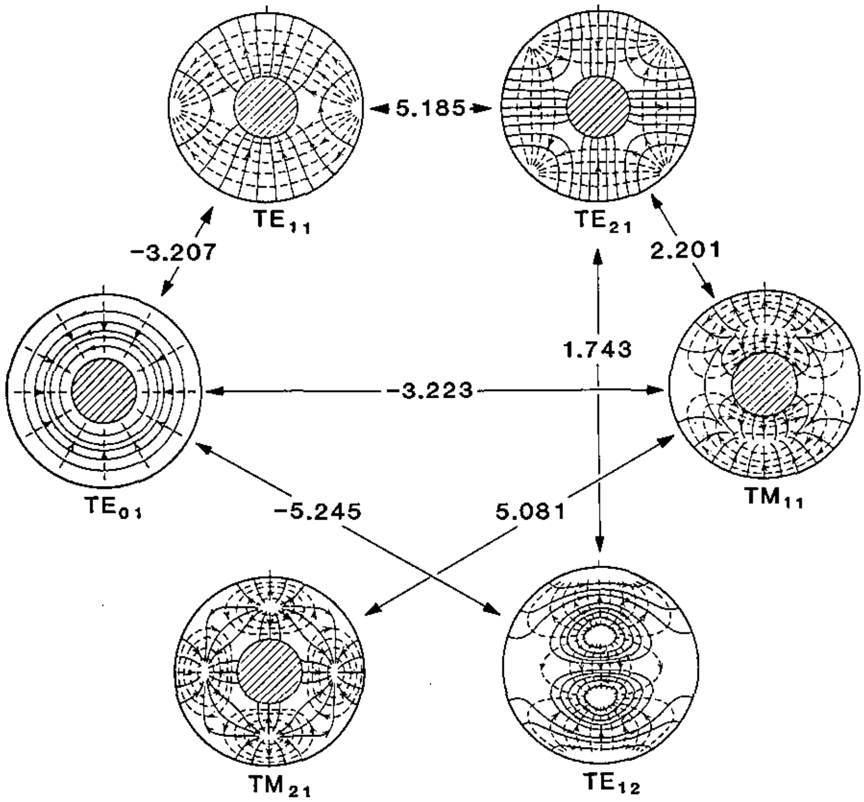


Fig. 8. Field pattern of modes of Table 1;
——— E-field, ------ H-field.

Table 2 gives the damping coefficients used (Collin [6]).

TABLE 2

Mode	attenuation constant copper
TE_{11}	5.3×10^{-3}
TE_{21}	9.8×10^{-3}
TE_{01}	0.6×10^{-3}
TM_{11}	12.6×10^{-3}
TM_{21}	12.9×10^{-3}
TE_{12}	1.7×10^{-3}

At each point in the converter the coupling coefficient from the Table is multiplied by the curvature at that point. As the IMSL routine (DVERK) used can handle only reals, the equations had to be separated into real and imaginary parts by putting:

$$\vec{V} = \vec{R} + j\vec{I} \; ; \quad \vec{C} = \vec{\alpha} + j\vec{\beta} \; . \tag{51}$$

The equations then become:

$$\frac{\partial \vec{R}}{\partial z} = \vec{\alpha} \cdot \vec{R} - \vec{\beta} \cdot \vec{I} \; , \tag{52}$$

$$\frac{\partial \vec{I}}{\partial z} = \vec{\beta} \cdot \vec{R} + \vec{\alpha} \cdot \vec{I} \; . \tag{53}$$

More insight into the simulation can be gained by looking at the circuit diagram (Fig. 9) of an analog computer [7], which could also be used for this type of problem. The second-order loops simulate the tuned circuits. The multipliers which cause the largest deviation from linearity in these circuits, occur only in the coupling. They are used to calculate the coupling as a function of t. A separate sinewave oscillator simulates the curvature.

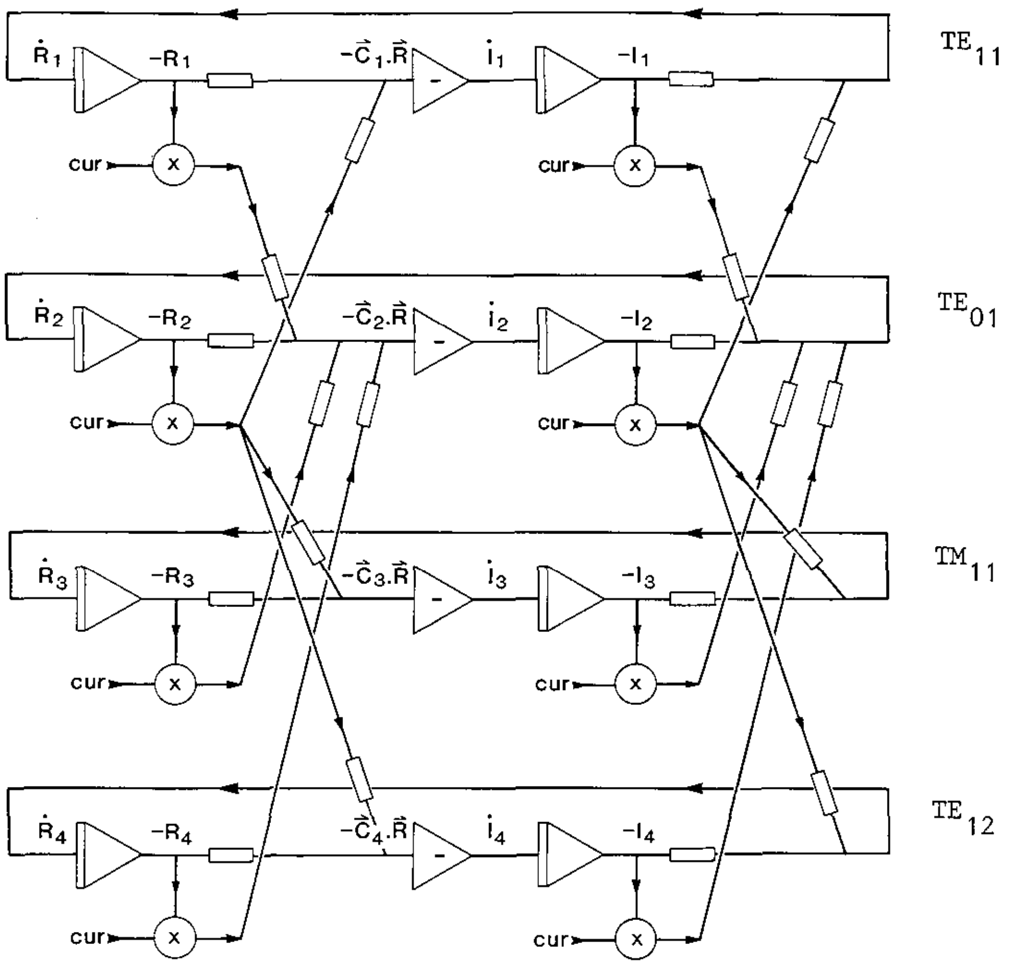


Fig. 9. Analog computer circuit diagram simulating four modes in the converter.

The optimum result for a converter using only one wiggle is shown in Fig. 10. The first-order approximations were certainly not allowed, but the results have didactic value. The drawn lines are the amplitudes of the four modes, that reach high amplitude, their scale is at the left. Their phase relative to mode TE_{01} is dotted with its scale at the right. The phase has been adjusted to lie within the range $-\pi \leq 0 \leq \pi$ by adding or subtracting a multiple of 2π if necessary.

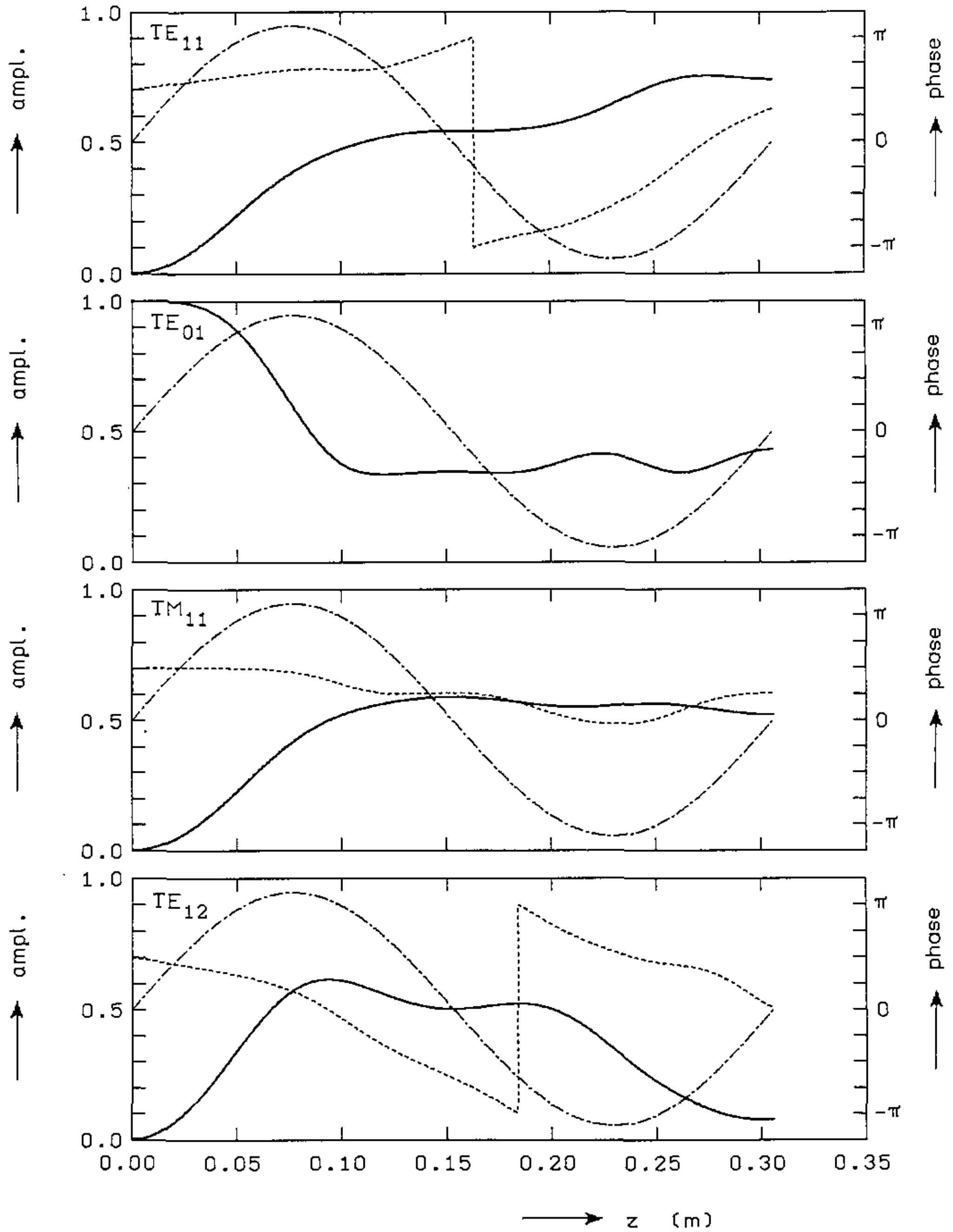


Fig. 10. Amplitude (———) and phase with respect to mode No. 2 (....) of four modes in hypothetical converter consisting of only one wiggle. Curvature (-.-.-) is without scale.

Note that the curvature is reversed when the phase of the desired mode passes a multiple of π. The phase of the degenerate mode TM_{11} which is $\pi/2$ in a gentle bend, is disturbed at strong curvature, so that it does not return to zero after the inverse bend.

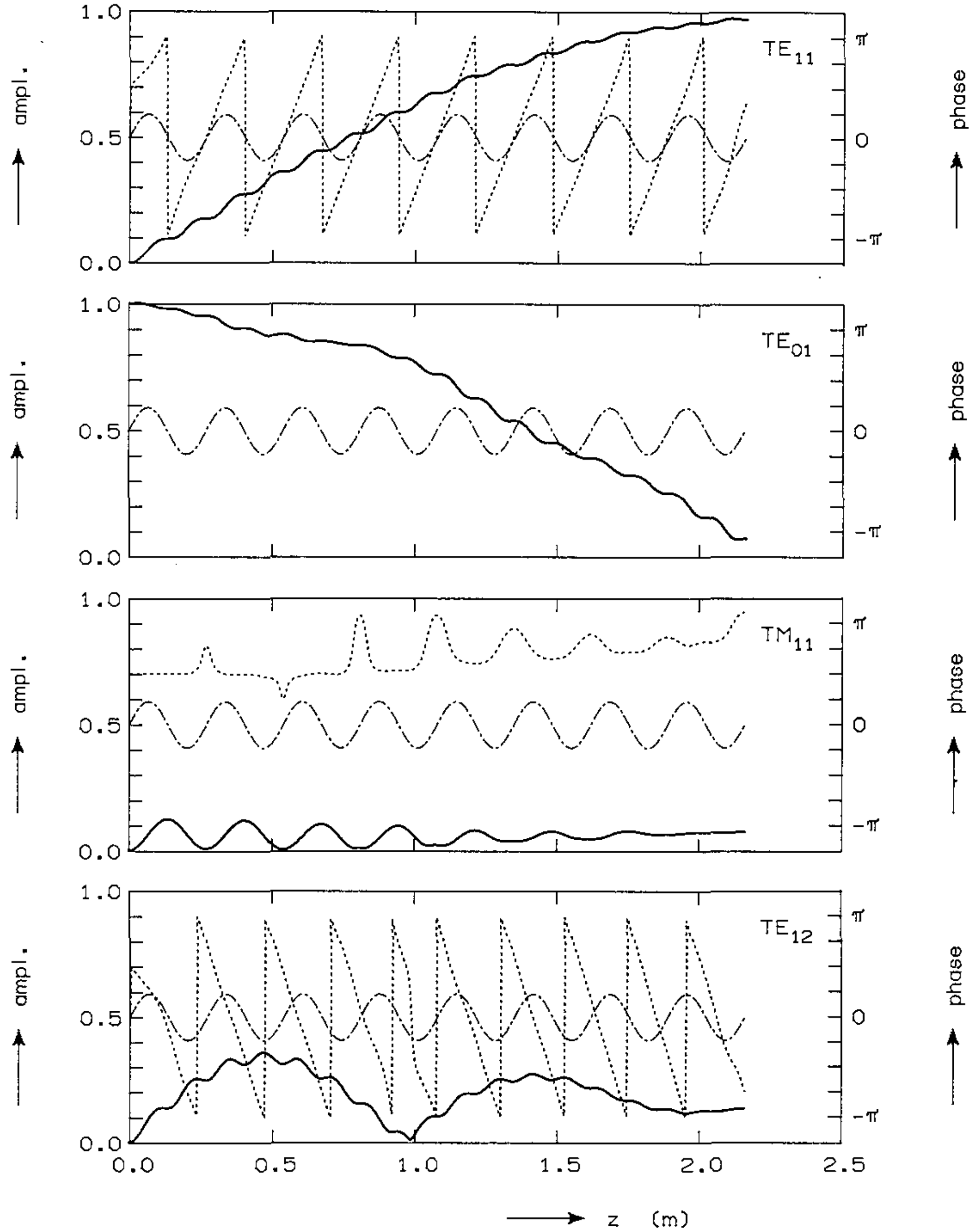


Fig. 11. Amplitude (——) and phase with respect to mode Nr. 2 (····) of four modes in converter consisting of 8 wiggles. Curvature (-.-.-.-) is without scale.

Figure 11 shows the optimum result for a converter using 8 wiggles. Here, the curvature is weak enough for the first-order approximation. The optimum values for cûr and wiggle length do not deviate much from the start values as shown in Table 3.

TABLE 3

	start value	optimum value
cûr	0.460	0.460
wiggles/metre	3.760	3.700
efficiency	90.70	95.20

Note in the upper graph that at each curvature reversal the phase of the desired mode TE_{11} is always a multiple of π. Phase difference grows slower at the beginning where the induced signal, which remains at $\pi/2$ is still relatively strong (see vector representation Fig. 6). The $\pi/2$ phase relation of the degenerate mode, TM_{11}, remains much better at these low curvature values. This results in a return to nearly zero during the opposite bend. Deviations of phase from $\pi/2$ are visible at the minima when sensitivity for coupling with other modes is at its highest.

The mode TE_{12} attains considerable amplitude (16% of power) after 2 and 5 wiggles. This means that it will not be wise to give the converter these numbers of wiggles. It can be seen that a multiple of 4 wiggles is a good choice. The efficiency vs the number of wiggles is given in Fig. 12.

They confirm that because of the interaction with the TE_{12}-mode the number of wiggles should be a multiple of 4.

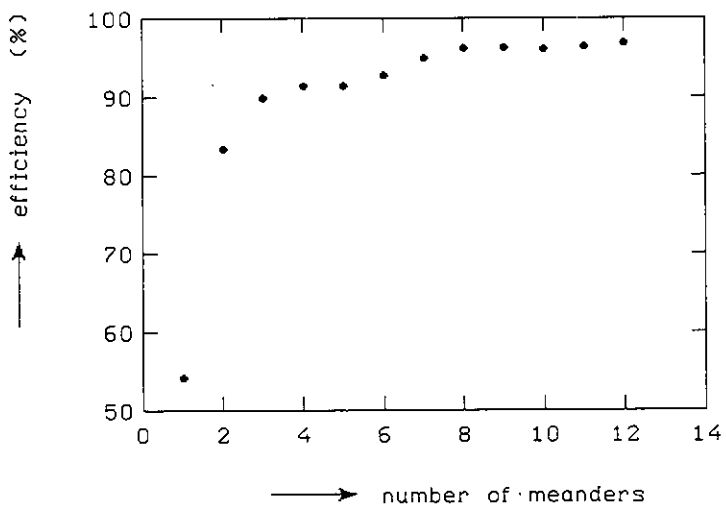


Fig. 12.

Calculated optimum efficiency of converters consisting of one to twelve wiggles.

Further improvement of efficiency can be obtained when the constraint for the wiggles to be identical is released. This can easily be done by replacing k_3z in Eq. (44) by the phase angle between the desired and the exciting mode. This results in only slightly different wiggles. The efficiency obtained in this way is 97.0% with 8 wiggles. The upper graph of Fig. 13 shows the mode composition and the lower graph the axis deviation of such a converter.

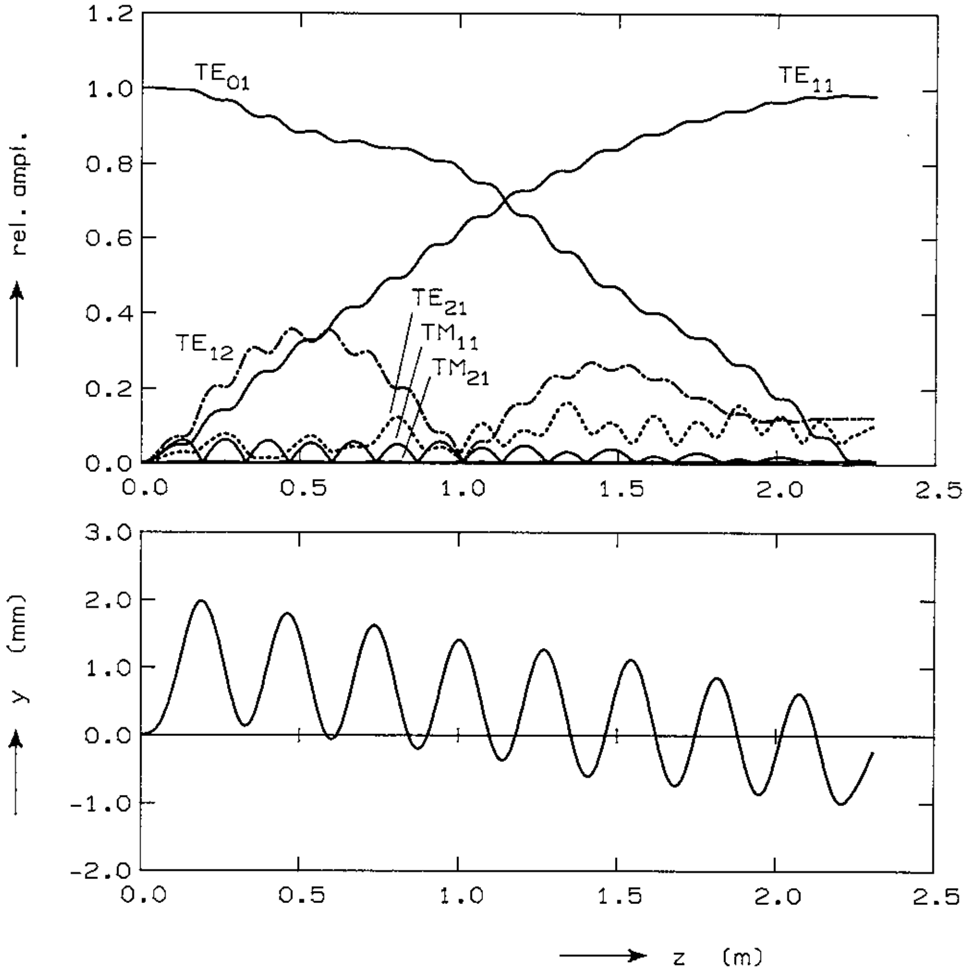


Fig. 13. (upper) Mode composition along axis of converter with non-identical wiggles.
(lower) Required bending (500× exagerated).

Practical result

We were able to compare our design with the mode converters designed and manufactured by Thomson-CSF. Their prototype TE_{01}-TE_{11} mode converter has 6 wiggles. The values chosen for curvature ($\hat{c}r$) and length are given in Table 4 with our calculated optimum values. The efficiency calculated for this converter and the calculated optimum efficiency differ by 2.6%. The measured efficiency is only 67%. It was concluded that the type of construction with a cut in a plane parallel to the z-axis introduced extra losses. The final converter with 8 wiggles was constructed with the cuts between the wiggles and perpendicular to the z-axis. The curvature chosen is 0.3% less than our calculated optimum value corresponding to a 1.5% lower efficiency. The measured efficiency was in total agreement with the theory.

TABLE 4

No. of wiggles	6		8	
$\hat{c}r$ prototype/opt (1/m)	0.594	0.608	0.445	0.460
length " (m)	1.600	1.623	2.156	2.162
calc. eff. " (%)	90.00	92.6	93.7	95.2
measured efficiency (%)	67 ± 2		96 ± 2	

Conclusion

The report demonstrates that once the coupling coefficients are known it is easy to design any mode converter. The microwave group at Rijnhuizen, foreseeing the need for other mode converters, has developed software to calculate coupling coefficients for stronger bends and will do so for dielectric rods introduced in the waveguide [8]. Solving Eq. (50) will be more complicated because more modes have to be taken into account and the off-diagonal elements of the matrix $\vec{C}$ will no longer be proportional to the curvature.

Acknowledgement

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APPENDIX

It is convenient to expand e_3E_3 and e_3H_3 instead of E_3 and H_3 in the set of functions ψ respectively ψ'. This is possible because e_3 has no poles or zeros in the interval $0 < \rho < a$, $0 < \phi < 2\pi$.

The components of the transverse field are expanded according to Eqs. (17), (18).

The coefficients $V_n V'_n, I_n, I'_n$, only depend on the third coordinate so that they can be taken through the operators ∇_1 and ∇_2. The expansions of the field components and the Maxwell equations they have to satisfy are:

$$E_1 = V_n \nabla_1 \psi_n + V'_n \nabla_2 \psi'_n \qquad A1$$

$$E_2 = V_n \nabla_2 \psi_n - V'_n \nabla_1 \psi'_n \qquad A2$$

$$E_3 = V_{3n} \; \psi_n / e_3 \qquad A3$$

$$H_1 = I'_n \nabla_1 \psi'_n + I_n \nabla_2 \psi_n \qquad A4$$

$$H_2 = I'_n \nabla_2 \psi'_n - I_n \nabla_1 \psi_n \qquad A5$$

$$H_3 = I_{3n'} \; \Psi'_n / e_3 \qquad A6$$

$$\frac{1}{e_3} \nabla_2 (e_3 E_3) - \frac{1}{e_2} \nabla_3 (e_2 E_2) = i\omega\mu H_1 \qquad ME\ 3\text{-}1$$

$$\frac{1}{e_1} \nabla_3 (e_1 E_1) - \frac{1}{e_3} \nabla_1 (e_3 E_3) = i\omega\mu H_2 \qquad ME\ 3\text{-}2$$

$$\frac{1}{e_2} \nabla_1 (e_2 E_2) - \frac{1}{e_1} \nabla_2 (e_1 E_1) = i\omega\mu H_3 \qquad ME\ 3\text{-}3$$

$$\frac{1}{e_3} \nabla_2 (e_3 H_3) - \frac{1}{e_2} \nabla_3 (e_2 H_2) = - i\omega\varepsilon E_1 \qquad ME\ 4\text{-}1$$

$$\frac{1}{e_1} \nabla_3 (e_1 H_1) - \frac{1}{e_3} \nabla_1 (e_3 H_3) = - i\omega\varepsilon E_2 \qquad ME\ 4\text{-}2$$

$$\frac{1}{e_2} \nabla_1 (e_2 H_2) - \frac{1}{e_1} \nabla_2 (e_1 H_1) = - i\omega\varepsilon E_3 \qquad ME\ 4\text{-}3$$

The summation convention is used. ME 3-1 is the first component of ME 3 etc. Substitution of the expansions A1, A2, A6 into ME 3-3 gives:

$$\frac{1}{e_2} \nabla_1 (e_2 V_n \nabla_2 \psi_n - e_2 V'_n \nabla_1 \psi'_n) - \frac{1}{e_1} \nabla_2 (e_1 V_n \nabla_1 \psi_n + e_1 V'_n \nabla_2 \psi'_n) = i\omega\mu \; I_{3n'} \frac{\psi'_n}{e_3} \; . \qquad A7$$

Rearranging terms and considering that the coefficients depend only on the third coordinate, we obtain:

$$- V'_n \left[\frac{1}{e_2} \nabla_1(e_2\nabla_1\psi'_n) + \frac{1}{e_1} \nabla_2(e_1\nabla_2\psi'_n)\right] + V_n \left[\frac{1}{e_2} \nabla_1(e_2\nabla_2\psi_n) - \frac{1}{e_1} \nabla_2(e_1\nabla_1\psi_n)\right] =$$

$$= I_{3n'} \; i\omega\mu \frac{\psi'_n}{e_3} . \qquad \text{A8}$$

But the ψ'_n satisfy the wave equation for the straight guide

$$\left[\frac{1}{e_2} \nabla_1(e_2\nabla_1) + \frac{1}{e_1} \nabla_2(e_1\nabla_2)\right] \psi' = - (k'_t)^2\psi' , \qquad \text{A9}$$

and the terms containing the V_n cancel.
This simplifies A8 to:

$$+ V'_n \, k^2_{tn'} \psi'_n = i\omega\mu \; I_{3n'} \frac{\psi'_n}{e_3} . \qquad \text{A10}$$

The ψ'_n are orthonormal according to

$$\int \psi'_r \psi'_s \; dS = \frac{\delta_{rs}}{(k'_{tr})^2} . \qquad \text{A11}$$

So that the projection of Eq. A10 onto the m-th component of the set gives

$$\int V'_n k^2_{tn'} \psi'_n \psi'_m \; dS = i\omega\mu \int I_{3n'} \frac{\psi'_n}{e_3} \psi'_m \; dS , \qquad \text{A12}$$

$$\rightarrow V'_m = I_{3n'} \, i\omega\mu \int \frac{\psi'_n \psi'_m}{e_3} \, dS . \qquad \text{A13}$$

This is a relation between the coefficients V' and I'_3 resulting from the fact that for each mode two coefficients suffice, one for E and one for H, or, in a different representation, one for the forward wave and one for the backward wave, while in the expansions we have three coefficients for each mode.

Equation A13, the relation between the V'_n and the $I_{3n'}$, can be written as

$$\vec{V}' = \vec{\vec{Z}}{}'_3 \cdot \vec{I}'_3 \quad , \text{ in which } \quad Z'_{3nm} = i\omega\mu \int \frac{\psi'_n \, \psi'_m}{e_3} , \qquad \text{A14}$$

or as

$$\vec{I}'_3 = \vec{\vec{Y}}'_3 \cdot \vec{V}' \text{ , in which } \vec{\vec{Y}}'_3 = (\vec{\vec{Z}}'_3)^{-1} \text{ .} \qquad \text{A15}$$

In a similar way substitution of A3, A4, A5 into ME 4-3 gives the relations between the V_{3n} and I_n

$$\vec{I} = \vec{\vec{Y}}_3 \cdot \vec{V}_3 \quad \text{in which } \vec{\vec{Y}}_3 = - i\omega\varepsilon \int \frac{\vec{\psi}\vec{\psi}}{e_3} dS \text{ ,} \qquad \text{A16}$$

or $$\vec{V}_3 = \vec{\vec{Z}}_3 \cdot \vec{I} \quad \text{in which } \vec{\vec{Z}}_3 = (\vec{\vec{Y}}_3)^{-1} \text{ .} \qquad \text{A17}$$

After substitution of the expansions A1, A2, A3, A4, A5, into ME 3-1, ME 3-2 the terms $\nabla_3(e_2E_2)$ and $\nabla_3(e_1E_1)$ will contain the dependence of the coefficients on the third coordinate we are looking for.

To eliminate all unknowns, except one i.e. $\partial V_m/\partial_3$ from the equations we take the following combination of equations

$$(\text{ME } 3\text{-}1)\ \nabla_2\psi_m - (\text{ME } 3\text{-}2)\nabla_1\psi_m \text{ ,} \qquad \text{A18}$$

which is in fact the evaluation of $-(\hat{z}\times\vec{\nabla}_t\psi_m) \cdot \vec{H}_t$.

Written out in detail this is

$$(\nabla_2\psi_m)\frac{1}{e_3}\nabla_2 V_{3n}\psi_n - (\nabla_2\psi_m)\frac{1}{e_2}\nabla_3(e_2V_n\nabla_2\psi_n - e_2V'_n\nabla_1\psi'_n)$$

$$- (\nabla_1\psi_m)\frac{1}{e_1}\nabla_3(e_1V_n\nabla_1\psi_n + e_1V'_n\nabla_2\psi'_n) + (\nabla_1\psi_m)\frac{1}{e_3}\nabla_1V_{3n}\psi_n =$$

$$= (\nabla_2\psi_m)\, i\omega\mu(I'_n\nabla_1\psi'_n + I_n\nabla_2\psi_n) - (\nabla_1\psi_m)\, i\omega\mu(I'_n\nabla_2\psi'_n - I_n\nabla_1\psi_n) \text{ .} \qquad \text{A19}$$

As ψ, e_1, e_2, e_3 depend only on ρ, ϕ and the coefficients depend only on z, A19 can be rearranged as follows:

$$\rightarrow (\nabla_3 V_n)[-\nabla_2\psi_m\nabla_2\psi_n - \nabla_1\psi_m\nabla_1\psi_n] + (\nabla_3 V_n')[\nabla_2\psi_m\nabla_1\psi_n' - \nabla_1\psi_m\nabla_2\psi_n']$$

$$+ \frac{V_{3n}}{e_3}[\nabla_2\psi_m\nabla_2\psi_n + \nabla_1\psi_m\nabla_1\psi_n] =$$

$$= i\omega\mu I_n[+\nabla_2\psi_m\nabla_2\psi_n + \nabla_1\psi_m\nabla_1\psi_n] + i\omega\mu I_n'[\nabla_2\psi_m\nabla_1\psi_n' - \nabla_1\psi_m\nabla_2\psi_n'] , \qquad \text{A20}$$

or as follows:

$$\rightarrow - (\nabla_3 V_n)(\vec{\nabla}_t\psi_m \cdot \vec{\nabla}_t\psi_n) - (\nabla_3 V_n')(\vec{\nabla}_t\psi_m \cdot (\vec{\nabla}_t\psi_n\times\hat{z})) + \frac{V_{3n}}{e_3}(\vec{\nabla}_t\psi_m \cdot \vec{\nabla}_t\psi_n) =$$

$$= + I_n\ i\omega\mu(\vec{\nabla}_t\psi_m \cdot \vec{\nabla}_t\psi_n) - I_n'\ i\omega\mu(\vec{\nabla}_t\psi_m \cdot (\vec{\nabla}_t\psi_n'\times\hat{z})) . \qquad \text{A21}$$

Equation A21 is multiplied by e_3, integrated over the cross-section and the following orthogonality relations between the ψ, ψ' functions are used

$$\int\vec{\nabla}_t\psi_m \cdot \vec{\nabla}_t\psi_n\ dS = \delta_{mn} ,$$

$$\int\vec{\nabla}_t\psi_m \cdot (\vec{\nabla}_t\psi_n\times\hat{z})\ dS = 0 .$$

This simplifies A21 to

$$\frac{\partial V_m}{\partial_3} - V_{3m} = - I_n\ i\omega\mu \int e_3\vec{\nabla}_t\psi_m \cdot \vec{\nabla}_t\psi_n\ dS + I_n'\ i\omega\mu \int e_3\vec{\nabla}_t\psi_m \cdot (\vec{\nabla}_t\psi_n'\times\hat{z})\ dS . \qquad \text{A22}$$

V_{3_m} can be expressed in the I_n with A16 yielding the final result:

$$\frac{\partial V_m}{\partial_3} = - I_n\left[i\omega\mu \int e_3\vec{\nabla}_t\psi_m \cdot \vec{\nabla}_t\psi_n\ dS + Z_{3mn}\right] +$$

$$+ I_n'\left[i\omega\mu \int e_3\vec{\nabla}_t\psi_m \cdot (\vec{\nabla}_t\psi_n'\times\hat{z})\ dS\right] .$$

$\partial V_m'/\partial_3$ is found in a similar way from Eq. (ME 3-1) $\nabla_1\psi_m'$ + (ME 3-2)$\nabla_2\psi_m'$, which is the evaluation of $\vec{\nabla}_t \psi'_m \cdot \vec{H}_t$, and to obtain $\partial I_m/\partial_3$ and $\partial I'_m/\partial_3$ ME 4-1, ME 4-2 are used for the evaluation of $\vec{\nabla}_t \psi_m \cdot \vec{E}_t$ and $(\vec{\nabla}_t \psi'_m \times \hat{z}) \cdot \vec{E}_t$ respectively.

Representation in terms of forward and backward travelling waves

The coefficients of the E- and the H-fields of any mode in the guide can always be interpreted as the result of a forward and a backward travelling wave of which the ratio between E and H is a constant, the wave impedance,

$$Z_{wmn} = \frac{k_{3nm}}{k_o} Z_o \quad ; \quad Z'_{wmn} = \frac{k_o}{k_{3nm}} Z_o \ .$$

Let F,F' be the amplitudes of the forward and B,B' those of the backward travelling waves of the modes, then

$$V_m = \sqrt{Z_{wm}}\,(F_m + B_m) \quad ; \quad V'_m = \sqrt{Z'_{wm}}\,(F'_m + B'_m)$$

$$I_m = \frac{1}{\sqrt{Z_{wm}}}\,(F_m - B_m) \quad ; \quad I'_m = \frac{1}{\sqrt{Z_{wm}}}\,(F'_m - B'_m) \ .$$

Substitution of this assumption into Eqs. 19 leads to an alternative set of GTE's. Mathematically speaking no advantage is gained.

The coupling coefficients between forward waves when no backward waves are assumed to exist, are found by substitution of the fixed ratio Z_w, Z'_w for V/I, V'/I' into the GTE's eliminating the I, I'

$$\frac{\partial V_m}{\partial_3} = \sum_n \frac{Z_{mn}}{Z_{wn}} V_n + \sum_{n'} \frac{Z_{mn'}}{Z_{wn'}} V_{n'} \ ,$$

$$\frac{\partial V_{m'}}{\partial_3} = \sum_n \frac{Z_{m'n}}{Z_{wn}} V_N + \sum_n \frac{Z_{m'n'}}{Z_{wn'}} V_{n'}$$