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Birth and Death in a Continuous Opinion Dynamics Model. The consensus case

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Abstract. We here discuss the process of opinion formation in an open community where agents are made to interact and consequently update their beliefs. New actors (birth) are assumed to replace individuals that abandon the community (deaths). This dynamics is simulated in the framework of a simplified model that accounts for mutual affinity between agents. A rich phenomenology is presented and discussed with reference to the original (closed group) setting. Numerical findings are supported by analytical calculations.

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1 Introduction

Opinion dynamics modeling represents a challenging field where ideas from statistical physics and non-linear science can be possibly applied to understand the emergence of collective behaviors, like consensus or polarization in so-cial groups. Several models have been proposed in the past to reproduce the key elements that supposedly drive the process of opinion making [1,2]. The problem of providing an adequate experimental backup to such developments is indeed an open one, and more work is certainly needed to eventually assess the interpretative ability of the proposed

mathematical formulations, following e.g. the guidelines of [3,4]. Models based on interacting agents display however a rich and intriguing dynamics which deserves to be fully unraveled.

Opinion dynamics models can be classified in two large groups. On the one hand, opinions are represented as discrete (spin–like) variables where the system behaves similarly to spin glasses models [5]. On the other, each individual bears a continuous opinion which span a pre–assigned range [6]. More recently, a new framework for a discrete but unbounded number of opinions was proposed in [2] and shown to nicely complement the picture. In all the above approaches, a closed system is generally assumed, meaning that the same pool of actors is made to interact during the evolution. This can be interpreted by assuming that the inherent dynamical timescales (e.g. opinion convergence time) are much faster than those associated to the processes (e.g. migration, birth/death) responsible for a modification of the group composition, these latter effects having being therefore so far neglected. Such an implicit assumption is certainly correct when the debate is bound to a small community of individuals, thus making it possible to eventually achieve a rapid convergence towards the final configuration. Conversely, it might prove inaccurate when applied to a large ensemble of interacting agents, as the process becomes considerably slower and external perturbations need to be accounted for. Given the above, it is therefore of interest to elucidate the open system setting, where the population is periodically renewed.

To this end, we refer to the model presented in [7], where the role of affinity among individuals is introduced as an additional ingredient. This novel quantity measures the degree of inter–personal intimacy and sharing, an effect of paramount importance in real social system [8] . Indeed, the outcome of an hypothetic binary interaction relies on the difference of opinions, previously postulated, but also on the quality of the mutual relationships. The affinity is dynamically coupled to the opinion, and, in this respect, it introduces a memory bias into the system: affinity between agents increases when their opinion tends to converge.

The aforementioned model is here modified to accommodate for a death/birth like process. In this formulation, M agents are randomly eliminated from the system, every T time steps. When an agent exits from the community (virtually, dies), he is immediately replaced by a new element, whose opinion and affinity with respect to the group are randomly assigned. As we shall see, the perturbation here prescribed alters dramatically the behavior of the system, with reference to the ideal close–system configuration. To understand such modifications via combined numerical and analytical tools, constitutes the object of the investigations here reported.

We shall mainly explore the parameters setting that would lead to an asymptotic consensus state (all agents eventually bearing the shared opinion 0.5), in absence of the external perturbation which is here object of investigations. The underlying model [7] displays however a richer phenomenology, exhibiting in particular stable polarized

states in the late time evolution and long-lived metastable regimes. Though it would be extremely interesting to extend the present analysis and hence cover those additional scenarios, we chose to only briefly touch upon this issue when commenting the details of the transition between single and fragmented phases.

The paper is structured as follows. We first introduce the model, then we present the obtained analytical and numerical results and, finally, we sum up and draw our conclusions.

2 The model

In the following, we will review the model previously introduced in [7] and present the additional features that are here under inspection. The interested reader can thus refer to the original paper [7] for an additional account on the model characteristics.

Consider a population of N agents and assume that at time t they bear a scalar opinion $O_i^t \in [0, 1]$. We also introduce the $N \times N$ time dependent matrix α^t, whose elements α_{ij}^t belong to the interval $[0, 1]$. The quantities α_{ij}^t specify the affinity of individual i vs. j, at time t: Larger values of α_{ij}^t are associated to more trustable relationships.

Both the affinity matrix and the agents opinions are randomly initialized time $t = 0$. At each time step t, two agents, say i and j, are chosen according to the following extraction rule: first the agent i is randomly identified, with a uniform probability. Then, the agent j which is closer to i in term of the social metric D_{ij}^η is selected for interaction. The quantity D_{ij}^η results from the linear

superposition of the so–called social distance, d_{ij}, and a stochastic contribution η_j, namely:

$$D_{ij}^\eta = d_{ij}^t + \eta_j(0, \sigma). \tag{1}$$

Here $\eta_j(0, \sigma)$ represents a normally distributed noise, with mean zero and variance σ, the latter being named social temperature. The social distance is instead defined as:

$$d_{ij}^t = \Delta O_{ij}^t(1 - \alpha_{ij}^t) \qquad j = 1, ..., N \qquad j \neq i, \tag{2}$$

with $\Delta O_{ij}^t = O_i^t - O_j^t$.

The smaller the value of d_{ij}^t the closer the agent j to i, both in term of affinity and opinion. The additive noise $\eta_j(0, \sigma)$ acts therefore on a fictitious 1D manifold, which is introduced to define the pseudo–particle (agent) interaction on the basis of a nearest neighbors selection mechanism and, in this respect, set the degree of mixing in the community.

When the two agents i and j are extracted on the basis of the recipe prescribed above, they interact and update their characteristics according to the following scheme [1]:

$$\begin{cases} O_i^{t+1} & = O_i^t - \frac{1}{2}\Delta O_{ij}^t \Gamma_1\left(\alpha_{ij}^t\right) \\ \alpha_{ij}^{t+1} & = \alpha_{ij}^t + \alpha_{ij}^t[1 - \alpha_{ij}^t]\Gamma_2\left(\Delta O_{ij}\right), \end{cases} \tag{3}$$

where the functions Γ_1 and Γ_2 respectively read:

$$\Gamma_1\left(\alpha_{ij}^t\right) = \Theta\left(\alpha_{ij}^t - \alpha_c\right) \tag{4}$$

and

$$\Gamma_2\left(\Delta O_{ij}\right) = \Theta\left(\Delta O_c - |\Delta O_{ij}^t|\right) - \Theta\left(|\Delta O_{ij}^t| - \Delta O_c\right) \tag{5}$$

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[1] The evolution of the quantities $O_j(t)$ and $\alpha_{ji}(t)$ is straightforwardly obtained by switching the labels i and j in the equations.

and the symbol $\Theta(\cdot)$ stands for the Heaviside step–function [2]. More specifically, Γ_1 is 0 or 1 while Γ_2 is -1 or 1, depending on the value of their respective arguments. In the above expressions α_c and ΔO_c are constant parameters. Notice that, for $\alpha_c \to 0$, the opinion is formally decoupled from affinity in (3) being $\Gamma_1 = 1$ irrespectively of the actual value of α_{ij}^t, and the former evolves following the Deffuant et al. scheme [6] with convergence rate $\mu = 0.5$ and interaction threshold $d = 1$ (confidence bound). The latter scheme pioneered the broad class of models inspired to the so–called bounded confidence hypothesis, an assumption which, though revisited, also enters the self-consistent scenario of Eqs.(3).

In [7], a preliminary analysis of the qualitative behavior of the model as a function of the involved parameters is reported. Asymptotic clusters of opinion are formed, each agglomeration being different in size and centered around distinct opinion values. Individuals sharing the same believes are also characterized by a large affinity scores, as it is exemplified in Fig. 1.

More quantitatively, the system is shown to undergo a continuous phase transition: above a critical value of the control parameter $(\sigma\alpha_c)^{-1/2}$ it fragments into several opinion clusters, otherwise convergence to a single group is

Fig. 1. Upper panel: Typical evolution of the opinion versus time, i.e. number of iterations and the asymptotic distribution of agents' opinion in a closed community. Bottom panel: Final affinity matrix, represented by the underlying network: node i is linked to node j if $\alpha_{ij} > \bar{\alpha} = 0.8$. Here $\sigma = 4 \cdot 10^{-4}$, $\Delta O_c = 0.5$, $\alpha_c = 0.5$ and $\rho = 0$, i.e. no agent can leave the group. Initial opinion are (random) uniformly distributed within the interval $[0, 1]$, while α_{ij}^0 is initialized with uniform (random) values between 0 an 0.5.

numerically shown to occur [3]. We shall here simply notice

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[2] In [7], the switchers Γ_1 and Γ_2 are smooth functions constructed from the hyperbolic tangent. We shall here limit the discussion to considering the Heaviside approximation, which is recovered by formally sending $\beta_{1,2}$ to infinity in Eqs. (3) and (4) of [7].
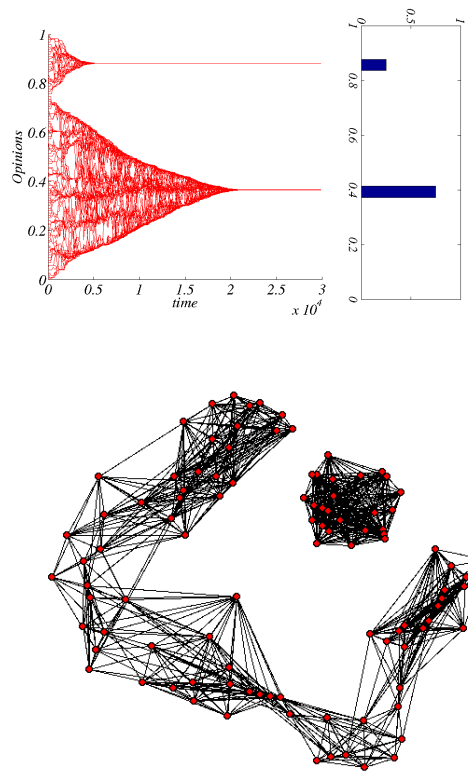
[3] Strictly speaking, it should be noted that the fragmented state is metastable, if the mean separation between the adjacent peaks is smaller than the interaction distance ΔO_c. There always exists in fact a finite, though small, probability of selecting two individuals which belong to different agglomerations. When the above condition applies, i.e. when the agents' opinions are closer than the threshold amount ΔO_c, such rare

that a significant degree of mixing (large social temperature σ) brings the system towards the single–cluster final configuration.

Starting from this setting, we introduce the birth/death process, which in turn amounts to place the system in contact with an external reservoir. The perturbation here hypothesized is periodic and leaves the total number of agent unchanged. Every T time steps (i.e. encounter events) M agents, randomly selected, are forced to abandon the system (death). Every removed individual is instantaneously replaced by a new element, whose initial opinion and affinity are randomly fished, with uniform probability, from the respective intervals $[0, 1]$ and $[0, \alpha_{max}]$. Further, we introduce $\rho = \frac{M}{T}$ to characterize the *departure density*, a crucial quantity that will play the role of the control parameter in our subsequent developments. As a final remark, it should be emphasized that no aging mechanisms are introduced: agents are mature enough to experience peer to peer encounters from the time they enter the system.

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encounters produce a gradual increase of the mutual affinity scores, a tendency which asymptotically drives a merging of the segregated clusters, as ruled by Eqs. (3). This final convergence is eventually achieved on extremely long time scales, diverging with the number of agents. Socially relevant dynamics are hence bound to the metastable phases, which are being investigated in Fig. 2.

3 Results

Numerical simulations are performed for a system of $N = 100$ individuals and its evolution monitored [4]. Qualitatively, the system shows the typical critical behavior as observed in the original formulation [7]. However, peculiar distinctions are found, some of those being addressed in the following discussion. First, an apparently smooth transition is also observed within this novel formulation, which divides the mono– and multi–clustered phases. Interestingly, the transition point is now sensitive to the departure density ρ. To further elucidate this point, we draw in Fig. 2 the average number of observed clusters versus a rescaled temperature. A clear transition towards an ordered (single-clustered) phase is observed, as the temperature increases. The parameter σ_c in Fig. 2 plays the role of an effective temperature, and it is numerically adjusted to make distinct curves collapse onto the same profile, which hence applies to all values of ρ. The inset of Fig. 2 shows that there is a linear correlation between σ_c and ρ.

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[4] The chosen value of N could be in principle considered too small to allow us extracting sound statistical information from the model at hand. As we shall however discuss, already at such relatively small value of N, one observes a satisfying matching between numerics and statistical based predictions. No substantial differences are detected when simulating a larger system, this observation motivating our choice to stick to the $N = 100$ case study. Notice also that potentially interesting applications in social sciences would often deal with a finite, possibly limited, number of agents, as for the case being addressed at present.

The larger the departure density ρ, the larger the effective temperature σ_c. In other words, when ρ is made to increase (i.e. the system is experiencing the effect of a more pronounced external perturbation), one needs to augment the degree of mixing, here controlled by the social temperature σ, if a convergence to the final mono–cluster is sought. The death/birth process is in fact acting against the thermal contribution, which brings into contact otherwise socially distant individuals. While this latter effect enhances the chances of convergence, the former favors the opposite tendency to spread.

To further elucidate the role of the external perturbation, we shall refer to the dynamical regime where the agents converge to a single cluster. When ρ is set to zero, the final shared opinion is 0.5 to which all agents eventually agree, see Fig. 3a. In other words, the final distribution is a Dirac delta, with the peak positioned in $O = 0.5$. Conversely, for positive, but small, values of ρ, the final distribution of opinions presents a clear spreading around the most probable value, still found to be 0.5. This scenario is clearly depicted in Fig. 3b. For larger ρ, when the birth-death perturbation becomes more frequent, the opinion profile cannot relax away from the initial distribution, the agents believes being uniformly scattered over the allowed interval, i.e. $[0, 1]$.

The associated standard deviation υ is deduced, from a series of simulations, and shown to depend on the selected value of ρ. The result of the analysis is reported in Fig. 4, where the calculated value of υ (symbols) is plotted versus the departure density amount ρ.
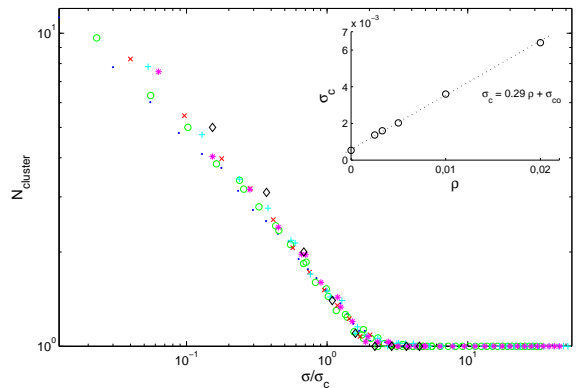

Fig. 2. Main plot: Average number of clusters as function of the rescaled quantity $\frac{\sigma}{\sigma_c}$ for different values of the density ρ. Simulations have been performed with parameter values: $\rho = 0$ ($\diamond$), $\rho = 0.0025$ ($*$), $\rho = 0.0033$ ($+$), $\rho = 0.005$ ($\times$), $\rho = 0.01$ ($\circ$) and $\rho = 0.05$ ($\blacklozenge$). In all simulations, here and after reported, unless otherwise specified, $\alpha_c = \Delta O_c = 0.5$; O_i^0 and α_{ij}^0 are random variables uniformly distributed in the intervals $[0, 1]$ and $[0, \alpha_{max}]$ – being $\alpha_{max} = 0.5$ – respectively. Inset : σ_c as a function of ρ. The open circles, $\circ$, represent the values calculated from the transitions shown in the main plot. The dotted lines represent the best linear fit : $\sigma_c = 0.3\rho + \sigma_{co}$, with $\sigma_{co} = 5.5 \cdot 10^{-4}$. Notice that $\sigma_c = \sigma_{co}$ is eventually recovered in the closed-system setting, which in turn corresponds to $\rho = 0$.

For small values of the control parameter ρ, the standard deviation υ of the cluster scales proportionally to $\sqrt{\rho}$, Fig. 3. Numerics indicate that the proportionality coefficient gets smaller, as α_c grows. In the opposite limit, namely for large values of the density ρ, the standard deviation υ rapidly saturates to a asymptotic value, υ_c. The latter is universal, meaning that it neither scales with α_c, nor it does with the social temperature σ. Our best numerical estimates returns, $\upsilon_c = 0.28 \simeq 1/\sqrt{12}$ which, as
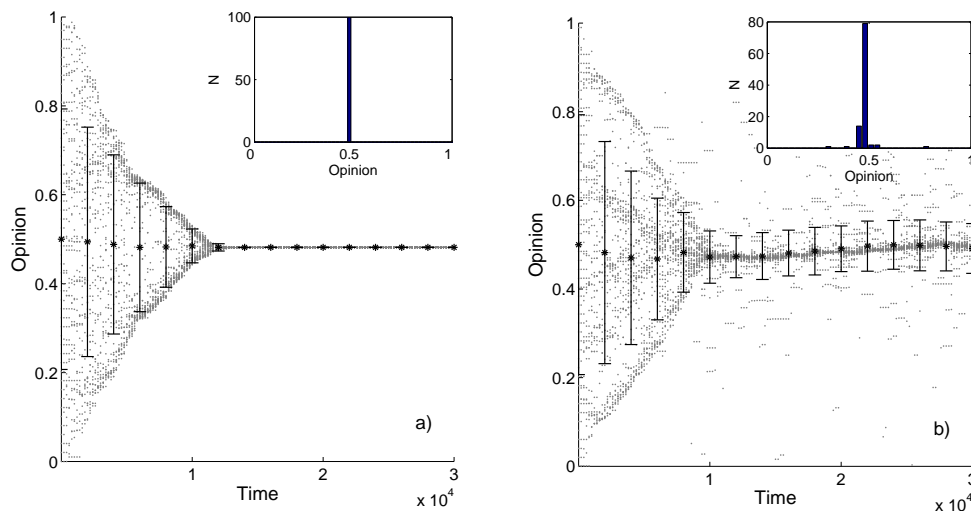
Fig. 3. Opinion evolution versus time, the latter being quantified through the number of iterations. The black stars represent the mean opinion and the error bar the standard deviation of the opinion distribution. In the insets, the histogram of asymptotic distribution of agents' opinion. In panel a) the system is closed (i.e. $\rho = 0$), in b) $\rho = 5 \cdot 10^{-3}$.

expected, corresponds to the standard deviation of the uniform distribution in the interval of length 1.

The solid lines in Fig. 4 represent the function :

$$v^2 = \frac{M}{12N[1 - \frac{N-M}{N}(\frac{T_c-T}{T_c})^2]},\qquad (6)$$

which straightforwardly follows from an analytical argument, developed hereafter. In the above expression T_c, stands for an effective estimate for time of convergence of the opinion cluster, and is deduced via numerical fit (see caption of Fig. 4 and [10] for further details). In [9], working within the Deffuant's scheme [6], i.e. closed community case without affinity, the time needed to form a coherent assembly from a sequence of binary encounters was shown to diverge with the population size N, with a super-linear scaling. Moreover, it was also proven that the affinity slows down the convergence rate, a fact that can be successfully captured by accounting for an additional

dependence of T_c over α_c: The larger α_c the longer the convergence time, as reported in [7]. A comprehensive discussion on the analytical derivation of $T_c(\alpha_c)$ falls outside the scope of the present discussion and will be presented in a forthcoming contribution [11].

Before turning to discuss the analytical derivation of Eq. (6), we wish to test its predictive adequacy with reference to the two limiting cases outlined above. Indeed, for $\rho << 1$ and $T << T_c$, Eq. (6) can be cast in the approximated form:

$$v = \sqrt{\frac{T_c(\alpha_c)}{24N}}\sqrt{\frac{M}{T}} = \gamma_t\sqrt{\rho},\qquad (7)$$

which presents the same dependence of v versus $\sqrt{\rho}$, as observed in the numerical experiments. Moreover, the coefficient γ_t is expected to decay when increasing the cutoff in affinity α_c, in agreement with the numerics. For $\rho >> 1$,
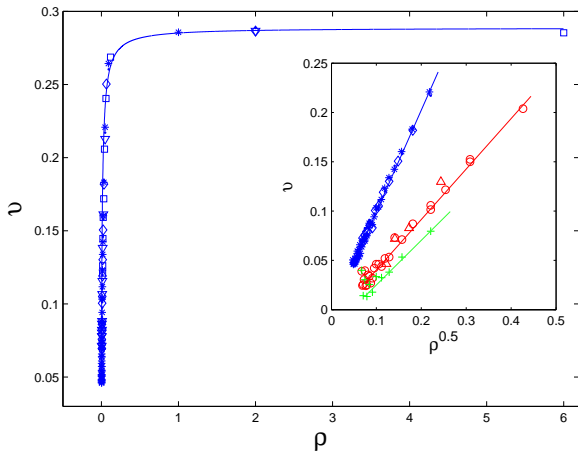
Fig. 4. Main: The standard deviation of the final mono–cluster υ as a function of the departure density ρ. Each point results from averaging out 100 independent runs, with $N = 100$. Symbols refer to numerical simulation performed with different values of α_c, M and σ : (∗) $\alpha_c = 0.5, M = 1, \sigma = 0.07$, (◇) $\alpha_c = 0.5$, $M = 2$, $\sigma = 0.07$, (□) $\alpha_c = 0.5$, $M = 6$, $\sigma = 0.07$, (●) $\alpha_c = 0.5$, $M = 2$, $\sigma = 0.25$, (▽) $\alpha_c = 0.5$, $M = 2$, $\sigma = 1$, (○) $\alpha_c = 0.3$, $M = 2$, $\sigma = 0.28$, (△) $\alpha_c = 0.3$, $M = 6$, $\sigma = 0.28$ and (×) $\alpha_c = 0$, $M = 2$, $\sigma = 0.28$). The solids line refers to the theoretical prediction (6): The free parameter T_c is numerically fitted and results in $T_c = 8030$ for $\alpha_c = 0.5$, $T_c = 1886$ for $\alpha_c = 0.3$ and $T_c = 1470$ for $\alpha_c = 0$. This values are in good agreement with the simulation results [10], which confirms the validity of the proposed analytical scheme. Inset: υ versus $\sqrt{\rho}$, for $\rho \in [0, 0.5]$. This zoomed view confirms the correctness of the scaling dependence derived in (7).

Eq. (6) implies:

$$\upsilon = \sqrt{\frac{1}{12}}, \qquad (8)$$

thus returning the correct result.

To derive Eq. (6) let us suppose that at time t the death/birth process takes place and the system experience an injection of new individuals. Label with υ_t the standard deviation of the agents opinion distribution $f^t(O)$, at time t. It is reasonable to assume that $f^t(O)$ is centered around $1/2$. After T interactions between agents, when the next perturbation will occur (M agents are randomly removed from the community and replaced by M new actors with random opinion and affinity scores) the distribution has been already modified, because of the underlying dynamical mechanism specified through Eqs. (3). More concretely, the opinions slightly converge around the peak value $1/2$, an effect that certainly translates into a reduction of the associated standard deviation. To provide a quantitative estimate of such phenomenon, we recall that in the relevant (O, t) plan, the convergence process fills an ideal *triangular* pattern, whose height measures T_c. This topological observation enables us to put forward the following linear ansatz:

$$\upsilon_{t+T}^{conv} = \upsilon_t \left(1 - \frac{T}{T_c} \right), \qquad (9)$$

where υ_{t+T}^{conv} labels the standard deviation just before the insertion of the next pool of incoming agents [5]

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[5] Numerical simulations (not reported here) show that in the closed model, the standard deviation of the opinions' distribution exhibits a exponential decay as a function of a power of time. This latter is approximately interpolated by the proposed

Recalling that the newly inserted elements are uniformly distributed, and labeling with $G(\cdot, \cdot)$ the opinion distribution (the two entries referring respectively to mean and the standard deviation), the updated variance is:

$$
\begin{aligned}
v_{t+T}^2 &= \frac{M}{N} \int_0^1 \left(O - \frac{1}{2} \right)^2 dO \\
&+ \frac{N-M}{N} \int_0^1 G\left(\frac{1}{2}, v_{t+T}^{conv} \right) \left(O - \frac{1}{2} \right)^2 dO \\
&= \frac{M}{12N} + \frac{N-M}{N} \left(1 - \frac{T}{T_c} \right)^2 v_t^2 \,.
\end{aligned}
$$

The asymptotic stationary solution correspond to $v_{t+T} = v_t$, a condition that immediately leads to Eq. (6) when plugged into (10). The above analysis also suggests linear relation (9), a choice which eventually allows us to carry out the analytical calculation resulting in expression (6) (see also the discussion in the appendix A). Formally, eq. (9) applies only for $t = 0$, when agents are populating the interval $[0, 1]$ with a uniform distribution. During the subsequent evolution, the convergence still gives rise to a macroscopic triangular pattern but, now, the associated triangle height T_c gets slightly reduced. At time t agents are still confined in the relevant interval $[0, 1]$ and experience a certain degree of spreading, effect of the perturbation externally imposed. However, and especially for intermediate values of ρ, the progressive bunching opposes the birth/death disturb (which would tend to restore the $t = 0$ variance) and drives an instantaneous reduction of T_c, as t grows. In the following, and to account for this self-consistent effect not captured by analytical framework, T_c is hence regarded as an effective parameter to be numerically adjusted: as commented below however, the best fit values of T_c correlate well with direct measurements of the convergence (aggregation) time of the unperturbed system, a finding which a posteriori confirms the plausibility of eq. (9).

that the final fate of the system is not affected by the time when the perturbation is first applied, t_{in}. This conclusion is also confirmed by direct numerical inspection: The asymptotic value of the standard deviation v does not depend on t_{in}, but solely on ρ. Even in the extreme condition, when the death/birth perturbation is switched on after the agents have already collapsed to the mean opinion 0.5, one observes that, after a transient, the cluster spreads and the measured value of v agrees with the theoretical prediction (6).

Aiming at further characterizing the system dynamics, we also studied the case where, initially, agents share the same belief O_0. The initial distribution of opinions is therefore a Dirac delta $f^0(O) = \delta(O - O_o)$. Such condition is a stationary solution, for any given O_o when the death/birth process is inactivated. Conversely, when the death/birth applies, the system evolves toward a state, characterized by a single cluster (localized, if ρ is small, uniformly spread over the allowed region as $\rho > 1$, see preceding discussion), centered in $O = 0.5$ and with standard deviation given by Eq. 6. It is also observed that the time needed by the system to complete the transition T_{conv} depends on the value of ρ and the critical affinity α_c, see Fig. 5. A simple theoretical argument enables us to quantitatively explain these findings. The initial distribution of opinions is modified after the first death/birth event as:

$$
f^1(O) = \frac{M}{N} + \frac{N-M}{N} \delta(O - O_o) \,. \tag{10}
$$

The first term refers to the freshly injected actors, while the second stands for the remaining Delta-distributed in-

dividuals. Hence, the mean opinion value reads:

$$\bar{O}_1 = \int_0^1 f^1(O)O dO = \frac{M}{2N} + \frac{N-M}{N}O_o. \qquad (11)$$

We can suppose that between the occurrence of two consecutive perturbations (separated by T iterations), the group average opinion does not significantly change. Notice that the probability of interaction of a newborn agent with another belonging to the main group is in fact proportional to M/N. Moreover several consecutive encounters of this type are necessary to induce a macroscopic change of the averaged opinion. Under this hypothesis the next death/birth event makes the average opinion change as:

$$\bar{O}_2 = \frac{M}{2N} + \frac{N-M}{N}\bar{O}_1. \qquad (12)$$

After n death/birth iterations, the opinion mean value reads :

$$\bar{O}_n = \frac{M}{2N} + \frac{N-M}{N}\bar{O}_{n-1}. \qquad (13)$$

From Eq. (13) one easily gets that the asymptotic equilibrium is reached for $\bar{O}_\infty = 0.5$, as found in our numerical experiments; in fact the following relation is straightforwardly obtained:

$$\bar{O}_n = \left(\frac{N-M}{N}\right)^n O_o + \sum_{l=0}^{n-1} \frac{M}{2N}\left(\frac{N-M}{N}\right)^l, \qquad (14)$$

being O_o the initial common believe. By setting $\alpha = \frac{N-M}{N}$ and $\beta = \frac{M}{2N}$, the solution of Eq. (14) reads:

$$\bar{O}_n = \beta\frac{1-\alpha^n}{1-\alpha} + \alpha^n O_o, \qquad (15)$$

whose asymptotic solution is given by $\bar{O}_n \to \bar{O}_\infty = 0.5$.

Expression (15) reproduces quite well the dynamics of the cluster mean, as seen in the simulations. The adequacy

of (15) is in fact clearly demonstrated in Fig. 5a. Let us define the convergence time T_{conv} as the number of iterations needed to bring the average opinion ϵ close to its asymptotic value $1/2$. Solving Eq. (15) for n and recalling that $T_{conv} = nT$ yield:

$$T_{conv} = T\log_\alpha\left[\frac{1}{|O_o - \frac{1}{2}|}(\epsilon - \frac{\beta}{1-\alpha})\right], \qquad (16)$$

The above estimate is in excellent agreement with the numerical results reported in Fig. 5b.
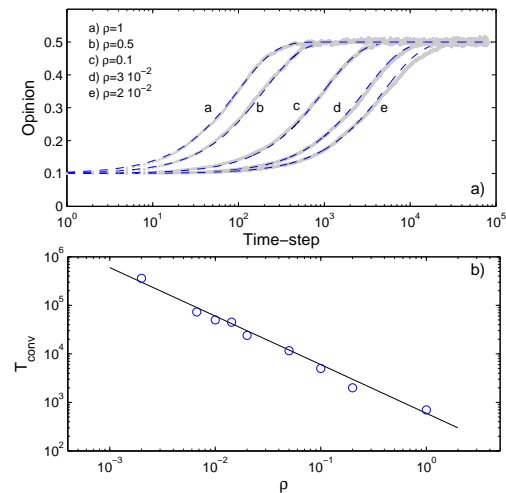

Fig. 5. a) The opinion as a function of time for the case where all the agents are initialized with the same opinion $O_o = 0.1$. The solid (red) line represent the numerical data while the segmented (blue) line represents the theoretical prediction from Eq. (15). b) The transition time T_{conv} as a function of the death density ρ. Symbols stand for the results of the simulations and the solid line represents the prediction (16). Here $N = 100$, $\alpha_c = \Delta O_c = 0.5$, $\epsilon = 0.001$ and $\sigma = 0.07$.

4 Conclusions

In this paper we have discussed the process of opinion making in an open group of interacting subjects. The model postulates the coupled dynamical evolution of both individuals' opinion and mutual affinity, according to the rules formulated in [7]. At variance with respect to the toy—model [7], the system is now open to contact with an external reservoir of potentially interacting candidates. Every T iterations the M agents are instantaneously replaced by newborn actors, whose opinion and affinity scores are randomly generated according to a pre–assigned (here uniform) probability distribution. The ratio $\rho = M/T$, here termed departure density plays the role of a control parameter. The occurrence of a transition is found which separates between two macroscopically different regimes: For large values of the so–called social temperature the system collapses to a single cluster in opinion, while in the opposite regime a fragmented phase is detected. The role of ρ is elucidated and shown to enter in the critical threshold as a linear contribution. Two phenomena are then addressed, with reference to the single clustered phase. On the one side, the external perturbation, here being hypothesized to mimic a death/birth process, induces a spreading of the final cluster. The associated variance is numerically shown to depend on the density amount ρ, the functional dependence being also analytically explained. On the other side, we also show that the birth/death events imposed at a constant pace can produce the progressive migration of a cluster, initially localized around a given opinion value. A theoretical argument is also developed

to clarify this finding. As a general comment, we should emphasize that the effect of opening up the system to external influences changes dramatically its intrinsic dynamics revealing peculiar, potentially interesting, features which deserves to be further explored.

A Appendix

This appendix is devoted to discussing a straightforward extension of the above analysis to the case of the original Deffuant et al. model, which is made open via a birth/death mechanism as outlined above. The interested reader can consult [6] for a detail account onto the closed model specifications. We shall here solely recall that the Deffuant's setting $\mu = 1/2$ and $d = 1$, is formally recovered by setting $\alpha_c = 0$ into the affinity model.

In the closed Deffuant's setting, assuming $d = 1$, the standard deviation of the opinion distribution decays as an exponential function [12], namely:

$$v(t) = e^{-t/\tau_c} v(0) , \qquad (17)$$

where τ_c plays the role of a *characteristic time*. Dedicated numerical simulations, relative to the case study $d = 1$, return the $\tau_c = 191.52$.

Assume now that every T–steps the system opens: M agents are randomly removed. New actors enter the systems, their opinions being randomly sampled from a uniform distribution in the interval $[0, 1]$. Let us denote $\rho = M/T$, the departure density. Furthermore, label with v_{t+T}^{conv} the standard deviation of the opinions just before the insertion of the next pool of incoming agents. Hence,

in analogy with the preceding discussion, one can straightforwardly write the recursive relation:

$$\upsilon_{t+T}^2 = \frac{M}{12N} + \left(1 - \frac{M}{N}\right) e^{-2T/\tau_c} \upsilon_t^2 \,, \qquad (18)$$

whose asymptotic stationary solution corresponds to

$$\upsilon^2 = \frac{M}{12N} \frac{1}{1 - (1 - M/N) e^{-2T/\tau_c}} \,, \qquad (19)$$

if T is small enough. This result can be compared to the result of direct numerical simulation, returning an excellent agreement, as displayed in Fig. 6.
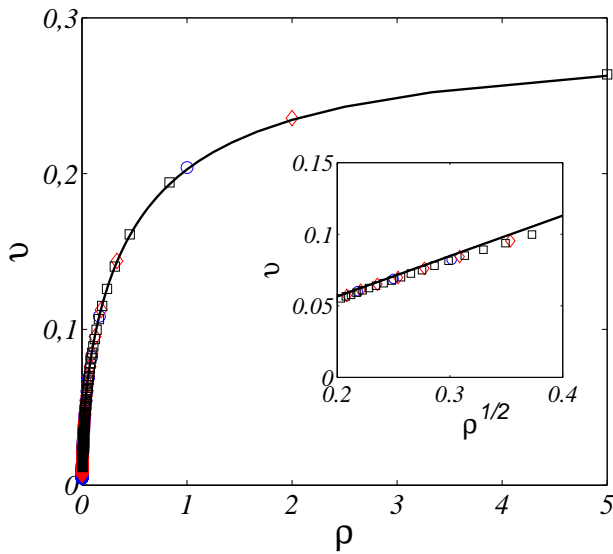

Fig. 6. Final standard deviation, υ, of the opinion cluster as a function of the departure density ρ. Each point results from averaging out 100 independent runs, each relative to $N = 100$ agents. Symbols refer to numerical simulations performed with $d = 1$, $\mu = 1/2$ and different M : ($\circ$) M=1, ($\diamond$) $M = 2$ and ($\square$) $M = 5$. The solids line refers to the theoretical prediction (19) with $\tau_c = 191.52$, this latter being independently estimated as discussed in the main text.

Remark 1 As a final remark, let us observe that in the general case, i.e. $\alpha_c > 0$, an equation formally analogous to (18) can be derived, by invoking the correct exponential ansatz (see main text). To obtain a closed analytical form for the asymptotic stationary standard deviation we however decided to resort to a linear approximation for the opinions convergence, as commented above.

Clearly, when starting from a preformed cluster of opinions the injection of new actors determines an effective migration of the mean, also ruled by Eq. (13) in the original Deffuant et al. scheme.

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10. To compare the fitted convergence time T_c with a direct measure of the aggregation time in absence of birth/death

perturbation, $\rho = 0$, we assumed the following stopping criterion. We monitored the standard deviation $\upsilon_{sim}(t)$, of the simulated opinion distribution and we measured the time when the latter gets smaller than a treshold amount ϵ_{thr} here set to 0.001 (i.e. 10 times smaller than the initial inter-particle separation.). The aggregation times thus obtained averaging over 50 indipendent runs, are $T_{agg} \sim 7100$ for $\alpha_c = 0.5$, $T_{agg} \sim 3000$ for $\alpha_c = 0.3$, and $T_{agg} \sim 1730$ for $\alpha_c = 0.0$, which, excepting the case $\alpha_c = 0.3$, agree with the values of the fitted parameters T_c.

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