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Dramé, Abdou; Dochain, Denis; Winkin, Joseph

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FUNDP
Faculté des Sciences
Département de Mathématique

Rempart de la Vierge, 8
B-5000 Namur Belgique

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A.K. DRAME, D. DOCHAIN and J. J. WINKIN

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Asymptotic behaviour and stability for solutions of a biochemical reactor distributed parameter model

A. K. Dramé, D. Dochain, and J. J. Winkin,

Abstract

This paper deals with the dynamical analysis of a tubular biochemical reactor. The existence of nonnegative state trajectories and the invariance of the set of all physically feasible state values under the dynamical equation as well as the convergence of the state trajectories to equilibrium profiles are proved. In addition, the existence of multiple equilibrium profiles is analyzed. It is proved that under physically meaningful conditions the system has two stable and one unstable equilibrium profiles.

Index Terms

state trajectories, asymptotic behaviour, multiple equilibrium profiles, stability, biochemical reactor.

I. Introduction

The dynamical analysis and control of tubular biochemical reactors have motivated many research activities over the last decades ([5], [6], [11], [13], [24], etc.). The dynamics of these reactors are described by distributed parameter systems and typically by nonlinear partial differential equations with e.g. Danckwerts type boundary conditions, see e.g. [5], [6]. The aim of this paper is to analyze the dynamical nonlinear model of a fixed bed tubular biochemical reactor with axial dispersion. The nonlinearity in the model arises from the substrate inhibition term in the model equations and is a specific rational function of the state components. The basis of the model under study is derived from the work performed on anaerobic digestion in the pilot fixed bed reactor of the LBE-INRA in Narbonne (France) and is mainly inspired from the dynamical models built and validated on the process ([3], [21]). This study follows the preliminary one performed in [13].
The existence and uniqueness of the state trajectories (limiting substrate and limiting biomass) as well as their asymptotic behaviour are analyzed for this model. First of all, it is proved that the trajectories exist on the whole (nonnegative real) time axis and the set of all physically feasible state values is invariant under the dynamical equation. This set takes into account the positivity of the state variables as well as a saturation condition on the substrate. Second, the asymptotic behaviour of the trajectories is investigated and it is proved that the trajectories converge to equilibrium solutions of the system.
In addition the existence of multiple equilibrium profiles is analyzed. The multiplicity is established together with the stability of equilibrium profiles by using phase plane analysis of ordinary differential equations.

The paper is organized as follows : in Section 2, the dynamical model is presented and some preliminary analysis is performed. In Section 3, the state trajectories of the tubular biochemical reactor dynamical model are analyzed. Sections 4 and 5 deal with the existence of multiple equilibrium profiles and their stability. Finally, some concluding remarks are given in Section 6.

II. The dynamical model

Applying the mass balance principles to the limiting substrate concentration $S(\tau,\zeta)$ and the living biomass concentration $X(\tau,\zeta)$ leads to the following nonlinear system of partial differential equations :

$$\frac{\partial S}{\partial \tau} = D\frac{\partial^2 S}{\partial \zeta^2} - \nu\frac{\partial S}{\partial \zeta} - k\mu(S,X)X \tag{II.1}$$

$$\frac{\partial X}{\partial \tau} = -k_d X + \mu(S,X)X, \tag{II.2}$$

with the boundary conditions : for all $\tau \geq 0$,

$$D\frac{\partial S}{\partial \zeta}(\tau,0) - \nu S(\tau,0) + \nu S_{in} = 0 \quad \text{and} \quad \frac{\partial S}{\partial \zeta}(\tau,L) = 0. \tag{II.3}$$

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A. K. Dramé and D. Dochain are with the Université catholique de Louvain, CESAME, 4-6 av. G. Lematre, B-1348 Louvain-La-Neuve, Belgium, drame@inma.ucl.ac.be, dochain@csam.ucl.ac.be
J. J. Winkin is with the University of Namur (FUNDP), Department of Mathematics, 8 Rempart de la Vierge, B-5000 Namur, Belgium, joseph.winkin@fundp.ac.be

The substrate inhibition is expressed via the following law :

$$\mu(S, X) = \mu_0 \frac{S}{K_S X + S + \frac{1}{K_i} S^2}. \tag{II.4}$$

In the equations above, D, k, k_d, K_S, K_i, S_{in}, ν and μ_0 are positive parameters. In particular, D, ν and k_d denote the axial dispersion coeficient, the superficial fluid velocity and the kinetic constant, respectively. The parameters k and K_S are dimensionless, whereas K_i has the dimension of a concentration. In addition the specific growth rate $\mu(S, X)$ has the dimension of the inverse of a time. Finally $\zeta \in [0,\, L]$ and $\tau \geq 0$ denote the spatial and time variables, respectively, and L denotes the length of the reactor.

Observe that the reaction considered here is autocatalytic, i.e. the biomass is not only a product of the reaction, but also a catalyst of that reaction. This feature is modelled by the last term in the right-hand sides (RHS) of equations (II.1) and (II.2). In addition the two first terms of the RHS of equation (II.1) are diffusion and convection terms, respectively. The removal-mortality phenomenon for the biomass is modelled by the first term of the RHS of equation (II.2). This corresponds to the observation that biomass aggregates reach a maximum size beyond which solid particles leave the reactor : this happens notably because of shear forces between the fluid (substrate) going through the reactor and the solid (biomass). The selection of the above model is largely motivated by its analogy with the single microbial growth model with Haldane kinetics in perfectly mixed reactors, for which the existence of multiple (stable and unstable) equilibria have been emphasized (see e.g. [1], [2], [4], [19]).

From a physical point of view, it is expected that the following saturation condition holds :

$$0 \leq S \leq S_{in}, \tag{II.5}$$

where S_{in} is the inlet limiting substrate concentration. One of the contributions of this paper is to prove that the physiscal set defined by inequalities (II.5) is invariant under the dynamic equation of the biochemical reactor model introduced above. As usual, the dynamical analysis of this model will be performed on an equivalent dimensionless infinite dimensional system description. By using the new state components

$$\tilde{x}_1 := \frac{S_{in} - S}{S_{in}} \quad \text{and} \quad \tilde{x}_2 := \frac{X}{S_{in}}$$

and the new spatial and time variables

$$z := \frac{\zeta}{L} \quad \text{and} \quad t := \frac{\nu}{L}\tau,$$

respectively, the model (II.1)-(II.2) can be rewritten as the following two PDE's :

$$\frac{\partial \tilde{x}_1}{\partial t} = \frac{1}{P_e}\frac{\partial^2 \tilde{x}_1}{\partial z^2} - \frac{\partial \tilde{x}_1}{\partial z} + k\tilde{\mu}(\tilde{x}_1, \tilde{x}_2)\tilde{x}_2 \tag{II.6}$$

$$\frac{\partial \tilde{x}_2}{\partial t} = -\gamma \tilde{x}_2 + \tilde{\mu}(\tilde{x}_1, \tilde{x}_2)\tilde{x}_2, \tag{II.7}$$

with the boundary conditions : for all $t \geq 0$,

$$\frac{1}{P_e}\frac{\partial \tilde{x}_1}{\partial z}(t, 0) - \tilde{x}_1(t, 0) = 0 \quad \text{and} \quad \frac{\partial \tilde{x}_1}{\partial z}(t, 1) = 0. \tag{II.8}$$

where the modified substrate inhibition law $\tilde{\mu}(\tilde{x}_1, \tilde{x}_2)$ is given by

$$\tilde{\mu} = \beta \frac{(1 - \tilde{x}_1)}{K_S \tilde{x}_2 + (1 - \tilde{x}_1) + \alpha(1 - \tilde{x}_1)^2}, \tag{II.9}$$

and where the constants α, β, γ and P_e (Peclet number) are given respectively by

$$\alpha := \frac{S_{in}}{K_i},\ \beta := \frac{L}{\nu}\mu_0,\ \gamma := \frac{L}{\nu}k_d \ \text{ and } \ P_e := \frac{\nu L}{D},$$

respectively. We also introduce the following change of variables in order to transform the equation (II.6)-(II.8) into a reaction-diffusion equation with a bounded perturbation. This transformation is for mathematical analysis purpose and we will use the resulting reaction-diffusion type equation to study asymptotic behaviour of solution of the system (II.6)-(II.8). This is a one-to-one transformation, so the two systems are equivalent.

$$x_1(z) := e^{-\frac{P_e}{2}z}\tilde{x}_1(z) \quad \text{and} \quad x_2(z) := e^{-\frac{P_e}{2}z}\tilde{x}_2(z) \text{ for all } z \in [0,\, 1], \quad \text{for all } \tilde{x_1},\, \tilde{x_2} \in C[0,\, 1].$$

Then, we have the following equations :

$$\frac{\partial x_1}{\partial t} = \frac{1}{P_e}\frac{\partial^2 x_1}{\partial z^2} - \frac{P_e}{4}x_1 + k\tilde{\tilde{\mu}}(x_1, x_2)x_2 \tag{II.10}$$

$$\frac{\partial x_2}{\partial t} = -\gamma x_2 + \tilde{\tilde{\mu}}(x_1, x_2)x_2, \tag{II.11}$$

with the boundary conditions : for all $t \geq 0$,

$$\frac{\partial x_1}{\partial z}(t,0) = \frac{P_e}{2}x_1(t,0) \quad \text{and} \quad \frac{\partial x_1}{\partial z}(t,1) = -\frac{P_e}{2}x_1(t,1), \tag{II.12}$$

where

$$\tilde{\tilde{\mu}}(z, x_1, x_2) = \tilde{\mu}(e^{\frac{P_e}{2}z}x_1, e^{\frac{P_e}{2}z}x_2).$$

Now the dimensionless model (II.10)-(II.12) can be given an infinite dimensional state space description as follows. Consider the Banach space $C[0,\,1]$ (equipped with the usual norm $\|.\|_{C^0}$) on which we define the unbounded linear operator A_1 by :

$$D(A_1) = \{x_1 \in C^2[0,1] : \frac{dx_1}{dz}(0) - \frac{P_e}{2}x_1(0) = 0,\ \frac{dx_1}{dz}(1) + \frac{P_e}{2}x_1(1) = 0\},$$
$$A_1 x_1 = \frac{1}{P_e}\frac{d^2x_1}{dz^2} - \frac{P_e}{4}x_1, \quad \forall x_1 \in D(A_1).$$

We also consider the bounded linear operator $A_2 = -\gamma I$ where I is the identity operator on $C[0,\,1]$. Therefore equations (II.10)-(II.12) can be rewritten as follows : for all $t \geq 0$,

$$\frac{dx(t)}{dt} = Ax(t) + N(x(t)),$$

where $x := (x_1, x_2)^T$ is a vector of the Banach space $Z := C[0,\,1] \oplus C[0,\,1]$ endowed with the natural norm $\|.\|_Z = \|.\|_{C^0} + \|.\|_{C^0}$, $A := \text{diag}(A_1, A_2)$ is defined on its domain $D(A) = D(A_1) \oplus C[0,\,1]$ and N is the nonlinear operator defined on X by $N(x) := (N_1(x), N_2(x))$ where

$$N_1 = kN_2 \quad \text{and} \quad N_2(x_1, x_2) = \tilde{\tilde{\mu}}(x_1, x_2)x_2.$$

By the arguments given in [7], the operator A_1 is the infinitesimal generator of an exponentially stable C_0-semigroup of bounded linear operators $T_1(t)$ on $C[0,\,1]$. It is easy to see that A_2 is the infinitesimal generator of an exponentially stable C_0-semigroup of bounded linear operators $T_2(t) = e^{-\gamma t}I$ on $C[0,\,1]$. Therefore A is the infinitesimal generator of an exponentially stable C_0-semigroup of bounded linear operators $T(t)$ on Z given by $T(t) = \text{diag}(T_1(t), T_2(t))$. Moreover, the semigroups $T_1(t)$ and $T_2(t)$ are analytic in $C[0,\,1]$ and $T_1(t)$ is compact in $C^1[0,\,1]$. Therefore, $T(t)$ is analytic in Z.

III. State trajectories

A. Global existence

The well-posedness and invariance properties of the state trajectories of the biochemical reactor dimensionless model (II.10)-(II.12) are now studied by using the description of an abstract semilinear Cauchy Problem of the form

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + N(x(t)), \\ \quad x(0) = x_0 \in \Omega, \end{cases} \tag{III.1}$$

where the operators A and N are defined in the previous section and Ω, motivated by (II.5), is the physically admissible set given by :

$$\Omega = \quad \{(x_1, x_2) \in Z : 0 \leq x_1(z) \leq e^{-\frac{P_e}{2}z}, \quad 0 \leq x_2(z) \quad \text{for all } \ z \in [0,\,1]\} \tag{III.2}$$

The main tool used in the analysis of this problem is the following theorem, which is an equivalent version of Theorem 5.1 of [16, p.355]. This Theorem gives sufficient conditions for the existence and the uniqueness of the mild solution of equations of type (III.1) on the whole interval $[0,\,+\infty)$ and for the invariance of the set Ω under the dynamical equations.

Theorem 3.1: [16, p. 355, Theorem 5.1] Let $(X, \|.\|)$ be a real Banach space and $\tilde{T}(t)$ a C_0-semigroup of bounded linear operators such that $\|\tilde{T}(t)\| \leq Me^{\delta t}$, for all $t \geq 0$, for some $\delta \in \mathbb{R}$ and $M \geq 1$. Let $\tilde{A}$ be the infinitesimal generator of $\tilde{T}(t)$ and $\tilde{D}$ be a closed subset of X. Assume also that $\tilde{N}$ is a continuous function from $\tilde{D}$ into X. Let us consider the following initial value problem

$$\begin{cases} \dfrac{dx(t)}{dt} = \tilde{A}x(t) + \tilde{N}(x(t)), \\ \quad x(0) = x_0 \in \tilde{D}. \end{cases} \tag{III.3}$$

Assume that

(i) $\tilde{D}$ is $\tilde{T}(t)$-invariant, i.e. for all $t \geq 0$, $\tilde{T}(t)\tilde{D} \subset \tilde{D}$;

(ii) For all $x \in \tilde{D}$,

$$\lim_{h \to 0^+} \frac{1}{h} d(x + h\tilde{N}(x); \tilde{D}) = 0;$$

(iii) $\tilde{N}$ is continuous on $\tilde{D}$ and there exists $l_{\tilde{N}} \in \mathbb{R}^+$ such that the operator $(\tilde{N} - l_{\tilde{N}} I)$ is dissipative on $\tilde{D}$.
Then for all $x_0 \in \tilde{D}$, (III.3) has a unique mild solution $x(t, x_0)$ on $[0,, \infty[$. Moreover, if $\mathbf{T}(t)$ is defined on $\tilde{D}$ by $\mathbf{T}(t)x_0 = x(t, x_0)$, for all $t \geq 0$ and $x_0 \in \tilde{D}$, it is a nonlinear semigroup on $\tilde{D}$, with $(\tilde{A} + \tilde{N})$ as its generator.
The following lemma holds.

Lemma 3.2: The semigroup $T(t)$ is positive and for all $t \geq 0$, $T(t)\Omega \subset \Omega$.
Proof : Let $x_0 = (x_{0,1}, x_{0,2}) \in \Omega$ and $x(t) = T(t)x_0 = (T_1(t)x_{0,1}, T_2(t)x_{0,2})^T$, for $t \geq 0$. Obviously, $x_2(t) = T_2(t)x_{0,2} \geq 0$, for all $t \geq 0$. We have to prove that $0 \leq x_1(t, z) = (T_1(t)x_{0,1})(z) \leq e^{-\frac{P_e}{2}z}$, for all $t \geq 0$ and $z \in [0, 1]$. Let us introduce $y_1(t) := -x_1(t)$ for all $t \geq 0$. Since the semigroup $T_1(t)$ is analytic in $C[0, 1]$, then y_1 satisfies the equation

$$\begin{cases} \dfrac{\partial y_1}{\partial t} & = & \dfrac{1}{P_e}\dfrac{\partial^2 y_1}{\partial z^2} - \dfrac{P_e}{4} y_1, \\ y_1(0) & = & -x_{0,1}, \\ \dfrac{\partial y_1}{\partial z}(t, 0) & = & \dfrac{P_e}{2} y_1(t, 0) \quad \text{and} \quad \dfrac{\partial y_1}{\partial z}(t, 1) = -\dfrac{P_e}{2} y_1(t, 1). \end{cases}$$

It follows by the comparison theorem (see eg. [20, Chap 3, Theorem 8]) that y_1 reaches its maximum at a boundary point $(0, z),\ z \in]0, 1[,\ (t, 0),\ t > 0$ or $(t, 1),\ t > 0$. If this maximum is reached at $(t, 0)$ (resp. at $(t, 1)$), then it follows from [20, Chap 3, Theorem 8] that

$$\frac{\partial y_1}{\partial z}(t, 0) < 0 \quad (\text{resp. } \frac{\partial y_1}{\partial z}(t, 1) > 0).$$

Hence, considering the boundary conditions, the solution y_1 cannot have a positive maximum at $(t, 0)$ and $(t, 1)$. It follows that its maximum is negative and therefore $0 \leq x_1(t)$, for all $t \geq 0$. Since $T_1(t)$ is a semigroup of contraction,

$$x_1(t, z) \leq \|x_1(t)\|_{C^0} \leq \|x_{0,1}\|_{C^0} \leq e^{-\frac{P_e}{2}z}, \text{ for all } z \in [0, 1] \text{ and } t \geq 0.$$

It follows that $T(t)x_0 \in \Omega$ that is : the semigroup $T(t)$ is positive and Ω is invariant under $T(t)$. □

Remark 3.1: It is easy to see that the function $N : \Omega \longrightarrow Z$ is Lipschitz continuous and that its Lipschitz constant, l_N, can be obtained by rather simple calculations.
Now let Λ be a closed interval of $\mathbb{R}$ and consider the set

$$K(\Lambda, C[0, 1]) := \{\phi \in C[0, 1] : \phi(z) \in \Lambda \text{ for all } z \in [0, 1]\}.$$

The proof of the following Lemma is similar to the one of [11, Proposition 3.1] is therefore omitted.

Lemma 3.3: Assume that $\Lambda = [a, b]$, $f_c : \Lambda \longrightarrow \mathbb{R}$ is a continuous function and $f_p : [0, 1] \times \Lambda \longrightarrow \mathbb{R}$ is a nonnegative bounded measurable function. If $f_c(a) \geq 0$ and $f_c(b) \leq 0$, then

$$\lim_{h \to 0^+} \frac{1}{h} d(\phi + h\mathcal{B}(\phi), K(\Lambda, C[0, 1])) = 0,$$

where the substitution operator $\mathcal{B}$ is defined on $K(\Lambda, C[0, 1])$ by

$$[\mathcal{B}(\phi)](z) := f_p(z, \phi(z)).f_c(\phi(z)),$$

for all $z \in [0, 1]$, for all $\phi \in K(\Lambda, C[0, 1])$.

Remark 3.2: By replacing $[a, b]$ by $[a, \infty)$ in Lemma 3.3 the conclusion holds if only the condition $f_c(a) \geq 0$ is satisfied.
The following Lemma can be deduced from Lemma 3.3 and Remark 3.2.

Lemma 3.4: For all $(x_1, x_2) \in \Omega$,

$$\lim_{h \to 0^+} \frac{1}{h} d((x_1, x_2) + hN((x_1, x_2)); \Omega) = 0.$$

Finally, we can deduce from Remark 3.1 that the operator $N - l_N$ is dissipative on Ω, where l_N is the Lipschitz constant of N.
In the following theorem, the global existence of state trajectories is reported. It follows from the Lemmas above and Remark 3.1, by using Theorem 3.1.

Theorem 3.5: For every $x_0 \in \Omega$, equation (III.1) has a unique mild solution $x(t, x_0)$ on the interval $[0, +\infty[$. Moreover, if one sets $\mathbf{T}(t)x := x(t, x_0)$ then, $(\mathbf{T}(t))_{t\geq 0}$ is a strongly continuous nonlinear semigroup on Ω, generated by the operator $A + N$.
Hence the state trajectories of the tubular biochemical reactor nonlinear model given by (III.1) (or equivalently (II.10)-(II.12)) are well-defined on the whole time interval $[0, +\infty)$. Moreover, the physically feasible set Ω is invariant under this model dynamics, i.e. for all $t \geq 0$, $\mathbf{T}(t)\Omega \subset \Omega$.

B. Asymptotic behaviour

This subsection deals with the asymptotic behaviour of solutions of (III.1). The main result is the convergence of the solutions to equilibrium profiles of the dynamical model of the tubular biochemical reactor. We will successively give some regularity and relative compactness result of the solutions before dealing with their asymptotic behaviour. For the sake of simplicity, we shall denote by $x(t)$ the solution $x(t, x_0)$ of (III.1). The proof of following Lemma is similar to the one of [7, Lemma 3.1] and is therefore omitted.

Lemma 3.6: (i) For any $x_0 \in \Omega$, the mild solution $x(t)$ of (III.1) is a classical solution, i.e. :

$$x \in C([0, +\infty[; Z) \cap C^1(]0, +\infty[; Z),$$

$$x(t) \in D(A), \text{ for all } t > 0 \text{ and } x(t) \text{ satisfies (III.1) in the usual sense.}$$

(ii) For any $t_0 > 0$, the subsets

$$\{Ax(t),\ t \geq t_0\} \quad \text{and} \quad \{\frac{\partial x(t)}{\partial t},\ t \geq 0\}$$

are bounded in Z. Moreover, the norms of $C[0, 1]$ and $C^1[0, 1]$ are equivalent on the subset $\{x_1(t),\ t \geq t_0\}$; i.e.

$$\exists\, M_0 > 0 \;/\; \forall\, t \geq t_0,\ \|x_1(t)\|_{C^0} \leq \|x_1(t)\|_{C^1} \leq M_0 \|x_1(t)\|_{C^0}.$$

In view of Lemma 3.6, we shall understand by solution of (III.1) a global classical solution.

Let us define now the following functionals, keeping in mind the idea of some kind of energy function.

$$J_1(x_1, x_2) = \int_0^1 \left(\frac{1}{2P_e} \left(\frac{\partial x_1}{\partial z} \right)^2 - \int_0^{x_1} F_1(z, u, x_2) du \right) dz + \frac{1}{4}(x_1^2(0) + x_1^2(1)),$$

$$J_2(x_1, x_2) = -\int_0^1 \int_0^{x_2} F_2(z, x_1, u) du dz,$$

where $F_1(z, x_1, x_2) = -\frac{P_e}{4} x_1 + k\tilde{\tilde{\mu}}(z, x_1, x_2) x_2$ and $F_2(z, x_1, x_2) = -\gamma x_2 + \tilde{\tilde{\mu}}(z, x_1, x_2) x_2$.
We finally define

$$J(x_1, x_2) = J_1(x_1, x_2) + J_2(x_1, x_2).$$

The functional J is well defined along the trajectories of equation (III.1) and the following statement holds.

Lemma 3.7: For any solution $x(t)$ of (III.1), we have

$$\begin{aligned} \frac{d}{dt} J(x(t)) \quad = \quad & - \int_0^1 \left(\left(\frac{\partial x_1}{\partial t} \right)^2 + \left(\frac{\partial x_2}{\partial t} \right)^2 \right) dz \\ & - \int_0^1 \left(\int_0^{x_1(t,z)} \frac{\partial}{\partial t} (F_1(z, u, x_2(t, z))) du + \int_0^{x_2(t,z)} \frac{\partial}{\partial t} (F_2(z, x_1(t, z), u)) du \right) dz. \end{aligned}$$

Proof : Let $x(t) = (x_1(t), x_2(t))$ be a solution of (III.1), then the component $x_1(t)$ satisfies the equation

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = \dfrac{1}{P_e} \dfrac{\partial^2 x_1}{\partial z^2} - \dfrac{P_e}{4} x_1 + N_1(x_1(t), x_2(t)), \\ \dfrac{\partial x_1}{\partial z}(t, 0) = \dfrac{P_e}{2} x_1(t, 0),\ \dfrac{\partial x_1}{\partial z}(t, 1) = \dfrac{P_e}{2} x_1(t, 1). \end{cases}$$

By differentiating this equation with respect to t and by using the comparison theorem [20, Chap 3, Theorem 8], one can prove that $\frac{\partial x_1}{\partial t}(t)$ belongs to $C^1[0,\,1]$ for all $t > 0$. The following calculation is then well founded.

$$\begin{aligned}\frac{d}{dt}J_1(x(t)) \quad &= \int_0^1 \left(\frac{1}{2P_e}\frac{\partial}{\partial t}\left(\frac{\partial x_1}{\partial z}\right)^2 - F_1(z,x_1,x_2)\frac{\partial x_1}{\partial t} - \int_0^{x_1(t,z)} \frac{\partial}{\partial t}(F_1(z,u,x_2(t,z)))du\right)dz \\ &\quad + \frac{1}{4}\frac{\partial}{\partial t}(x_1^2(t,0) + x_1^2(t,1)) \\ &= \int_0^1 \left(\frac{1}{P_e}\frac{\partial x_1}{\partial z}\frac{\partial}{\partial t}\left(\frac{\partial x_1}{\partial z}\right) - F_1(z,x_1,x_2)\frac{\partial x_1}{\partial t} - \int_0^{x_1(t,z)} \frac{\partial}{\partial t}(F_1(z,u,x_2(t,z)))du\right)dz \\ &\quad + \frac{1}{4}\frac{\partial}{\partial t}(x_1^2(t,0) + x_1^2(t,1)) \\ &= -\int_0^1 \left(\frac{1}{P_e}\frac{\partial^2 x_1}{\partial z^2} + F_1(z,x_1,x_2)\right)\frac{\partial x_1}{\partial t}dz - \int_0^1\int_0^{x_1(t,z)} \frac{\partial}{\partial t}(F_1(z,u,x_2(t,z)))dudz \\ &\quad + \frac{1}{P_e}\left(\frac{\partial x_1}{\partial t}\frac{\partial x_1}{\partial z}|_{z=1} - \frac{\partial x_1}{\partial t}\frac{\partial x_1}{\partial z}|_{z=0}\right) + \frac{1}{4}\frac{\partial}{\partial t}(x_1^2(t,0) + x_1^2(t,1)).\end{aligned}$$

Using the boundary conditions in (II.12) and the fact that $x_1(t)$ satisfies the PDE above, it follows that

$$\frac{d}{dt}J_1(x(t)) = -\int_0^1 \left(\frac{\partial x_1}{\partial t}\right)^2 dz - \int_0^1\int_0^{x_1(t,z)} \frac{\partial}{\partial t}(F_1(z,u,x_2(t,z)))dudz.$$

On the other hand,

$$\frac{d}{dt}J_2(x(t)) = -\int_0^1 \left(\left(\frac{\partial x_2}{\partial t}\right)^2 - \int_0^{x_2(t,z)} \frac{\partial}{\partial t}(F_2(z,x_1(t,z),u))du\right)dz.$$

Hence

$$\begin{aligned}\frac{d}{dt}J(x(t)) \quad &= \quad -\int_0^1 \left(\left(\frac{\partial x_1}{\partial t}\right)^2 + \left(\frac{\partial x_2}{\partial t}\right)^2\right)dz \\ &\quad - \quad \int_0^1 \left(\int_0^{x_1(t,z)} \frac{\partial}{\partial t}(F_1(z,u,x_2(t,z)))du + \int_0^{x_2(t,z)} \frac{\partial}{\partial t}(F_2(z,x_1(t,z),u))du\right)dz.\end{aligned}$$

□

Let us denote by $K(\tau)$, for any $\tau \geq 0$, the subset of Z given by

$$K(\tau) = \{x(t),\ t \geq \tau\}.$$

The following Lemma states a relative compactness of the solutions of (III.1). Note that the invariance of the set Ω implies the boundedness of solutions of (III.1). Note also that N_2 is Lipschitz continuous in $C[0,\,1]$ with constant l_{N_2}.

Lemma 3.8: Assume that $l_{N_2} < \gamma$ and let $x_0 = (x_{0,1}, x_{0,2}) \in \Omega$ and $x(t)$ be the solution of (III.1), then $K(0)$ is relatively compact in Z. Moreover, for any $t_0 > 0$, $K(t_0)$ is relatively compact in $C^1[0,\,1] \oplus C[0,\,1]$.

Proof : 1. **Relative compactness in** Z

Observe that $x_1(t)$ satisfies the integral equation

$$x_1(t) = T_1(t)x_{0,1} + \int_0^t T_1(t-s)N_1(x_1(s),x_2(s))ds$$

and recall that the semigroup $T_1(t)$ is compact in $C[0,\,1]$ and $N_1(x_1(t),x_2(t))$ is bounded in $C[0,\,1]$. Then, it follows from [18, p. 236, Lemma 2.4] that $x_1(t)$ has a compact closure in $C[0,\,1]$.

Let us now prove that $x_2(t)$ is relatively compact in $C[0,\,1]$ by applying Ascoli-Arzela's Theorem. To apply this Theorem, it is sufficient to prove that $x_2(t)$ is equicontinuous since it is bounded in $C[0,\,1]$ and $x_2(t)(z) \in \mathbb{R}$, for all $t \geq 0$, $z \in [0,\,1]$.

Let $z_0, z \in [0,\, 1]$, we have

$$\begin{aligned}
x_2(t)(z_0) - x_2(t)(z) &= (T_2(t)x_{0,2})(z_0) - (T_2(t)x_{0,2})(z) \\
&+ \int_0^t (T_2(t-s)N_2(x_1(s), x_2(s)))(z_0) - (T_2(t-s)N_2(x_1(s), x_2(s)))(z)ds \\
&= e^{-\gamma t}(x_{0,2}(z_0) - x_{0,2}(z)) \\
&+ \int_0^t e^{-\gamma(t-s)}\left(N_2(x_1(s), x_2(s)))(z_0) - N_2(x_1(s), x_2(s)))(z)\right) ds.
\end{aligned}$$

Since N_2 is Lipschitz continuous with constant l_{N_2}, we have

$$\begin{aligned}
|x_2(t)(z_0) - x_2(t)(z)| &\leq e^{-\gamma t}|x_{0,2}(z_0) - x_{0,2}(z)| \\
&+ l_{N_2}\int_0^t e^{-\gamma(t-s)}\left(|x_1(s)(z_0) - x_1(s)(z)| + |x_2(s)(z_0) - x_2(s)(z)|\right) ds.
\end{aligned}$$

Let $\varepsilon > 0$. Since $x_1(t)$ is relatively compact in $C[0,\, 1]$ then, there exists $\delta_0 > 0$ such that

$$|z - z_0| \leq \delta_0 \Longrightarrow |x_1(s)(z_0) - x_1(s)(z)| < \varepsilon,\ \forall\ s \geq 0.$$

Then, for $z \in [0,\, 1]$ satisfying $|z - z_0| \leq \delta_0$, we have

$$e^{\gamma t}|x_2(t)(z_0) - x_2(t)(z)| \leq (1 + \frac{l_{N_2}}{\gamma})\varepsilon e^{\gamma t} + l_{N_2}\int_0^t e^{\gamma s}|x_2(s)(z_0) - x_2(s)(z)|ds.$$

By applying Gronwall's inequality, we have

$$\begin{aligned}
e^{\gamma t}|x_2(t)(z_0) - x_2(t)(z)| &\leq (1 + \frac{l_{N_2}}{\gamma})\varepsilon e^{\gamma t} + l_{N_2}(1 + \frac{l_{N_2}}{\gamma})\varepsilon\int_0^t e^{l_{N_2}(t-s)}e^{\gamma s}ds \\
&\leq (1 + \frac{l_{N_2}}{\gamma})\varepsilon e^{\gamma t} + \frac{l_{N_2}(1 + \frac{l_{N_2}}{\gamma})}{\gamma - l_{N_2}}\varepsilon e^{l_{N_2}t}e^{(\gamma - l_{N_2})t} \\
&\leq \frac{\gamma(1 + \frac{l_{N_2}}{\gamma})e^{\gamma t}}{\gamma - l_{N_2}}\varepsilon.
\end{aligned}$$

So, as $l_{N_2} < \gamma$, let ε_0 be such that

$$\varepsilon = \frac{\gamma - l_{N_2}}{\gamma(1 + \frac{l_{N_2}}{\gamma})}\varepsilon_0.$$

By using ε_0 in the role of ε in the previous calculations, we find that for $z \in [0,\, 1]$

$$|z - z_0| \leq \delta_0 \Longrightarrow |x_2(t)(z) - x_2(t)(z_0)| < \varepsilon,\ \forall\ t \geq 0.$$

It follows that $x_2(t)$ is equicontinuous in $C[0,\, 1]$ and by Ascoli-Arzela's Theorem, $x_2(t)$ has compact closure in $C[0,\, 1]$.

2. **Relative compactness in** $C^1[0,\, 1] \oplus C[0,,\, 1]$
By Lemma 3.6, the set $\mathcal{K}_0 = \{x_1(t),\ t \geq t_0\}$ is bounded in $C^1[0,\, 1]$. Observe also that for each $z \in [0,\, 1]$, $\{x_1(t, z)),\ t \geq t_0\}$ has a compact closure in $\mathbb{R}$. Once again, by Lemma 3.6, the norms of $C[0,\, 1]$ and $C^1[0,\, 1]$ are equivalent on $\mathcal{K}_0$. Since $x_1(t)$ is equicontinuous in $C[0,\, 1]$ then $\mathcal{K}_0$ is also equicontinuous in $C^1[0,\, 1]$. It follows from Ascoli-Arzela's Theorem that $K(t_0)$ has a compact closure in $C^1[0,\, 1] \oplus C[0,\, 1]$. This completes the proof. □

As the solutions of (III.1) are defined for all time, we can define their ω-limit in the usual way as for dynamical systems.

Definition 3.1: The ω-limit set of a solution $x(t)$ of (III.1) with respect to the topology of Z is the set defined by

$$\omega(x_0/Z) = \{\varphi \in Z:\ \exists(t_n)_{n\in\mathbb{N}},\ t_n \underset{n\to+\infty}{\longrightarrow} +\infty \text{ and } x(t_n) \underset{n\to+\infty}{\longrightarrow} \varphi \text{ in } Z\}.$$

The following statement holds for equation (III.1).

Lemma 3.9: Let $x(t)=(x_1(t),x_2(t))$ be a solution of (III.1). Then, the ω-limit set of $x(t)$ with respect to the topology of Z coincides with its ω-limit set with respect to the topology of $C^1[0,\,1]\oplus C[0,\,1]$; i.e.

$$\omega(x_0/Z)=\omega(x_0/C^1[0,\,1]\oplus C[0,\,1]).$$

Proof : Observe that

$$\omega(x_0/C^1[0,\,1]\oplus C[0,\,1])=\omega(x_{01}/C^1[0,\,1])\oplus\omega(x_{02}/C[0,\,1]).$$

We have only to prove the identity for the component $x_1(t)$ of $x(t)$. Obviously,

$$\omega(x_{0,1}/C^1[0,\,1])\subset\omega(x_{0,1}/C[0,\,1]).$$

Let $\varphi\in\omega(x_{0,1}/C[0,\,1])$, then there exists $(t_n)_{n\in\mathbb{N}}$ such that

$$t_n\underset{n\to+\infty}{\longrightarrow}+\infty\quad\text{and}\quad x_1(t_n)\underset{n\to+\infty}{\longrightarrow}\varphi\quad\text{in}\ \ C[0,\,1].$$

Without loss of generality, we can assume that $(t_n)_{n\geq 0}$ is increasing and that $t_0>0$. From Lemma 3.8, $x_1(t)$ is eventually relatively compact in $C^1[0,\,1]$. Then there exists a subsequence $(t_{n_k})_{k\geq 0}$ of $(t_n)_{n\geq 0}$ and a function $\tilde{\varphi}\in C^1[0,\,1]$ such that

$$t_{n_k}\underset{k\to+\infty}{\longrightarrow}+\infty\ \ \text{and}\ \ x_1(t_{n_k})\underset{k\to+\infty}{\longrightarrow}\tilde{\varphi}\quad\text{in}\ \ C^1[0,\,1].$$

It follows from the uniqueness of the limit in $C[0,\,1]$ that $\varphi=\tilde{\varphi}$ and $\varphi\in C^1[0,\,1]$, since the above convergence also holds in $C[0,\,1]$. Hence

$$\omega(x_{0,1}/C[0,\,1])\subset\omega(x_{0,1}/C^1[0,\,1]).$$

This completes the proof. □

In view of this Lemma, the ω-limit set of a solution $x(t)$ will be simply denoted by $\omega(x_0)$.
The following Theorem is one of the main results related to the asymptotic behaviour of solutions of (III.1) established here.

Theorem 3.10: Assume that $l_{N_2}<\gamma$ and let $x_0=(x_{0,1},x_{0,2})\in\Omega$ and $x(t)$ be the solution of (III.1) with $x(0)=x_0$ and $\omega(x_0)$ its ω-limit set. Then $\omega(x_0)$ is non empty and is contained in $D(A)$. Moreover $\omega(x_0)$ consists of equilibrium profiles of (III.1).
Proof : Let $x_0\in\Omega$, from Lemma 3.8, $\omega(x_0)$ is non empty. Therefore, there exist $\varphi\in\omega(x_0)$ and a sequence $(t_n)_{n\geq 0}$ such that

$$t_n\underset{n\to+\infty}{\longrightarrow}+\infty\quad\text{and}\quad x(t_n)\underset{n\to+\infty}{\longrightarrow}\varphi\quad\text{in}\ Z.$$

Let us introduce $\bar{x}_n:=x(t_n)$ and $\bar{y}_n(t):=x(t+t_n)$ for all $n\in\mathbb{N}$ and $t\geq 0$. Then, $\bar{y}_n(t)$ satisfies the integral equation

$$\bar{y}_n(t)=T(t)\bar{x}_n+\int_0^t T(t-s)N(\bar{y}_n(s))ds. \tag{III.4}$$

Recall that $T(t)$ is a C_0-semigroup of contractions and N is Lipschitz continuous with constant l_N. Then,

$$\|\bar{y}_n(t)-\bar{y}_m(t)\|_Z\leq\|\bar{x}_n-\bar{x}_m\|_Z+l_N\int_0^t\|\bar{y}_n(s)-\bar{y}_m(s)\|_Z ds,\quad\text{for all } t\geq 0 \text{ and } n,m\in\mathbb{N}.$$

Hence, by using Gronwall's inequality : for any $t_0>0$, there exists a constant $C>0$ such that

$$\sup_{0\leq t\leq t_0}\|\bar{y}_n(t)-\bar{y}_m(t)\|_Z\leq C\|\bar{x}_n-\bar{x}_m\|_Z,\quad\forall\, m,n\in\mathbb{N}.$$

Then considered as a subset of $C([0,\,t_0];\,Z)$, $(\bar{y}_n)_{n\geq 0}$ is a Cauchy sequence. It follows that there exists a continuous function $y:[0,\,\infty[\longrightarrow Z$ such that

$$\forall\, t_0>0,\quad\lim_{n\to\infty}\sup_{0\leq t\leq t_0}\|\bar{y}_n(t)-y(t)\|_Z=0.$$

Taking the limit in (III.4) when $n\to\infty$, leads to

$$y(t)=T(t)\varphi+\int_0^t T(t-s)N(y(s))ds,\ \text{for all } t\geq 0.$$

Then $y(t)$ is a mild solution of (III.1) and by Lemma 3.6, $y(t)$ is a classical solution of (III.1).
Moreover, $\bar{y}_n(t)\in D(A)$ and $AT(t)\in L(Z)$ for all $t\geq 0$ and $n\in\mathbb{N}$, and

$$A\bar{y}_n(t)=AT(t)\bar{x}_n+\int_0^t AT(t-s)N(\bar{y}_n(s))ds.$$

Taking the limit, when $n \to \infty$,

$$\lim_{n\to\infty} A\bar{y}_n(t) = AT(t)\varphi + \int_0^t AT(t-s)N(y(s))ds = Ay(t).$$

Since the function N is continuous, one has

$$\lim_{n\to\infty} \frac{d\bar{y}_n}{dt}(t) = \frac{dy(t)}{dt}.$$

By Lemma 3.7, for any $t_0 > 0$

$$\begin{aligned} J(x(t_0)) - J(x(t)) &= \int_{t_0}^t \int_0^1 \left(\left(\frac{\partial x_1}{\partial \tau}\right)^2 + \left(\frac{\partial x_2}{\partial \tau}\right)^2 \right) dz d\tau \\ &- \int_{t_0}^t \int_0^1 \left(\int_0^{x_1(\tau,z)} \frac{\partial}{\partial \tau} \left(F_1(z,u,x_2(\tau,z))\right) du + \int_0^{x_2(\tau,z)} \frac{\partial}{\partial \tau} \left(F_2(z,x_1(\tau,z),u)\right) du \right) dz d\tau. \end{aligned}$$

As $x(t)$ is bounded in Z (there exists M_0 such that $0 \le \|x(t)\|_Z \le M_0$, for all $t \ge 0$), we define the functions

$$G_1(z,\tau,u) = \begin{cases} F_1(z,u,x_2(\tau,z)), & \text{if} \quad 0 \le u \le x_1(\tau,z) \\ 0 & \text{if} \quad u \ge x_1(\tau,z) \end{cases}$$

$$G_2(z,s,u) = \begin{cases} F_2(z,x_1(\tau,z),u), & \text{if} \quad 0 \le u \le x_2(\tau,z) \\ 0 & \text{if} \quad u \ge x_2(\tau,z) \end{cases}$$

Then,

$$\begin{aligned} \int_{t_0}^t \int_0^{x_1(\tau,z)} \frac{\partial}{\partial \tau}(F_1(z,u,x_2(\tau,z)))du d\tau &= \int_{t_0}^t \int_0^{M_0} \frac{\partial}{\partial \tau}(G_1(z,\tau,u))du d\tau \\ &= \int_0^{M_0} \int_{t_0}^t \frac{\partial}{\partial \tau}(G_1(z,\tau,u))du d\tau \\ &= \int_0^{M_0} \left(G_1(z,t,u) - G_1(z,t_0,u)\right) du \\ &= \int_0^{x_1(t,z)} \left(F_1(z,u,x_2(t)) - F_1(z,u,x_2(t_0))\right) du. \end{aligned}$$

As $x_1(t)$ is bounded and F_1 is continuous, then the integral

$$\int_0^1 \int_{t_0}^t \int_0^{x_1(\tau,z)} \frac{\partial}{\partial \tau}(F_1(z,u,x_2(\tau,z)))du d\tau dz, \quad t \ge t_0$$

is bounded.
Similarly, by using G_2, we show that the integral

$$\int_0^1 \int_{t_0}^t \int_0^{x_1(\tau,z)} \frac{\partial}{\partial \tau}(F_2(z,x_1(\tau,z),u))du d\tau dz, \quad t \ge t_0$$

is bounded.

By Lemma 3.6, $J(x(t))$ is bounded in Z. Hence,

$$\int_{t_0}^\infty \int_0^1 \left(\left(\frac{\partial x_1}{\partial \tau}\right)^2 + \left(\frac{\partial x_2}{\partial \tau}\right)^2 \right) dz d\tau < \infty.$$

Taking any $0 < t_0 < t_1 < \infty$, one gets the following identity

$$\lim_{n\to\infty} \int_{t_0}^{t_1} \int_0^1 \left(\left(\frac{\partial \bar{y}_{n,1}}{\partial t}\right)^2 + \left(\frac{\partial \bar{y}_{n,2}}{\partial t}\right)^2 \right) dz dt = \lim_{n\to\infty} \int_{t_0+t_n}^{t_1+t_n} \int_0^1 \left(\left(\frac{\partial x_1}{\partial t}\right)^2 + \left(\frac{\partial x_2}{\partial t}\right)^2 \right) dz dt = 0.$$

It follows that

$$\int_{t_0}^{t_1} \int_0^1 \left(\left(\frac{\partial y_1}{\partial t}\right)^2 + \left(\frac{\partial y_2}{\partial t}\right)^2 \right) dz dt = 0.$$

Hence, y is independent of t and $y(t) = \varphi$, for all $t \geq 0$. $y(t)$ being a classical solution of (III.1), it follows that

$$\varphi \in D(A) \quad \text{and} \quad A\varphi + N(\varphi) = 0.$$

□

The convergence of solutions of (III.1) to its equilibrium profiles is reported now.

Theorem 3.11: Asssume that $l_{N_1} < \gamma$ and let $x_0 = (x_{0,1}, x_{0,2}) \in \Omega$ and $x(t)$ be the solution of (III.1) with $x(0) = x_0$. Then there exists an equilibrium profile $\bar{x}$ of (III.1) such that

$$\lim_{t\to\infty} x(t) = \bar{x} \quad \text{in} \quad C^1[0,\, 1] \oplus C[0,\, 1].$$

Proof : Let $x(t) = (x_1(t), x_2(t))$ be the solution of (III.1) and $\omega(x_0)$ its ω-limit set. Let us define

$$f(t, z, w) = N_1(w, x_2(t, z)),$$

Then, the x_1-equation in (III.1) can be written as follows

$$\left\{\begin{array}{c} \dfrac{\partial x_1}{\partial t} = \dfrac{1}{P_e}\dfrac{\partial^2 x_1}{\partial z^2} - \dfrac{P_e}{4}x_1 + f(t, z, x_1) \\ x_1(0, z) = x_{0,1}(z) \\ \dfrac{\partial x_1}{\partial z}(t, 0) = \dfrac{P_e}{2}x_1(t, 0); \quad \dfrac{\partial x_1}{\partial z}(t, 1) = -\dfrac{P_e}{2}x_1(t, 1), \end{array}\right. \tag{III.5}$$

and we denote by $\omega(x_{0,1})$ its ω-limit set, that is : $\omega(x_0) = \omega(x_{0,1}) \oplus \omega(x_{0,2})$. By Theorem 3.10, $\omega(x_0)$ consists of equilibrium solutions. Then, $\omega(x_{0,1})$ consists of solutions of ordinary differential equations and the convergence of subsequences of $x_1(t)$ holds in norm $C^1[0,\, 1]$. Then, by Theorem A in [14], $\omega(x_{0,1})$ contains at most one element $\bar{x}_1$. As $\omega(x_{0,1})$ is non empty, then it contains exactly one element. It follows that $x_1(t)$ converges towards $\bar{x}_1$ in norm $C^1[0,\, 1]$. Therefore, $x_1(t), x_2(t)$ converges to an equilibrium $(\bar{x}_1, \bar{x}_2)$ in $C^1[0,\, 1] \oplus C[0,\, 1]$. □

IV. Multiple equilibrium profiles

In this section, the equilibrium profiles analysis of the biochemical reactor dimensionless model (II.6)-(II.9) is performed by using a perturbation theory. The equilibrium profiles $(\bar{x}_1, \bar{x}_2)$ are solutions of

$$\left\{\begin{array}{rcl} \dfrac{1}{P_e}\bar{x}_1^{''} - \bar{x}_1^{'} + k\tilde{\mu}(\bar{x}_1, \bar{x}_2)\bar{x}_2 & = & 0, \\ -\gamma\bar{x}_2 + \tilde{\mu}(\bar{x}_1, \bar{x}_2)\bar{x}_2 & = & 0, \\ \dfrac{1}{P_e}\bar{x}_1^{'}(0) - \bar{x}_1(0) = \bar{x}_1^{'}(1) & = & 0. \end{array}\right. \tag{IV.1}$$

Obviously, the system (IV.1) has a trivial solution $(\bar{x}_1, \bar{x}_2) = (0, 0)$ which corresponds to the reactor washout $(\bar{S}, \bar{X}) = (S_{in}, 0)$ when considering the initial state variables. In the following, we shall be interested in solution of (IV.1) which satisfy $\tilde{\mu}(\bar{x}_1, \bar{x}_2) = \gamma$. Recall that $P_e = \dfrac{\nu L}{D}$. We give here a result of the existence of multiple equilibrium profiles whose proof is based on pertubation theory and is more simple. The following statement holds

Proposition 4.1: There exists $D^* > 0$ sufficiently large and $\nu^* > 0$ such that for all $D \geq D^*$ the system (IV.1) has

(i) at least two non trivial solutions if the parameter ν satisfies $0 \leq \nu < \nu^*$,

(ii) at least one nontrivial solution for $\nu = \nu^*$,

Proof : From the second equation of (IV.1), we have $\tilde{\mu}(\bar{x}_1, \bar{x}_2) = \gamma$ which implies that

$$\bar{x}_2 = \frac{(1 - \bar{x}_1)(M + \alpha k_d \bar{x}_1)}{k_d K_S}, \quad \text{where } M = \mu_0 - k_d - \alpha k_d < 0.$$

Let us consider the real valued function g defined by

$$g(\bar{x}_1) = \frac{kL(1 - \bar{x}_1)(M + \alpha k_d \bar{x}_1)}{K_S}.$$

The system (IV.1) is equivalent to the following equation

$$\left\{\begin{array}{rcl} D\bar{x}_1^{''} - \nu\bar{x}_1^{'} + g(\bar{x}_1) & = & 0, \\ D\bar{x}_1^{'}(0) - \nu\bar{x}_1(0) = \bar{x}_1^{'}(1) & = & 0. \end{array}\right. \tag{IV.2}$$

Let us introduce $u(z) := \bar{x}_1(1-z)$ and $v(z) := \bar{x}_1^{'}(1-z)$ for all $0 \leq z \leq 1$. Then equation (IV.2) becomes

$$\left\{\begin{array}{rcl} u' & = & -v, \\ v' & = & -\dfrac{1}{D}(\nu v - g(u)), \\ u(0) & = & a,\ v(0) = 0 \ \text{ and } v(1) = \dfrac{\nu}{D}u(1). \end{array}\right. \tag{IV.3}$$

Note that (IV.2) can be solved by finding a parameter $\nu = \nu(a, D)$ (depending on a and D), whenever a and D are given, such that the solution (u, v) of the Cauchy Problem in (IV.3) satisfies the final condition

$$v(1) = \frac{\nu}{D}u(1).$$

Therefore, if there are $a_1 \neq a_2$ and $D > 0$ such that $\nu(a_2, D) = \nu(a_2, D)$, then (IV.2) has at least two solutions. So, the existence of multiple equilibrium profiles is equivalent to the existence of $a_1, a_2, \ldots,$ and $D > 0$ such that $\nu(a_i, D) = \nu(a_j, D)$, for all i and j.

Now we assume that D is large enough and we introduce $\varepsilon = \dfrac{1}{D}$, $u_\varepsilon = u$ and $v_\varepsilon = \dfrac{1}{\varepsilon}v$ and we regard ν as a function of ε ($\nu = \nu(a, \varepsilon)$) instead of a function of D. This leads to the following regular perturbation problem

$$\left\{\begin{array}{rcl} u_\varepsilon^{'} & = & -\varepsilon v_\varepsilon \\ v_\varepsilon^{'} & = & -(\nu\varepsilon v_\varepsilon - g(u_\varepsilon)) \\ u_\varepsilon(0) & = & a,\ v_\varepsilon(0) = 0 \ \text{ and } v_\varepsilon(1) = \nu u_\varepsilon(1). \end{array}\right. \tag{IV.4}$$

Considering the non-perturbed problem,

$$u_0 \equiv a,\ v_0(1) = g(u_0), \quad \text{whence} \quad \nu a = g(a).$$

Then, for $\varepsilon = 0$ we have

$$\nu(a, 0) = \frac{g(a)}{a}.$$

From direct computations, one can check that

$$\frac{\partial \nu(a,0)}{\partial a} = -\frac{kL}{K_S a^2}\left(M + \alpha k_d a^2\right) \quad \text{ and } \quad \frac{\partial^2 \nu(a,0)}{\partial a^2} = \frac{2kLM}{K_S a^3} < 0.$$

It follows that the function $a \longrightarrow \nu(a, 0)$ is concave and has the following form (Figure 1)

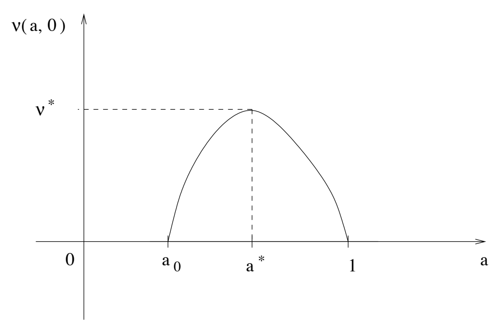

Fig. 1. Graph of $\nu(., 0)$

From the Theorem of dependence of the solutions of ordinary differential equations on initial conditions,

$$\lim_{\varepsilon \to 0} \nu(a, \varepsilon) = \nu(a, 0) \quad \text{ in } C^2[0,\, 1].$$

It follows that there exists $\varepsilon^* > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon^*$, the equation (IV.4) has

(i) at least two non trivial solutions for $0 \leq \nu < \nu^*$,

(ii) at least one non trivial solution for $\nu = \nu^*$,

□

A. Numerical illustration

Numerical simulations have been performed in order to illustrate the existence of multiple equilibrium profiles for the tubular biochemical reactor model. The following values of the model's parameters are considered :

$$S_{in} = 10;\ K_i = 5;\ \mu_0 = 0.65;\ k_d = 0.26;\ D = 1.3;\ L = 1; k = 7;\ K_S = 3;\ \text{ and } \nu = 0.03.$$

It is shown that besides the trivial equilibrium $(0, 0)$, there exist two non trivial equilibrium profiles $(\bar{x}_1, \bar{x}_2)$ and (x_1^*, x_2^*) represented in Figure 2 and Figure 3. The equilibrium profiles satisfy the inequalities

$$0 \leq \bar{x}_1(z) \leq x_1^*(z), \quad \text{for all } z \in [0,\ 1]$$

Recall that

$$\bar{x}_1 = \frac{S_{in} - \bar{S}}{S_{in}},$$

where $\bar{S}$ is the corresponding substrate concentration at equilibrium. So, in both Figures 2 and 3, the substrate concentration is decreasing along the reactor. In Figure 3, the biomass concentration X^* is also decreasing along the reactor. Indeed, in this case the substrate concentration S^* is too small and so the death effect becomes more important than the growth for the biomass.

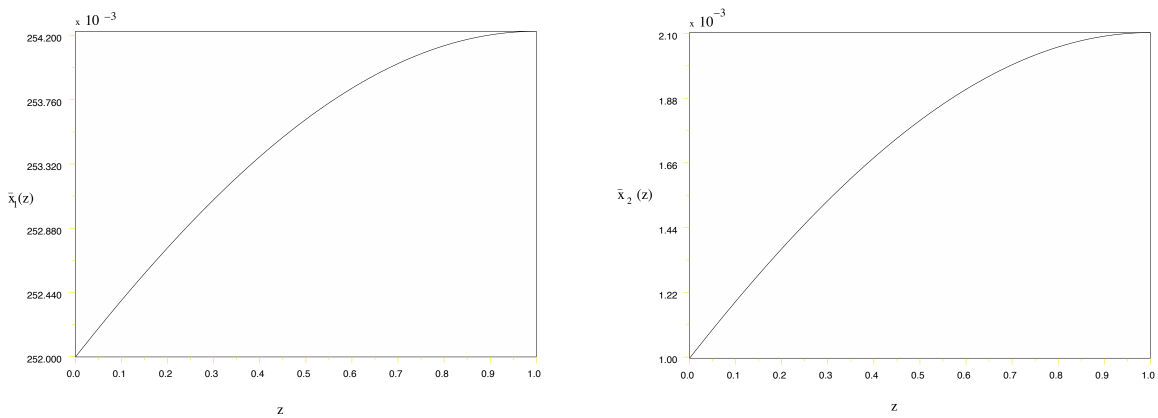

Fig. 2. Nontrivial equilibrium $(\bar{x}_1, \bar{x}_2)$

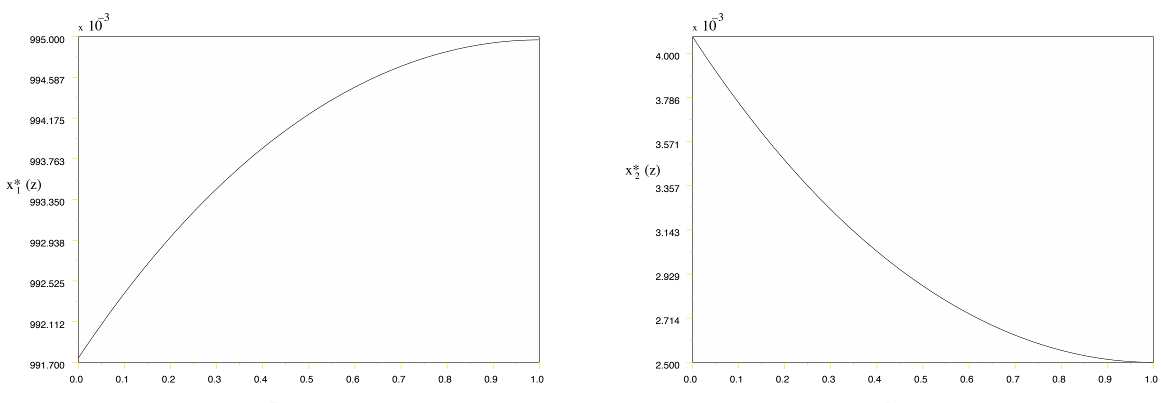

Fig. 3. Nontrivial equilibrium (x_1^*, x_2^*)

V. Stability of equilibrium profiles

The stability of equilibrium profiles of the biochemical reactor dimensionless model (II.6)-(II.9) is analyzed here. This analysis is based on the linearized equations at the equilibrium. As in Lemma 3.8, one can see that the semigroup associated with the linearized equations is compact and by [23, Theorems 11.20 and 11.22], the stability of equilibria is determined by the eigenvalue problem. We first prove that the trivial equilibrium $(0, 0)$ is locally asymptotically stable. Among the multiple equilibrium profiles, it is shown that the middle one $(\bar{x}_1, \bar{x}_2)$ is unstable whereas the third equilibrium profile (x_1^*, x_2^*) is locally

asymptotically stable.
Let us consider an equilibrium solution $(\bar{x}_1, \bar{x}_2)$ of (II.6)-(II.9). Recall that

$$\bar{x}_2 = \frac{(1-\bar{x}_1)(M+\alpha k_d \bar{x}_1)}{k_d K_S} \text{ and } \tilde{\mu}(\bar{x}_1, \bar{x}_2) = \frac{\mu_0}{\beta}\frac{(1-\bar{x}_1)}{K_S\bar{x}_2 + (1-\bar{x}_1) + \alpha(1-\bar{x}_1)^2}, \quad \text{for all } z \in [0,\, 1]. \tag{V.1}$$

Let us introduce for $i = 1, 2$,

$$\varphi_i(z) = \frac{\beta}{\mu_0}\frac{\partial}{\partial u_i}\left(\tilde{\mu}(u_1, u_2)u_2\right)|_{(u_1,u_2)=(\bar{x}_1(z),\bar{x}_2(z))}.$$

It follows from direct computations, by using (V.1), that

$$\varphi_1(z) = \frac{1}{\mu_0^2 K_S}(M+\alpha k_d \bar{x}_1(z))(k_d - \mu_0 + 2\alpha k_d(1-\bar{x}_1(z))), \quad \text{for all } z \in [0,\, 1]$$

and

$$\varphi_2(z) = \frac{k_d}{\mu_0^2}(k_d + \alpha k_d(1-\bar{x}_1(z))), \quad \text{for all } z \in [0,\, 1].$$

The linearized system of (II.6)-(II.9) at $(\bar{x}_1, \bar{x}_2)$ is given by

$$\left\{\begin{array}{rcl} \dfrac{\partial x_1}{\partial t} & = & \dfrac{1}{P_e}\dfrac{\partial^2 x_1}{\partial z^2} - \dfrac{\partial x_1}{\partial z} + \dfrac{k\mu_0}{\beta}\varphi_1(z)x_1 + \dfrac{k\mu_0}{\beta}\varphi_2(z)x_2, \\ \dfrac{\partial x_2}{\partial t} & = & -\gamma x_2 + \dfrac{\mu_0}{\beta}\varphi_1(z)x_1 + \dfrac{\mu_0}{\beta}\varphi_2(z)x_2, \\ \dfrac{\partial x_1}{\partial z}(t,0) & = & P_e x_1(t,0) \text{ and } \dfrac{\partial x_1}{\partial z}(t,1) = 0. \end{array}\right. \tag{V.2}$$

The stability of the equilibria is determined by the eigenvalue problem

$$\left\{\begin{array}{rcl} \dfrac{1}{P_e}x_1^{''} - x_1^{'} + \dfrac{k\mu_0}{\beta}\varphi_1(z)x_1 + \dfrac{k\mu_0}{\beta}\varphi_2(z)x_2 & = & \lambda x_1, \\ -\gamma x_2 + \dfrac{\mu_0}{\beta}\varphi_1(z)x_1 + \dfrac{\mu_0}{\beta}\varphi_2(z)x_2 & = & \lambda x_2, \\ \dfrac{1}{P_e}x_1^{'}(0) = x_1(0) \text{ and } x_1^{'}(1) & = & 0. \end{array}\right. \tag{V.3}$$

A. Stability of the trivial equilibrium $(0,0)$

For the trivial equilibrium, the functions φ_1 and φ_2 are constant and given by

$$\tilde{\varphi_1} = \varphi_1(z) = \frac{M(\alpha k_d - M)}{\mu_0^2 K_S} < 0, \quad \text{for all } z \in [0,\, 1],$$

$$\tilde{\varphi_2} = \varphi_2(z) = \frac{k_d^2(1+\alpha)}{\mu_0^2} > 0, \quad \text{for all } z \in [0,\, 1].$$

The parameters are assumed to satisfy the following condition.

A1 : $k_d(1+\alpha) < \mu_0^2\left(\dfrac{L}{\nu^*}\right)^2.$

Hence, the following result holds.

Proposition 5.1: Assume that **A1** holds. Then the trivial equilibrium $(0,0)$ is locally asymptotically stable.
Proof : First of all, observe that if

$$\lambda \leq -\gamma + \frac{\mu_0}{\beta}\tilde{\varphi_2}$$

then under assumption **A1**, $\lambda < 0$ and therefore the equilibrium profile $(0,0)$ is locally asymptotically stable.
Now assume that

$$\lambda > -\gamma + \frac{\mu_0}{\beta}\tilde{\varphi_2}. \tag{V.4}$$

Then from the second equation of (V.3), x_2 can be expressed as follows

$$x_2 = \frac{\frac{\mu_0}{\beta}\tilde{\varphi}_1}{\lambda + \gamma - \frac{\mu_0}{\beta}\tilde{\varphi}_2} x_1 = \tilde{b} x_1.$$

We set also

$$a = \frac{k\mu_0}{\beta}\tilde{\varphi}_1 < 0 \quad \text{and} \quad b = \frac{k\mu_0}{\beta}\tilde{b} < 0.$$

The problem (V.3) is reduced to the following one :

$$\begin{cases} \frac{1}{P_e} x_1^{''} - x_1^{'} + (a + b - \lambda) x_1 &= 0, \\ \frac{1}{P_e} x_1^{'}(0) = x_1(0) \text{ and } x_1^{'}(1) &= 0. \end{cases}$$

Let us introduce the following variables

$$u(z) := e^{-\frac{P_e}{2}z} x_1(z), \quad v(z) := u'(z), \quad \text{for all } z \in [0,\, 1].$$

The equations above are equivalent to the following ones

$$\begin{cases} u' &= v, \\ v' &= P_e\left(\frac{P_e}{4} + \lambda - a - b\right) u, \\ v(0) &= \frac{P_e}{2} u(0) \text{ and } v(1) = -\frac{P_e}{2} u(1). \end{cases} \tag{V.5}$$

A necessary condition for equations (V.5) to have a solution is that

$$\frac{P_e}{4} + \lambda - a - b < 0.$$

It turns out that λ must satisfy

$$\lambda < -\frac{P_e}{4} + a + b < 0.$$

It follows that the trivial equilibrium profile $(0, 0)$ is locally asymptotically stable. □

Now we shall consider that there exist two nontrivial equilibrium profiles $(\bar{x}_1, \bar{x}_2)$ and (x_1^*, x_2^*) as determined by Proposition 4.1. Note that the equilibrium profiles satisfy

$$0 \leq \bar{x}_1(z) \leq x_1^*(z), \ \forall\, z \in [0,\, 1].$$

We shall first deal with the the stability of the nontrivial equilibrium profile (x_1^*, x_2^*).

B. Stability of the nontrivial equilibrium (x_1^*, x_2^*)

Observe that the function φ_2 is positive and decreasing in $[0,\, 1]$. Also,

$$\varphi_1(z) > 0, \quad \text{for all } z \in [0,\, 1] \text{ such that } x_1^*(z) \in \left[\frac{-M}{\alpha k_d},\, \frac{1}{2} - \frac{M}{2\alpha k_d}\right[,$$

$$\varphi_1(z) < 0, \quad \text{for all } z \in [0,\, 1] \text{ such that } x_1^*(z) \in \left[\frac{1}{2} - \frac{M}{2\alpha k_d},\, 1\right]$$

and if $x_1^*(z) \in [\frac{1}{4} - \frac{3M}{4\alpha k_d},\, 1]$ for all $z \in [0,\, 1]$, then φ_1 is decreasing.

Let us consider the eigenvalues problem (V.3). Observe that if there exists $z_0 \in [0,\, 1]$ such that

$$\lambda + \gamma - \frac{\mu_0}{\beta}\varphi_2(z_0) \leq 0$$

then, by using assumption **A1**, one has $\lambda < 0$. Also, if $\lambda + \gamma \leq 0$ then $\lambda < 0$. Therefore, we shall assume that

$$\lambda + \gamma > 0 \ \text{ and } \ \lambda + \gamma - \frac{\mu_0}{\beta}\varphi_2(z) > 0, \quad \forall\, z \in [0,\, 1]. \tag{V.6}$$

It follows from equation (V.3) that

$$x_2 = \frac{\mu_0}{\beta} \frac{\varphi_1(z)}{\lambda + \gamma - \dfrac{\mu_0}{\beta}\varphi_2(z)} x_1 \tag{V.7}$$

and

$$\left\{ \begin{array}{rcl} \dfrac{1}{P_e} x_1^{''} - x_1^{'} + \dfrac{\mu_0 k}{\beta}\varphi_1(z)x_1 + \dfrac{\mu_0 k}{\beta}\varphi_2(z)x_2 & = & \lambda x_1, \\ \\ \dfrac{1}{P_e} x_1^{'}(0) = x_1(0), \quad x_1^{'}(1) & = & 0. \end{array} \right.$$

One can deduce from the Krein-Rutman theory that x_1 never changes sign. Without loss of generality, we can assume that x_1 is positive and as (x_1, x_2) is an eigenvector then x_1 is strictly positive. Let us introduce the following variables

$$u(z) := e^{\frac{-P_e}{2}z} x_1(z) \text{ and } v(z) := u^{'}(z), \qquad \text{for all } z \in [0,\, 1].$$

Then (considering also (V.7))

$$\left\{ \begin{array}{rcl} u^{'} & = & v, \\ v^{'} & = & P_e \left(\dfrac{P_e}{4} + \lambda - \dfrac{\mu_0 k}{\beta} \dfrac{\lambda + \gamma}{\lambda + \gamma - \frac{\mu_0}{\beta}\varphi_2(z)} \varphi_1(z) \right) u, \\ v(0) & = & \dfrac{P_e}{2} u(0), \quad v(1) = -\dfrac{P_e}{2} u(1). \end{array} \right. \tag{V.8}$$

First : Observe that if x_1^* is in the interval $[\frac{1}{2} - \frac{M}{2\alpha k_d},\, 1]$ then $\varphi_1(z)$ is negative for all $z \in [0,\, 1]$. Hence, reminding (V.6), a necessary condition for (V.8) to have a solution is $\lambda < 0$.

Let us consider now the opposite case and let us make the following assumptions.

A2 : $\dfrac{1}{4} - \dfrac{3M}{4\alpha k_d} \leq \sqrt{\dfrac{-M}{\alpha k_d}}$ and $\dfrac{1}{4} - \dfrac{3M}{4\alpha k_d} \leq x_1^*(z)$, for all $z \in [0,\, 1]$.

Observe that if λ is a very large positive number then $v^{'}(z)$ is always strictly positive and (V.8) cannot have a solution. Therefore, there exists λ_{max} such that $\lambda + \gamma \leq \lambda_{max} + \gamma$. Also, from (V.6), there exists a strictly positive number C_0 such that $\lambda + \gamma - \frac{\mu_0}{\beta}\varphi_2(z) \geq C_0$ for all $z \in [0,\, 1]$.

Second : Observe that if x_1^* is such that $\varphi_1(z)$ is positive for all $z \in [0,\, 1]$ i.e. $x_1^*(z) \in [\frac{1}{4} - \frac{3M}{4\alpha d},\, \frac{1}{2} - \frac{M}{2\alpha d}]$ for all $z \in [0,\, 1]$, then the function

$$z \longrightarrow -\frac{\mu_0 k}{\beta} \left(\frac{\lambda + \gamma}{\lambda + \gamma - \frac{\mu_0}{\beta}\varphi_2(z)} \right) \varphi_1(z)$$

is increasing in $[0,\, 1]$. Hence, a necessary condition for (V.8) to have a solution is

$$\frac{P_e}{4} + \lambda - \frac{\mu_0 k}{\beta} \left((\frac{\lambda + \gamma}{\lambda + \gamma - \frac{\mu_0}{\beta}\varphi_2(0)} \right) \varphi_1(0) < 0.$$

There exists a positive number $\bar{A}$ (depending on x_1^*) such that

$$\frac{P_e}{4} + \lambda - \frac{\mu_0 k}{\beta} \left((\frac{\lambda + \gamma}{\lambda + \gamma - \frac{\mu_0}{\beta}\varphi_2(0)} \right) \varphi_1(0) - \bar{A} < 0.$$

We make the following assumptions

A3 : $\bar{A} \geq \max\limits_{i=1,2} \left\{ a_0 \displaystyle\int_0^1 |\varphi_1^{'}(z)| dz + a_i \int_0^1 |\varphi_1(z)\varphi_2^{'}(z)| dz \right\},$

A4 : $a_0 \displaystyle\int_0^1 |\varphi_1(z)| dz < 1,$

where

$$a_0 = \frac{\mu_0 k}{\beta} \frac{\lambda_{max} + \gamma}{C_0}, \quad a_1 = \frac{\mu_0^2 k}{\beta^2} \frac{\lambda_{max} + \gamma}{C_0^2}, \quad a_2 = \frac{\mu_0^2 k}{\beta^2} \frac{1}{\lambda_{max} + \gamma}.$$

Under assumption **A3**, the solution (u, v) of (V.8) satisfies

$$\left|\frac{v(z)}{u(z)}\right| \leq \frac{P_e}{2},\ \forall\, z \in [0,\, 1].$$

Introducing $\theta := \dfrac{v}{u}$ leads to

$$|\theta(z)| \leq \frac{P_e}{2},\ \forall\, z \in [0,\, 1] \tag{V.9}$$

and (by using (V.7))

$$\left\{\begin{array}{rcl} \theta' & = & P_e\left(\dfrac{P_e}{4} + \lambda - \dfrac{\mu_0 k}{\beta}\dfrac{\lambda+\gamma}{\lambda+\gamma-\frac{\mu_0}{\beta}\varphi_2(z)}\varphi_1(z)\right) - \theta^2, \\ \theta(0) & = & \dfrac{P_e}{2}, \quad \theta(1) = -\dfrac{P_e}{2}, \end{array}\right. \tag{V.10}$$

Integrating this equation leads to

$$-P_e = \frac{P_e^2}{4} - \int_0^1 \theta^2(z)dz + P_e\left(\lambda - \frac{\mu_0 k}{\beta}\int_0^1 \varphi_1(z)\left(\frac{\lambda+\gamma}{\lambda+\gamma-\frac{\mu_0}{\beta}\varphi_2(z)}\right)dz\right).$$

By using (V.9), one has

$$P_e\lambda \leq P_e\left(-1 + \frac{\mu_0 k}{\beta}\int_0^1 \varphi_1(z)\left(\frac{\lambda+\gamma}{\lambda+\gamma-\frac{\mu_0}{\beta}\varphi_2(z)}\right)dz\right).$$

Then, by assumptions **A3-A4**, it follows that $\lambda < 0$.

Third : Let us consider now the case when there exists $z_0 \in]0,\, 1[$ such that

$$x_1^*(z) \in \left[\frac{1}{4} - \frac{3M}{4\alpha k_d},\, \frac{1}{2} - \frac{M}{2\alpha k_d}\right] \text{ for all } z \in [0,\, z_0] \text{ and } x_1^*(z) \geq \frac{1}{2} - \frac{M}{2\alpha k_d},\ \text{for all } z \in [z_0,\, 1].$$

Two situations are possible. If $v^{'}(0) < 0$ then by using assumptions **A3-A4** and the arguments given in the previous case, one has $\lambda < 0$. On the opposite case, we have $v^{'}(z) \geq 0$ for all $z \in [0,\, z_0]$ and as in the first part (of the subsection) a necessary condition for (V.8) to have a solution is $\lambda < 0$ since

$$\frac{P_e}{4} - \frac{\mu_0 k}{\beta}\frac{\lambda+\gamma}{\lambda+\gamma-\frac{\mu_0}{\beta}\varphi_2(z)}\varphi_1(z) > 0,\ \text{for all } z \in [z_0,\, 1].$$

In view of the arguments above, the following statement holds :

Proposition 5.2: Assume that **A1-A4** hold. Then the nontrivial equilibrium profile (x_1^*, x_2^*) is locally asymptotically stable.

C. Instability of the nontrivial equilibrium profile $(\bar{x}_1, \bar{x}_2)$

Let us consider the functionals

$$Q_1(x_1, x_2) = \frac{\displaystyle\int_0^1 \left(-\frac{1}{P_e}|x_1^{'}|^2 + \frac{\mu_0 k}{\beta}\varphi_1(z)x_1^2 + \frac{\mu_0 k}{\beta}\varphi_2(z)x_1x_2\right)dz - \frac{1}{2}\left(x_1^2(0) + x_1^2(1)\right)}{\|x_1\|_2^2},$$

$$Q_2(x_1, x_2) = \frac{\displaystyle\int_0^1 \left(-\gamma|x_2|^2 + \frac{\mu_0}{\beta}\varphi_2(z)x_2^2 + \frac{\mu_0}{\beta}\varphi_1(z)x_1x_2\right)dz}{\|x_2\|_2^2},$$

where $\|x_i\|_2$ is the L_2-norm of x_i on $[0,\, 1]$. Observe that if (x_1, x_2) is an eigenvector of (V.3) then necessarily $Q_1(x_1, x_2) = Q_2(x_1, x_2)$ and is the corresponding eigenvalue. As in the idea of [23, Theorem 11.6], one can see that the principal eigenvalue λ_1 is an increasing function of P_e. Then $\lambda_1(P_e) \geq \lambda_1(0)$. Therefore, one has to consider the case $P_e = 0$. Let us first introduce

$$\varepsilon = P_e, \quad u_\varepsilon(z) := x_1(z), \quad v_\varepsilon(z) := \frac{1}{\varepsilon}x_1^{'}(z),\ w_\varepsilon(z) := x_2(z), \quad \text{for all } z \in [0,\, 1].$$

One can deduce from (V.3) that

$$\begin{cases} u_\varepsilon' &= \quad \varepsilon v_\varepsilon, \\ v_\varepsilon' &= \left(\varepsilon v_\varepsilon - \dfrac{\mu_0 k}{\beta}\varphi_1(z)u_\varepsilon - \dfrac{\mu_0 k}{\beta}\varphi_2(z)w_\varepsilon + \lambda u_\varepsilon\right), \\ v_\varepsilon(0) &= u_\varepsilon(0), \quad v_\varepsilon(1) = 0. \end{cases}$$

For $\varepsilon = 0$, the equilibria as well as the eigenfunctions are scalar and by integrating the equation above, one has :

$$-u_\varepsilon(0) = -\frac{\mu_0 k}{\beta}\varphi_1(0)u_\varepsilon(0) - \frac{\mu_0 k}{\beta}\varphi_2(0)w_\varepsilon(0) + \lambda u_\varepsilon(0).$$

Therefore,

$$\lambda(0)u_\varepsilon(0) = (\frac{\mu_0 k}{\beta}\varphi_1(0) - 1)u_\varepsilon(0) + \frac{\mu_0 k}{\beta}\varphi_2(0)x_2(0).$$

It is easy to see that $\bar{x}_1$ is a zero of the function

$$a \in \left[-\frac{M}{\alpha k_d}, 1\right] \to \frac{k\tilde{\mu}(a, \bar{x}_2)\bar{x}_2}{a} - 1$$

and that the derivative of this function at $\bar{x}_1$ is strictly positive. Hence, $\frac{\mu_0 k}{\beta}\varphi_1(0) - 1 > 0$. On the other hand, one can see from the expression of $\varphi_2(z)$ that it is always positive. It follows that the eigenvalue $\lambda_1(0)$ is strictly positive and this holds for all $P_e > 0$.
In view of the arguments above, the following statement holds

Proposition 5.3: The non trivial equilibrium profile $(\bar{x}_1, \bar{x}_2)$ is unstable.

VI. Concluding remarks

This paper was devoted to the dynamical analysis of a tubular biochemical reactor nonlinear model. The dynamics of these reactors are described by distributed parameter systems. More specifically, a system of nonlinear partial differential equations was studied. The existence of the state trajectories and the invariance of the set of physically feasible state values under the dynamics of the reactor as well as the convergence of the state trajectories to equilibrium profiles of the dynamical model were proved. The multiple existence and stability of equilibrium profiles were analyzed under physically meaningfull conditions. It was proved that, depending on the axial dispersion parameter, the dynamical model can have two stable and one unstable equilibrium profiles. In this study, it is shown that, for large values of axial dispersion parameter, the tubular biochemical behaves like a stirred tank reactor. And, as for completely mixed systems, the system studied here can have an unstable equilibrium which can coincide with an interesting operating point in practical viewpoint. So, as perspectives for further researches, we shall be interested in the control of this system which is nonlinear and infinite dimensional.

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