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We study a formal extension of the Dirac equation in the framework of a non-commutative two-sheeted space-time. It is shown that this approach naturally extends the classical Dirac theory by doubling the number of fermionic states, which can then be identified as matter and hidden-matter states. Our model exhibit several interesting features that could have observational consequences. Among them, we predict a small electromagnetic coupling between matter and hidden matter universes which should lead to matter/hidden matter oscillations in presence of intense electromagnetic vector potentials.

I. INTRODUCTION

The concept of hidden matter traces back to 1956 when Lee & Yang first noticed that the parity violation problem involved in the weak interactions could be solved by enlarging the particle content to include a mirror sector [1]. The underlying idea of a mirror sector is to duplicate the standard model and allow opposite symmetry breaking in the two sectors. Thus, for each left-handed particle there would be a right-handed mirror partner to restore parity.

This idea was further extended by a number of authors over the years and there is now a huge collection of papers devoted to this topic. Usually, it is assumed that usual matter and matter from the hidden world can not interact through ordinary interactions except gravitation. As a consequence, hidden matter made of hidden atoms could exist with exactly the same internal properties as ordinary matter but would be completely undetectable for us through electromagnetic means. In recent years however, Foot & Volkas have suggested to extend the original idea to allow a possible coupling of matter and mirror matter at the quantum level. This coupling involves some specific kind of interactions including for instance photon-mirror photon kinetic mixing [2] and also neutrino-mirror neutrino mass mixing [3]. These authors conclude that even if those interactions are tiny, the experimental consequences could be dramatic. Several possible astrophysical, cosmological and physical implications of mirror matter are extensively reviewed in [2].

Many other theoretical approaches use the phenomenological power of a hidden sector. One of the most important concerns superstring theories with $E8 \times E8$ symmetry group [5]. This approach assumes that particles are associated with the endpoints of open strings which are attached to D-branes. Ordinary and hidden-sector particles live then on different branes embedded in the bulk of a higher-dimensional space.

More recently, A. Connes proposed a two-sheeted space-time using non-commutative geometry (NCG) [6,7]. In his work, left and right-handed fermions are assumed to live on the two different sheets, which are coupled by a scalar field representing the Higgs field. Thus, in the low energy limit, this theory predicts the existence of two copies of space-time associated with a double Hilbert space. The cornerstone of this approach is to restrict the five dimensional space to a finite number of discrete points, generally two [8-12]. In several aspects, the two-sheeted space-time represents a discretized version of Kaluza-Klein theory in which, the fifth compact circular dimension is replaced by discrete points. This theory also presents some specific advantages like for instance a possible explanation of the huge difference between the electroweak and the Planck scales.

In the present paper, the NCG developed by A.Connes will be used to extend further the idea of a hidden sector embedded in a 5D bulk. Our study focuses on the dynamics of a massive particle in a two-sheeted space-time using relevant extensions of the Dirac and Pauli equations. The results of this model differ from previous works of literature essentially by the way of the particle mass is introduced into the model. It is shown that this approach leads to

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several interesting phenomena that could have strong observational consequences. The most noticeable ones concern two-sheeted oscillations of massive fermions in presence of an electromagnetic vector potential and a possible increase of the electric charge with the particle velocity.

The paper is organized as follows. In Sec. 2, we will first develop the minimal mathematical knowledge required to introduce the NC two-sheeted space-time. Then, we shall propose a formal extension of the Dirac equation and we will show that it allows for exact diagonalization in flat space-time. In Sec. 3, the non-relativistic limit of the NC Dirac equation in presence of a two-sheeted gauge field will be derived. It will be shown that it leads to a system of coupled Pauli equations relative to both sheets. Finally, we will determine the effect on the particle dynamics of 1) a constant magnetic vector potential solely or 2) coupled with a magnetic field. The last section closes by discussing some physical implications of the model.

II. Z2 NON-COMMUTATIVE DIRAC EQUATION IN FLAT SPACE-TIME – MATHEMATICAL FRAMEWORK

Let us introduce a Bi-Euclidean space X defined as the product of a 4 continuous manifold by a discrete two points space, i.e. $X = M \times Z_2$. Any smooth function in this spacetime belong to the algebra $A = C^\infty(M) \oplus C^\infty(M)$ and can be adequately represented by a 2x2 diagonal matrix F such that :

$$F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \tag{1}$$

The expression of the exterior derivative $D = d + Q$ where d acts on M and Q on the Z_2 internal variable has been given by A. Connes [6] : $D : (f_1, f_2) \rightarrow (df1, df2, \delta.(f_2 - f_1), \delta.(f_1 - f_2))$. Viet has proposed a representation of D acting as a derivative operator and fulfilling the above requirements [9] Following this author we set :

$$D_\mu = \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix}, \ \mu = 0, 1, 2, 3 \text{ and } D_5 = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} \tag{2}$$

Throughout this paper, we work in units with $\hbar = c = 1$.

The parameter δ has the dimension of mass in order to give the fifth component of space-time the same dimension as the other components. Its acts as a finite difference operator along the discrete dimension and corresponds formally to the distance between the two sheets. Using (2), one can build the Dirac operator defined as

$$\slashed{D} = \Gamma^N D_N = \Gamma^\mu D_\mu + \Gamma^5 D_5 \tag{3}$$

By considering the following extension of the gamma matrices (we are working in the Hilbert space of spinors, see [7])

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix} \text{ and } \Gamma^5 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix} \tag{4}$$

it can be easily shown that the Dirac operator given by eq.3 has the following self adjoint realization :

$$\slashed{D} = \begin{pmatrix} \slashed{D}_+ & \delta\gamma^5 \\ \delta\gamma^5 & \slashed{D}_- \end{pmatrix} = \begin{pmatrix} \gamma^\mu\partial_\mu & \delta\gamma^5 \\ \delta\gamma^5 & \gamma^\mu\partial_\mu \end{pmatrix} \tag{5}$$

Usually in Z2 non commutative geometry, one considers that the off diagonal term proportional to δ (which is a matrix in the most generalized case) is related to the particle mass through the Higgs field. In this work we make a different choice and consider that δ is constant and take the same value for every particles. To take into account different particles, it is thus necessary to introduce a mass term as in the classical approach of Dirac's equation, e.g:

$$M = m \begin{pmatrix} I_4 & 0 \\ 0 & I_4 \end{pmatrix} \tag{6}$$

whereas we keep the scalar field δ constant.

By analogy with the classical approach, one can then construct a 2-sheeted Dirac equation such that

$$D_{irac}\Psi = (i \slashed{D} - M) \Psi = 0 \tag{7}$$

with $\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ the two sheeted wave function. In this notation, the indices "+" and "−" are purely conventional and simply allow to discriminate the two sheets embedded in the 5D bulk.

It can be easily shown that the equation (7) can be derived from the Lagrangian **L** such that

$$\boldsymbol{L} = \bar{\Psi} \left(i \, \slashed{D} - M \right) \Psi \tag{8}$$

Where $\bar{\Psi} = \left(\bar{\psi}_+, \bar{\psi}_- \right)$ is the two sheeted spinor adjoint to Ψ and with $\bar{\psi}_+$ and $\bar{\psi}_-$ the spinors conjugated respectively to ψ_+ and ψ_-.

From (7), one gets

$$\left(i \, \slashed{D} - M \right) \Psi = \left(i\Gamma^N D_N - M \right) \Psi = \begin{pmatrix} i\gamma^\mu \partial_\mu - m & i\delta\gamma^5 \\ i\delta\gamma^5 & i\gamma^\mu \partial_\mu - m \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{9}$$

Thus the equation of motion for the two-component field $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ becomes

$$i\gamma^\mu \partial_\mu \psi_+ - m\psi_+ + i\delta\gamma^5 \psi_- = 0 \tag{10}$$

$$i\gamma^\mu \partial_\mu \psi_- - m\psi_- + i\delta\gamma^5 \psi_+ = 0 \tag{11}$$

The coupling between the two sheets arises from the presence of the δ term. It disappears completely for $\delta = 0$, which corresponds to infinitely, separated sheets. Note that the system derived here is similar to the one discussed in [13] to explain the flavor oscillation of neutrinos. We will return to this in the last section. It is worth noticing that if the mass m equals to zero, then the equations 10 and 11 turn out to be the standard equations of the Z2 non-commutative spacetime [6]. In that case, the indices $+$ and $-$ can be substituted by the indices L and R which refer to actual left and right parity states.

III. FREE FIELD SOLUTION OF THE Z2-DIRAC EQUATION

To solve the system (9), we introduce the auxiliary field Φ such that

$$\Psi = \left\{ i\Gamma^N D_N + M \right\} \Phi \tag{12}$$

It is then straightforward to show that Φ satisfies

$$\left\{ \Box + m^2 + \delta^2 \right\} \Phi = 0 \tag{13}$$

Let us make the following plane wave solution ansatz: $\Phi = \Phi_0 \exp\left[-i\varepsilon \left(Et - \vec{p} \cdot \vec{x} \right) \right]$ with $\varepsilon = \pm 1$ to take into account the positive and negative energy solutions.

Then eq.13 gives

$$\left[-E^2 + p^2 + m^2 + \delta^2 \right] \Phi = 0 \tag{14}$$

such that the energy eigenvalues are $E = \sqrt{p^2 + m^2 + \delta^2}$ with $\varepsilon = \pm 1$. Note that the distance between the two sheets appears here as a simple correction to the particle rest mass. So, even in the case where $m = 0$, the mass of the fermion can never be completely equals to zero. Perhaps, the smallness of neutrino masses could be explained that way.

For a momentum

$$\vec{p} = p \left(\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta \right) \tag{15}$$

we now define two component eigenstates of the matrix $\vec{\sigma} \cdot \vec{p}$ for later convenience

$$\chi_{1/2} \left(\vec{p} \right) = \begin{pmatrix} \cos\left(\theta/2\right) \\ \sin\left(\theta/2\right) \exp\left[i\phi\right] \end{pmatrix} \text{ and } \chi_{-1/2} \left(\vec{p} \right) = \begin{pmatrix} -\sin\left(\theta/2\right) \exp\left[-i\phi\right] \\ \cos\left(\theta/2\right) \end{pmatrix} \tag{16}$$

For the positive energy, we then look for solutions of the form

$$\Phi_\lambda = \begin{bmatrix} \chi_\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \Phi_\lambda = \begin{bmatrix} 0 \\ 0 \\ \chi_\lambda \\ 0 \end{bmatrix} \tag{17}$$

whereas for the negative energy, we consider

$$\Phi_\lambda = \begin{bmatrix} 0 \\ \chi_\lambda \\ 0 \\ 0 \end{bmatrix} \text{ or } \Phi_\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \chi_\lambda \end{bmatrix} \tag{18}$$

with $\lambda = \pm 1/2$.

Using the following representation of the Dirac matrices

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \ \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \ \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{19}$$

with $\left(\gamma^5\right)^2 = 1$ and $\{\gamma^5, \gamma^\mu\} = 0$, it can be shown that eq. 12 leads to

$$\Psi = \begin{bmatrix} E+m & -\vec{\sigma}\cdot\vec{p} & 0 & i\delta \\ \vec{\sigma}\cdot\vec{p} & -E+m & i\delta & 0 \\ 0 & i\delta & E+m & -\vec{\sigma}\cdot\vec{p} \\ i\delta & 0 & \vec{\sigma}\cdot\vec{p} & -E+m \end{bmatrix} \Phi \tag{20}$$

Using eq. 17 and 18, the solutions for Ψ can be easily derived. After normalization, one gets for the positive energy $(\varepsilon = +1)$

$$U_\lambda = \frac{1}{\sqrt{2E\left(E+m\right)}} \begin{bmatrix} (E+m)\chi_\lambda \\ \vec{\sigma}\cdot\vec{p}\chi_\lambda \\ 0 \\ i\delta\chi_\lambda \end{bmatrix}, \ \tilde{U}_\lambda = \frac{1}{\sqrt{2E\left(E+m\right)}} \begin{bmatrix} 0 \\ i\delta\chi_\lambda \\ (E+m)\chi_\lambda \\ \vec{\sigma}\cdot\vec{p}\chi_\lambda \end{bmatrix} \tag{21}$$

and for the negative energy $(\varepsilon = -1)$

$$V_\lambda = \frac{1}{\sqrt{2E\left(E+m\right)}} \begin{bmatrix} \vec{\sigma}\cdot\vec{p}\chi_\lambda \\ (E+m)\chi_\lambda \\ i\delta\chi_\lambda \\ 0 \end{bmatrix}, \ \tilde{V}_\lambda = \frac{1}{\sqrt{2E\left(E+m\right)}} \begin{bmatrix} i\delta\chi_\lambda \\ 0 \\ \vec{\sigma}\cdot\vec{p}\chi_\lambda \\ (E+m)\chi_\lambda \end{bmatrix} \tag{22}$$

It is instructive to compare those solutions with the usual ones given by the standard Dirac equation, i.e. for the positive energy

$$u_\lambda = \frac{1}{\sqrt{2E\left(E+m\right)}} \begin{bmatrix} (E+m)\chi_\lambda \\ \vec{\sigma}\cdot\vec{p}\chi_\lambda \end{bmatrix} \tag{23}$$

and for the negative energy

$$v_\lambda = \frac{1}{\sqrt{2E\left(E+m\right)}} \begin{bmatrix} \vec{\sigma}\cdot\vec{p}\chi_\lambda \\ (E+m)\chi_\lambda \end{bmatrix} \tag{24}$$

One can see that the two-sheeted solutions can be identified with the classical ones provided that $\delta \to 0$. So, for a very small δ, the difference between the standard and the two sheeted Dirac theory is not expected to be significant. In our approach the positive and negative energy solutions are assumed to correspond to particle and antiparticle respectively, as in the classical Dirac theory.

The form of solutions Ψ indicates that the fermion doubling is related to the two possible localization of the particles in 5D, i.e. in one or the other sheet. For illustrative purpose, let us consider the case of a positive energy particle. A particle mainly located in the first sheet can be written as a linear combination of U_λ solutions, i.e.

$$\Psi = \frac{1}{\sqrt{V}} \sum_\lambda N_\lambda U_\lambda \exp\left[-i\left(Et - \vec{p}\cdot\vec{x}\right)\right] \tag{25}$$

with N_λ such that $\sum_\lambda |N_\lambda|^2 = 1$.

Let also P_+ and P_- be the probability for the particle to be in the first (noted $+$) and second (noted $-$) sheet respectively. Then, considering the integrated value of $|\psi_+|^2$ and $|\psi_-|^2$ (given by the four first and four last components of Ψ), it can be shown that

$$P_+ = 1 - K \tag{26}$$

$$P_- = K \tag{27}$$

with $K = \delta^2 / [2E (E + m)]$. Provided that δ is small enough, i.e. $K << 1$, one verifies that the particle is mainly in the sheet "$+$". In the same way, a particle corresponding to a wave function Ψ written as a linear combination of $\bar{U}_\lambda$ solutions is mainly located in the sheet "$-$". Notice that since the "confinement" of the particle within the sheet increases with the energy of the particle (decreasing K), the apparent electric charge of a particle should follow the same behavior and this could be a possible way to experimentally check out the validity of the model. We note that if we consider, for instance, the case of an electron in the rest of frame and if we suppose a distance between both sheet on the order of 1 mm, the related value of δ leads to $K \sim 4 \cdot 10^{-20}$. Even if the distance decreases to one angstrom, then $K \sim 4 \cdot 10^{-6}$. The electric charge values $q = eK$ then obtained are close but out of the range of measurements obtained in the current experiments [14].

So, in the two-sheeted spacetime, positive energy fermions can be localized indifferently in one or the other sheet. A similar consideration holds of course for a negative energy particle.

IV. ROLE OF THE ELECTROMAGNETIC FIELD AND THE PAULI EQUATION IN THE TWO SHEETED SPACE-TIME

The most general form for a two-sheeted gauge field is given by

$$A = \begin{pmatrix} A_+ & \chi \\ \chi & A_- \end{pmatrix} \tag{28}$$

A gauge field on such a generalized space-time consists of the usual gauge field A_+, A_- in the two Euclidean manifolds supplemented by a scalar field χ, usually identified as the Higgs field. In this paper, we will limit ourselves to the more restrictive gauge with $\chi = 0$ to concentrate only on the effect of the photon fields.

By construction, A_+ is coupled with the four first components of the spinor Ψ (i.e. ψ_+) whereas A_- is coupled with the four last components (i.e. ψ_-). The minimal coupling of the gauge field with the Dirac fields yields to

$$(i (\slashed{D} + A) - M) \Psi = \begin{pmatrix} i\gamma^\mu (\partial_\mu + iA_{+,\mu}) - m & i\delta\gamma^5 \\ i\delta\gamma^5 & i\gamma^\mu (\partial_\mu + iA_{-,\mu}) - m \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{29}$$

It is instructive to note that the operator $\Gamma^5 D_5$ in $\slashed{D}$ does not commute with A since

$$[\Gamma^5 D_5, A] = \begin{pmatrix} 0 & 2i\delta\gamma^\mu\gamma^5 (A_{+\mu} - A_{-\mu}) \\ 2i\delta\gamma^\mu\gamma^5 (A_{-\mu} - A_{+\mu}) & 0 \end{pmatrix} \tag{30}$$

This point is noteworthy as the off-diagonal terms suggests the existence of electromagnetic coupling between matter and hidden matter sectors. Notice that in order to avoid a breaking of the electromagnetic gauge invariance, we are forced to consider that the same gauge transformation applies simultaneously to both sheets. As the value of δ is not dictated by any obvious physical considerations, the coupling strength might be strong enough to affect quantum phenomena. The last part of this paper addresses this important issue.

To clarify the effect of the coupling term, it is suitable to derive the non-relativistic limit of the Dirac equation. We start with

$$i\gamma^\mu (\partial_\mu + iA_{+,\mu}) \psi_+ - m\psi_+ + i\delta\gamma^5\psi_- = 0 \tag{31}$$

$$i\gamma^\mu (\partial_\mu + iA_{-,\mu}) \psi_- - m\psi_- + i\delta\gamma^5\psi_+ = 0 \tag{32}$$

Following the standard procedure to derive Pauli's equation from Dirac's one, one can easily show that :

$$i\frac{\partial \varphi_1}{\partial t} = \left[\frac{1}{2m}\left[\left(-i\vec{\nabla}-\vec{A}_+\right)^2+\delta^2\right]+A_{+,0}-\left(\frac{\vec{\sigma}\cdot\vec{B}_+}{2m}\right)\right]\varphi_1 + i\frac{\delta}{2m}\left[\vec{\sigma}\cdot\left(\vec{A}_+-\vec{A}_-\right)\right]\chi_1 \tag{33}$$

$$i\frac{\partial \chi_1}{\partial t} = \left[\frac{1}{2m}\left[\left(-i\vec{\nabla}-\vec{A}_-\right)^2+\delta^2\right]+A_{-,0}-\left(\frac{\vec{\sigma}\cdot\vec{B}_-}{2m}\right)\right]\chi_1 + i\frac{\delta}{2m}\left[\vec{\sigma}\cdot\left(\vec{A}_--\vec{A}_+\right)\right]\varphi_1 \tag{34}$$

with $\vec{B}_+$ and $\vec{B}_-$ the magnetic fields on the two sheets, and where we have used $\psi_+ = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $\psi_- = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$.
To derive 33 and 34, we have assumed that the mass term prevails on the kinetic and coulomb energies.

So, we get two Pauli equations coupled through the magnetic vector potentials of both sheets. We stress that the wave functions on the two sheets correspond to the "large components" of the Dirac equation as in the standard approach.

Clearly, those equations allow a non gravitational interaction between the two sectors. The nature of the coupling is however different from the one discussed in earlier papers as we will see hereafter [3]. Once again, we stress that for $A_+ = A_- = 0$, the coupling disappears and the two copies of space-time are completely non-interacting (in flat geometry). In that case, the field theory treatment reduces to the standard quantum mechanics with two independent Pauli equations.

V.   EFFECT OF A CONSTANT VECTOR POTENTIAL ON A MASSIVE FERMION

To investigate the behavior of a particle belonging to this "Z2 space-time", we now look at a special but physically instructive case of coupling between both sheets. We first rewrite the equations (33) and (34) in a much simpler form:

$$i\partial_0 \begin{pmatrix} \varphi_1 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} H_+ & W \\ -W & H_- \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \chi_1 \end{pmatrix} = H_{Z2}\begin{pmatrix} \varphi_1 \\ \chi_1 \end{pmatrix} \tag{35}$$

with H_{Z2} the two sheeted Hamiltonian with components

$$H_+ = \frac{1}{2m}\left[\left(-i\vec{\nabla}-\vec{A}_+\right)^2+\delta^2\right]+A_{+,0}-\left(\frac{\vec{\sigma}\cdot\vec{B}_+}{2m}\right) \tag{36}$$

$$H_- = \frac{1}{2m}\left[\left(-i\vec{\nabla}-\vec{A}_-\right)^2+\delta^2\right]+A_{-,0}-\left(\frac{\vec{\sigma}\cdot\vec{B}_-}{2m}\right) \tag{37}$$

$$W = i\frac{\delta}{2m}\left[\vec{\sigma}\cdot\left(\vec{A}_+-\vec{A}_-\right)\right] \tag{38}$$

To focus specifically on the coupling between both sheets, let us consider the more restrictive Hamiltonian

$$H_{Z2} = \begin{pmatrix} 0 & W \\ -W & 0 \end{pmatrix} \tag{39}$$

The calculation can be further simplified by taking zero scalar potentials on both folds, just keeping the vector potential in the + sheet, i.e.

$$A_{+,z} = A_0, \ A_{+,j\neq z} = 0 \text{ and } A_{-,z} = 0, \ A_{-,j\neq z} = 0 \tag{40}$$

In that case, the previous system of Pauli equations leads to

$$\partial_0 \varphi_1 = \frac{\delta}{2m}\sigma^z A_0 \chi_1 \tag{41}$$

$$\partial_0 \chi_1 = -\frac{\delta}{2m}\sigma^z A_0 \varphi_1 \tag{42}$$

Using $\varphi_1 = \begin{pmatrix} \varphi_1^+ \\ \varphi_1^- \end{pmatrix}$ and $\chi_1 = \begin{pmatrix} \chi_1^+ \\ \chi_1^- \end{pmatrix}$ with $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, one gets

$$\frac{\partial \varphi_1^+}{\partial t} = \frac{\delta}{2m}A_0\chi_1^+, \ \frac{\partial \varphi_1^-}{\partial t} = -\frac{\delta}{2m}A_0\chi_1^-, \ \frac{\partial \chi_1^+}{\partial t} = -\frac{\delta}{2m}A_0\varphi_1^+, \ \frac{\partial \chi_1^-}{\partial t} = +\frac{\delta}{2m}A_0\varphi_1^- \tag{43}$$

The general solution can be readily derived

$$\varphi_1^+(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t)$$

$$\varphi_1^-(t) = C_4 \cos(\omega t) + C_3 \sin(\omega t) \tag{44}$$

$$\chi_1^+(t) = C_1 \cos(\omega t) - C_2 \sin(\omega t)$$

$$\chi_1^-(t) = -C_3 \cos(\omega t) + C_4 \sin(\omega t)$$

where we have set $\omega = \frac{\delta A_0}{2m}$.

Assume that at $t = 0$, we have $\varphi_1^+(0) = 1$, $\varphi_1^-(0) = 0$, $\chi_1^+(0) = 0$, $\chi_1^-(0) = 0$ (the particle is located in the "+" sheet with a spin up) then we get

$$\varphi_1^+(t) = \cos(\omega t) \ \text{and} \ \chi_1^+(t) = -\sin(\omega t) \tag{45}$$

whereas the other components of the wave function vanish.

Hence, the probabilities of finding the particle in the "+" or "−" sheet are respectively

$$P_+ = \left|\varphi_1^+(t)\right|^2 = \cos^2(\omega t) \ \text{and} \ P_- = \left|\chi_1^+(t)\right|^2 = \sin^2(\omega t) \tag{46}$$

Therefore, we find that the particle oscillates between the two sectors with a time periodicity depending on the intensity of the magnetic vector potential. This result is the most striking one of this paper as it suggests a possible exchange of matter between the two sheets. It is however a very simplified model that does not take into account the perturbative effects exerted by the environment.

VI. COMBINATION OF A CONSTANT VECTOR POTENTIAL AND A MAGNETIC FIELD

We can get some insight onto the question of the environmental effects by considering the case of a static magnetic field $B_+ = B_0\vec{e}_z$ superimposed to the vector potential (40). In this approach, no magnetic field is postulated in the "−" sheet. Moreover, one considers that the vector potential due to the magnetic field B_+ is much smaller than A_+ such that its contribution to the overall vector potential can be completely neglected. Under those assumptions, it can be easily shown that the equations (33) and (34) lead to

$$i\partial_0\varphi_1 = -\frac{\sigma^z B_0}{2m} + \frac{i\delta}{2m}\sigma^z A_0 \chi_1 \tag{47}$$

$$i\partial_0\chi_1 = -\frac{i\delta}{2m}\sigma^z A_0 \varphi_1 \tag{48}$$

A procedure similar to the one developed in the preceding paragraph allows then to find the solution

$$\varphi_1^+(t) = \exp\left[i\frac{K}{2}t\right]\left\{\cos\left[\frac{1}{2}\sqrt{K^2 + 4\omega^2}t\right] + \frac{K}{\sqrt{K^2 + 4\omega^2}}\sin\left[\frac{1}{2}\sqrt{K^2 + 4\omega^2}t\right]\right\} \tag{49}$$

$$\chi_1^+(t) = -\exp\left[i\frac{K}{2}t\right]\frac{2\omega}{\sqrt{K^2+4\omega^2}}\sin\left[\frac{1}{2}\sqrt{K^2+4\omega^2}t\right] \tag{50}$$

with $K = B_0/2m$. The other components of the field vanish. To derive eq.49 and eq.50, we have used the initial conditions of the third paragraph at time $t = 0$.

Thus, in presence of a magnetic field, the probabilities P_+, P_- become

$$P_+ = \left|\varphi_1^+(t)\right|^2 = 1 - \frac{4\omega^2}{4\omega^2+K^2}\sin^2\left[\frac{1}{2}\sqrt{K^2+4\omega^2}t\right],\ \ P_- = 1 - P_+ \tag{51}$$

So, it appears that the period and the amplitude of the oscillations decrease with the magnetic field. As a consequence, the particle is confined to the sheet with strongly suppressed oscillations. One can easily convince oneself that a similar consideration holds when the magnetic field is substituted by a scalar potential.

VII. DISCUSSION

Several models in brane theories, predict that massive particles are able to leave the brane and propagate freely in the 5D bulk. However, it is usually assumed that only highly energetic particles can travel that way. Contrarily, in our approach, a low energy particle can move in the 5D bulk as well by doing oscillations between both spacetime sheets. Still should we explain how locality and energy conservation could be satisfied in such circumstances. Indeed, from the point of view of a "one-sheeted" observer, as we are, the behavior of such a particle would be in conflict with every known physical principles, the most noticeable ones being locality and energy conservation. However, when rescaled at a "two-sheeted" level, the problem disappears since the sum of the energies of both sheets remains then constant during the oscillations. So, for an hypothetical observer, able to see both sheets simultaneously, the particle never disappears from the 5D bulk and the apparent energy violation problem in 4D is only an artifact of low dimensionality.

In our model, the oscillation frequency depends on the particle mass. Therefore, one may wonder what would happen for an ensemble of particles embedded in a region of high vector potential. If all interactions can be neglected, our model predicts that each particle will undergo oscillations at a specific frequency depending on its mass. The lightest particles will oscillate first followed by the heavier particles. In the case of strongly interacting particles however, the situation is completely different. The presence of the other neighboring particles must be taken into account to describe adequately what happens. A complete treatment of this problem is of course far beyond the scope of this paper. Nevertheless, the results obtained by considering the effect of a magnetic field are a first step to mimic the environmental effects. Our results suggest that the oscillations are strongly suppressed in presence of an applied field similar to the one that could be generated by neighboring particles. In fact, each collision or exchange of particles results likely in a damping of the amplitude of the oscillations as in the familiar quantum Zeno effect. In such circumstances, the particle remains perfectly localized in its sheet with a frozen oscillatory behavior.

Despite the complications due to environmental effects, it seems possible to design an experimental set-up to evidence the oscillatory behavior. A first critical condition to be satisfied is to limit the perturbations due to environment. That implies to avoid the presence of any electromagnetic fields and to study isolated particles only. Very intense vector potentials must also be used to enhance the oscillation frequency between the two sheets. All these very specific conditions can be met, for example, in a hollow cylinder with an inner flow of electrical current, operating under a high vacuum. In this system, no magnetic nor electric field will be generated but nevertheless a constant vector potential will appear in the hollow part of the cylinder. A source emitting charged particles of low energy can then be placed at one side of the system; ideally this source should emit particles one by one to prevent their mutual interaction. At the opposite side of the cylinder, a detector should be placed to collect every particle emitted by the source. If the vector potential is fixed at a sufficiently high value, particles oscillations should occur and the detector should record a lack of events.

If our model has a physical reality, various reasons can be proposed to explain why "two-sheeted oscillations" have not been observed so far

1. The damping of oscillations by collisional processes is so huge that the fermions are completely trapped in their own space-time sheet,

2. The intensity of the usual electromagnetic vector potentials are much too low to lead to observable oscillations,

3. The value of δ is so small that even for very intense potentials, the oscillations remain very difficult to be detected. Notice that since the distance between the two sheets d varies like $1/\delta$, this would suggest that both space-time-sheets are separated by a cosmological distance in the discrete 5D bulk.

Obviously, the last hypothesis is the most stimulating one. Indeed, a cosmological model could shed a new light on several puzzling phenomena such as the missing mass matter, the acceleration of the cosmic expansion...and could also provide an assessment of the distance between the two sheets, necessary for determining the typical values of ω. However, the study of such a cosmological model is left for future prospects.

VIII.   CONCLUSION

In this paper, we have considered the quantum dynamics of massive particles in a two-sheeted space-time using the non-commutative geometry. The two-sheeted counterparts of the usual Dirac and Pauli equations have been derived and the free field solutions of the Z2-Dirac equation have been explicitly given. It is shown that our approach provides a possible description of matter and hidden sector matter, which are then localized on different sheets in the discrete 5D bulk. The model predicts that each isolated massive particle oscillates between the two space-time sheets in presence of a constant magnetic vector potential.

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