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Scaling law in the standard map critical function. Interpolating Hamiltonian and frequency map analysis

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Abstract. We study the behaviour of the standard map critical function in a neighbourhood of a fixed resonance, that is the scaling law at the fixed resonance. We prove that for the fundamental resonance the scaling law is linear. We show numerical evidence that for the other resonances p/q, $q \geqslant 2$, $p \neq 0$ and p and q relatively prime, the scaling law follows a power-law with exponent $1/q$.

AMS classification scheme numbers: 37C55, 37E40, 70K43, 34C28

1. Introduction

The standard map [15, 21] is the area-preserving twist map of the cylinder: $\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}$ given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T_\epsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + \epsilon \sin x \\ y + \epsilon \sin x \end{pmatrix} \tag{1.1}$$

where ϵ is a real parameter. Note that for all values of $\epsilon \in \mathbb{R}$ one has $S \circ T_\epsilon \circ S = T_{-\epsilon}$ where S is the involution of the cylinder $S\binom{x}{y} = \binom{x+\pi}{y}$. This allows us to restrict the study of T_ϵ to ϵ non-negative.

The dynamics of the standard map depends critically on the parameter ϵ. For $\epsilon = 0$ the curves $y_0 =$ constant are preserved by the map and the motion on these curves is a rotation of frequency $\omega = y_0$. For $\epsilon > 0$ not all these curves persist. Given an initial condition (x_0, y_0) we define the *rotation number* ω_T by

$$2\pi\omega_T\,(x_0, y_0, \epsilon) = \lim_{n\to\infty} \frac{\pi_1\left(T_\epsilon^n\,(x_0, y_0)\right) - x_0}{n} \tag{1.2}$$

when the limit exists†, where π_1 is the projection on the first coordinate: $\pi_1(x, y) = x$. For $\epsilon > 0$ by the Poincaré–Birkhoff theorem (see [10, 11, 22, 42] and [37] for a different proof) homotopically non-trivial invariant curves with rational rotation number do not exist: the invariant curves of rotation number $\omega_T(x, y, 0) = p/q$, for integer p, q, $q > 0$ relatively

† To be precise the previous formula holds replacing T_ϵ with a lift to the universal cover of the cylinder, in such a way that $x_n = \pi_1\left(T_\epsilon^n(x_0, y_0)\right)$ belongs to $\mathbb{R}$ for all n.

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prime, split into an even multiple of q points when $\epsilon > 0$. This set of points is divided into two subsets of points: the *elliptic* points and the *hyperbolic* ones.

One can define

$$\epsilon_{crit}(\omega) = \sup\{\epsilon > 0 : \forall \epsilon' \in [0, \epsilon], \text{ there exists a } \mathcal{C}^1, \text{ homotopically non-trivial, invariant circle of } T_{\epsilon'} \text{ whose rotation number is } \omega\}$$

and $\epsilon_{crit}(\omega) = 0$ if $\omega \in \mathbb{Q}$. The function $\omega \mapsto \epsilon_{crit}(\omega)$ is called the *critical function* for the standard map. The definition of $\epsilon_{crit}(\omega)$ actually depends on the smoothness required for the invariant curve. However, whatever is the regularity of the curve (say $\mathcal{C}^1$, $\mathcal{C}^k$, $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$) this function is very irregular: it is almost everywhere discontinuous and zero on a dense G_δ-set. Moreover, Mather proved [36] that $\sup_{\omega\in\mathbb{R}} \epsilon_{crit}(\omega) \leqslant \frac{4}{3}$ and it is conjectured that $\sup_{\omega\in\mathbb{R}} \epsilon_{crit}(\omega) = \epsilon_{crit}\left(\frac{\sqrt{5}-1}{2}\right) = 0.971\,635\ldots$ [21, 27].

For suitable irrational ω, KAM theory [2, 23, 38] ensures that for sufficiently small $|\epsilon|$ there exists an analytic invariant circle upon which the dynamics of T_ϵ, restricted on this curve, is analytically conjugated to the rotation $R_\omega(\theta) = \theta + 2\pi\omega$. This means that there exists an analytic (with respect to θ on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $|\epsilon| < \epsilon_0$, for some $\epsilon_0 > 0$) function $u(\theta, \omega, \epsilon)$ such that

$$x = \theta + u(\theta, \omega, \epsilon) \qquad 1 + \frac{\partial u}{\partial \theta}(\theta, \omega, \epsilon) > 0 \qquad \forall \theta \in \mathbb{T}$$

and $\theta' = \theta + 2\pi\omega$, namely u reparametrizes the invariant curve. Using perturbation theory we can expand this function into a power series with respect to ϵ and Fourier series with respect to θ

$$u(\theta, \omega, \epsilon) = \sum_{n\geqslant 1} \epsilon^n u_n(\theta, \omega) = \sum_{n\geqslant 1} \epsilon^n \sum_{k\in\mathbb{Z}\setminus\{0\}, |k|\leqslant n} \hat{u}_n^{(k)}(\omega)\, \mathrm{e}^{ik\theta} \tag{1.3}$$

which gives the so-called *Lindstedt series* [41]. One may define the *critical radius of convergence* of such series as

$$\rho_{crit}(\omega) = \left[\sup_{\theta\in\mathbb{T}} \limsup_{n\to\infty} |u_n(\theta, \omega)|^{1/n}\right]^{-1}. \tag{1.4}$$

It is clear that $\rho_{crit}(\omega) \leqslant \epsilon_{crit}(\omega)$, but it is an interesting open question to decide whether the two functions are equal or not on some set of frequencies. See [5, 6] for a positive numerical answer and [17] for a negative one. This will imply that the function defined for $\epsilon \in \mathbb{D}_{\rho_{crit}(\omega)}$ as the sum of the series (1.3) can be analytically continued outside its disc of convergence.

In [34] Marmi and Stark studied the relation between the $\epsilon_{crit}(\omega)$ and a 'universal function' depending only on the frequency: the Brjuno function† $B(\omega)$ (see [12, 35, 45]). Using the Greene residue criterion [21], Marmi and Stark find *numerically* evidence that it exists for $\beta > 0$ and a positive constant C, such that

$$\left|\ln \epsilon_{crit}(\omega) + \beta B(\omega)\right| \leqslant C \tag{1.5}$$

uniformly in ω, as ω ranges over some set of quadratic irrationals, with β approximately equal to 0.9. Comparing their results with similar problems in [33, 45] they expected the value $\beta = 2$.

In [9], and more recently in [7], the behaviour of $\rho_{crit}(\omega)$ when ω goes to a rational value has been studied; in both cases *complex frequencies* were used. It is well known that if the

† Let $A : (0, 1) \to (0, 1)$ be the Gauss map $A(\theta) = \theta^{-1} + \lfloor\theta^{-1}\rfloor$. To every $\theta \in \mathbb{R} \setminus \mathbb{Q}$ we associate two sequences, $(\beta_k)_{k\geqslant -1}$ and $(\theta_k)_{k\geqslant 0}$, defined by $\beta_{-1} = 1$, $\theta_0 = \theta - \lfloor\theta\rfloor$ and $\beta_k = \prod_{i=0}^k \theta_i$, $\theta_{k+1} = A(\theta_k)$ for all $k \geqslant 0$. Then the Brjuno function is defined by $B(\theta) = \sum_{k\geqslant 0} \beta_{k-1} \ln \theta_k^{-1}$ if the series converges, otherwise we set $B(\theta) = +\infty$.

imaginary part of ω is not zero the series (1.3) converges because there are no small divisors. In these papers the limit $\omega \to \frac{p}{q}$ of $\rho_{crit}(\omega)$ when $\omega = \frac{p}{q} + \mathrm{i}\eta$ for some real $\eta \to 0$ was studied. The result is the following scaling law:

$$\rho_{crit}(\omega) \sim C_{p/q} \left| \omega - \frac{p}{q} \right|^{2/q} \tag{1.6}$$

for some positive constant $C_{p/q}$ depending on the resonance.

Finally, very recently Davie [16] and Berretti and Gentile [8] proved that

$$\left| \ln \rho_{crit}(\omega) + 2 B(\omega) \right| \leqslant C' \tag{1.7}$$

for some positive constant C', uniformly with respect to $\omega \in \mathbb{R}$. This implies that the series given in (1.3) converges if and only if ω is a Brjuno number (i.e. $B(\omega) < \infty$).

The scaling law is related to this result which makes a link between the rotation number and the Brjuno function. One can prove [14] that

$$\liminf_{\substack{\omega \to p/q \\ \omega \text{ is a Brjuno number}}} \left(B(\omega) + \frac{1}{q} \ln \left| \omega - \frac{p}{q} \right| \right) = c'_{p/q} \tag{1.8}$$

with $c'_{p/q}$ some non-negative finite constant depending on the resonance involved†, then the result of Berretti and Gentile [8] implies the value $\beta = 2$.

To end this short review about standard map critical functions and their scaling laws we cite the very recent work of Treschev and Zubelevich [44], where they studied (quite general) area-preserving twist maps of the standard cylinder into itself, ϵ-close to an integrable ones. They proved that under some assumptions: given any rational number $p/q \in \mathbb{Q}$, there exist two positive constants c and ϵ_0 such that for all $|\epsilon| < \epsilon_0$, there exist two invariant homotopically non-trivial $\mathcal{C}^1$ curves with frequencies ω_l and ω_b and

$$\omega_l < \frac{p}{q} < \omega_b \qquad \text{and} \qquad |\omega_b - \omega_l| < c\epsilon.$$

The theorem they prove (theorem 3, p 76 [44]) is based on a strong hypothesis (in particular assumption 2, which depends on the involved resonance) which is not verified by the standard map, except for the resonance 0/1.

We consider the fundamental resonance 0/1, we fix $\epsilon > 0$ and we define

$$\begin{aligned}\omega^+(\epsilon) = \inf\{\omega > 0 : \; & T_\epsilon \text{ has a } \mathcal{C}^1\text{, homotopically non-trivial,} \\ & \text{invariant curve of rotation number } \omega\}\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\omega^-(\epsilon) = \sup\{\omega < 0 : \; & T_\epsilon \text{ has a } \mathcal{C}^1\text{, homotopically non-trivial,} \\ & \text{invariant curve of rotation number } \omega\}\end{aligned} \tag{1.10}$$

note that $\omega^+(\epsilon) = -\omega^-(\epsilon)$. These definitions are similar to the one of $\mu(\omega)$ given by MacKay [32] in the study of the Hamiltonian of Escande. The existence of a stochastic layer around the hyperbolic fixed point $x = y = 0$, implies $\omega^+(\epsilon) \neq 0$, namely there is a 'gap' in the curve

† We point out that the previous result is false if we replace lim inf with lim. In fact, considering the fundamental resonance, for simplicity, and taking the sequence of Brjuno's numbers $(\omega_n)_{n \geqslant 0}$, $\omega_n^{-1} = a_1(n) + \frac{1}{a_2(n)+\xi}$, with $a_i : \mathbb{N} \to \mathbb{N}^*$ strictly monotonically increasing functions for $i = 1, 2$ and ξ any Brjuno number. Then we have $\lim_{n\to+\infty}(B(\omega_n) + \ln \omega_n) = 0$ if $\ln a_2(n) = \mathrm{o}(a_1(n))$ for $n \to +\infty$, whereas if $a_1(n) = \mathrm{o}(\ln a_2(n))$ for $n \to +\infty$, then $\lim_{n\to+\infty}(B(\omega_n) + \ln \omega_n) = +\infty$.

which associates to each invariant circle its rotation number (frequency curve). The main result of the first part of this paper is the following theorem†.

Theorem 1.1. *There exist two positive constant $\Omega_{0/1}$ and $c_{0/1}$ such that for all $|\omega| < \Omega_{0/1}$ of Brjuno's type, then*

$$\limsup_{\substack{\omega\to 0\\ \omega \text{ is a Brjuno number}}} \left| \ln \epsilon_{crit}(\omega) - \ln|\omega| \right| \leqslant c_{0/1}.$$

Following [44], to prove our theorem we need an upper bound for $\ln \epsilon_{crit}(\omega) - \ln|\omega|$. We obtain this bound combining ideas taken from two different domains: the exponential smallness of the splitting of the separatrices and the interpolation of diffeomorphisms by vector fields.

The splitting of the separatrices for the standard map has been widely studied. Using the results of Lazutkin and Gelfreich [19, 29–31] we obtain a lower bound to the action corresponding to the last invariant torus. More precisely for a fixed ϵ we define $y^+(\epsilon)$, respectively, $y^-(\epsilon)$, as the intersection point of the homotopically non-trivial invariant $\mathcal{C}^1$ curve of frequency $\omega^+(\epsilon)$, respectively, $\omega^-(\epsilon)$, with the y-axis, note that $y^+(\epsilon) = -y^-(\epsilon)$. Then we prove (proposition 2.2) that

$$y^+(\epsilon) \geqslant c\mathrm{e}^{-\pi^2/2\sqrt{\epsilon}}$$

for some positive constant c and ϵ small enough.

At this point we have to estimate the standard map rotation number of a rotation orbit with initial data (x_0, y_0) where $x_0 = 0$ and $y_0 = \mathcal{O}\big(\mathrm{e}^{-\pi^2/2\sqrt{\epsilon}}\big)$. To do this we use the following result due to Benettin and Giorgilli [4], which allows us to construct a Hamiltonian system whose time-1 flow interpolates the standard map and so to exhibit an asymptotic development for the standard map rotation number (proposition 2.13) which, joint with the bound on $y^+(\epsilon)$, leads to the desired upper bound. We note that to prove our result it will be sufficient to find an interpolating Hamiltonian system $\epsilon^{3/2}$-close to the standard map; we nevertheless give the asymptotic development because to the best of our knowledge this result was not know and, secondly, the construction of an interpolating vector field exponentially close leads to the same amount of difficulty as the construction of an ϵ^k-close vector field, for any power k.

In the second part (section 3) we present the *frequency map analysis* (FMA) method of Laskar [18, 24, 25, 27] and its application to the numerical investigation of the standard map critical function.

For the fundamental resonance 0/1, we find numerical results in agreement with the analytical ones presented in section 2. For the other resonances we find the behaviour

$$\epsilon_{crit}(\omega) \sim c_{p/q}\left|\omega - \frac{p}{q}\right|^{1/q} \tag{1.11}$$

for a Brjuno ω in a neighbourhood of p/q and $c_{p/q} > 0$. Equation (1.11) deserves some reflections. As already stated the critical function is very irregular and how $\epsilon_{crit}(\omega)$ goes to zero when ω goes to a rational value, depends on the arithmetical properties of ω. So roughly speaking with equation (1.11) we mean that the function $\ln \epsilon_{crit}(\omega)$ has a 'main' singularity of the type $\frac{1}{q}\ln|\omega - \frac{p}{q}|$, as $\omega \to \frac{p}{q}$. In mathematical language we can say that‡

† We point out that already in [32] for a one-degree-of-freedom Hamiltonian system depending periodically on time, logarithmic singularities were found for the break-up of invariant tori with frequencies closer and closer to rational ones.

‡ See footnote on page 2035 for a rigorous result concerning the use of the lim sup and the Brjuno function.

that the lim sup $\epsilon_{crit}(\omega)$ behaves as $\frac{1}{q}\ln|\omega-\frac{p}{q}|$ when $\omega\to\frac{p}{q}$ through a sequence of Brjuno number.

Using the FMA tool we deal with *real frequencies* and we can perform investigations directly on the critical function and go deeply into the resonant region. Actually, we deal with frequencies that are only 6×10^{-4} far from the resonant value (in [34] the use of the Greene residue criterion gives frequencies that are 10^{-2} close to the resonant value). The result we show in this paper for the critical function is in agreement with that of [34].

Supported by considerations on both methods (analytical and numerical) we make the following conjecture:

Conjecture. For all rational p/q, we can find positive constant $c_{p/q}$, such that when a Brjuno ω is in a small *real neighbourhood* of p/q, then

$$\epsilon_{crit}(\omega)\sim c_{p/q}\left|\omega-\frac{p}{q}\right|^{1/q}. \tag{1.12}$$

The same result as given after equation (1.11) applies.

Comparing this conjecture with [8, 9] we see that the domain of analyticity with respect to $\epsilon\in\mathbb{C}$ of the function defined by the series (1.3) is quite different from a circle as ω tends to a resonance [5, 6]. It is an open question to completely understand its geometry: it could be an ellipse whose major semi-axis is in the direction of real ϵ with length proportional to the square root of the minor semi-axis, or probably a more complicated curve with the nearest singularity to the origin with positive imaginary part.

2. Scaling law for the fundamental resonance

The goal of this section is to prove our main result: theorem 1.1. We note that from the definitions of $\epsilon_{crit}(\omega)$ and $\omega^{\pm}(\epsilon)$ we have $\epsilon_{crit}\left(\omega^{\pm}(\epsilon)\right)=\epsilon$, then using the fact that the Brjuno condition is necessary and sufficient to have an invariant circle for the standard map, theorem 1.1 is equivalent to

Theorem 2.1. *Let $\omega^{\pm}(\epsilon)$ be the functions of ϵ defined in (1.9) and (1.10). Then there exist $\epsilon_0'>0$ and $c_0'>0$ such that for all $0<\epsilon<\epsilon_0'$ we have*

$$\left|\ln|\omega^{\pm}(\epsilon)|-\ln\epsilon\right|\leqslant c_0'.$$

Using [44, theorem 3, p 76] we can prove that there exist $c>0$ and $\epsilon_0>0$ such that for all $0<\epsilon<\epsilon_0$ there exist two invariant homotopically non-trivial $\mathcal{C}^1$ curves with frequencies ω_l and ω_b and

$$\omega_l<0<\omega_b \qquad \text{and} \qquad |\omega_b-\omega_l|<c\epsilon.$$

Recalling (1.9) and (1.10) we obtain the lower bound

$$\ln\epsilon-\ln|\omega^{\pm}(\epsilon)|>\ln 2-\ln c$$

for ϵ small enough. Thus the proof of theorem 2.1 reduces to prove the following bound:

$$\ln\epsilon-\ln|\omega^{\pm}(\epsilon)|<c'' \tag{2.1}$$

for some $c''>0$ and ϵ small enough. This will be done in proposition 2.4. Now we present the two results we need to prove proposition 2.4.

Let us fix $\epsilon>0$ and recall the definitions of $y^{\pm}(\epsilon)$: $y^{+}(\epsilon)$, respectively, $y^{-}(\epsilon)$, are the intersection point of the homotopically non-trivial invariant $\mathcal{C}^1$ curve of frequency $\omega^{+}(\epsilon)$, respectively, $\omega^{-}(\epsilon)$, with the y-axis. Then from the exponentially smallness of the splitting of the separatrices we deduce

Proposition 2.2. *There exist $c > 0$ and $\epsilon_0'' > 0$ such that for all $0 < \epsilon < \epsilon_0''$ we have*

$$|y^{\pm}(\epsilon)| \geqslant c\mathrm{e}^{-\pi^2/2\sqrt{\epsilon}}. \tag{2.2}$$

We leave the proof of this proposition to subsection 2.1 and we introduce the second element we need to prove our result. For $\epsilon > 0$, we rewrite the standard map as a perturbation of the identity map, namely

$$S_\mu \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix} + \mu \begin{pmatrix} z \\ \sin x \end{pmatrix} + \mu^2 \begin{pmatrix} \sin x \\ 0 \end{pmatrix} \tag{2.3}$$

we can pass from (2.3) to (1.1) putting $\mu = \sqrt{\epsilon}$ and $\mu z = y$. And we define ω_S to be its rotation number, namely

$$\omega_S(x, z, \mu) = \lim_{n\to\infty} \frac{\pi_1\left(S_\mu^n(x, z)\right) - x}{n} \tag{2.4}$$

when this limit exists†. Note that this limit exists if and only if the limit for ω_T on the corresponding orbit, exists, in fact they are related by

$$\omega_T\,(x, y, \epsilon) = \sqrt{\epsilon}\omega_S\left(x, \frac{y}{\sqrt{\epsilon}}, \sqrt{\epsilon}\right). \tag{2.5}$$

Following [4] we construct (propositions 2.8 and 2.9) an *integrable Hamiltonian system* whose time-1 flow differs from the standard map in the formulation (2.3) for an exponentially small term: $\mathcal{O}\left(\mathrm{e}^{-\frac{1}{\sqrt{\epsilon}}}\right)$ for $\epsilon \to 0^+$. This allows us to prove (proposition 2.13) the existence of an asymptotic development of the standard map rotation number in terms of $\sqrt{\epsilon}$, whose first-order term is the pendulum frequency for rotation orbits. In fact, the best interpolating Hamiltonian system is an integrable perturbation of order $\mathcal{O}\left(\sqrt{\epsilon}\right)$ of the pendulum $H_0(x, z) = \frac{z^2}{2} + \cos x$. This implies the following lemma:

Lemma 2.3. *Let h_0 be a real number strictly greater than 1. Let ω_T be the standard map rotation number and let ω_{pend} be the frequency of a rotation orbit of energy $h_0 > 1$ for the pendulum $H_p(x, z) = \frac{z^2}{2} + \cos x$. For $\epsilon > 0$ we consider the homotopically non-trivial invariant circle of the standard map of initial conditions (x_0, y_0), $x_0 = 0$, and the corresponding rotation orbit of the pendulum with initial data (x_0, z_0), where $x_0 = 0$, $z_0 = \frac{y_0}{\sqrt{\epsilon}}$ and $h_0 = \frac{z_0^2}{2} + 1$. Then for ϵ sufficiently small we have*

$$\left|\omega_T(0, y_0, \epsilon) - \sqrt{\epsilon}\omega_{pend}(h_0)\right| = \mathcal{O}\left(|\epsilon|^{3/2}\right).$$

We can now prove the bound (2.1).

Proposition 2.4 (upper bound). *Let $\omega^{\pm}(\epsilon)$ be the functions of ϵ defined in (1.9) and (1.10). Then there exist $\bar{\epsilon} > 0$ and $c'' > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$ we have*

$$\ln \epsilon - \ln |\omega^{\pm}(\epsilon)| \leqslant c''.$$

Proof. Lemma 2.3 (or its refinement given by proposition 2.13) says that the standard map rotation number is 'well approximated' by the pendulum frequency. Consider a rotation orbit of the pendulum starting at $(0, z_0)$. Its frequency is given by

$$\omega_{pend}(h_0) = \frac{\pi}{k} \frac{1}{\mathcal{K}(k)}$$

† See the first footnote in this paper for a comment about this definition.

where $k^2 = \left(1 + \frac{z_0^2}{4}\right)^{-1}$, $h_0 = 1 + \frac{z_0^2}{2}$ and

$$\mathcal{K}(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

is the complete elliptic integral of first kind. For $k = 1$ (that is $z_0 = 0$) the integral diverges logarithmically and (see [1], formula 17.3.26, p 591)

$$\lim_{z_0 \to 0} \left[\mathcal{K} - \frac{1}{2} \ln \frac{4 + z_0^2}{z_0^2}\right] = 2 \ln 2.$$

Therefore, in the limit $z_0 \to 0$, ω_{pend} behaves as

$$\left|\omega_{pend} - \frac{\pi}{\ln 2/|z_0|}\right| = \mathcal{O}\left(\frac{z_0^2}{\ln |z_0|}\right). \tag{2.6}$$

Assume now that $\omega_T(0, y_0, \epsilon)$ exists for ϵ small enough, where $y_0 = z_0\sqrt{\epsilon}$. By (2.5), lemma 2.3 and (2.6) one has

$$|\omega_T(0, y_0, \epsilon)| \geqslant \sqrt{\epsilon} \frac{\pi}{\ln \frac{8\sqrt{\epsilon}}{|y_0|}} + \mathcal{O}\big(|\epsilon|^{3/2}\big). \tag{2.7}$$

If y_0 is the intersection of the last invariant torus with the axis $x = 0$ (so for this circle the rotation number ω_T exists), namely $y_0 = y^+(\epsilon)$ and $\omega_T(0, y_0, \epsilon) = \omega^+(\epsilon)$, then the bound (2.2) gives

$$|\omega^+(\epsilon)| \geqslant \sqrt{\epsilon} \frac{\pi}{C + \frac{1}{2} \ln \epsilon + \frac{\pi^2}{2\sqrt{\epsilon}}} + \mathcal{O}\big(|\epsilon|^{3/2}\big)$$

for some positive constant C. Then for ϵ small enough we conclude that there exists another positive constant c such that

$$|\omega^+(\epsilon)| \geqslant c\epsilon.$$

Recalling that $\omega^-(\epsilon) = -\omega^+(\epsilon)$, the proposition is proved. □

2.1. Splitting of the separatrices

The aim of this subsection is to introduce some results concerning the splitting of separatrices for the standard map necessary to prove proposition 2.2. The splitting of the separatrices has been studied by many authors, here we use the notation and the presentation given in [19].

For $\epsilon = 0$ the circle $y = 0$ is formed by fixed points of the standard map, for $\epsilon > 0$ only two fixed points survive: $(0, 0)$ and $(\pi, 0)$. The first one is hyperbolic (for these values of ϵ) and the other one elliptic. The linearized map at the origin is

$$\begin{pmatrix} 1 + \epsilon & 1 \\ \epsilon & 1 \end{pmatrix}$$

and its eigenvalues are λ and λ^{-1} where

$$\lambda = 1 + \frac{\epsilon}{2} + \sqrt{\epsilon}\sqrt{1 + \frac{\epsilon}{4}}.$$

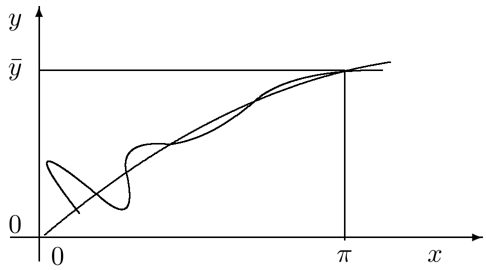

Figure 1. The stable and unstable manifolds for the standard map with some of their infinitely many intersection points.

The stable, $\mathcal{W}^s$, and unstable, $\mathcal{W}^u$, manifolds of this fixed point are analytic curves passing through $(0, 0)$. We introduce the parameter $\delta = \ln \lambda$ and we note that $\epsilon \sim \delta^2$ for small ϵ. We parametrize the part of the unstable separatrix, $\mathcal{W}_1^u$, which grows upwards from the origin, with $\left(x^-(t), y^-(t)\right)$, imposing the boundary conditions†

$$\lim_{t\to-\infty} x^-(t) = 0 \qquad x^-(0) = \pi.$$

We assume that $t = 0$ is the first intersection of $\mathcal{W}_1^u$ with the vertical line $x = \pi$. The parametrization of $\mathcal{W}_1^s$ is given by

$$\left(x^+(t), y^+(t)\right) = \left(2\pi - x^-(-t), y^-(-t) + \epsilon \sin x^-(-t)\right).$$

One can show that

$$\lim_{t\to+\infty} x^+(t) = 0 \qquad x^+(0) = \pi$$

i.e. $t = 0$ corresponds to a homoclinic point. Lazutkin [30] proposed to study the *homoclinic invariant* defined by

$$\Omega = \det \begin{pmatrix} \dot{x}^-(0) & \dot{x}^+(0) \\ \dot{y}^-(0) & \dot{y}^+(0) \end{pmatrix}.$$

The homoclinic invariant has the same value for all points of one homoclinic trajectory and it is invariant with respect to symplectic coordinate changes. Gelfreich in [19], proposition 2.1, gives an asymptotic expansion in power of δ of the parametrization $x^{\pm}(t)$. Its first terms are

$$\begin{aligned} &x_0(t) + x_1(t) + x_2(t) + \cdots \\ &\quad = 4 \arctan \mathrm{e}^t + \delta^2 \frac{1}{4} \frac{\sinh t}{(\cosh t)^2} + \delta^4 \left[-\frac{41}{1728} \frac{\sinh t}{(\cosh t)^2} + \frac{91}{864} \frac{\sinh t}{(\cosh t)^4} \right] + \cdots . \end{aligned} \tag{2.8}$$

Of course the zero-order term is nothing but the parametrization of the separatrix of the pendulum.

The main result of [19] is $\Omega \geqslant \frac{c}{\delta^2} \mathrm{e}^{-\frac{\pi^2}{\delta}}$ for some positive constant c and δ small enough. Using this result we obtain an estimation of $y^{\pm}(\epsilon)$ in proposition 2.2.

Proposition 2.2. *There exist $c > 0$ and $\epsilon_0'' > 0$ such that for all $\epsilon < \epsilon_0''$ we have*

$$|y^{\pm}(\epsilon)| \geqslant c\mathrm{e}^{-\pi^2/2\sqrt{\epsilon}}.$$

† $y(t)$ can be found using $y(t + \delta) = x(t + \delta) - x(t)$.

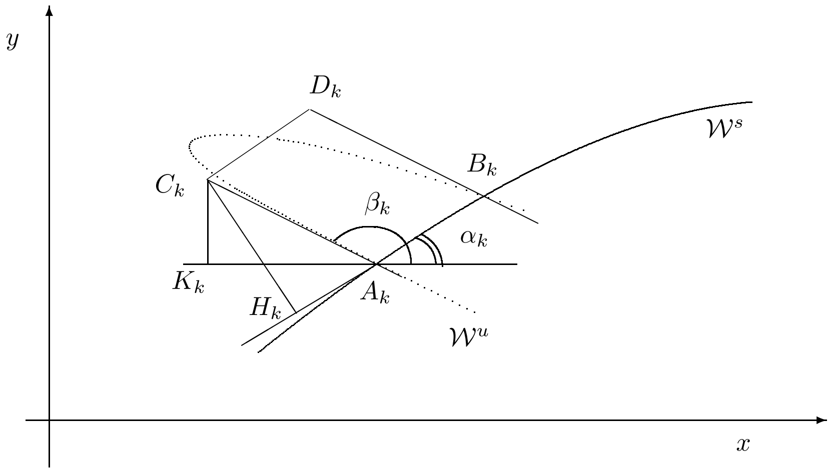

Figure 2. Two intersection points of the stable, $\mathcal{W}^s$, and unstable, $\mathcal{W}^u$ (dotted curve), manifolds, producing the kth lobe.

Remark 2.5. Compare with figure A6 where we plot $\ln|y^+(\epsilon_{crit}) - y^-(\epsilon_{crit})|$ against ϵ_{crit}, with data obtained numerically using the FMA.

Proof. For a fixed value of $\epsilon > 0$, the stable and unstable manifolds intersect firstly at $(\pi, \bar{y}(\epsilon))$ and then infinitely many times. We call a *lobe* the arc of a manifold between two successive intersections and the *lobe area* the area bounded by the lobe. Each lobe will be distinguished by an integer: the number of iterates (of T_ϵ) needed to put the point $(\pi, \bar{y})$ onto the right point base of the lobe.

The hyperbolic fixed point $(0, 0)$ stretches the unstable manifold along the dilating direction (corresponding to the eigenvalue λ), so there exists an integer $\hat{k}$ such that the $\hat{k}$th lobe intersects the y-axis at some point $y_{int}(\epsilon)$. Because the invariant circles of the standard map are at least Lipschitz graph, we cannot have homotopically non-trivial invariant circles, passing by $(0, \bar{y}')$ with $\bar{y}' < y_{int}(\epsilon)$, thus $y^+(\epsilon) \geqslant y_{int}(\epsilon)$. The lobe area is conserved under iteration of the standard map and by [19], corollary 1.3, this area is† for δ small enough

$$A_{lobe} \sim \frac{2\omega_0}{\pi} \mathrm{e}^{-\frac{\pi^2}{\delta}}$$

where $\omega_0 = 1118.827\,706\ldots$ and $\delta = \ln\lambda \sim \sqrt{\epsilon}$ for ϵ small.

In figure 2 we represent the geometry of the generic kth lobe: the points $A_k = (x_{-k}, y_{-k})$ and $B_k = \left(x_{-(k-1)}, y_{-(k-1)}\right)$ are the iterates of the standard map

$$\begin{cases} x_{-(k-1)} = x_{-k} + y_{-(k-1)} \\ y_{-(k-1)} = y_{-k} + \delta^2 \sin x_{-k} \end{cases}$$

and the segment of line $\overline{A_k C_k}$ is tangent to the unstable manifold at A_k.

† In the following we will often use the standard notation (see [20, no 4, ch II]) $f(t) \sim g(t)$ for $t \to 0$, this means that f and g are equivalent when t is close enough to 0. We can restate it by saying that $f(t) = g(t) + \mathrm{o}(g(t))$ when $t \to 0$.

The angle α_k is defined by

$$\tan\alpha_k = \frac{y_{-(k-1)} - y_{-k}}{x_{-(k-1)} - x_{-k}} = \frac{\delta^2 \sin x_{-k}}{\Delta x_{-k}}.$$

Let us consider the lobe which intersects for the first time the y-axis and let us call y_{int} the intersection point. We do not know exactly the analytical form of the arc of unstable manifold joining $A_{\hat{k}} = (x_{-\hat{k}}, y_{-\hat{k}})$ to $(0, y_{int})$, so we cannot calculate y_{int} exactly. Then we approximate this arc with its tangent at $A_{\hat{k}}$:

$$y = y_{-\hat{k}} + (\tan\beta_{\hat{k}})\,(x - x_{-\hat{k}}). \tag{2.9}$$

This straight line intersect the y-axis in some point $\tilde{y}$ and $|\tilde{y} - y_{int}| = \mathcal{O}(x^2_{-\hat{k}})$.

We approximate the lobe with a parallelogram of sides $A_k B_k$ and $A_k C_k$ where:

(a) the straight line through A_k and C_k is tangent to the lobe at A_k;
(b) the area, $\overline{A_k B_k} \cdot \overline{A_k C_k} \sin(\beta_k - \alpha_k)$, of the parallelogram is equal to A_{lobe}.

The condition of intersection with the y-axis is now $\overline{A_{\hat{k}} K_{\hat{k}}} = x_{-\hat{k}}$: which defines $\hat{k}$ implicitly.

Let H_k denote the orthogonal projection of C_k on the straight line through A_k and B_k. Then $A_{lobe} = \overline{A_k B_k} \cdot \overline{C_k H_k} = \frac{\Delta x_{-k}}{\cos\alpha_k} \overline{C_k A_k} \sin(\beta_k - \alpha_k)$. Since $\overline{A_k K_k} = -\overline{C_k A_k} \cos\beta_k$, the intersection condition becomes

$$\overline{A_{\hat{k}} K_{\hat{k}}} = \frac{-\cos\alpha_{\hat{k}} \cos\beta_{\hat{k}}}{\Delta x_{-\hat{k}} \sin(\beta_{\hat{k}} - \alpha_{\hat{k}})} A_{lobe} = x_{-\hat{k}}. \tag{2.10}$$

From (2.8), using $t = -\hat{k}\delta$ one finds

$$x_{\hat{k}} \sim 4\mathrm{e}^{-\hat{k}\delta} \qquad \Delta x_{-\hat{k}} \sim 4\delta \mathrm{e}^{-\hat{k}\delta}$$

thus

$$\tan\alpha_{\hat{k}} \sim \delta(1 + \mathrm{e}^{-2\hat{k}\delta}) \qquad \sin\alpha_{\hat{k}} \sim \delta(1 + \mathrm{e}^{-2\hat{k}\delta}) \qquad \cos\alpha_{\hat{k}} \sim 1.$$

To estimate $\beta_{\hat{k}}$ we again use the fact that the intersection will happen very close to the origin: the tangent to the unstable manifold at $A_{\hat{k}}$ is very close to the expanding direction at the origin, so we obtain

$$\tan\beta_{\hat{k}} \sim -\delta\left(1 + \frac{\delta}{2}\right) \qquad \cos\beta_{\hat{k}} \sim 1 - \delta^2.$$

Using these approximations we can rewrite (2.10) as

$$(1 - \delta^2)\frac{2\omega_0}{\pi}\mathrm{e}^{-\frac{\pi^2}{\delta}} = 4\mathrm{e}^{-\hat{k}\delta}\frac{4\delta \mathrm{e}^{-\hat{k}\delta}}{1 + \mathrm{e}^{-2\hat{k}\delta}} 2\delta + \mathrm{o}(\delta^2 \mathrm{e}^{-2\hat{k}\delta})$$

from which we obtain

$$\delta^2 \mathrm{e}^{-2\hat{k}\delta} = C_1 \mathrm{e}^{-\frac{\pi^2}{2\delta}} + \mathrm{O}(\mathrm{e}^{-\frac{\pi^2}{\delta}})$$

for some positive constant C_1. This allows us to compute $\hat{k}$ explicitly. To determine $\tilde{y}$, we recall (2.9) and using the previous value of $\hat{k}$ we obtain

$$\tilde{y} = y_{-\hat{k}} - x_{-\hat{k}} \tan\beta_{\hat{k}} \sim 4\delta \mathrm{e}^{-\hat{k}\delta} + 4\mathrm{e}^{-\hat{k}\delta}\delta\left(1 + \frac{\delta}{2}\right) \sim C_1' \mathrm{e}^{-\frac{\pi^2}{2\delta}}$$

for some new positive constant C_1'. Finally, since $|\tilde{y} - y_{int}| = \mathcal{O}(x^2_{-\hat{k}}) = \mathcal{O}(\mathrm{e}^{-\frac{\pi^2}{\sqrt{\epsilon}}})$, we conclude

$$-y^-(\epsilon) = y^+(\epsilon) \geqslant c\mathrm{e}^{-\pi^2/2\sqrt{\epsilon}}$$

for some positive constant c. $\square$

2.2. Interpolating vector fields

Once we have determined a lower bound for $y^+(\epsilon)$ in terms of the intersection of the lobe with the y-axis, we need to compute a lower bound for the rotation number of the corresponding invariant circle. To this end, following [4] we consider the problem of the construction of a vector field interpolating a given diffeomorphism. With this subsection and the following one we prove the existence of an interpolating Hamiltonian system exponentially close to the map (2.3) (propositions 2.8 and 2.9). These results will be used in section 2.4 to prove the existence of an asymptotic development for the standard map rotation number (proposition 2.13) from which lemma 2.3 immediately follows.

Let D be a bounded subset of $\mathbb{R}^m$ for $m \in \mathbb{N}^*$, I a real interval containing the origin and let us consider the one-parameter family of diffeomorphisms Ψ

$$\Psi : D \times I \to \mathbb{R}^m$$

$$(x, \mu) \mapsto \Psi(x, \mu)$$

such that for all $x \in D$ and $\mu \in I$ it has the form

$$\Psi(x, \mu) = x + \sum_{n \geqslant 1} \Psi_n(x)\mu^n. \tag{2.11}$$

The problem consists in finding a one-parameter family of $F : D \times I \to \mathbb{R}^m$ vector fields such that the time-1 flow of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, \mu)$$

coincides with the diffeomorphism Ψ.

We assume that $\Psi_n(x)$, defined in (2.11), are real analytic functions of x in some complex neighbourhood $\mathcal{D}_\rho$ of D, defined as follows. Given $\rho = (\rho_1, \dots, \rho_m) \in \mathbb{R}^m_+$ then

$$\mathcal{D}_\rho = \bigcup_{x \in D} \{z \in \mathbb{C}^m : |z_i - x_i| < \rho_i, 1 \leqslant i \leqslant m\}.$$

We introduce standard norms on complex analytic functions. Let f be a scalar analytic function on some domain $D \subset \mathbb{R}^m$, let $\rho \in \mathbb{R}^m_+$ and consider its complex extension $\mathcal{D}_\rho$, then

$$\|f\|_\rho = \sup_{z \in \mathcal{D}_\rho} |f(z)| \tag{2.12}$$

defines a norm. If F is a vector-valued analytic function $F = (f_1, \dots, f_m)$ on $\mathcal{D}_\rho$, for some $\rho = (\rho_1, \dots, \rho_m) \in \mathbb{R}^m_+$, then we define a norm (noted as in the scalar case)

$$\|F\|_\rho = \max_{1 \leqslant i \leqslant m} \|f_i\|_\rho.$$

If Ψ is a formal μ-series we can look for a formal interpolating vector field $F(x, \mu) = \sum_{n \geqslant 1} F_n(x)\mu^n$ whose time-1 flow coincides with the diffeomorphism, as we show with the following proposition.

Proposition 2.6. *Let $\Psi(x, \mu)$ be a diffeomorphism of the form (2.11) and assume that $\Psi_n(x)$ is analytic in $\mathcal{D}_\rho$ for all $n \geqslant 1$ and for some $\rho \in \mathbb{R}^m_+$. Then there exists a formal vector field F_μ of the form*

$$F_\mu(x) = \sum_{n \geqslant 1} F_n(x)\mu^n \tag{2.13}$$

such that

$$\mathrm{e}^{L_{F_\mu}}(x) = \Psi(x, \mu) \tag{2.14}$$

for all $x \in \mathcal{D}_\rho$.

Proof. The proof is a very natural application of the Lie series. We give it for the sake of completeness.

By definition of the exponential of the Lie derivative

$$e^{L_{F_\mu}}(x) = x + \sum_{n\geqslant 1} \frac{1}{n!} L^n_{F_\mu}(x).$$

Looking for F_μ in the form (2.13) and using the linearity of the Lie operator we have

$$L_{F_\mu} = \sum_{n\geqslant 1} \mu^n L_{F_n}.$$

For all integer $n \geqslant 1$ and $2 \leqslant m \leqslant n$ we define

$$\mathcal{L}_{1,n}(x) = L_{F_n}(x) = F_n(x) \qquad \mathcal{L}_{m,n}(x) = \sum_{j=1}^{n-m+1} L_{F_j}\mathcal{L}_{m-1,n-j}(x)$$

and it is easy to prove that

$$L^m_{F_\mu} = \sum_{n\geqslant m} \mu^n \mathcal{L}_{m,n}.$$

Then by (2.14) and (2.11) we conclude that for all $k \geqslant 1$

$$\sum_{n=1}^{k} \frac{1}{n!}\mathcal{L}_{n,k}(x) = \Psi_k(x)$$

namely

$$L_{F_k}(x) = \Psi_k(x) - \sum_{n=2}^{k} \frac{1}{n!}\mathcal{L}_{n,k}(x). \tag{2.15}$$

Let $\Psi_k = \left(\Psi_k^{(1)}, \dots, \Psi_k^{(m)}\right)$ and let $G = \sum_{n=2}^{k} \frac{1}{n!}\mathcal{L}_{n,k}(x) = (G_1, \dots, G_m)$. Note that G depends on Ψ_j and F_{j-1} for $j \leqslant k$. Using (2.15) we have for all $1 \leqslant l \leqslant m$

$$F_k^{(l)}(x) = \Psi_k^{(l)} + G_l$$

from which we obtain $F_k(x) = \left(F_k^{(1)}(x), \dots, F_k^{(m)}(x)\right)$. □

If Ψ is analytic with respect to μ then we have the following result [4, proposition 1, p 1122]

Theorem 2.7. *Let $\Psi(x,\mu)$ be a diffeomorphism of the form (2.11) with $\Psi_n(x)$ real analytic in some complex domain $\mathcal{D}_\rho$ for $\rho \in \mathbb{R}^m_+$, and let F_μ be the formal interpolating vector field obtained in proposition 2.6. Assume that for all k*

$$\|\Psi_k\|_\rho \leqslant \Gamma\gamma^{k-1}$$

for some positive constants Γ and γ then

$$\|F_1\|_\rho \leqslant \Gamma \qquad \forall k \geqslant 2 \quad \|F_k\|_{\frac{\rho}{2}} < \tfrac{1}{2}k^{k-1}\beta^{k-1}\Gamma$$

with $\beta = 4\max\{\gamma, \Gamma\}$.

Moreover, if $\Psi(x,\mu)$ is symplectic then for all n $F_n(x)$ are locally Hamiltonian.

In the form (2.3) the standard map verifies the hypothesis of theorem 2.7, but to obtain optimal estimates we state a Hamiltonian version of theorem 2.7 adapted to the standard map.

Proposition 2.8. *Let $\bar{z} > 0$ and consider the interval $I = (-\bar{z}, \bar{z})$. Fixed $R_0 > 0$ and $S_0 > 0$ we define the complex cylinder*

$$\mathcal{D}_{(R_0,S_0)} = \left\{ x \in \mathbb{C}/2\pi\mathbb{Z} : |\operatorname{Im} x| < S_0 \right\} \times \bigcup_{z \in I} \{ z' \in \mathbb{C} : |z - z'| < R_0 \bigg\}.$$

Then we can find a formal Hamiltonian $H(x, z, \mu) = \sum_{n=1}^{\infty} H_{n-1}(x, z)\mu^n$ such that

$$\forall (x, z) \in \mathcal{D}_{(R_0,S_0)} \qquad \mathrm{e}^{L_H}(x, z) = S_\mu(x, z).$$

Moreover, for all $0 < d < 1$ there exist $A = 2S_0 \|H_0\|_{(R_0,S_0)}$ and $B = \frac{20}{3} \frac{\|H_0\|_{(R_0,S_0)}}{R_0 d^2}$ such that

$$\forall k \quad \|H_k\|_{(1-d)(R_0,S_0)} \leqslant AB^k (k + 1)!$$

namely H is 1-Gevrey with respect to μ.

With the norm defined in (2.12) it results that $\|H_0\|_{(R_0,S_0)} = \frac{R_0^2}{2} + \mathrm{e}^{S_0}$.

The proof of proposition 2.8 is an adaptation of the proof of theorem 2.7 and we omit it.

We write down the first few terms of H

$$H(x, z, \mu) = \frac{z^2}{2} + \cos x + \mu \frac{z}{2} \sin x + \mathcal{O}\big(\mu^2\big) \tag{2.16}$$

and as we stated this is a perturbation of order $\mathcal{O}(\mu) = \mathcal{O}\big(\sqrt{\epsilon}\big)$ of the pendulum.

2.3. Exponentially small estimates

Proposition 2.8 gives us a Hamiltonian function represented as a (maybe divergent) series. It is then interesting to define a truncation of this series, namely take $N \in \mathbb{N}^*$ define $H^{(N)}(x, z, \mu)$ by

$$H^{(N)}(x, z, \mu) = \sum_{n=1}^{N} H_{n-1}(x, z)\, \mu^n$$

and to ask for the relations between $S_\mu(x, z)$ and $\Psi^{(N)}(x, z, \mu)$, where the latter is defined by

$$\Psi^{(N)}(x, z, \mu) = \mathrm{e}^{L_{H^{(N)}}}(x, z).$$

We can prove that we can chose N such that $S_\mu(x, z)$ and $\Psi^{(N)}(x, z, \mu)$ are exponentially close. More exactly:

Proposition 2.9. *Let $H^{(N)}(x, z, \mu)$ be the truncation of the 1-Gevrey interpolating Hamiltonian system given by proposition 2.8, and let $\Psi^{(N)}(x, z, \mu)$ be its time-1 flow. Then we can find a positive integer N^* such that for all $0 < d < \frac{2}{3}$ and for all $|\mu| \leqslant \frac{eR_0S_0d^2}{32A}$*

$$\|S_\mu(x, z) - \Psi^{(N^*)}(x, z, \mu)\|_{\left(1-\frac{3}{2}d\right)(R_0,S_0)} \leqslant |\mu| D \mathrm{e}^{-\frac{D'}{|\mu|}}$$

with $D = M + \frac{16Am}{eR_0S_0d^2}$, $D' = (2eB)^{-1}$, $M = \max\{R_0 + |\mu|\mathrm{e}^{S_0}, \mathrm{e}^{S_0}\}$ and $m = \max\{R_0, \mathrm{e}^{S_0}\}$.

This proposition is an adaptation of corollary 1, p 1124 of [4] and we do not prove it.

In the following we will note that $H^*(x, z, \mu) = H^{(N^*)}(x, z, \mu)$: the 'best' truncation of the interpolating Hamiltonian.

Let us introduce the frequency† of the rotation orbit of H^*:

$$2\pi\omega_*(x, z, \mu) = \lim_{n\to\infty} \frac{\pi_1\left(\mathrm{e}^{nL_{H^*}}(x, z)\right) - x}{n}. \tag{2.17}$$

† See the first footnote in this paper for a comment concerning this definition.

We remark that for $|\mu|$ sufficiently small the previous limit exists and its value is the same as the one given by the integral (2.21). This is because H^* is a one-degree-of-freedom integrable system and the surface of fixed energy is a graph of a 2π-periodic function in the plane x, z.

From proposition 2.9 we deduce the following corollary:

Corollary 2.10. *Let ω_S be the rotation number of a homotopically non-trivial curve of the map S_μ defined in (2.3). Let $H^*(x, z, \mu)$ be the best truncation of the interpolating Hamiltonian system given by proposition 2.9 and let $2\pi\omega_*$ be the frequency of one of its rotation curves. Then for $|\mu|$ sufficiently small, we have*

$$|\omega_S(x, z, \mu) - \omega_*(x, z, \mu)| \leqslant D|\mu|e^{-D'/|\mu|}. \tag{2.18}$$

To prove it we need an estimation for the difference between the successive iterates of the maps S_μ and $e^{L_{H^*}}$, which is given by the following algebraic lemma.

Lemma 2.11. *Let A and B be two elements of a non-commutative algebra, then for all $k \geqslant 1$ we have*

$$A^k - B^k = \sum_{l=0}^{k-1} B^{k-1-l}(A - B)A^l. \tag{2.19}$$

Proof. The proof by induction on k it is trivial once we remark that

$$A^{k+1} - B^{k+1} = B(A^k - B^k) + (A - B)A^k. \qquad \square$$

We can finally prove corollary 2.10.

Proof. The previous lemma implies

$$S_\mu^k(x, z) - e^{kL_{H^*}}(x, z) = \sum_{l=0}^{k-1} e^{(k-1-l)L_{H^*}} \left(S_\mu - e^{L_{H^*}}\right) S_\mu^l(x, z). \tag{2.20}$$

For real x, z and μ, the map S_μ is real valued so $\xi_l = S_\mu^l(x, z)$ is in $\mathbb{T} \times \mathbb{R}$ for all $l \geqslant 0$, moreover for small μ the iterates of z stay close to the initial value. A similar statement holds for the flow of H^*, in fact using the energy conservation and the implicit function theorem (see (2.22)) then $x(t) \in \mathbb{T}$ and $z(t)$ stay close to $z(0)$ for all t. Actually, the rotation numbers ω_S (see (2.4)) and ω_* (see (2.17)) are defined when the maps are replaced with some lift to the universal cover of $\mathbb{T} \times \mathbb{R}$. This is a minor technical fact and in the following we will denote with the same symbol the maps and their lifts.

Let us define the function $\Delta_\mu = S_\mu - e^{L_{H^*}}$, then proposition 2.9 implies that if $|\mu|$ is sufficiently small, the image under Δ_μ of a complex domain containing $\mathbb{T} \times I$, is exponentially small in $|\mu|$: $\xi_l' = \Delta_\mu(\xi_l)$ and $|\xi_l'| \leqslant D|\mu|e^{-D'/|\mu|}$ for $(x, z) \in \mathcal{D}_{(1-\frac{3}{2}d)(R_0,S_0)}$ and $|\mu|$ sufficiently small.

Such a D depends on the domain through d, for some fixed S_0 and R_0, but using the previous remark on the closeness of ξ_l with respect to the initial data, we can fix once and for all a domain, and so a constant D, in such a way that ξ_l belongs for all l to this domain. If $|\xi_l'|$ is small, then for all n: $|e^{nL_{H^*}}\left(\xi_l'\right)| \leqslant D|\mu|e^{-D'/|\mu|}$, because $e^{nL_{H^*}}\left(\xi_l'\right) = \xi_l' + \mathcal{O}\left(|\xi_l'|\right)$.

We conclude that

$$\left|\left[\pi_1\left(S_\mu^k(x,z)\right)-x\right]-\left[\pi_1\left(\mathrm{e}^{kL_{H^*}}(x,z)\right)-x\right]\right|\leqslant k|\mu|D\mathrm{e}^{-D'/|\mu|}$$

from which (2.18) follows. □

In the following paragraph we actually prove that ω_* exists and for $|\mu|$ sufficiently small it is $|\mu|^2$-close to the frequency of the curve with the same initial data, of the pendulum whose Hamiltonian is $H(x,z)=\frac{z^2}{2}+\cos x$.

Remark 2.12. Concerning the width of the interval I of definition of H^* (see proposition 2.8) with respect to the variable z. This interval must be big enough to contains the value z_0 corresponding to the invariant rotation circle 'closest' to the separatrix. If we take $I=(-\bar{z},\bar{z})$ with $\bar{z}=\mathcal{O}(1)$ as $\epsilon\to 0$, recalling that $z=y/\sqrt{\epsilon}$ we have $y=\mathcal{O}(\sqrt{\epsilon})$. From (2.7) we know that if $|y_0|\sim\sqrt{\epsilon}$ then $\omega_T(0,y_0,\epsilon)\sim\sqrt{\epsilon}$ which is bigger than the rotation number of the 'last' invariant torus which is order ϵ. Because the frequency map (for a fixed value of x and ϵ) is monotonically increasing with respect to y (or z) we are sure that with this choice of I the initial datum z_0 corresponding to the invariant rotation circle 'closest' to the separatrix, is contained in I.

2.4. Asymptotic development

We can now prove the existence of an asymptotic development for the standard map rotation number in a neighbourhood of $\epsilon=0$.

The Hamiltonian H^* defines an integrable (it has only one degree of freedom) Hamiltonian system whose period is given by

$$T=\frac{2\pi}{\omega_*}=\int_0^{2\pi}\frac{\mathrm{d}x}{\partial H^*/\partial z\big|_{h_0}} \tag{2.21}$$

where h_0 is an energy level corresponding to rotation orbits.

Let us fix $h_0>1$. We are interested in finding curves on the energy level $H^*(x,z,\mu)=h_0$. Take $z_0>0$ and consider the point $P_0=(x_0,z_0,0)$ such that $P_0\in H^{*-1}\{h_0\}$, from (2.16) it follows that $\frac{\partial H^*}{\partial z}(P_0)=z_0>0$. We can then apply the implicit function theorem which ensures that it exists a function $\hat{z}=\hat{z}(x,\mu)$ such that locally it satisfies

$$H^*\left(x,\hat{z},\mu\right)=h_0 \tag{2.22}$$

moreover this function is smooth. We compute the first terms and we obtain $\hat{z}(x,\mu)=\sqrt{2(h_0-\cos x)}-\frac{\mu}{2}\sin x+\mathcal{O}(|\mu|^2)$ and

$$\begin{aligned}&\frac{\partial H^*}{\partial z}\left(x,\hat{z},\mu\right)-\sqrt{2(h_0-\cos x)}\\&\qquad=\hat{z}+\frac{\mu}{2}\sin x+\sum_{n=2}^{N^*}\frac{\partial H_n}{\partial z}\left(x,\hat{z}\right)\mu^n-\sqrt{2(h_0-\cos x)}=\mathcal{O}(|\mu|^2).\end{aligned} \tag{2.23}$$

Recalling (2.5) we prove the following proposition:

Proposition 2.13. *Let $\omega_T(x,y,\epsilon)$ be the rotation number of the standard map in the formulation (1.1), then it exists a formal $\sqrt{\epsilon}$-power series $\sum_{n=0}^{\infty}B_n\left(\sqrt{\epsilon}\right)^n$ asymptotic to $\omega_T(x,y,\epsilon)$ for $\epsilon\to 0$. Namely, for all positive integer N we have*

$$\lim_{\epsilon\to 0}\left(\sqrt{\epsilon}\right)^{-(N+1)}\left|\omega_T-\sum_{n=0}^{N}B_n\left(\sqrt{\epsilon}\right)^{n+1}\right|=0. \tag{2.24}$$

Proof. The solution, $\hat{z}$, of (2.22) given by the implicit function theorem is analytic with respect to $x \in \mathbb{T}$ and μ in a neighbourhood of 0, $\partial H^*/\partial z\big|_{h_0}(x, 0)$ is positive for $h_0 > 1$, then for μ close enough to the origin $\partial H^*/\partial z\big|_{h_0}(x, \mu)$ is positive and analytic with respect to $x \in \mathbb{T}$ and μ. We can then write the following Taylor development:

$$\left(\frac{\partial H^*}{\partial z}\bigg|_{h_0}(x, \mu)\right)^{-1} = \sum_{n\geqslant 0} K_n\,(x, h_0)\,\mu^n \tag{2.25}$$

where $(K_n)_{n\geqslant 0}$ are some known functions of (x, h_0). Using (2.23) we obtain

$$K_0\,(x, h_0) = \frac{1}{\sqrt{2\,(h_0 - \cos x)}} \qquad \text{and} \qquad K_1\,(x, h_0) = 0 \tag{2.26}$$

we left to remark 2.14 an iterative scheme to calculate K_n.

For all $n \geqslant 0$ we can integrate the $K_n\,(x, h_0)$ and we rewrite (2.21) as

$$\frac{2\pi}{\omega_*} = \sum_{n\geqslant 0} C_n(h_0)\mu^n$$

from (2.26) we obtain

$$C_0(h_0) = \frac{2\pi}{\omega_{pend}(h_0)} = \int_0^{2\pi} \frac{\mathrm{d}x}{\sqrt{2\,(h_0 - \cos x)}} \qquad \text{and} \qquad C_1(h_0) = 0$$

where $\omega_{pend}(h_0)$ is the frequency of a rotation orbit of energy h_0 of the pendulum $H_p(x, z) = \frac{z^2}{2} + \cos x$.

Clearly, for all positive integer N we have

$$\lim_{\mu\to 0} |\mu|^{-N}\left|\frac{2\pi}{\omega_*(h_0)} - \sum_{n=0}^{N} C_n(h_0)\mu^n\right| = 0. \tag{2.27}$$

We show now that we can obtain a similar estimate for ω_*. Let us set, for all positive integer N, $S_N\,(\mu, h_0) = \sum_{n=0}^{N} C_n(h_0)\mu^n$. Because $S_N\,(0, h_0) \neq 0$ we can express $1/S_N$ as a μ-power series

$$\begin{aligned}\frac{1}{S_N\,(\mu, h_0)} &= \frac{1}{C_0(h_0)} + \sum_{n\geqslant 2}\mu^n \sum_{m=\lfloor \frac{n}{N}\rfloor+1}^{\lfloor \frac{n}{2}\rfloor} \frac{(-1)^m}{C_0^{m+1}(h_0)} \sum_{\substack{n_1+\cdots+n_m=n\\ 2\leqslant n_i\leqslant N}} C_{n_1}(h_0)\ldots C_{n_m}(h_0)\\ &= \frac{1}{2\pi}\sum_{n\geqslant 0} B_n(h_0)\mu^n \end{aligned} \tag{2.28}$$

where the second line is the definition of B_n, $B_0(h_0) = C_0(h_0)^{-1}$ and $B_1(h_0) = 0$. Recalling (2.27) and (2.28) we can then conclude that for all positive integer N

$$\left||\mu|^{-N}\left(\omega_* - \sum_{n=0}^{N}\mu^n B_n(h_0)\right) - |\mu|^{-N}\sum_{n\geqslant N+1}\mu^n B_n(h_0)\right| = \frac{2\pi\,\omega_*}{S_N\,(\mu, h_0)}\mathrm{o}\left(|\mu|^N\right)$$

and

$$\lim_{\mu\to 0} |\mu|^{-N}\left|\omega_* - \sum_{n=0}^{N}\mu^n B_n(h_0)\right| = 0.$$

From (2.5) and corollary 2.10 we obtain for all positive integers N

$$\left|\omega_T - \sum_{n=0}^{N} B_n(h_0)\left(\sqrt{\epsilon}\right)^{n+1}\right| \leqslant \epsilon D \mathrm{e}^{-\frac{D'}{\sqrt{\epsilon}}} + \sqrt{\epsilon}\,\mathrm{o}\left(\sqrt{\epsilon}\right)^{N}$$

dividing by $\left(\sqrt{\epsilon}\right)^{N+1}$ and passing to the limit $\epsilon \to 0$ we obtain (2.24). □

From this proposition, lemma 2.3 follows immediately.

Remark 2.14. The calculation of K_n defined by (2.25) is a little cumbersome but easy, it needs only some algebraic manipulations of power series. We assume that the solution, $\hat{z}\,(x,\mu,h_0)$, of (2.22) given by the implicit function theorem is known, and using the analyticity hypothesis we write it as a μ-power series

$$\hat{z}\,(x,\mu,h_0) = \hat{z}_0\,(x,h_0) + \sum_{n\geqslant 1} \hat{z}_n\,(x,h_0)\,\mu^n.$$

Starting from H^* given by (2.16), we calculate its partial derivative with respect to z, then we substitute $z = \hat{z}$ and we reorder the powers of μ to obtain

$$\frac{\partial H^*}{\partial z}\left(x,\hat{z},\mu\right) = \hat{z}_0\,(x,h_0) + \sum_{l\geqslant 2}\hat{z}_l\mu^l + \sum_{l\geqslant 3}\mu^l \sum_{\substack{n+m=l\\ m\geqslant 1, 2\leqslant n\leqslant N}} \sum_{k=1}^{m}\frac{1}{k!}\frac{\partial^{k+1}H_n}{\partial z}\left(x,\hat{z}_0\right)$$
$$\times \sum_{\substack{m_1+\cdots+m_k=m\\ m_i\geqslant 1}} \hat{z}_{m_1}\ldots\hat{z}_{m_k} = \hat{z}_0(x,h_0) + \sum_{l\geqslant 2}\mathcal{H}_l(x,h_0)\mu^l.$$

This series is invertible in the field of the formal μ-power series, then (2.25) correctly defines a μ-power series, which is given by

$$\left(\frac{\partial H^*}{\partial z}\left(x,\hat{z},\mu\right)\right)^{-1} = \frac{1}{\hat{z}_0} + \sum_{l\geqslant 2}\mu^l \sum_{m=1}^{\lfloor \frac{l}{2}\rfloor}\frac{(-1)^m}{\hat{z}_0^{m+1}} \sum_{\substack{l_1+\cdots+l_m=l\\ l_i\geqslant 2}} \mathcal{H}_{l_1}\ldots\mathcal{H}_{l_m}$$
$$= K_0\,(x,h_0) + \sum_{l\geqslant 2} K_l\,(x,h_0)\,\mu^l.$$

3. Numerical analysis of the scaling law at the resonance p/q

3.1. Frequency map analysis

The frequency map analysis of Laskar is a numerical method which allows us to obtain a global view of the dynamics of Hamiltonian systems by studying the properties of the frequency map, numerically defined from the action-like variables to the frequency space using adapted Fourier techniques. Thanks to its precision it was used for the study of stability questions and/or diffusion properties of a large class of dynamical systems: solar system [25], particles accelerator [28], galactic dynamics [39, 40], standard map [24, 27]. We present here the outlines of the method, following [26].

3.1.1. Frequency maps. Considering an n-degrees-of-freedom quasi-integrable Hamiltonian system in the form

$$H(J,\theta;\epsilon) = H_0(J) + \epsilon H_1(J,\theta) \tag{3.1}$$

where H is real analytic for $(J, \theta) \in B \times \mathbb{T}^n$, with B being an open domain in $\mathbb{R}^n$, and ϵ is a real 'small' parameter. For $\epsilon = 0$ this system reduces to an integrable one. The motion takes place on invariant tori $J_j = J_j(0)$ described at constant velocity $\nu_j(J) = \partial H_0/\partial J_j\big|_{J(0)}$, for $j = 1, \ldots, n$. Assuming a non-degeneration condition on H_0 the frequency map $F : B \to \mathbb{R}^n$

$$F : J \mapsto F(J) = \nu$$

is a diffeomorphism on its image Ω. In this case KAM theory [2, 23, 38] ensures that for sufficiently small values of ϵ, there exists a Cantor set $\Omega_\epsilon \subset \Omega$ of frequency vectors satisfying a Diophantine condition

$$|\langle k, \nu\rangle| > \frac{C_\epsilon}{|k|^m}$$

for some positive constants C_ϵ and m, for which the quasi-integrable system (3.1) still possess smooth invariant tori, ϵ-close to the tori of the unperturbed system, with linear flow $t \mapsto \nu_j t + \theta_j(0) \bmod 2\pi$ for $j = 1, \ldots, n$. Moreover, according to Pöschel [43] there exists a diffeomorphism $\Psi : \mathbb{T}^n \times \Omega \to \mathbb{T}^n \times B$

$$\Psi : (\phi, \nu) \mapsto (\theta, J)$$

which is analytic with respect to ϕ in $\mathbb{T}^n$ and $\mathcal{C}^\infty$ with respect to ν in Ω_ϵ, and which transforms the Hamiltonian system (3.1) into

$$\begin{cases} \dfrac{\mathrm{d}\nu_j}{\mathrm{d}t}(t) = 0 \\ \dfrac{\mathrm{d}\phi_j}{\mathrm{d}t}(t) = \nu_j. \end{cases}$$

For frequency vectors $\nu \in \Omega_\epsilon$ the invariant torus can be represented in the complex variables $(z_j = J_j \mathrm{e}^{\mathrm{i}\theta_j})_{j=1,n}$ by a quasiperiodic expression

$$z_j(t) = z_j(0)\,\mathrm{e}^{\mathrm{i}\nu_j t} + \sum_m a_{j,m}(\nu)\,\mathrm{e}^{\mathrm{i}\langle m,\nu\rangle t}. \tag{3.2}$$

If we take the section $\theta = \theta_0$, for some $\theta_0 \in \mathbb{T}^n$, of the phase space, we obtain a frequency map $F_{\theta_0} : B \to \Omega$

$$F_{\theta_0} : J \mapsto \pi_2\left(\Psi^{-1}(\theta_0, J)\right) \tag{3.3}$$

where $\pi_2(\psi, \nu) = \nu$ is the projection on Ω. For sufficiently small ϵ the non-degeneracy condition ensures that F_{θ_0} is a smooth diffeomorphism.

3.1.2. Quasiperiodic approximations. If we have the numerical values of a complex signal over a finite time span $[-T, T]$ and we think that it has a quasiperiodic structure represented by a quasiperiodic function $f(t)$, we can construct a quasi-periodic approximation, $f'(t)$, as follows. We represent the given signal as

$$f(t) = \mathrm{e}^{\mathrm{i}\nu_1 t} + \sum_{k\in\mathbb{Z}^n\setminus(1,0,\ldots,0)} a_k \mathrm{e}^{\mathrm{i}\langle k,\nu\rangle t} \qquad a_k \in \mathbb{C} \tag{3.4}$$

and the signal reconstructed by FMA as $f'(t) = \sum_{k=1}^N a'_k \mathrm{e}^{\mathrm{i}\omega'_k t}$. Frequencies ω'_k and amplitudes a'_k are determined with an iterative scheme. We first determine the first frequency ω'_1 as the value of σ which maximizes

$$\phi(\sigma) = \left|\frac{1}{2T}\int_{-T}^{T} f(t)\,\mathrm{e}^{-\mathrm{i}\sigma t}\chi(t)\,\mathrm{d}t\right|$$

where $\chi(t)$ is some weight function†. Once the first frequency is found the associated complex amplitude a_1' is obtained by orthogonal projection. We then iterate the scheme to the function: $f_1(t) = f(t) - a_1' e^{i\omega_1' t}$.

Given the signal (3.4) over a time span $[-T, T]$, if we assume some arithmetic properties of the frequencies (Diophantine numbers, for example), then one can prove [26] that, using the Hanning filter of order p: $\chi_p(t) = \frac{2^p (p!)^2}{(2p)!}(1+\cos\pi t)^p$, FMA converges towards the first frequency ν_1^T with the asymptotic expression for $T \to +\infty$:

$$\nu - \nu_1^T = \frac{(-1)^{p+1}\pi^{2p}(p!)^2}{A_p T^{2p+2}} \sum_k \frac{\operatorname{Re} a_k}{\Omega_k^{2p+1}} \cos(\Omega_k T) + o\left(\frac{1}{T^{2p+2}}\right) \tag{3.5}$$

where $\Omega_k = \langle k, \nu \rangle - \nu_1$ and $A_p = -\frac{2}{\pi^2}\left(\frac{\pi^2}{6} - \sum_{k=1}^{p} \frac{1}{k^2}\right)$. We can thus note that the accuracy of the determination of the frequency is $\mathcal{O}\big(1/T^{2p+2}\big)$, which will usually be several order of magnitude better that with usual FFT ($\mathcal{O}(1/T)$).

3.1.3. Frequency map analysis. With the previous algorithm, it is possible to construct numerically a frequency map in the following way: (a) fix all values of the angles $\theta_j = \theta_{j0}$; (b) for any value J_0 of the action variables, integrate numerically the trajectories of initial condition (J_0, θ_0) over the time span T; (c) search for a quasiperiodic approximation of the trajectory with the algorithm of the previous section; (d) identify the fundamental frequencies ν of this quasiperiodic approximation. The frequency map is then

$$F_{\theta_0}^T : J \mapsto \nu \tag{3.6}$$

and from the previous section, we know that for $T \longrightarrow +\infty$, on the set of regular KAM solutions, $F_{\theta_0}^T \longrightarrow F_{\theta_0}$. In particular, $F_{\theta_0}^T$ should converge towards a smooth function of the set of invariant KAM curves. The destruction of these invariant curves can thus be identified by the non-regularity of the frequency map $F_{\theta_0}^T$.

3.2. Application to the standard map

We are interested in the study of homotopically non-trivial invariant orbits of the standard map, in particular, in the relation between the rotation number ω and the critical function $\epsilon_{crit}(\omega)$. Given a couple of initial conditions (x_0, y_0) the FMA gives us approximate amplitudes and frequencies of the resulting orbit. For $\epsilon = 0$ all invariant curves are transverse to the section $x =$ constant, for ϵ sufficiently small this fact still holds, that is the transversality property. For $\epsilon > 0$, 'curves' of rational frequency do not persist and are replaced by a 'set of periodic fixed points' such that if $\omega(x_0, y_0) = p/q$ then this set contains an even multiple of q of periodic points, one-half of which are hyperbolic (HFP) and the other elliptic (EFP).

Near an HFP appears a stochastic layer, i.e. there are no invariant curves with frequency 'very close' to the frequency of the HFP. This is shown in figure 3 where the stochastic layer is revealed by the non-regularity of the frequency map around the frequency value $1/3$.

For a fixed value of ϵ and an HFP of rotation number p/q, we study the frequency curve $y \mapsto \omega(x, y)$ for y in some neighbourhood, of y_{HFP}, and $x = x_{HFP}$ fixed. We generalize in

† A weight function is a positive and even function such that

$$\frac{1}{2T}\int_{-T}^{T} \chi(t)\,dt = 1.$$

In all the computations we will take the Hanning filter: $\chi_1(t) = 1 + \cos\pi t/T$.

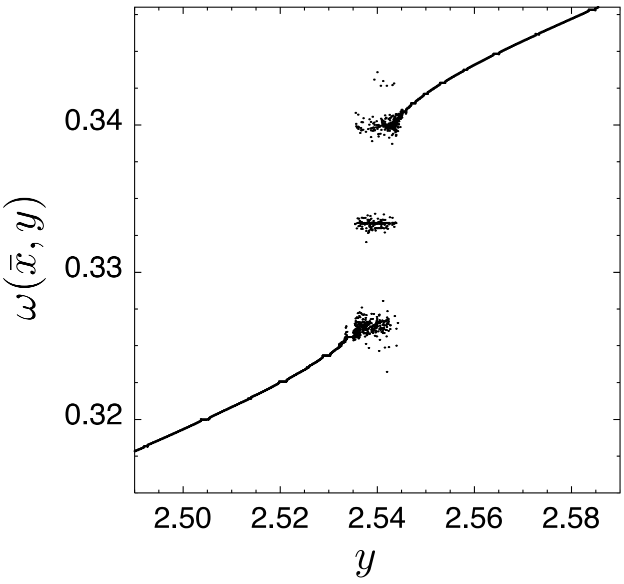

Figure 3. Detailed view of a nearby zone of an HFP, section $\bar{x} = 4.408$ and $\epsilon = 0.7$ (neighbourhood of the $1/3$ resonance). Note that for $y \in [2.535, 2.545]$ the frequency curve looks very irregular.

an obvious way the definitions (1.9) and (1.10) of $\omega^+(\epsilon)$ and $\omega^-(\epsilon)$ defined in the vicinity of rotation number 0 for invariant curves in the vicinity of a general HPF of rotation number p/q as $\omega^+_{p/q}(\epsilon)$, $\omega^-_{p/q}(\epsilon)$ and $\Delta\omega_{p/q} = \omega^+_{p/q} - \omega^-_{p/q}$. Similarly, we extend the definitions of $y^\pm(\epsilon)$ as $y^\pm_{p/q}(\epsilon)$.

Numerically, $y^+_{p/q}(\epsilon)$ is determined as the smallest value of $y(0)$, larger than y_{HFP}, for which the frequency curve is still regular (see the appendix), and similarly for $y^-_{p/q}(\epsilon)$. Then we assume $\omega^\pm_{p/q}(\epsilon) = \omega\big(x_{HFP}, y^\pm_{p/q}(\epsilon)\big)$. Studying $\Delta\omega_{p/q}(\epsilon)$ as a function of ϵ we are then able to reconstruct the curve $\epsilon_{crit}(\Delta\omega_{p/q})$ for a fixed resonance. In fact, the non-existence of invariant curves, for a fixed value of ϵ, with frequency $\tilde{\omega}$, between $\omega^+_{p/q}$ and $\omega^-_{p/q}$ means that $\epsilon_{crit}\big(\tilde{\omega}\big) > \epsilon$ and $\epsilon_{crit}\big(\omega^\pm_{p/q}\big) = \epsilon$.

3.3. Presentation of the numerical results

Using the FMA we studied the behaviour of $\epsilon_{crit}\left(\Delta\omega_{p/q}\right)$ in neighbourhood of the resonances $0/1, 1/5, 1/4, 2/5, 1/3, 1/2$. The results are presented in table 1 and in figure 4, where the plot of $\ln\epsilon_{crit}$ as a function of $\ln\Delta\omega_{p/q}$ is shown. All numerical calculations are done with 100 digit accuracy using the Fortran 90 multi-precision package *mpfun* [3].

Numerical fitted values of the slopes $A_{p/q}$ are given in table 1. When fitted over all values of ϵ_{crit} the relative differences between the determined slopes and the conjectured ones are still significant. When only the smallest values of ϵ_{crit} are used for the fit, which correspond to the closest approach to the asymptotic regime (because these points are closest to the resonances, $\Delta\omega_{p/q}$ are small), the values of the numerical slopes, $\tilde{A}_{p/q}$, improve significantly (compare columns two, three and five of table 1). We expect that further numerical computations in this asymptotic regime should give numerical slopes closer to the conjectured ones. We are

Table 1. $\ln \epsilon_{crit} = A_{p/q} \ln \Delta\omega_{p/q} + C_{p/q}$. The coefficients $A_{p/q}$ and $C_{p/q}$ are determined by a linear least-squares fit to all the FMA results. The coefficients $\tilde{A}_{p/q}$ and $\tilde{C}_{p/q}$ are determined by a linear least-squares fit to the set of data (usually one-half) closest to the resonances (i.e. for which $\Delta\omega_{p/q}$ is small). The last column contains the conjectured slopes, $\hat{A}_{p/q} = 1/q$.

Resonance p/q	$A_{p/q}$	$C_{p/q}$	$\tilde{A}_{p/q}$	$\tilde{C}_{p/q}$	$\hat{A}_{p/q}$
0/1	0.937	1.066	0.962	1.228	1
1/5	0.175	0.489	0.184	0.557	0.2
1/4	0.216	0.663	0.225	0.725	0.25
2/5	0.176	0.714	0.180	0.742	0.2
1/3	0.312	1.000	0.335	1.175	$0.\bar{3}$
1/2	0.427	0.957	0.455	1.094	0.5

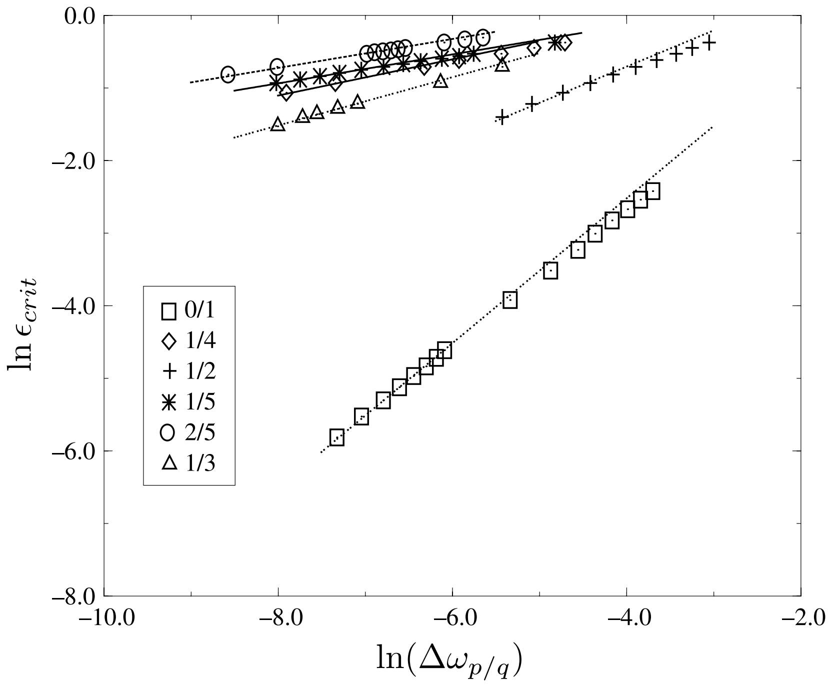

Figure 4. Plot of $\ln \epsilon_{crit}$ versus $\ln(\Delta\omega_{p/q})$ for $p/q \in \{0/1, 1/5, 1/4, 2/5, 1/3, 1/2\}$. The plotted lines: $\ln \epsilon_{crit} = \hat{A}_{p/q} \ln \Delta\omega_{p/q} + \hat{C}_{p/q}$, are obtained fixing the slopes to the conjectured values, whereas the $\hat{C}_{p/q}$s are determined by a linear least-squares fit with one free parameter on the points closest to the resonances (namely for which $\Delta\omega_{p/q}$ is small).

nevertheless confident in our results because from table 1 and figure 4 this asymptotic behaviour is already evident: the closer the points are to the resonances (namely points for which $\Delta\omega_{p/q}$ is small) the better the agreement of the calculated slopes, $\tilde{A}_{p/q}$, is with respect to the conjectured values $A_{p/q}$.

These results suggest the following geometry of the ϵ_{crit}–ω-plane. Consider only the resonance p/q, then for ω close to p/q we have

$$\epsilon_{crit}(\omega) \sim C'_{p/q} \left| \omega - \frac{p}{q} \right|^{1/q}. \tag{3.7}$$

Because of the density of the rational numbers on the real line, close to p/q there are infinitely

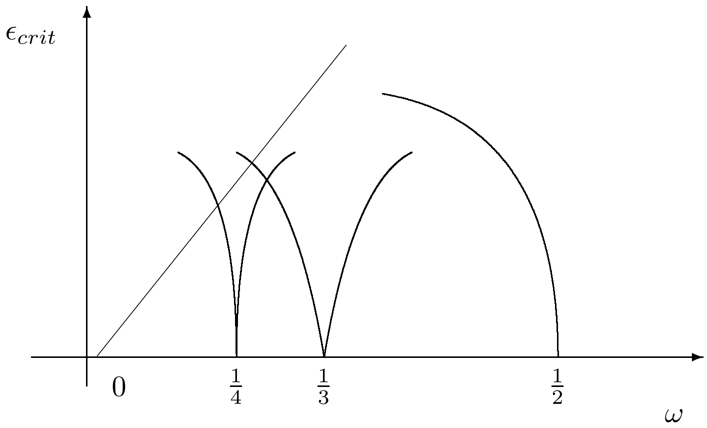

Figure 5. The resonant domains in the ϵ_{crit}–ω-plane.

many rational numbers p'/q', with q' bigger and bigger as p'/q' is closer to p/q. The relation between ϵ_{crit} and ω is not the simple one given by (3.7), even in a very small neighbourhood of p/q, but we must add the contributions of all p'/q' near p/q. These contributions are nevertheless negligible because for very big q' the right-hand side of (3.7) is almost equal to one. Then the local description given by the (3.7) should be a very good approximation of the relation between ϵ_{crit} and ω (see figure 5).

4. Conclusions

We have studied both analytically and numerically the standard map critical function in a neighbourhood of a fixed resonance. For the fundamental resonance we find a linear scaling law for the critical function, whereas in [7, 8] a quadratic scaling law was found for the critical radius. This shows, in particular, that for Brjuno's values of the rotation number ω, sufficiently close to zero, the critical function $\epsilon_{crit}(\omega)$ and the critical radius of convergence $\rho_{crit}(\omega)$ are different. More precisely $\epsilon_{crit}(\omega) > \epsilon^2_{crit}(\omega) \sim \rho_{crit}(\omega) > 0$.

We stress again that this result implies that the conjugating function u defined as the sum of the standard map Lindstedt series (1.3) for $|\epsilon| < \rho_{crit}(\omega)$, can be analytically continued outside its disc of convergence at least for ϵ real and that the domain of convergence, in the ϵ-complex plane, of the Lindstedt series can be very different from a disc; it will be interesting to study its geometry, in particular its boundary. It could be a fractal curve as in the case of the Schröder–Siegel centre problem [13], we only know that the nearest singularity to the origin has positive imaginary part.

A second possibility can happen near the resonances the standard map with parameter ϵ, such that $\rho_{crit}(\omega) < \epsilon < \epsilon_{crit}(\omega)$ still has invariant tori of rotation number ω which can be non-analytic ones.

The proof of theorem 1.1 is different from that of [7]. The FMA method does not allow us (at least for the moment) to use a complex map so we could not verify the results of [7] directly. It would be interesting to investigate the scaling law for the generic resonance p/q with our methods, but the proof of theorem 1.1 is specific to the fundamental resonance: we do not have a bound of the initial data for the last invariant torus for a generic resonance as we proved in proposition 2.2 for the resonance $0/1$. Supported by the numerical results of section 3, we can nevertheless make the following conjecture†:

Conjecture 1. *For all rational p/q, we can find a positive constant $c_{p/q}$, such that for all ω of Brjuno type, in a small* real neighbourhood *of p/q*

$$\epsilon_{crit}(\omega) \sim c_{p/q} \left| \omega - \frac{p}{q} \right|^{1/q}. \tag{4.1}$$

We can make a link with the Brjuno function. Equation (1.8) with theorem 1.1 implies that there exists a neighbourhood of the origin where $B(\omega) + \ln \epsilon_{crit}(\omega)$ is uniformly bounded. If we are able to prove the previous conjecture, a similar bound will hold in the neighbourhood of every rational, that is we could prove $B(\omega) + \ln \epsilon_{crit}(\omega)$ is uniformly bounded for all frequencies.

In this paper we were interested in finding the optimal scaling law, namely the optimality of the exponent $1/q$ in (4.1), neglecting the behaviour of the proportionality constants $c_{p/q}$ as a function of p/q. It will be interesting to investigate (at least numerically) this function, in particular to test its continuity.

Acknowledgments

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Appendix. Numerical precision of the results

Here we analyse more precisely the numerical results, pointing out some important remarks concerning the precision of the calculations.

The destruction of invariant curve is identified by the non-regularity of the frequency map. In general, this non-regularity is clearly visible (figure 3), and the destruction of invariant curves is determined with no ambiguity. However, when one searches for very fine details, which correspond to small values of the perturbing coefficient ϵ, one needs to be able to distinguish between non-regularity of the frequency map and the apparent non-regularity resulting from numerical errors in the computations.

One way to work around this problem is to always search for a regular part of the frequency curve. The existence of this regular part is the indication that we have not yet reached the threshold for numerical round-off error. An illustration is given in figure A1, where the frequency curve is plotted for $\epsilon = 0.1$, $T = 90\,000$, for both double- and quadruple-precision computations. It is clear that with double-precision computations, we are no longer able to distinguish precisely between regular and non-regular behaviour of the frequency curve, while it is still possible with quadruple-precision arithmetics.

As we know that on the regular curves, the FMA algorithm converges towards the true frequency as $1/T^4$ with the Hanning window of order 1 ($p = 1$ in equation (3.5)), we can check the errors of the FMA method by increasing T. If the variation of frequencies increases instead of decreasing, and still presents an irregular behaviour as in figure A1(a), it should come from numerical noise, and needs to be checked further.

† See page 2036 for the meaning of the symbol $\sim$.

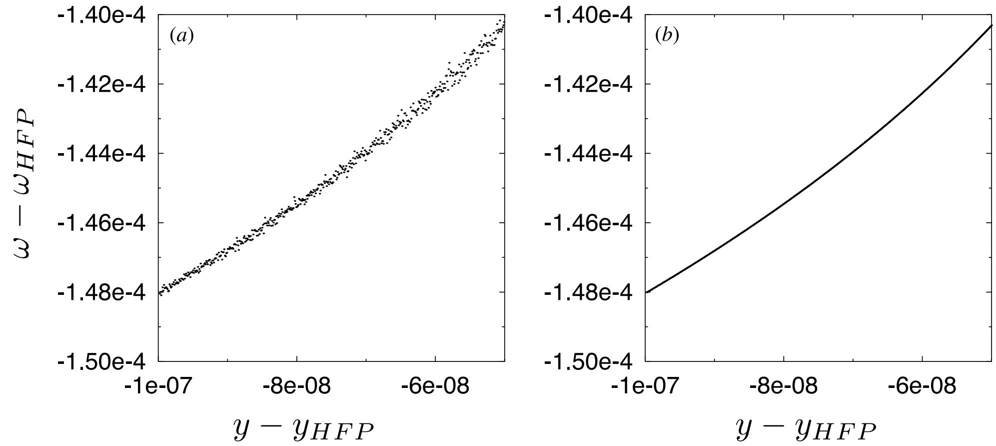

Figure A1. Frequency curve calculated in (*a*) double precision and (*b*) quadruple precision. Section $x = 4.408$ for the standard map for $\epsilon = 0.1$, integration time $T = 90\,000$. The regular set $\mathcal{I}_{reg}$ is represented.

A.1. False interpretation of a first result

In fact, the numerical accuracy for the iteration of the map is very important when one searches for very small values of ϵ, and if not taken properly into account can lead to false results. Indeed, in figure A2, we show sections $x = x_{HFP}$ of the standard map with $\epsilon = 0.1$ for double-precision (figure A2(*a*)) and quadruple-precision (figure A2(*b*)) calculations. One can note a very large change in the y-range, which implies that the values of $y^-_{p/q}$ and $y^+_{p/q}$ obtained using double precision are largely overestimated. In this case, we must use at least the quadruple calculation, while for the final results of the present paper, we had to use 100-digits multi-precision using the Fortran 90 package *mpfun*.

Because of the finite-precision arithmetic there is a threshold x_{thr} such that for all x below the threshold then $\sin x = x$. When we start with initial condition $(x_0, y_0) = o(x_{thr})$ then instead of iterating the standard map (1.1) we iterate the linear map T^{lin}_ϵ

$$T^{lin}_\epsilon : \begin{cases} y' = y + \epsilon x \\ x' = x + y' \qquad \mod 2\pi. \end{cases}$$

The iteration of T^{lin}_ϵ produces a strange result which is misleading. Using data below the threshold value x_{thr}, the iteration of T^{lin}_ϵ and then the numerical analysis of the signal gives the following scaling law, there exists $0 < \epsilon^* < 1$ such that,

(a) if $\epsilon_{crit}(\omega) > \epsilon^*$ then: $\ln \epsilon_{crit}(\Delta\omega_{p/q}) \sim \frac{1}{q} \ln \Delta\omega_{p/q} + c_1$;

(b) if $\epsilon_{crit}(\omega) < \epsilon^*$ then: $\ln \epsilon_{crit}(\Delta\omega_{p/q}) \sim \frac{2}{q} \ln \Delta\omega_{p/q} + c_2$;

namely it seems to be a discontinuity in the scaling law as figure A3 shows.

Looking at [7] and comparing with the results of [34], using double precision, we interpreted the result of figure A3 as follows:

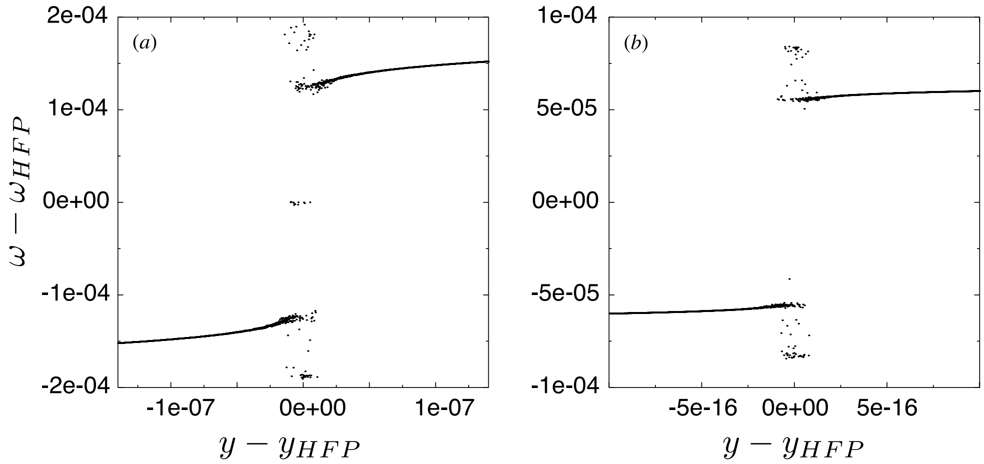

Figure A2. Double-precision calculations (a) and Quadruple-precision calculations (n) for the section $x = x_{HFP}$ for the standard map with $\epsilon = 0.1$.

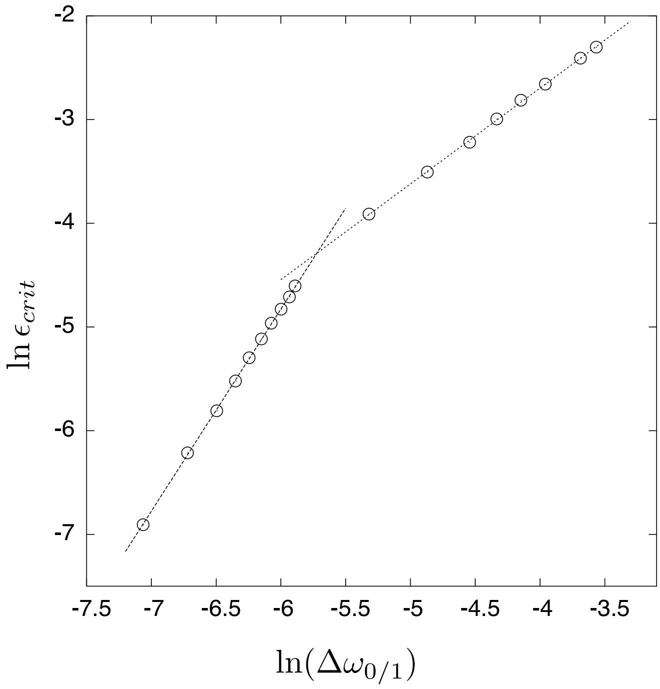

Figure A3. Change of slope phenomenon for the resonance $0/1$. $\ln \epsilon_{crit}$ is plotted versus $\ln \Delta\omega_{0/1}$ using quadruple precision. The linear fits $\ln \epsilon_{crit} = 1.948 \ln \Delta\omega_{0/1} + 6.863$ and $\ln \epsilon_{crit} = 0.92 \ln \Delta\omega_{0/1} + 0.998$ are also plotted.

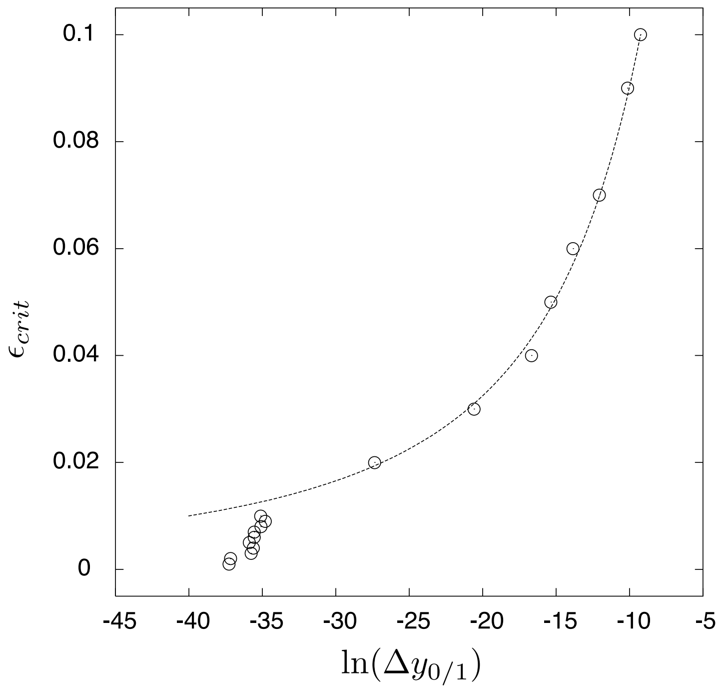

Figure A4. $\ln \Delta y_{0/1}$ as a function of ϵ_{crit} for the fundamental resonance using quadruple precision. It is evident that for $\epsilon \leqslant 0.01$, $\ln \Delta y_{0/1} \sim -35$.

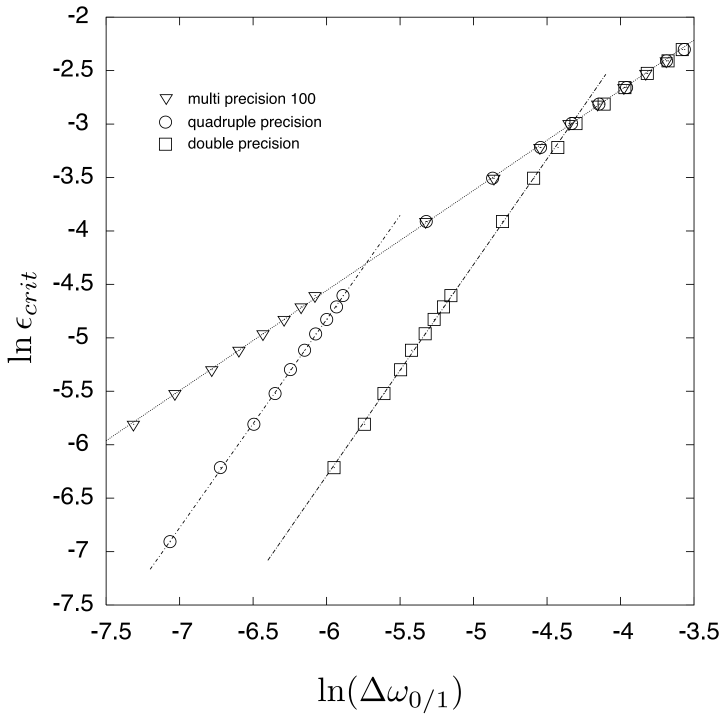

Figure A5. Plot of $\ln \epsilon_{crit}$ as a function of $\Delta\omega_{0/1}$ using multiple-precision 100, quadruple precision and double precision.

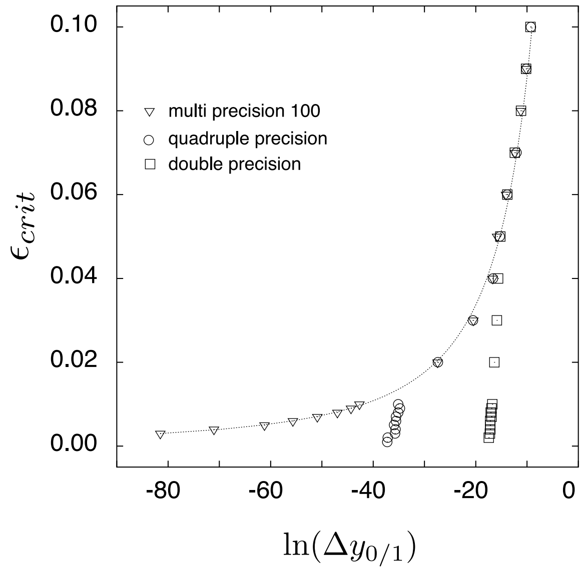

Figure A6. Plot of ϵ_{crit} versus $\ln \Delta y_{0/1}$ for different precisions. The full curve is the fit of the multiprecision-100 data: $\ln \Delta y_{0/1} = a - \frac{b}{\sqrt{\epsilon_{crit}}}$, with $a = 6.177$ and $b = 4.804$. To compare with the analytical result of proposition 2.2 which gives $\ln \Delta y_{0/1} \geqslant C - \frac{\pi^2}{2\sqrt{\epsilon_{crit}}}$. We do not have a good estimation for C but we can compare b with $\pi^2/2 \sim 4.935$, exhibiting a good agreement between the numerical results and the analytical one.

The scaling law at resonances for the standard map is that presented in [7]. The different result of [34] can be justified by saying that they did not go deep enough inside the resonant region and the nonlinearity of the model added some distortion. We are in the presence of two behaviours: close to the resonant region we find the proper law and far away the distorted one.

In studying further the reasons for this transition, we convinced ourselves that this interpretation was not correct. We understood that this behaviour was due to the finite-precision arithmetic of the iteration of the map. We also observed that for values of ϵ smaller than a critical value ϵ^*, there was a saturation of the measured value of $\Delta y_{0/1} = y^+(\epsilon) - y^-(\epsilon)$, which was also a sign of numerical problems (figure A4). This was confirmed by the analytic result of theorem 1.1 for the resonance $0/1$.

A.2. New interpretation

With lemma 2.3 we proved that the rotation number of the standard map is very well approximated by the pendulum frequency. Rewriting (2.7) as follows:

$$\ln \epsilon_{crit} \sim 2 \ln \Delta\omega_{0/1} + C + 2 \ln \ln \frac{8\sqrt{\epsilon_{crit}}}{\Delta y_{0/1}} \tag{A.1}$$

and using the previous remark on $\Delta y_{0/1}$, it follows that the term $\ln \ln \frac{8\sqrt{\epsilon_{crit}}}{\Delta y_{0/1}}$ in (A.1) is almost constant in the interval of variation of ϵ, say for example $\epsilon \in [0.001, 0.01]$. A similar result holds for every resonance p/q; then for a fixed numerical precision we obtain the following scaling law:

$$\ln \epsilon_{crit} \sim \frac{2}{q} \ln \Delta\omega_{p/q} + c.$$

To prove that this change of slope is 'not real' we increased the precision of the calculations (i.e. passing from double- to quadruple-precision and then to multi-precision-100), obtaining that the threshold ϵ^* decreases (see figure A5), showing that in the limit of exact arithmetic this threshold does not exist.

Figure A6 shows $\Delta y_{0/1}$ as a function of ϵ_{crit} for different machine precisions. Also in this case it is clear that the threshold value under which Δy is nearly constant goes to zero when we increase the numerical precision.

For the quadruple precision the threshold is $x_{thr} \sim 10^{-12}$ and it goes down to order 10^{-32} using the multi-precision package 100-digits. This value is sufficiently small to do precise calculations in reasonable CPU times. In fact, the threshold value for the multi-precision package 1000-digits is order 10^{-333} but the CPU time for the calculations increases by several orders of magnitude, and this was not necessary to obtain good confidence with our present results.

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