A note on existence and uniqueness of limit cycles for Liénard systems

Carletti, Timoteo; Villari, Gabriele

Published in:
Journal of Mathematical Analysis and Applications

Publication date:
2005

Document Version
Early version, also known as pre-print

Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 307 (2005) 763–773
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Note

A note on existence and uniqueness of limit cycles for Liénard systems

Timoteo Carletti [a,*], Gabriele Villari [b]

[a] *Scuola Normale Superiore, piazza dei Cavalieri 7, 56126 Pisa, Italy*
[b] *Dipartimento Matematica U. Dini, viale Morgagni 67/A, 50134 Firenze, Italy*

Received 29 July 2003
Available online 17 March 2005
Submitted by Z.S. Athanassov

Abstract

We consider the Liénard equation and we give a sufficient condition to ensure existence and uniqueness of limit cycles. We compare our result with some other existing ones and we give some applications.

Keywords: Liénard equation; Limit cycles; Existence and uniqueness

1. Introduction

In this paper we consider the Liénard equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$, with particular attention to the existence and uniqueness of limit cycles. This is a classical problem of non-linear oscillation for second order differential equations. Different assumptions on f and g and different methods used to study the problem, gave rise to a large amount of literature on this topic; for a review of results and

* Corresponding author.
E-mail addresses: t.carletti@sns.it (T. Carletti), villari@math.unifi.it (G. Villari).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.01.054

methods, reader can consult [9], [15, Chapter IV] or [14]. In the following we will give some more references.

We make the following assumptions on f and g:

(A) f is a continuous function and g verifies a locally Lipschitz condition.
(B) $f(0) < 0$, $f(x) > 0$ for $|x| > \delta$, for some $\delta > 0$, and $xg(x) > 0$ for $x \neq 0$.

Condition (A) assures existence and uniqueness of the Cauchy initial value problem for the Liénard equation. In fact, passing to the *Liénard plane* the second order differential equation is *equivalent* to the following first order system:

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{2}$$

where $F(x) = \int_0^x f(\xi)\,d\xi$. Hence assuming hypothesis (A) the right-hand side of (2) is Lipschitz continuous, from which the claim follows.

Assumption (B) guarantees that the origin is the only singular point of the system, which results a repeller. Moreover, orbits of (2) turn clockwise around it. Hypothesis on the sign of $f(0)$ can be weakened by asking $xF(x) < 0$ for $|x|$ small. We nevertheless prefer the former formulation (A) because of the applications we will give in the last part of the paper.

Assuming assumptions (A) and (B) on f and g, our main result will be the following theorem.

Theorem 1.1. *Let $G(x) = \int_0^x g(\xi)\,d\xi$ and suppose that F and G verify*:

(C) *F has only three real transversal zeros, located at $x_0 = 0$, $x_2 < 0 < x_1$. Assume, moreover, that F is monotone increasing outside the interval $[x_2, x_1]$.*
(D) $G(x_1) = G(x_2)$.
(E) $\limsup_{x\to+\infty}[G(x) + F(x)] = +\infty$ *and* $\limsup_{x\to-\infty}[G(x) - F(x)] = +\infty$.

Then system (2) *has a unique periodic orbit in the (x, y)-plane which is stable.*

Because of the equivalence of Eq. (1) and system (2) the former has a unique limit cycle if Theorem 1.1 applies.

We postpone the proof of the Theorem 1.1 to the next section. In Section 3 we will discuss the role of our hypotheses and compare this result with other existence and uniqueness results concerning Liénard systems.

Our result follows from investigating the geometry of limit cycles, in particular their (eventual) intersections with the lines $x = x_1$ and $x = x_2$. With Proposition 2.1 we give sufficient conditions to ensure intersection of limit cycles with one or both lines $x = x_1$ and $x = x_2$. Our result will then follow joining these informations with the result of [11, Theorem 1].

First of all we stress that assumptions are quite standard ones. Hypotheses (A), (B), (C) and (E) guarantee existence of limit cycles as it will be shown in Section 2.1. Hypotheses on F and the equality for G at roots of $F(x) = 0$ are *fundamental* for our proof. While we can already find in literature such hypotheses of F, the link between zeros of F and values of G at these points are new, as far as we know. We remark that hypothesis (C) can be weakened by allowing F to have zeros inside (x_2, x_1), other than $x_0 = 0$, where it does not change sign.

We already gave some bibliography of results concerning existence and/or uniqueness of limit cycles for Liénard equations; we do not try to compare our result with all the existing ones, we will restrict ourselves to emphasize the strong point of our Theorem and to compare it with some general results.

First of all we *do not assume any parity conditions* on F and/or g, on the contrary if F and g are odd, then Theorem 1.1 contains the Levinson–Smith result [4] as particular case: let $x_1 = -x_2$ be the non-zeros root of $F(x) = 0$, $G(x)$ is even because of oddness of g, and then $G(x_1) = G(-x_2)$.

The monotonicity on F is required only outside the interval determined by the smallest and largest zeros, namely its derivative $F'(x) = f(x)$ can have several zeros inside this interval, this is a more general situation than the results of Massera [5] and Sansone [7]. The last one follows from our result by remarking that if $g(x) = x$, then $G(x) = x^2/2$ and let $\Delta > 0$ be such that $F(\Delta) = F(-\Delta) = 0$, we get $G(\Delta) = G(-\Delta)$.

The second remark concerns the hypothesis (D): it is easy to verify if this condition on G holds, just compare the function at two points. We do not need to use the inversion of any function as in the Filippov case [3] (and in all results inspired by his method), or to impose conditions on functions obtained by composition and inversion. These facts make our theorem easily applicable as the results of Section 3 will show.

2. Main result

The aim of this section is to prove our main result, Theorem 1.1. The proof is divided in two steps, presented in Sections 2.2 and 2.3. Before let us introduce two preliminary results, first, Proposition 2.1, whose role is to give information about the geometry of limit cycles with respect to lines $x = x_i$, where x_i are non-zero roots of $F(x) = 0$. Second, give a proof (Section 2.1) of existence of limit cycles assuming hypotheses (A), (B), (C) and (E), as claimed in the Introduction.

Proposition 2.1. *Let f and g verify hypotheses* (A) *and* (B). *Let $F(x) = \int_0^x f(\xi)\,d\xi$ $G(x) = \int_0^x g(\xi)\,d\xi$ and assume $F(x)$ verify hypothesis* (C). *Then*

- *if $G(x_1) \geqslant G(x_2)$ all* (*eventual*) *limit cycles of* (2) *will intersect the line $x = x_2$;*
- *whereas if $G(x_1) \leqslant G(x_2)$ all* (*eventual*) *limit cycles of* (2) *will intersect the line $x = x_1$.*

Proof. Let us denote by $X_{\mathcal{L}}(x, y) = (y - F(x), -g(x))$ the Liénard field associate to (2) and let us consider the family of ovals given by $\mathcal{E}_N = \{(x, y) \in \mathbb{R}^2\colon\ y^2/2 + G(x) - N = 0\}$.

Let us consider the case $G(x_1) \geqslant G(x_2)$, the other can be handle similarly and we will omit it. The oval $\mathcal{E}_{G(x_2)}$ does not intersect the line $x = x_1$, whereas $\mathcal{E}_{G(x_1)}$ passes through points $(x_2, \pm\sqrt{2(G(x_1) - G(x_2))})$. Namely $\mathcal{E}_{G(x_1)}$ contains in its interior $\mathcal{E}_{G(x_2)}$ which contains the origin in its interior.

The flow of Liénard system (2) is transversal to $\mathcal{E}_{G(x_2)}$ (more precisely it points outward with respect to $\mathcal{E}_{G(x_2)}$):

$$\left\langle \nabla \mathcal{E}_{G(x_2)}, X_{\mathcal{L}}(x,y)\big|_{\mathcal{E}_{G(x_2)}}\right\rangle = -F(x)g(x) \geqslant 0,$$

equality holds only for $x = 0$ and $x = x_2$. Let us call $(x_1^*, 0)$ the unique intersection point of $\mathcal{E}_{G(x_2)}$ with the positive x-axis.

Hence from Poincaré–Bendixson theorem no-limit cycle can be completely contained in the strip $[x_2, x_1^*) \times \mathbb{R}$. Moreover, orbits of (2) spiral outward leaving $\mathcal{E}_{G(x_2)}$. Thus any (eventual) limit cycle must intersect the line $x = x_2$. □

2.1. Existence of limit cycles

Let us investigate the existence of limit cycles. Consider assumption (E), then if $\lim_{x\to\pm\infty} G(x) = +\infty$, we observe that assumption (C) guarantees the existence of $\epsilon > 0$ and $\alpha < 0 < \beta$ such that $\int_\alpha^\beta f(\xi)\,d\xi > \epsilon$. Moreover, $f(x) > 0$ for $x \notin [\alpha, \beta]$. We can then apply [10, Theorem 1] to obtain existence of limit cycles.

On the other hand, let us assume $\lim_{x\to+\infty} G(x) < +\infty$ (the case $\lim_{x\to-\infty} G(x) < +\infty$ can be handle similarly and we omit it). Then using [12, Theorem 3] we complete the proof of the existence of limit cycles.

2.2. Uniqueness: Step I

In [11] the following result has been proved.

Theorem 2.2. *Let f and g verify hypotheses* (A), (B) *and let F verify hypothesis* (C). *Let $x_2 < 0 < x_1$ be the non-zero roots of $F(x) = 0$. Assume that all limit cycles of* (2) *intersect the lines $x = x_2$ and $x = x_1$. Then system* (2) *has at most one limit cycle, if it exists it is stable.*

Let us give its proof for completeness.

Proof. We claim that for any limit cycle, γ, of system (2) we have:

$$\oint_\gamma g(x)\,dt = 0, \qquad \oint_\gamma g(x)y\,dt = 0 \quad \text{and} \quad \oint_\gamma g(x)\big[y - F(x)\big]\,dt = 0.$$

This can be proved easily by remarking that $g(x)y = \frac{d}{dt}\left(\frac{1}{2}y^2\right)$. Hence:

$$\oint_\gamma g(x)F(x)\,dt = 0. \tag{3}$$

Hypotheses (B) and (C) give $F(x)g(x) < 0$ for all $x \in (x_2, 0) \cup (0, x_1)$. Then using the monotonicity of F outside $[x_2, x_1]$ and the hypothesis that all limit cycles intersect both the lines $x = x_1$ and $x = x_2$, we conclude that if γ_1 and γ_2 are two limit cycles of (2), with γ_1 contained in the interior of γ_2, one has

$$\oint_{\gamma_1} g(x)F(x)\,dt < \oint_{\gamma_2} g(x)F(x)\,dt,$$

which contradicts (3) and so the number of limit cycles is at most one. □

The weak point of this result is the assumption that all limit cycles must intersect both the lines $x = x_1$ and $x = x_2$. In general, this is not true and, moreover, it may be difficult to verify. With our result we give sufficient hypotheses to ensure this fact. Theorem 1.1 is based on a slightly generalization of Theorem 2.2 that we state here without proof, which can be obtained following closely the previous one.

Theorem 2.3. *Assume* (A), (B) *and* (C) *of Theorem* 1.1 *hold, let* N_{x_1,x_2} *denote the number of limit cycles of system* (2) *which intersect both the lines* $x = x_i$, $i = 1, 2$. *Then* $N_{x_1,x_2} \leqslant 1$.

Now we are able to prove the main part of our result.

2.3. Uniqueness: Step II

The number of limit cycles of system (2) is by definition $N_{\mathrm{lc}} = N_{x_1,x_2} + N_{x_1} + N_{x_2}$, being N_{x_i} the number of limit cycles which intersect only the line $x = x_i$. So to prove our main result we only need to control N_{x_i}.

From Proposition 2.1 and assumption (D) we know that all limit cycles must intersect *both* the lines $x = x_i$, $i = 1, 2$. Namely $N_{x_i} = 0$, $i = 1, 2$.

As already remarked in Section 2.1 our hypotheses imply existence of at least one limit cycle, $N_{\mathrm{lc}} \geqslant 1$, thus we finish our proof by recalling that Theorem 2.3 gives $N_{\mathrm{lc}} \leqslant 1$.

Before passing to the applications of our theorem, let us consider in the next paragraph what can happen when we do not assume hypothesis (D).

2.4. Removing the assumption $G(x_1) = G(x_2)$

The first remark is that assumption (D) cannot be removed without avoiding cases with more than one limit cycle, as the following example shows.

Remark 2.4 (A case with $G(x_1) < G(x_2)$). Starting from a classical counterexample of Duff and Levinson [2] to the H. Serbin conjecture [8], we exhibit a polynomial system where all hypotheses (A)–(E) are verified but (D), which has 3 limit cycles.

Let us consider the equation:

$$\ddot{x} + \epsilon f(x)\dot{x} + g(x) = 0, \tag{4}$$

where ϵ is a small parameter, $g(x) = x$ and f is a polynomial of degree 6, $f(x) = \sum_{l=0}^{3} a_{2l}x^{2l} + Ax + Bx^3$, where $a_0 I_0 = -4/81$, $a_2 I_2 = 49/81$, $a_4 I_4 = -14/9$, $a_6 I_6 = 1$, $I_{2k} = \int_0^{2\pi} \sin^2\theta \cos^{2k}\theta\, d\theta$ and A, B to be determined. Coefficients $(a_{2l})_{2l}$ are fixed in such a way that, passing to polar coordinates, for ϵ small enough and $A, B = 0$, system (4) has three limit cycles.

In fact, let us introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then (4) can be rewritten as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - \epsilon f(x) y. \end{cases}$$

Thus

$$\frac{dr}{d\theta} = \frac{\epsilon r f(r\cos\theta)\sin^2\theta}{1 + \epsilon f(r\cos\theta)\sin\theta\cos\theta}.$$

If r and $|\epsilon|$ are small enough, we can rewrite the previous equation as

$$\frac{dr}{d\theta} = \epsilon\big[H_0(r,\theta) + \epsilon H_1(r,\theta) + \epsilon^2 H_2(r,\theta,\epsilon)\big], \tag{5}$$

where H_i are analytic functions of r, θ and ϵ. Let $\rho > 0$ and let us denote by $r(\theta, \rho, \epsilon)$ the solution of (5) with initial datum $r = \rho$. Then our system has a limit cycle if and only if ρ is an isolated positive root of $r(2\pi, \rho, \epsilon) - \rho = 0$. Integrating (5) we get

$$r(2\pi, \rho, \epsilon) - \rho = \epsilon \bar{F}(\rho) + \epsilon^2 R_2(\rho, \epsilon), \tag{6}$$

where $\bar{F}(\rho) = \int_0^{2\pi} \rho f(\rho\cos\theta)\sin^2\theta\, d\theta$ and $R_2(\rho, \epsilon)$ is some analytic remainder function. With our choice of $(a_{2l})_{0\leqslant l\leqslant 3}$ we obtain $\bar{F}(\rho) = \rho(\rho^2 - 1/9)(\rho^2 - 4/9)(\rho^2 - 1)$, and then from (6) we conclude that if $|\epsilon|$ is sufficiently small, $r(2\pi, \rho, \epsilon) - \rho$ has three positive isolated simple roots, ϵ-close to $1/3$, $2/3$ and 1.

The method used to find the number of limit cycle does not involve the values of A, B. We claim that we can vary these parameters in such a way that $F(x) = \int_0^x f(\xi)\, d\xi$ verifies hypothesis (C), with $|x_2| > x_1$ and then $G(x) = x^2/2$ does not verify hypothesis (D). Just as an example consider

$$F(x) = \frac{x}{\pi}\left(-\frac{4}{81} + \frac{196}{81}\frac{x^2}{3} - \frac{112}{9}\frac{x^4}{5} + \frac{64}{5}\frac{x^6}{7} + \frac{1}{200}x + \frac{1}{2}x^3\right),$$

which has three real zeros $x_0 = 0$, $x_2 < 0 < x_1$ and its monotone increasing outside (x_2, x_1). Moreover, $f(x) = F'(x)$ has four zeros in the same interval.[1]

To conclude this part let us remark that adding further assumptions on $F(x)$, one can ensure that all limit cycles must intersect both lines $x = x_1$, $x = x_2$, thus obtaining a *existence and uniqueness* result for (2). For instance one can prove the following theorem.

Theorem 2.5. *Assume f and g verify hypotheses* (A) *and* (B). *Let F and G be the primitives of f and g vanishing at $x = 0$ and assume they verify hypotheses* (C) *and* (E). *Assume one of the following conditions holds*:

[1] Using Sturm's method to find real roots of polynomials we obtain that the zeros of F belong to the intervals, $x_2 \in [-1.130, -1.129]$ and $x_1 \in [0.247, 0.248]$, whereas zeros of f verify $x'_4 \in [-0.969, -0.9688]$, $x'_3 \in [-0.343, -0.342]$, $x'_2 \in [-0.173, -0.172]$ and $x'_1 \in [0.139, 0.140]$.

(D′) *$G(x_1) > G(x_2)$ and there exists $x_2^* \in (x_2, 0)$ such that $F(x_2^*) \geqslant \sqrt{2G(x_1)}$;*
(D″) *$G(x_1) < G(x_2)$ and there exists $x_1^* \in (0, x_1)$ such that $F(x_1^*) \leqslant -\sqrt{2G(x_2)}$.*

Then Liénard system (2) *has one and only one limit cycle.*

Proof. We only prove the theorem assuming (D′), being the other case very similar. Let us assume $G(x_1) > G(x_2)$ and that there exists $x_2^* \in (x_2, 0)$ such that $F(x_2^*) \geqslant \sqrt{2G(x_1)}$, we will prove that any orbit which intersects the line $x = x_2^*$ must intersect also the line $x = x_1$.

Considering the oval $\mathcal{E}_{G(x_1)} = \{(x, y) \in \mathbb{R}^2\colon\ y^2/2 + G(x) - G(x_1) = 0\}$ one realizes that there exists a unique point $(0, y_A)$ with $y_A < \sqrt{2G(x_1)}$, whose future orbit will intersect the line $x = x_1$ at the point $(x_1, 0)$.

Let us consider now a point (x_2^*, y_B), with $y_B \geqslant F(x_2^*)$, we claim that its future orbit will intersect the y-axis at some $(0, y_{B'})$ such that $y_{B'} > \sqrt{2G(x_1)}$. This can be proved by considering the evolution of the function $\Lambda(x, y) = y^2/2 + G(x)$ under the flow of the Liénard system.

Summarizing, the orbit of all points of the form (x_2^*, y_B), $y_B > F(x_2^*)$, will intersect the line $x = x_1$ with positive coordinate y. This concludes the proof once we remark that orbits of points (x_2^*, y'), $y' < F(x_2^*)$, turn clockwise and will intersect again the line $x = x_2^*$ at the some point (x_2^*, y'') with $y'' \geqslant F(x_2^*)$.

To complete the proof of the theorem one remark that by Proposition 2.1 all limit cycles must intersect the line $x = x_2$. Hence they must intersect the line $x = x_2^*$, being $x_2 < x_2^*$. By the first part these limit cycles intersect also the line $x = x_1$ and then applying Theorem 2.2 we conclude the proof. □

3. Some applications

In this section we give some applications of Theorem 1.1. The first application concerns Liénard's systems (2), where F and G verify all hypotheses of Theorem 1.1 but (D) (Sections 3.1 and 3.2). Our aim is to show that we can find a new Liénard system (slightly modified version of the original one) for which Theorem 1.1 holds, exhibiting one and only a limit cycle. The second application is of different nature, starting with a given Liénard system, which does not verify assumptions of Theorem 1.1, we prove existence and uniqueness of limit cycles for a new system obtained from the first one just by introducing two parameters. We will consider the polynomial case (Section 3.3) and a more general one (Section 3.4).

3.1. Case I: Deform g

Let us recall that F has three real zeros, $x_0 = 0$ and $x_2 < 0 < x_1$, let us assume $G(x_1) \neq G(x_2)$. Let us introduce the 1-parameter family of functions:

$$g_\lambda(x) = \begin{cases} g(x) & \text{if } x \geqslant 0, \\ \lambda g(x) & \text{if } x < 0. \end{cases} \tag{7}$$

Then $(g_\lambda)_\lambda$ verifies hypotheses (A) and (B) of Theorem 1.1, provided $\lambda > 0$. Let us define $G_\lambda(x) = \int_0^x g_\lambda(\xi)\,d\xi$. Let $\lambda_* = G(x_1)/G(x_2) > 0$, then $G_{\lambda_*}(x_1) = G_{\lambda_*}(x_2)$. Hence also hypotheses (D) and (E) hold and the differential equation

$$\ddot{x} + f(x)\dot{x} + g_{\lambda_*}(x) = 0,$$

has a unique isolated periodic solution.

3.2. Case II: Deform F

Let us assume $G(x_1) < G(x_2)$. The idea is now to modify the roots of F in such a way hypothesis (D) holds. We do this in a simple way, more sophisticated ones are possible.

Let $\lambda > 0$ and let us introduce the 1-parameter family of functions $(F_\lambda)_\lambda$, defined by

$$F_\lambda(x) = \begin{cases} F(x) & \text{if } x \geqslant 0, \\ F(\lambda x) & \text{if } x < 0. \end{cases}$$

Clearly $(F_\lambda)_\lambda$ verifies hypothesis (E) if F does; $(F_\lambda)_\lambda$ is no longer Lipschitz at $x = 0$ but existence and uniqueness of the Cauchy problem are still verified.

Thanks to the form of g and hypothesis on G, there exists the unique $x_2^* < 0$ such that $G(x_2^*) = G(x_1)$, moreover $x_2 < x_2^*$. Let $\lambda_* = |x_2|/|x_2^*|$ and $\bar{x}_{\lambda_*} = x_2/\lambda_*$. We claim that $\bar{x}_{\lambda_*}$ is the unique negative zero of $F_\lambda(x)$ and hence hypotheses (C) and (D) hold. F_λ, in fact, has three zeros, x_0, $x_1 > 0$ (as F does) and $\bar{x}_\lambda$, moreover, $G(\bar{x}_\lambda) = G(x_2^*) = G(x_1)$. Hence

$$\begin{cases} \dot{x} = y - F_{\lambda_*}(x), \\ \dot{y} = -g(x), \end{cases}$$

has the unique limit cycle.

3.3. Polynomial case

Let us consider a polynomial $P_{2n+1}(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x$. Assume $n \geqslant 1$, $a_{2n+1} > 0$ and hypothesis (C) does not hold. We claim that we can introduce a modified polynomial $P_\lambda(x) = P_{2n+1}(x) - \lambda x$ and a function g verifying hypotheses (A), (B) and (D) such that

$$\begin{cases} \dot{x} = y - P_\lambda(x), \\ \dot{y} = -g(x), \end{cases} \tag{8}$$

has the unique limit cycle.

$P_{2n+1}(x)$ has at most $2n$ local maxima and minima, so let us define:

$$\xi_+ = \min\bigl\{x > 0\colon \forall y > x \ P'_{2n+1}(y) > 0 \text{ and } P''_{2n+1}(y) > 0\bigr\},$$
$$\xi_- = \max\bigl\{x < 0\colon \forall y < x \ P'_{2n+1}(y) > 0 \text{ and } P''_{2n+1}(y) > 0\bigr\}.$$

Let us consider $\lambda_\pm \geqslant 0$ such that

$$P_{2n+1}(x) \leqslant \lambda_+ x \quad \text{for all } 0 < x < \xi_+ \quad \text{and}$$
$$P_{2n+1}(x) \geqslant \lambda_- x \quad \text{for all } \xi_- < x < 0.$$

Such the $\lambda_\pm$ can be obtained as follows. Consider the straight lines $y = \mu x$ tangent to $y = P_{2n+1}(x)$ for $x \in (0, \xi_+)$. They are in finite number, so one can take $\lambda_+ = \max|\mu_i|$; if $P_{2n+1}(x) < 0$ on $(0, \xi_+)$ we set $\lambda_+ = 0$. A similar construction can be done for λ_-.

Let $\bar{\lambda} = \max\{\lambda_+, \lambda_-\}$. We claim that for all $\lambda > \bar{\lambda}$, $P_\lambda(x) = P_{2n+1}(x) - \lambda x$ satisfies hypothesis (C). By construction, $P_\lambda(x) < 0$ for all $x \in (0, \xi_+)$ and $P_\lambda(x) > 0$ for all $x \in (\xi_-, 0)$. Because of $a_{2n+1} > 0$ for sufficiently large $|x|$, $P_\lambda(x)$ has the same sign as x. Then for $x > 0$ large enough, $P_\lambda(x) > 0$ and hence there is at least one zero of $P_\lambda(x)$. Actually this will be the only one. On contrary, suppose there are more zeros[2] and call them $\bar{x}_1 < \bar{x}_2 < \bar{x}_3$. By construction for all $x \in (\bar{x}_1, \bar{x}_2)$ we have $P_{2n+1}(x) > \lambda x$ whereas $P_{2n+1}(x) < \lambda x$ for $x \in (\bar{x}_2, \bar{x}_3)$. This implies $P_{2n+1}(x)$ non-convex for $x > \xi_+$, against the definition of ξ_+. The case for negative x can be handle in a similar way. Let us call x_1 the positive zeros and x_2 the negative one. Summarizing: $P_\lambda(x)$ has three real zeros: $x_0 = 0$, $x_2 < 0 < x_1$, moreover, $P_\lambda(x) < 0$ for $0 < x < x_1$, and $P_\lambda(x) > 0$ for $x_2 < x < 0$. Remark that $x_1 > \xi_+$ and $x_2 < \xi_-$, namely $P_\lambda(x)$ is monotone increasing outside $[x_2, x_1]$.

Let g be any locally Lipschitz function such that $xg(x) > 0$ for $x \neq 0$ and $\int_{x_2}^{x_1} g(\xi)\, d\xi = 0$, then Theorem 1.1 applies and (8) has a unique limit cycle.

3.4. *Generalization of the polynomial case*

In this section we will generalize the result of the previous section, by proving an existence and uniqueness result for the Liénard equation.

Theorem 3.1. *Let us consider the Liénard equation*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{9}$$

where f and g verify:

(A) *f is continuous and g is locally Lipschitz*;
(B′) $\lim_{x\to\pm\infty} f(x) = +\infty$ *and* $xg(x) > 0$ *for all* $x \neq 0$.

Then there exists $\hat{\lambda}$, such that for all $\lambda \geqslant \hat{\lambda}$ there exists $\mu = \mu(\lambda)$ and system

$$\ddot{x} + f_\lambda(x)\dot{x} + g_\mu(x) = 0, \tag{10}$$

has the unique limit cycle, where $f_\lambda(x) = f(x) - \lambda$ and g_μ are defined in (11).

Remark 3.2. Hypothesis (B′) is a strong one, even though it is verified for the important class of polynomial Liénard equations. It can be relaxed by assuming $\lim_{x\to\pm\infty} F(x) = \pm\infty$ and F to be monotone increasing outside some interval containing the origin, where as usual $F(x) = \int_0^x f(\xi)\, d\xi$.

Proof. For any $\lambda_1 > f(0)$, system (9) where $f_{\lambda_1}(x) = f(x) - \lambda_1$ replaces $f(x)$, has at least a limit cycle (see [12, Theorem 3]). Then one can find a $\hat{\lambda} \geqslant \lambda_1$ such that for all $\lambda \geqslant \hat{\lambda}$,

[2] They will be at least three, if transversal, because $P_\lambda(\xi_+) < 0$ and $P_\lambda(x) > 0$ for x large enough. Non-transversal zeros can be removed by small increment of λ.

$F_\lambda(x) = -\lambda x + \int_0^x f(\xi)\,d\xi$ verifies hypotheses of Theorem 1.1. Just use monotonicity of F, as we did in the previous section for the polynomial case, to ensure that with λ large enough, F_λ has only two non-zeros roots and it is monotone increasing outside the interval whose boundary is formed by the two non-zeros roots.

Let us call $x_2(\lambda) < 0 < x_1(\lambda)$, the non-zeros roots of $F_\lambda(x) = 0$. Then we can modify g (for instance as we did in Section 3.1) by introducing

$$g_\mu(x) = \begin{cases} g(x) & \text{if } x \geqslant 0; \\ \mu g(x) & \text{if } x < 0, \end{cases} \tag{11}$$

in a way $\int_{x_2}^{x_1} g(\xi)\,d\xi = 0$. Namely also hypothesis (D) of Theorem 1.1 holds, and so system (10) has the unique limit cycle. □

The role of f and g in the previous theorem may be in some sense inverted. More precisely, one can prove the following result.

Remark 3.3. Let us consider the global center system

$$\ddot{x} + g(x) = 0, \tag{12}$$

with g locally Lipschitz, $xg(x) > 0$ for $x \neq 0$, $G(x) = \int_0^x g(\xi)\,d\xi$ and assume $\lim_{x\to\pm\infty} G(x) = +\infty$. Take any $x_2 < 0 < x_1$ such that $G(x_2) = G(x_1)$. Then we can *perturb* (12) by adding *any continuous friction term* $f(x)\dot{x}$, such that $F(x) = \int_0^x f(\xi)\,d\xi$ verifies $F(x_1) = F(x_2) = 0$ and $F(x)$ is monotone increasing outside the interval $[x_2, x_1]$, obtaining a Liénard system $\ddot{x} + f(x)\dot{x} + g(x) = 0$ with one and only one limit cycle.

After this paper has been submitted for publication on JMAA, we learned that a result in the same framework appeared on [13]. However, despite some obvious similarities in the proof, the papers are independent and have different applications. We also would like to mention that the present result has been already generalized in two directions [1,6].

References

[1] T. Carletti, Uniqueness of limit cycles for a class of planar vector fields, QTDS (2005), in press.

[2] G.F.D. Duff, N. Levinson, On the non-uniqueness of periodic solutions for an asymmetric Liénard equation, Quart. Appl. Math. 10 (1952) 86–88.

[3] A.F. Filippov, A sufficient condition for the existence of a stable limit cycle for a second order equation, Mat. Sb. 30 (1952) 171–180 (in Russian).

[4] N. Levinson, O.K. Smith, A general equation for relation oscillations, Duke Math. J. 9 (1942) 382–403.

[5] J.L. Massera, Sur un théorème de G. Sansone sur l'équation de Liénard, Boll. Un. Mat. Ital. 9 (1954) 367–369.

[6] M. Sabatini, G. Villari, About limit cycle's uniqueness for a class of generalized Liénard systems, preprint number 672, Univ. Trento, 2004.

[7] G. Sansone, Sopra l'equazione di A. Liénard delle oscillazioni di rilassamento, Ann. Mat. Pura Appl. (4) 28 (1949) 153–181.

[8] H. Serbin, Periodic motions of non-linear dynamical system, Quart. Appl. Math. 8-A (1950) 296–303.

[9] U. Staude, Uniqueness of periodic solutions of the Liénard equation, in: Recent Advances in Differential Equations, Academic Press, 1981, pp. 421–429.

[10] G. Villari, On the existence of periodic solutions for Liénard equation, Nonlinear Anal. 7 (1983) 71–78.

[11] G. Villari, Some remarks on the uniqueness of the periodic solutions for Liénard's equation, Boll. Unione Mat. Ital., Anal. Funz. Appl. C (6) IV (1985) 173–182.
[12] G. Villari, On the qualitative behaviour of solutions of Liénard equation, J. Differential Equations 67 (1987) 269–277.
[13] D. Xiao, Z. Zhang, On the uniqueness and nonexistence of limit cycles for predator–prey systems, Nonlinearity 16 (2003) 1185–1201.
[14] Y.Q. Ye, et al., Theory of Limit Cycles, Transl. Math. Monogr., vol. 66, Amer. Math. Soc., Providence, RI, 1986.
[15] Z. Zhi-fen, et al., Qualitative Theory of Differential Equations, Transl. Math. Monogr., vol. 101, Amer. Math. Soc., Providence, RI, 1992.