Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

On criticality for competing influences of boundary and external field in the Ising model

Priscilla E. Greenwood, Jiaming Sun

submitted: 10th October 1996

Department of Mathematics
University of British Columbia
Vancouver, BC, V6T 1Z2
Canada
e-mail: pgreenw@math.ubc.ca and
sunj@math.ubc.ca

Preprint No. 272
Berlin 1996

1991 Mathematics Subject Classification. 60K35, 82B27.
Key words and phrases. Competing influences, Ising model, Gibbs measures.

This research was supported by the Natural Sciences and Engineering Research Council of Canada.
The authors are grateful for the hospitality of the WIAS.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
e–mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
e–mail (Internet): preprint@wias-berlin.de

1 Introduction

Consider the Gibb's measures $\mu_{\Lambda(1/h),-,s}$ (defined below) of the Ising model, in a box $\Lambda(1/h)$ in Z^d with side length $1/h$, with external field s and negative boundary condition at a temperature $T < T_c$. It is well known that when $s = 0$, namely no external field, $\mu_{\Lambda(1/h),-,0}$ converges weakly to the pure $(-)$-phase μ_- as $h \searrow 0$. And when $s \neq 0$ is fixed, $\mu_{\Lambda(1/h),-,s}$ converges weakly to a measure μ_s which does not depend on the boundary conditons (Ellis (1985)). But if one lets the external field s decrease as h goes to zero in such a way that it competes with the negative boundary (in particular, if $s = Bh$), then one may obtain different limits in different ranges of B. This phenomenon of competing influences has been investigated by several authors. Martirosyan (1987) first proved that at low temperature T and with large B, the Gibb's measure $\mu_{\Lambda(1/h),-,Bh}$ converges weakly to the pure $(+)$-phase μ_+. Schonmann (1994) (referred to in the sequel as [Sch]) showed that at low temperature T, there are values $B_1(T) \leq B_2(T)$ such that when $B < B_1(T)$, $\mu_{\Lambda(1/h),-,Bh}$ converges weakly to μ_- and when $B > B_2(T)$, the limit is μ_+. This says that the negative boundary conditon dominates in the limit when $B < B_1(T)$ whereas the small external field dominates when $B > B_2(T)$. The question, then, is whether there exists a critical value $B_0 = B_0(T) = B_1(T) = B_2(T)$ for all $T < T_c$ such that $\mu_{\Lambda(1/h),-,Bh}$ converges to μ_- when $B < B_0$ and to μ_+ when $B > B_0$. In the case of $d = 2$, this question was completely solved by Schonmann and Shlosman (1996), using large deviation results and techniques. For higher dimensions, Greenwood and Sun (1997) ([GS] hereafter) proved the criticality of a certain value B_0 for all $T < T_c$, but only in terms of the convergence of average spins rather than in terms of weak convergence. This paper extends these results by showing that for low temperature and the same critical value B_0, $\mu_{\Lambda(1/h),-,Bh}$ converges weakly to μ_- when $B < B_0$ and to μ_+ when $B > B_0$.

In [Sch], the main results are about the relaxation time of a *stochastic* Ising model in relation to an external field h. He shows that the relaxation time blows up when $h \searrow 0$ as $\exp(\lambda/h^{d-1})$. In fact he obtains upper and lower bounds for $\lambda = \lambda(T)$, which are derived from his $B_1(T)$, $B_2(T)$ and his estimate of the spectral gap of the generator of the evolution. One might hope to obtain a critical value of λ using Schonmann's methods and the critical value B_0. This indeed again gives bounds for λ but not a critical value. A reason is that estimation of the spectral gap is involved.

2 Notations and Results

Let $\Omega = \{-1,1\}^{Z^d}$ be the configuration space. Let $\mathcal{F}$ be the set of all finite subsets of Z^d. For any $\Lambda \in \mathcal{F}$, define $\Omega(\Lambda) = \{-1,1\}^{\Lambda}$. Typical configurations in Ω or $\Omega(\Lambda)$

are denoted by $\sigma, \eta, \cdots$. Denote the value of σ at $x \in Z^d$ by σ_x. For two sites x and y in Z^d, define $|x-y| = \max\{|x_1 - y_1|, \cdots, |x_d - y_d|\}$. The energy function of the Ising model in Λ, with boundary condition η and external field s is

$$H_{\Lambda,\eta,s}(\sigma) := -\frac{1}{2} \sum_{\{x,y\}\subset\Lambda, |x-y|=1} \sigma_x \sigma_y - \frac{1}{2} \sum_{x\in\Lambda, y\in\Lambda^c, |x-y|=1} \sigma_x \eta_y - \frac{s}{2} \sum_{x\in\Lambda} \sigma_x, \tag{2.1}$$

for $\sigma \in \Omega$. Given a set $\Lambda \in \mathcal{F}$ and a configuration η, we introduce

$$\Omega_{\Lambda,\eta} := \{\sigma \in \Omega : \quad \sigma_x = \eta_x \text{ for all } x \notin \Lambda\}.$$

The Gibbs measure in Λ with boundary condition η and external field s at the temperature $T = 1/\beta$ is defined on Ω as

$$\mu_{\Lambda,\eta,s}(\sigma) := \begin{cases} \exp(-\beta H_{\Lambda,\eta,s}(\sigma))/Z_{\Lambda,\eta,s} & \text{if } \sigma \in \Omega_{\Lambda,\eta} \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

where $Z_{\Lambda,\eta,s}$ is the normalizing constant given by

$$Z_{\Lambda,\eta,s} := \sum_{\sigma\in\Omega_{\Lambda,\eta}} \exp(-\beta H_{\Lambda,\eta,s}(\sigma)). \tag{2.3}$$

Expectation with respect to $\mu_{\Lambda,\eta,s}$ is denoted by $E_{\Lambda,\eta,s}$. When $\eta \equiv -1$ or $+1$, we replace η by $-$ or $+$. It is well known that $\mu_{\Lambda,-,0}$ ($\mu_{\Lambda,+,0}$) converges weakly to the pure ($-$)-phase μ_- (($+$)-phase μ_+) as Λ grows to Z^d. For $d \geq 2$, there is a critical temperature $T_c > 0$ such that $\mu_- = \mu_+$ if $T > T_c$ and $\mu_- \neq \mu_+$ if $T < T_c$, in which case a phase transition occurs. The spantaneous magnetization $m_T^* := E_+[\sigma_0]$ ($= -E_-[\sigma_0]$) is zero when $T > T_c$ and positive when $T < T_c$.

To study the competing influences of boundary condition and external field, let $\Lambda(1/h)$ be the cube in Z^d with side length $1/h$ and centered at the origin for all $h > 0$. The average of σ in $\Lambda(1/h)$, called the average spin, is defined by

$$X_{\Lambda(1/h)}(\sigma) := (1/h)^{-d} \sum_{x\in\Lambda(1/h)} \sigma_x, \quad \forall \sigma \in \Omega. \tag{2.4}$$

It also well known that $X_{\Lambda(1/h)}$ converges a.s. as $h \searrow 0$ to $-m_T^*$ (m_T^*) under μ_- (μ_+).

Define, for all $t \in R$,

$$\overline{\Phi}(t) = \limsup_{h\searrow 0} h^{d-1} \log E_{\Lambda(1/h),-,0}[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})]. \tag{2.5}$$

The function $\overline{\Phi}(t)$ is continuous and convex on R by Hölder's inequality. Define

$$B_0 = B_0(T) := 2T \sup\{t : \overline{\Phi}(t) = -m_T^* t\}. \tag{2.6}$$

For all $T < T_c$, B_0 is nonnegative and for small T, it is positive ([GS]). It is still an open problem whether $B_0 > 0$ for all $T < T_c$. At the end of the next section we see that B_0 converges to $2d$ as T goes to zero (Remark 2). We know from [GS] that for all $T < T_c$, if $B < B_0$ then for all $\epsilon > 0$,

$$\mu_{\Lambda(1/h),-,Bh}(|X_{\Lambda(1/h)} - (-m_T^*)| \geq \epsilon) \to 0, \text{ as } h \searrow 0, \tag{2.7}$$

whereas if $B > B_0$ then the average spin $X_{\Lambda(1/h)}$ no longer converges to $-m_T^*$ under $\mu_{\Lambda(1/h),-,Bh}$. Now we state our main results.

Theorem 1. For all $T < T_c$, (a) $0 \leq B_0 \leq 2d/m_T^*$, and (b) $\mu_{\Lambda(1/h),-,Bh}$ converges weakly to μ_- as $h \searrow 0$ if $B < B_0$.

Theorem 2. For small temperature T and all $B > B_0$, $\mu_{\Lambda(1/h_n),-,Bh_n}$ converges weakly to μ_+ along a subsequence $h_n \searrow 0$. Moreover, $\mu_{\Lambda(1/h),-,Bh}$ converges weakly to μ_+ along the whole sequence $h \searrow 0$ if $B > 2d/m_T^*$.

Theorems 1 and 2 together yield that B_0 is a critical value of the balance parameter B for low temperature. Note that in Theorem 2 the weak convergence of $\mu_{\Lambda(1/h),-,Bh}$ to μ_+ is proved only along a subsequence $h_n \searrow 0$ for $B \in (B_0, 2d/m_T^*]$. It is reasonable that this should hold along the whole sequence $h \searrow 0$, but our method does not prove this.

3 The Proofs

To prove Theorem 1(b) we first show that any weak limit μ of the Gibbs measures $\mu_{\Lambda(1/h),-,Bh}$ has the same one dimensional marginal first moments as μ_-. Then we use the monotonicity $\mu_- \leq \mu$, to conclude that $\mu_- = \mu$.

Lemma 1. For all $T < T_c$, $B < B_0$ and $x \in Z^d$,

$$\lim_{h \searrow 0} E_{\Lambda(1/h),-,Bh}[\sigma_x] = -m_T^* \quad (= E_{\mu_-}[\sigma_x]). \tag{3.8}$$

Proof. We will use the result about average spin (2.7) to prove (3.8). Suppose that $B < B_0$. The FKG inequality implies that

$$-m_T^* = \lim_{h \searrow 0} E_{\Lambda(1/h),-,0}[\sigma_x] \leq \liminf_{h \searrow 0} E_{\Lambda(1/h),-,Bh}[\sigma_x], \tag{3.9}$$

for all $x \in Z^d$. So we need only prove that

$$\limsup_{h \searrow 0} E_{\Lambda(1/h),-,Bh}[\sigma_x] \leq -m_T^*, \tag{3.10}$$

for all $x \in Z^d$. We first show (3.10) for $x = 0$ and all $B < B_0$ and then extend it to all $x \in Z^d$. Fix $B_1 \in (B, B_0)$ and let $B/B_1 = 1 - \epsilon$. Note that, for each x in the box $\Lambda(\epsilon/h)$, there is a box $\Lambda_x((1-\epsilon)/h)$ inside $\Lambda(1/h)$ and centered at x. By the FKG inequality and the translation invariance of Gibbs measures,

$$
\begin{aligned}
&E_{\Lambda(1/h),-,B_1 h}[\sum_{x\in\Lambda(1/h)} \sigma_x] \\
&\quad \geq E_{\Lambda(1/h),-,B_1 h}[\sum_{x\in\Lambda(1/h)\setminus\Lambda(\epsilon/h)} \sigma_x] + \sum_{x\in\Lambda(\epsilon/h)} E_{\Lambda_x((1-\epsilon)/h),-,B_1 h}[\sigma_x] \\
&\quad \geq E_{\Lambda(1/h),-,0}[\sum_{x\in\Lambda(1/h)\setminus\Lambda(\epsilon/h)} \sigma_x] + |\Lambda(\epsilon/h)| \cdot E_{\Lambda((1-\epsilon)/h),-,B_1 h}[\sigma_0] \\
&\quad = E_{\Lambda(1/h),-,0}[\sum_{x\in\Lambda(1/h)} \sigma_x] - E_{\Lambda(1/h),-,0}[\sum_{x\in\Lambda(\epsilon/h)} \sigma_x] + \\
&\qquad + |\Lambda(\epsilon/h)| \cdot E_{\Lambda((1-\epsilon)/h),-,Bh/(1-\epsilon)}[\sigma_0] \\
&\quad \geq E_{\Lambda(1/h),-,0}[\sum_{x\in\Lambda(1/h)} \sigma_x] - |\Lambda(\epsilon/h)|(-m_T^*) + |\Lambda(\epsilon/h)| \cdot E_{\Lambda((1-\epsilon)/h),-,Bh/(1-\epsilon)}[\sigma_0].
\end{aligned}
$$

In the last step we used that, by FKG, $E_{\Lambda(1/h),-,0}[\sigma_x]$ increases to $-m_T^*$ as $h \searrow 0$. Dividing the above inequalities by $|\Lambda(1/h)|$ and taking the superior limit, we get

$$
\limsup_{h\searrow 0} E_{\Lambda(1/h),-,B_1 h}[X_{\Lambda(1/h)}] \geq -m_T^* + \epsilon^d m_T^* + \epsilon^d \limsup_{h\searrow 0} E_{\Lambda(1/h),-,Bh}[\sigma_0].
$$

By (2.7), since $B_1 < B_0$, the LHS equals $-m_T^*$. This proves (3.10) for $x = 0$ and all $B < B_0$. Now let $x \in Z^d$ be arbitrary. Then $x \in \Lambda(1/h)$ for all small $h > 0$. Define $h' > 0$ such that $\frac{1}{h'} = 2|x| + \frac{1}{h}$. Then $h' < h$, $\frac{h}{h'} \to 1$ as $h \searrow 0$ and the box $\Lambda_x(1/h')$ centered at x contains $\Lambda(1/h)$. Choose $B_1 \in (B, B_0)$. Note that $B\frac{h}{h'} \leq B_1$ for small $h > 0$. By the FKG inequality,

$$
\begin{aligned}
E_{\Lambda(1/h),-,Bh}[\sigma_x] &\leq E_{\Lambda_x(1/h'),-,Bh}[\sigma_x] \\
&= E_{\Lambda(1/h'),-,B\frac{h}{h'}h'}[\sigma_0] \\
&\leq E_{\Lambda(1/h'),-,B_1 h'}[\sigma_0],
\end{aligned}
$$

for small $h > 0$. Combining this with (3.10) for $x = 0$, we obtain (3.10) for $x \in Z^d$.□

For $\eta, \xi \in \Omega$, we define $\eta \leq \xi$ if $\eta(x) \leq \xi(x)$ for all $x \in Z^d$. A function f on Ω is said to be increasing if $f(\eta) \leq f(\xi)$ whenever $\eta \leq \xi$. For any two probability measures μ_1 and μ_2, define $\mu_1 \leq \mu_2$ if $E_{\mu_1}[f] \leq E_{\mu_2}[f]$ for all bounded local increasing functions f on Ω.

Lemma 2. Let μ_1 and μ_2 be two probability measures on Ω such that $\mu_1 \leq \mu_2$. Suppose they have the same one dimensional marginal first moments, that is, $E_{\mu_1}[\sigma_x] = E_{\mu_2}[\sigma_x]$ for all $x \in Z^d$. Then $\mu_1 = \mu_2$.

Proof. Recall that $\Omega = \{-1,1\}^{Z^d}$. It is not difficult to see that for $\mu_1 = \mu_2$ one needs only

$$\mu_1(\sigma_{x_1} = \cdots = \sigma_{x_n} = 1) = \mu_2(\sigma_{x_1} = \cdots = \sigma_{x_n} = 1), \tag{3.11}$$

for any $x_1, \cdots, x_n \in Z^d$, $n \geq 1$. We prove (3.11) by induction. When $n = 1$, (3.11) holds because μ_1 and μ_2 have the same first moments. Now assume (3.11) is true for any $n-1$ sites $x_1, \cdots, x_{n-1} \in Z^d$. Consider any $x_1, \cdots, x_n \in Z^d$. Define

$$A = \{\sigma \in \Omega : \quad \sigma_{x_1} + \sigma_{x_2} \geq 0, \text{ and } \sigma_{x_3} = \cdots = \sigma_{x_n} = 1\}.$$

Then the indicator of A is increasing. By monotonicity, $\mu_1(A) \leq \mu_2(A)$. Define

$$\Delta = \mu_2(\sigma_{x_1} = \cdots = \sigma_{x_n} = 1) - \mu_1(\sigma_{x_1} = \cdots = \sigma_{x_n} = 1),$$

which is nonnegative since $\mu_2 \geq \mu_1$. Then

$$\begin{aligned}
&\mu_2(\sigma_{x_1} = -1, \sigma_{x_2} = \cdots = \sigma_{x_n} = 1) \\
&\quad = \mu_2(\sigma_{x_2} = \cdots = \sigma_{x_n} = 1) - \mu_2(\sigma_{x_1} = \cdots = \sigma_{x_n} = 1) \\
&\quad = \mu_1(\sigma_{x_2} = \cdots = \sigma_{x_n} = 1) - \mu_2(\sigma_{x_1} = \cdots = \sigma_{x_n} = 1) \\
&\quad = \mu_1(\sigma_{x_1} = -1, \sigma_{x_2} = \cdots = \sigma_{x_n} = 1) - \Delta.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mu_2(\sigma_{x_1} = 1, \sigma_{x_2} = -1, \sigma_{x_3} = \cdots = \sigma_{x_n} = 1) \\
&\quad = \mu_1(\sigma_{x_1} = 1, \sigma_{x_2} = -1, \sigma_{x_3} = \cdots = \sigma_{x_n} = 1) - \Delta.
\end{aligned}$$

So we have

$$\begin{aligned}
\mu_2(A) \;=\; & \mu_2(\sigma_{x_1} = \cdots = \sigma_{x_n} = 1) + \mu_2(\sigma_{x_1} = -1, \sigma_{x_2} = \cdots = \sigma_{x_n} = 1) + \\
& + \mu_2(\sigma_{x_1} = 1, \sigma_{x_2} = -1, \sigma_{x_3} = \cdots = \sigma_{x_n} = 1) = \mu_1(A) - \Delta.
\end{aligned}$$

Therefore $\Delta \leq 0$, and hence $\Delta = 0$, which completes the proof.□

Proof of Theorem 1. (a) It was proved in [GS] that $\overline{\Phi}(t) = -m_T^* t$ for $t \leq 0$. Hence $B_0 \geq 0$. Changing the negative boundary condition to positive and using the fact that $Z_{\Lambda(1/h),-,0} = Z_{\Lambda(1/h),+,0}$ and Jensen's inequality, one has

$$\begin{aligned}
&\log E_{\Lambda(1/h),-,0}[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})] \\
&\quad \geq \log[\exp(-2\beta d(1/h)^{d-1}) \cdot E_{\Lambda(1/h),+,0}[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})] \\
&\quad \geq -2\beta d(1/h)^{d-1} + t(1/h)^{d-1} E_{\Lambda(1/h),+,0}[X_{\Lambda(1/h)}].
\end{aligned} \tag{3.12}$$

Hence $\overline{\Phi}(t) \geq -2\beta d + m_T^* t$ for all $t > 0$. By the definition of B_0, it cannot exceed $2T$ times the value of t such that $-m_T^* t = -2\beta d + m_T^* t$, that is, $B_0 \leq 2d/m_T^*$.

(b) The Gibb's measures $\mu_{\Lambda(1/h),-,Bh}$ are weakly relatively compact since Ω is compact. Let μ be any weak limit of $\mu_{\Lambda(1/h),-,Bh}$ along a subsequence. The FKG inequality implies that $\mu_- \leq \mu$. By Lemma 1, μ and μ_- have the same one dimensional marginal first moments. Lemma 2 implies that $\mu = \mu_-$, and we have weak convergence of $\mu_{\Lambda(1/h),-,Bh}$ to μ_- as $h \searrow 0$.□

Theorem 2 is an extension of Theorem 2 ii) of [Sch] in which he proved the weak convergence of $\mu_{\Lambda(1/h),-,Bh}$ to μ_+ as $h \searrow 0$ for low temperature T and B exceeding some constant $B_2(T)$. We will use his strategy to prove Theorem 2.

The idea is to show that for $B > B_0$, with high probability under $\mu_{\Lambda(1/h),-,Bh}$ there will appear a large contour in $\Lambda(1/h)$, and then to show that as $h \searrow 0$, this large contour will eventually cover the whole space, and hence the Gibbs measures $\mu_{\Lambda(1/h),-,Bh}$ converge to μ_+ as $h \searrow 0$.

Contours are defined as usual ([Sch]). For any contour γ, let $\Theta(\gamma)$ be the set of sites inside of γ. To study the occurance of large contours, we denote by $A_{h,\epsilon}$ the set of configurations in $\Omega_{\Lambda(1/h),-}$ which have at least one contour surrounding a number of sites larger than the volume of $\Lambda(\epsilon/h)$, i.e.,

$$A_{h,\epsilon} = \{\sigma \in \Omega_{\Lambda(1/h),-} : \ \exists \text{ a contour } \gamma \text{ in } \sigma \text{ such that } |\Theta(\gamma)| > |\Lambda(\epsilon/h)|\}.$$

Note that $A_{h,\epsilon}$ (with h, ϵ fixed) is increasing, in the sense that its indicator is an increasing function as defined earlier, because of the negative boundary condition. The first step is to show that the probability $\mu_{\Lambda(1/h),-,Bh}(A_{h,\epsilon})$ converges to 1 as $h \searrow 0$ for $B > B_0$ and ϵ in a certain range.

Let b be the combinatorial constant defined by [Sch] in his expression (4). It appears in a bound for the number of families of contours satisfying certain constraints which we need not set out here. We will use inequality (44) of [Sch], which requires the quantity $\beta' := \beta - \log b$ to be positive. Later we will use that $\beta'/\beta \to 1$ as T goes to 0.

Lemma 3. Suppose $\beta > \log b$.

(a) For each $B \in (B_0, 2d/m_T^*]$, there exists a subsequence $h_n \searrow 0$ such that for all $\epsilon < 2d(\beta'/\beta)/B$ there exists a $c > 0$ such that,

$$\mu_{\Lambda(1/h_n),-,Bh_n}(A_{h_n,\epsilon}) \geq 1 - e^{-(1/h_n)^{d-1}(c+o(1))}. \tag{3.13}$$

(b) If $B > 2d/m_T^*$, then for all $\epsilon < m_T^*\beta'/\beta$ there exists a $c > 0$ such that

$$\mu_{\Lambda(1/h),-,Bh}(A_{h,\epsilon}) \geq 1 - e^{-(1/h)^{d-1}(c+o(1))}. \tag{3.14}$$

Remark 1. If $B \in (B_0, 2d/m_T^*]$, we have $2d(\beta'/\beta)/B \geq m_T^*\beta'/\beta$. Then the ϵ in both (a) and (b) can take values up to $m_T^*\beta'/\beta$, which depends only on the temperature T and converges to 1 when T goes to 0, since $\lim_{T\searrow 0} m_T^* = 1$ (Ellis (1985)).

Proof. The proof is a modification of that of Lemma 8 in [Sch]. Let $A = \epsilon B$. For any subset E of Ω, define

$$Z_{\Lambda,-,s}(E) = \sum_{\sigma \in E \bigcap \Omega_{\Lambda,-}} \exp(-\beta H_{\Lambda,-,s}(\sigma)).$$

Let $\mathcal{R}_{\epsilon/h}$ denote the complement of $A_{h,\epsilon}$. Then

$$\begin{aligned}
\mu_{\Lambda(1/h),-,Bh}(A^c_{h,\epsilon}) &= \mu_{\Lambda(1/h),-,Bh}(\mathcal{R}_{\epsilon/h}) \\
&= \frac{Z_{\Lambda(1/h),-,Bh}(\mathcal{R}_{\epsilon/h})}{Z_{\Lambda(1/h),-,0}(\mathcal{R}_{\epsilon/h})} \frac{Z_{\Lambda(1/h),-,0}(\mathcal{R}_{\epsilon/h})}{Z_{\Lambda(1/h),-,0}} \frac{Z_{\Lambda(1/h),-,0}}{Z_{\Lambda(1/h),-,Bh}} \\
&\leq \frac{Z_{\Lambda(1/h),-,Bh}(\mathcal{R}_{\epsilon/h})}{Z_{\Lambda(1/h),-,0}(\mathcal{R}_{\epsilon/h})} \frac{Z_{\Lambda(1/h),-,0}}{Z_{\Lambda(1/h),-,Bh}} \\
&= \frac{Z_{\Lambda(B/h'),-,h'}(\mathcal{R}_{A/h'})}{Z_{\Lambda(B/h'),-,0}(\mathcal{R}_{A/h'})} \frac{Z_{\Lambda(1/h),-,0}}{Z_{\Lambda(1/h),-,Bh}}, \quad (h' = Bh).
\end{aligned} \tag{3.15}$$

Under our assumption, $A = \epsilon B < 2d\beta'/\beta$. From inequality (44) of [Sch], which holds for all $B > 0$,

$$\limsup_{h'\searrow 0} (h')^{d-1} \log \frac{Z_{\Lambda(B/h'),-,h'}(\mathcal{R}_{A/h'})}{Z_{\Lambda(B/h'),-,0}(\mathcal{R}_{A/h'})} \leq -(\beta/2)m_T^* B^d, \tag{3.16}$$

for $\epsilon < 2d(\beta'/\beta)/B$. Note that $h' = hB$. So

$$\limsup_{h\searrow 0} (h)^{d-1} \log \frac{Z_{\Lambda(B/h'),-,h'}(\mathcal{R}_{A/h'})}{Z_{\Lambda(B/h'),-,0}(\mathcal{R}_{A/h'})} \leq -m_T^* \beta B/2, \tag{3.17}$$

for $\epsilon < 2d(\beta'/\beta)/B$. On the other hand, it is not difficult to see that for $h > 0$,

$$\frac{Z_{\Lambda(1/h),-,Bh}}{Z_{\Lambda(1/h),-,0}} = E_{\Lambda(1/h),-,0}[\exp(\frac{\beta B}{2}(1/h)^{d-1} X_{\Lambda(1/h)})]. \tag{3.18}$$

Now choose a subsequence so that

$$\overline{\Phi}(\frac{\beta B}{2}) = \lim_{n\to\infty} h_n^{d-1} \log E_{\Lambda(1/h_n),-,0}[\exp(\frac{\beta B}{2}(1/h_n)^{d-1} X_{\Lambda(1/h_n)})]. \tag{3.19}$$

The definition of B_0 says that $\overline{\Phi}(\beta B/2) > -m_T^*\beta B/2$ for all $B > B_0$. From (3.15), (3.17) and (3.19) (note that the limit exists in (3.19)), we have

$$\lim_{h_n\searrow 0} h_n^{d-1} \log \mu_{\Lambda(1/h_n),-,Bh_n}(A^c_{h_n,\epsilon}) \leq -m_T^*\beta B/2 - \overline{\Phi}(\beta B/2) < 0.$$

This proves (3.13).

To prove (b), let $B > 2d/m_T^*$. Using (3.18) and (3.12), we obtain that

$$h^{d-1}\log\frac{Z_{\Lambda(1/h),-,Bh}}{Z_{\Lambda(1/h),-,0}} \geq -2\beta d + \frac{\beta B}{2}E_{\Lambda(1/h),+,0}[X_{\Lambda(1/h)}].$$

By (3.15) and (3.17),

$$\begin{aligned}\lim_{h\searrow 0} h^{d-1}\log\mu_{\Lambda(1/h),-,Bh}(A^c_{h,\epsilon}) &\leq -m_T^*\beta B/2 + 2\beta d - m_T^*\beta B/2 \\ &= -m_T^*\beta(B - 2d/m_T^*) < 0, \end{aligned} \qquad (3.20)$$

for $\epsilon < 2d(\beta'/\beta)/B$. This proves (3.14) for $\epsilon < 2d(\beta'/\beta)/B$. Finally, if $\epsilon_1 < m_T^*\beta'/\beta$, choose B_1 such that $B > B_1 > 2d/m_T^*$ and $\epsilon_1 < 2d(\beta'/\beta)/B_1$. Note that $\mu_{\Lambda(1/h),-,Bh}(A_{h,\epsilon_1}) \geq \mu_{\Lambda(1/h),-,B_1h}(A_{h,\epsilon_1})$ since A_{h,ϵ_1} is an increasing set. Inequality (3.14) for ϵ_1 follows from (3.20) for $B = B_1$ and $\epsilon = \epsilon_1$.□

Proof of Theorem 2. The proof is much like the proof of Theorem 2 ii) in [Sch]. Let $\mathcal{B}$ be the set of all $\sigma \in \Omega_{\Lambda(1/h),-}$ such that the box $\Lambda(1/(2h))$ intersects the infinite cluster of negative spins in σ. The objective is to show that $\mu_{\Lambda(1/h),-,Bh}(\mathcal{B})$ converges to zero as $h \searrow 0$. Now let $\sigma \in \Omega$ and let $\mathcal{C}(\sigma)$ be the set of all sites in Z^d which belong to infinite clusters of negative spins in σ. In the setting of Lemma 10 of [Sch], replace the three boxes $\Lambda(d/h)$, $\Lambda(3d/(2h))$ and $\Lambda(2d/h)$ by $\Lambda(1/(2h))$, $\Lambda(3/(4h))$ and $\Lambda(1/h)$, respectively. Then following [Sch], for all $\alpha > 0$, one can define a subset $\mathcal{B}^\alpha$ of $\mathcal{B}$ such that for small $h > 0$,

$$\mathcal{B} \subset \mathcal{B}^\alpha \bigcup \{\sigma : |\mathcal{C}(\sigma) \bigcap \Lambda(1/h)| \geq (\frac{\alpha}{2^d - 1})^d (\frac{1}{16h})^d\}. \qquad (3.21)$$

We indicate briefly the structure of the set $\mathcal{B}^\alpha$, how the argument of [Sch] works and where our Lemma 3 enters the argument. For each $\sigma \in \mathcal{B}$, denote the infinite cluster of negative spins in σ which intersects the box $\Lambda(1/(2h))$ by N_σ. Now $\mathcal{B}$ is divided into two parts according to the shape of N_σ: $\mathcal{B}^\alpha$ consists of configurations σ such that N_σ contains, somewhere between its intersections with the boundary of $\Lambda(3/(4h))$ and that of $\Lambda(1/h)$, a relatively small set of negative spins which, when changed to positive, produce one or more new large contours which intersect both the box $\Lambda(1/(2h))$ and $\Lambda(3/(4h))$. The definition of $\mathcal{B}^\alpha$ ensures that this change of spins can be made "at little cost", compared with the probability of the changed configuration, which in turn, is exponentially small because it contains these large contours. On the other hand, if $\sigma \in \mathcal{B} \setminus \mathcal{B}^\alpha$, i.e., if N_σ does not have this shape, then σ must be in the second set of the union in (3.21), which we treat by Lemma 3.

With the modifications above, the delicate proof of Lemma 11 ([Sch]) can be amended to show that

$$\lim_{h \searrow 0} \mu_{\Lambda(1/h),-,Bh}(\mathcal{B}^{\alpha}) = 0, \tag{3.22}$$

for all $\alpha < (\beta/2 - \log b - \log 2)/(2\beta d)$ and $B > 0$, because the change of external field does not effect the proof.

Now we apply Lemma 3. Fix $l > 0$. Let $\epsilon^d = 1 - l^d$. For T small enough so that $m_T^* \beta'/\beta > \epsilon$,

$$\mu_{\Lambda(1/h),-,Bh}(|\mathcal{C}(\sigma) \bigcap \Lambda(1/h)| \geq (\frac{l}{h})^d) \leq \mu_{\Lambda(1/h),-,Bh}(A_{h,\epsilon}^c) \to 0, \tag{3.23}$$

as $h \searrow 0$ if $B > 2d/m_T^*$, and as $h \searrow 0$ along a subsequence if $B \in (B_0, 2d/m_T^*]$.

Now choose $\alpha > 0$ so that $\alpha < (\beta/2 - \log b - \log 2)/(2\beta d)$ for all T small and let l in (3.23) be the constant $\frac{\alpha}{16(2^d-1)}$ as given in (3.21). Then from (3.21), (3.22) and (3.23) we conclude that for small temperature T,

$$\mu_{\Lambda(1/h),-,Bh}(\mathcal{B}) \to 0, \tag{3.24}$$

as $h \searrow 0$ if $B > 2d/m_T^*$, and as $h \searrow 0$ along the subsequence if $B \in (B_0, 2d/m_T^*]$.

Note that for each $\sigma \in \Omega_{\Lambda(1/h),-} \setminus \mathcal{B}$, there exists a contour in σ which surrounds the box $\Lambda(1/(2h))$ and such that the spins at the inner boundary of the contour are all positive. Therefore, the argument of [Sch], p19, conditioning on the contours and then using the Markov property of Gibbs measures and the FKG inequality, gives that for any increasing local function f on Ω,

$$E_{\Lambda(1/h),-,Bh}[f] \to E_{\mu_+}[f], \tag{3.25}$$

as $h \searrow 0$ if $B > 2d/m_T^*$, and as $h \searrow 0$ along the subsequence if $B \in (B_0, 2d/m_T^*]$. Theorem 2 now follows.□

Remark 2. Theorem 2 i) of [Sch] says that $\mu_{\Lambda(1/h),-,Bh}$ converges weakly to μ_- for all $\beta > \log b$ and all $B < 2d\beta'/\beta$. By our Theorems 1 and 2, one can conclude that for low temperatures

$$2d\beta'/\beta \leq B_0 \leq 2d/m_T^*. \tag{3.26}$$

In particular, $\lim_{T \searrow 0} B_0 = 2d$, as one expects (see the comments following Theorem 2 of [Sch]).

References

R. S. Ellis, Entropy, Large Deviations, and Statistical Mechanics (Springer-Verlag, 1985).

P. Greenwood and J. Sun, Equivalences of the large deviation principle for Gibbs measures and critical balance in the Ising model, J. Stat. Phys. (1997) to appear.

D. G. Martirosyan, Theorems on strips in the classical Ising model, Soviet Journal Contemporary Mathematical Analysis, 22 (1987) 59-83.

R. H. Schonmann, Slow droplet-driven relaxation of stochastic Ising models in the vicinity of the phase coexistence region, Commun. Math. Phys. 161 (1994) 1-49.

R. H. Schonmann and S. B. Shlosman, Constrained variational problem with applications to the Ising model, J. Stat. Phys. (1996) to appear.

Recent publications of the
Weierstraß–Institut für Angewandte Analysis und Stochastik

Preprints 1996

243. Alexey K. Lopatin: Oscillations and dynamical systems: Normalization procedures and averaging.

244. Grigori N. Milstein: Stability index for invariant manifolds of stochastic systems.

245. Luis Barreira, Yakov Pesin, Jörg Schmeling: Dimension of hyperbolic measures – A proof of the Eckmann–Ruelle conjecture.

246. Leonid M. Fridman, Rainer J. Rumpel: On the asymptotic analysis of singularly perturbed systems with sliding mode.

247. Björn Sandstede: Instability of localised buckling modes in a one–dimensional strut model.

248. Björn Sandstede, Christopher K.R.T. Jones, James C. Alexander: Existence and stability of N–pulses on optical fibers with phase–sensitive amplifiers.

249. Vladimir Maz'ya, Gunther Schmidt: Approximate wavelets and the approximation of pseudodifferential operators.

250. Gottfried Bruckner, Sybille Handrock–Meyer, Hartmut Langmach: On the identification of soil transmissivity from measurements of the groundwater level.

251. Michael Schwarz: Phase transitions of shape memory alloys in soft and hard loading devices.

252. Gottfried Bruckner, Masahiro Yamamoto: On the determination of point sources by boundary observations: uniqueness, stability and reconstruction.

253. Anton Bovier, Véronique Gayrard: Hopfield models as generalized random mean field models.

254. Matthias Löwe: On the storage capacity of the Hopfield model.

255. Grigori N. Milstein: Random walk for elliptic equations and boundary layer.

256. Lutz Recke, Daniela Peterhof: Abstract forced symmetry breaking.

257. Lutz Recke, Daniela Peterhof: Forced frequency locking in $\mathbf{S}^1$–equivariant differential equations.

258. Udo Krause: Idealkristalle als Abelsche Varietäten.

259. Nikolaus Bubner, Jürgen Sprekels: Optimal control of martensitic phase transitions in a deformation–driven experiment on shape memory alloys.

260. Christof Külske: Metastates in disordered mean field models: random field and Hopfield models.

261. Donald A. Dawson, Klaus Fleischmann: Longtime behavior of a branching process controlled by branching catalysts.

262. Tino Michael, Jürgen Borchardt: Convergence criteria for waveform iteration methods applied to partitioned DAE systems in chemical process simulation.

263. Michael H. Neumann, Jens–Peter Kreiss: Bootstrap confidence bands for the autoregression function.

264. Silvia Caprino, Mario Pulvirenti, Wolfgang Wagner: Stationary particle systems approximating stationary solutions to the Boltzmann equation.

265. Wolfgang Dahmen, Angela Kunoth, Karsten Urban: Biorthogonal spline–wavelets on the interval – Stability and moment conditions.

266. Daniela Peterhof, Björn Sandstede, Arnd Scheel: Exponential dichotomies for solitary–wave solutions of semilinear elliptic equations on infinite cylinders.

267. Andreas Rathsfeld: A wavelet algorithm for the solution of a singular integral equation over a smooth two–dimensional manifold.

268. Jörg Schmeling, Serge E. Troubetzkoy: Dimension and invertibility of hyperbolic endomorphisms with singularities.

269. Erwin Bolthausen, Dmitry Ioffe: Harmonic crystal on the wall: a microscopic approach.

270. Nikolai N. Nefedov, Klaus R. Schneider: Delayed exchange of stabilities in singularly perturbed systems.

271. Michael S. Ermakov: On large and moderate large deviations of empirical bootstrap measure.