Abstract: We rigorously investigate the size dependence of disordered mean field models with finite local spin space in terms of metastates. Thereby we provide an illustration of the framework of metastates for systems of randomly competing Gibbs measures. In particular we consider the thermodynamic limit of the empirical metastate $1/N \sum_{n=1}^{N} \delta_{\mu_n(\eta)}$ where $\mu_n(\eta)$ is the Gibbs measure in the finite volume $\{1, \dots, n\}$ and the frozen disorder variable η is fixed. We treat explicitely the Hopfield model with finitely many patterns and the Curie Weiss Random Field Ising model. In both examples in the phase transition regime the empirical metastate is dispersed for large N. Moreover it does not converge for a.e. η but rather in distribution for whose limits we give explicit expressions. We also discuss another notion of metastates, due to Aizenman and Wehr.

I. Introduction

In a recent series of papers [NS1],[NS2],[NS3], the interesting role of the volume dependence in disordered systems having more than one infinite volume Gibbs states was stressed. In a particularly interesting article [NS3] the notion of metastates, being probability measures on the states of the systems, was introduced to describe the volume dependence of system with frozen disorder. (See therein and the discussion with [P] for implications on the theory of spinglasses and the relation to the phenomena of replica symmetry breaking and non self averaging.) It is the aim of this paper to provide a rigorous step into the investigation of size dependence by metastates by our investigation of examples of random mean field systems.

In the general case of disordered lattice spin systems in the presence of phase transitions, the problem of size dependence is the following. To start with a nontrivial situation, assume that the system admits more than one infinite volume Gibbs state. We consider the finite volume Gibbs measures $\mu_{\Lambda_N}(\eta)$, for fixed realization of the disorder η, in the finite volume Λ_N. We want to study a situation where the boundary conditions for the measures $\mu_{\Lambda_N}(\eta)$ are such that they do not preselect one of the infinite volume Gibbs measures. (There are many natural situations, where it is (practically) impossible (or not of interest) to select Gibbs measures by boundary conditions. This is the case in spin glasses, where the Gibbs measures are not explicitely known. Note moreover that in mean field systems it is impossible to put boundary conditions at all.)

To be concrete, we imagine that, for large N, the state of the system will be close to a mixture of random infinite volume Gibbs measures. That is, a good approximation for the finite volume Gibbs measures will often be

$$\mu_{\Lambda_N}(\eta) \approx \sum_m p_N^m(\eta) \mu_\infty^m(\eta) \tag{1.1}$$

where $(\mu_\infty^m(\eta))_{m\in\mathcal{M}}$ are the (supposedly countably many) extremal Gibbs measures in the infinite volume.

The problem of size dependence is: *Characterize the behavior of* $\mu_{\Lambda_N}(\eta)$ *along the sequence* Λ_N. This has some analogy with studying the orbit of a dynamical system with 'time' N (see [NS3]). Possible 'extremes' that could occur here, are e.g. 1) convergence to one infini volume Gibbs measure or 2) an 'erratic' sequence of states, a behavior that was named chaotic size dependence in [NS3].

A first question one may ask is: What Gibbs measures can be constructed through any subsequences Λ_{N_k} at all? More interesting even, lead by the dynamical system analogy, the following object was introduced in [NS3] to describe the 'trajectory' $N \mapsto \mu_{\Lambda_N}(\eta)$ in more detail:.

$$\kappa_N(\eta) := \frac{1}{N}\sum_{n=1}^{N}\delta_{\mu_{\Lambda_n}(\eta)} \tag{1.2}$$

We will refer to $\kappa_N(\eta)$ as the 'empirical metastate' and it will the main object of our study. Note that κ_N is a random measure (through its η-dependence) on the states of the system. For large N it will effectively be centered on the infinite volume Gibbs measures.

There are various scenarios for the large N-behavior of $\kappa_N(\eta)$. If the system admits just one infinite volume Gibbs measure $\mu_\infty(\eta)$, the situation is easy: $\kappa_N(\eta)$ will converge to $\delta_{\mu_\infty(\eta)}$. But note that also in the presence of phase transitions κ_N can converge to a δ-measure. (Take as an example the ordinary ferromagnetic $2d$ Ising model without disorder, at low temperatures and put periodic boundary conditions. Then $\mu_N \to \frac{1}{2}(\mu_\infty^+ + \mu_\infty^-)$ with $N\uparrow\infty$. Consequently $\kappa_N \to \delta_{\frac{1}{2}(\mu_\infty^+ + \mu_\infty^-)}$.)

Nondegeneracy for the metastate can arise for random systems because, for fixed realization of the disorder, the finite volume fluctuations of the underlying random quantities could favor one of different phases even when they are equivalent in the average. While the structure of the phase diagram is nonrandom, the degeneracy between the phases in the finite volume would then be lifted in a random fashion. The information about how this is done lies in the $p_N^m(\eta)$. A variety of large-N behavior is then possible: κ_N can be the dirac measure on a mixture of states, it can be a mixture of dirac measures on pure states, it can be a mixture of dirac measures on mixtures.

The second aspect is that κ_N itself is a random object: In what way will the behavior of κ_N depend on the realization? One could be tempted to expect that, as a generic behavior, $\kappa_N(\eta)$ will converge at (almost) all fixed η (see [NS3] for a conjecture in that direction for certain systems). This were the case if the random objects $\mu_{\Lambda_N}(\eta)$ lost memory very rapidly along the path $N \to \mu_N(\eta)$. In this paper we provide examples where this is *not* the case. Nevertheless, the limiting behavior of the empirical metastate can be described in our examples in two ways: First, it is possible to give *pathwise* approximation results, that hold for all typical realizations. Second, we suggest to study the behavior of the empirical metastate in *law*. This idea extends [APZ] wheconvergence of the Gibbs measures themselves was considered in law. We will see that, in our examples, infinite volume limits exist in law and give interesting information about the asymptotical behavior of the system along the path.

In order to make sense out of this, one has to speak about notions of convergence of κ_N with $N \uparrow \infty$. As it is common practice, we will choose the weak topologies that are inherited on the space of states and on the space of metastates when we choose as a starting point the product topology on the spin space (see Chapter 2). It makes the two spaces polish. The physical content of this notion of convergence is that convergence is checked locally on all levels.

In the first part of this paper we will outline the general treatment of random mean field models with quadratic interaction and finite state space. Then we will consider two representatives of this class in detail. The advantage our mean field models is that they allow for explicite expressions for the weights $p_N^m(\eta)$ and good enough approximations (1). Our two examples are:

(i) The Curie Weiss Random Field Ising Model (CWRFIM):

Denote $\Omega := \{1,-1\}^{I\!N}$ the space of Ising spin configurations $\sigma = (\sigma_i)_{i\in I\!N}$. We will denote the set of states (which is the set of probability measures on Ω) by $\mathcal{P}(\Omega)$. Let $\eta = (\eta_i)_{i\in I\!N}$, denote a sequence of i.i.d. Bernoulli variables taking the values $\epsilon, -\epsilon$ with probability $\frac{1}{2}$. For the inverse temperature β define the Gibbs measures

$$\mu_N(\eta)[(\sigma_i)_{i=1,\dots,N}] := \frac{1}{Norm.} \exp\left(\frac{\beta}{2N}\sum_{1\le i,j\le N}\sigma_i\sigma_j + \beta\sum_{1\le i\le N}\eta_i\sigma_i\right) \tag{1.3}$$

in the finite volume[1] $\{1,\dots,N\}$. The phase diagram of the system is well known (see [SW],[APZ]). At low temperatures and small ϵ the model is ferromagnetic, i.e. there exist two 'pure' phases, a ferromagnetic $+$ phase $\mu_\infty^+(\eta)$ and a $-$ phase $\mu_\infty^-(\eta)$. We restrict our interest to this region of the phase diagram.

(ii) The Hopfield model with finite number of patterns:

Let Ω be the space of Ising spins as above. Let $\xi = (\xi_i^\mu)_{i\in I\!N,\mu=1,\dots,M}$ denote i.i.d. Bernoulli variables taking the values $1,-1$ with probability $\frac{1}{2}$. $\xi^\mu = (\xi_i^\mu)_{i\in I\!N}$ are the *patterns* the model

[1] As usual they can also be viewed as measures on Ω by tensoring with arbitrary product measures for the spins at sites $i > N$.

is supposed to learn ([Ho]). For $\beta > 0$ define the finite volume Gibbs measures

$$\mu_N(\xi)[(\sigma_i)_{i=1,\dots,N}] := \frac{1}{Norm.} \exp\left(\frac{\beta}{2N} \sum_{1\le i,j\le N} \sum_{1\le\nu\le M} \xi_i^\nu \xi_j^\nu \sigma_i \sigma_j\right) \tag{1.4}$$

Due to our restriction on the number of patterns to remain fixed when $N \uparrow \infty$, we are deep inside the 'region of perfect memory' if $\beta > 1$. This means that, for large N, the system is approximately in a mixture of the M 'Mattis states' $\mu_\infty^\nu(\xi)$. The latter is a state, associated the ν-th pattern, s.t. the overlap vector $\left(\frac{1}{N}\sum_{i=1}^N \xi_i^\rho \sigma_i\right)_{\rho=1,\dots,M}$ is centered around $\pm m^*(\beta) a^\nu$, where a^ν is the ν-th unity vector in $I\!R^M$. ($m^*(\beta)$ is the solution of the ordinary Curie Weiss Mean Field equation.) For precise statements, see e.g. [BGP],[BG1]. For the state of the art in the Hopfield model with $\lim_{N\uparrow\infty} \frac{M(N)}{N} = \alpha$ small and positive we refer to [BG2] where a beautiful proof of the validity of the replica symmetric solution is given. One reason for treating the Hopfield model here is of course, that it can be viewed as an interpolation between a ferromagnet and a spinglass.

For the limiting distribution of the empirical metastates in these two models we will prove the following theorems. (For the pathwise approximation results and related information, see Theorems 1' and 2'). These show that even in these simple models there is some richness in the empirical metastate.

Theorem 1: *For all bounded continuous functions $F : \mathcal{P}(\Omega) \mapsto I\!R$ we have the limit in law*

$$\lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^N F(\mu_n(\eta)) =^{law} n_\infty F\left(\mu_\infty^+(\eta)\right) + (1-n_\infty) F\left(\mu_\infty^-(\eta)\right) \tag{1.5}$$

where n_∞ is a random variable, independent of η on the r.h.s, distributed according to

$$I\!P\left[n_\infty < x\right] = \frac{2}{\pi} \arcsin\sqrt{x} \tag{1.6}$$

Thus, the empirical metastate is centered on the two 'extremal' Gibbs measures with random weights. The occurence of the arcsin-law will be explained by the fact that in this simple model the weights $p_N^+(\eta), p_N^-(\eta)$ are in fact functions of the random walk $N \mapsto \sum_{i=1}^N \eta_i$.

An analogous, but more involved, result holds for the Hopfield model:

Theorem 2: *For all bounded continuous functions $F : \mathcal{P}(\Omega) \mapsto I\!R$ we have the limit in law*

$$\lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^N F(\mu_n(\xi)) =^{law} \int_0^1 dt F\left(\sum_{\nu=1}^M p^\nu\left(\frac{W_t}{\sqrt{t}}\right) \mu_\infty^\nu(\xi)\right) \tag{1.7}$$

where $(W_t)_{0\leq t\leq 1}$ is a $M(M-1)/2$-dimensional Brownian motion starting at the origin, independent of ξ on the r.h.s. For the definition of the function $p^\nu\left(\frac{W_t}{\sqrt{t}}\right)$, see (5.5) and thereafter.

Here, the empirical metastate is richer, in that it is in fact a random mixture with support on mixtures of Gibbs measures. The occurence of the Brownian motion in Theorem 2 will be explained by an invariance principle for the underlying disorder variables; the time t is nothing but the rescaled system size.

Remark: We see explicitely that, in both cases, the empirical metastate $\kappa_N(\eta)$ does *not* converge (see Theorems 1',2') for fixed realization. Thus, having a limit for $\kappa_N(\eta)$ is only possible when it is viewed as a random variable. This is expressed by the fact that the n_∞ respectively $p^\nu\left(\frac{W_t}{\sqrt{t}}\right)$ are random variables with nondegenerate distributions.

We would like to mention that, apart from the empirical metastate, there is a second notion of metastates, whose construction is due to [AW]. We will discuss its relation to the former; as we will see, it contains less information. It will be obtained by the r.h.s. of (1.5) (respectively (1.6)) by integration over n_∞ (resp. W_t).

Its precise definition will be given in Chapter 2, where we also state some straightforward approximation properties that are valid for lattice systems as well as for mean field systems. We describe the role of sets of regular realizations of the disorder at a general level here, since dealing with such sets is typical for disordered systems. In Chapter 3 we introduce disordered mean field models with quadratic interaction. We give approximation criteria and describe general features of the behavior expected in these models. We also discuss the relation between the conditioned and the empirical metastate. In Chapter 4 we consider specifically the CWRFIM and prove Theorems 1 and 1'. In Chapter 5 we consider the Hopfield model and prove Theorem 2 and 2'.

2. Notations and Generalities about metastates

The following considerations are true for general random spin systems with finite local spin space S. We assume the state space is a countable product of S over the lattice points, in practice $\Omega = S^{\mathbb{Z}^d}$ or $\Omega = S^{\mathbb{N}}$. Spin variables will be denoted by σ, their projections on finite volumes Λ by σ_Λ; when necessary to dintinguish between spin variables and their values, we denote the latter by ω.

Some topological remarks are in order (see also [AW] appendix,[NS3],[N],[Se]). Ω is equipped with the product topology. We denote the space of probability measures on Ω (the set of states) by $\mathcal{P}(\Omega)$. It is equipped with the weak topology which coincides with the local topology in our case; that is, convergence of measures is checked on functions that depend only on finitely many spins. It is metrizable and to be explicit we choose the metric

$$d(\mu,\mu')=\sum_{k=1}^{\infty}2^{-k}\sum_{\omega_{\Lambda_k}\in\Omega_{\Lambda_k}}|\mu[\sigma_{\Lambda_k}=\omega_{\Lambda_k}]-\mu'[\sigma_{\Lambda_k}=\omega_{\Lambda_k}]| \tag{2.1}$$

where Λ_k is an enumeration of all finite subsets of lattice points (See [Geo], p.60). Given two sequences μ_N and μ'_N, $\lim_{N\uparrow\infty}d(\mu_N,\mu'_N)=0$ is thus equivalent with the condition $\lim_{N\uparrow\infty}|\mu_N[\sigma_\Lambda=\omega_\Lambda]-\mu'_N[\sigma_\Lambda=\omega_\Lambda]|=0$ for all finite subsets of sites Λ, for all $\omega_\Lambda\in\Omega_\Lambda$.

We denote the set of probability measures on $\mathcal{P}(\Omega)$ (the set of metastates) by $\mathcal{P}(\mathcal{P}(\Omega))$. In the same spirit, it will be equipped with its weak topology, inherited from the topology on $\mathcal{P}(\Omega)$ (as in [AW]). Thus convergence is checked on bounded continuous functions on the states, which means more concretely that convergence needs to be checked on functions of the type

$$F(\mu)=\tilde{F}\left(\mu(f_1),\dots,\mu(f_l)\right) \tag{2.2}$$

where $\tilde{F}:\mathbb{R}^l\to\mathbb{R}$ is a polynomial, $l=1,2,\dots$ and $f_1,\dots,f_l$ are local functions on Ω. The topology can be metrized with the aid of such functions. In the Ising case one may restrict to functions f_j of the form $\prod_{i\in I}\sigma_i$ with a finite set of lattice points I. Both spaces $\mathcal{P}(\Omega)$, $\mathcal{P}(\mathcal{P}(\Omega))$ are then compact polish (i.e. complete, separable, metric) spaces. All spaces we consider carry automatically the associated Borel σ-algebras generated by the open sets.

Note that, for fixed η, the empirical metastate $\kappa_N(\eta)$, as defined in (1.2) will always possess limit points, due to the compactness of $\mathcal{P}(\mathcal{P}(\Omega))$. We remark that the definition of the empirical metastate in (1.2) depends a priori (and in reality!) on the sequence of volumes Λ_n one is interested in. In mean field models there is the natural choice of volumes $\Lambda_n=\{1,\dots,n\}$ that we will stick to. In generalization of the definition (1.2) one could even want to study the objects $\int\rho_N(d\Lambda)\delta_{\mu_\Lambda}$ with some sequence of measures ρ_N on the set of finite subsets of the lattice, s.t. $\rho_N(\{\Lambda\})\to 0$ for all finite Λ with $N\uparrow\infty$. We don't treat this general case here.

We will generically denote the probability space of the random variables η describing the quenched disorder by $(\mathcal{H},\mathcal{F},\mathbb{P})$, and expectation w.r.t $\mathbb{P}$ will be denoted by $\mathbb{E}$. We assume that $\mathcal{H}$ is a product of a polish space over the lattice points. $\mathcal{H}$, too, is equipped with the product topology. We can now consider the *skew space* $\mathcal{H}\times\Omega$ (see [Se]), equipped with the product topology.

There is another notion of metastate, introduced by [AW], that we will refer to as the *conditioned metastate*. For its introduction it will be necessary to consider, one level higher, the space $\mathcal{H}\times\mathcal{P}(\Omega)$, equipped with the product topology. Assume that we are given a measurable sequence of random states $\mu_N(\eta)$. We will focus on the random elements $\delta_{\mu_N(\eta)}$ in $\mathcal{P}(\mathcal{P}(\Omega))$ and view these as kernels from $\mathcal{H}$ to $\mathcal{P}(\Omega)$. Then we consider the associated probability measures on the space $\mathcal{H}\times\mathcal{P}(\Omega)$, given by $I\!E[G(\mu_N(\eta),\eta)]$, for a bounded continuous function G on $\mathcal{H}\times\mathcal{P}(\Omega)$.

Assume now, that the sequence $\mu_N(\eta)$ is such that, for any bounded continuous G, the limits

$$\lim_{N\uparrow\infty} I\!E\left[G(\mu_N(\eta),\eta)\right] =: \int K(d\mu,d\eta)G(\mu,\eta) \tag{2.3}$$

exist and define a probability measure $K\in\mathcal{P}(\mathcal{P}(\Omega)\times\mathcal{H})$. Then, the *conditioned metastate* $\bar\kappa(\eta)(d\mu)$ will be the regular conditional probability of K given η. It is thus the measurable map $\bar\kappa:\mathcal{H}\to\mathcal{P}(\mathcal{P}(\Omega))$ s.t. $\int K(d\mu,d\eta)G(\mu,\eta) = I\!E[\bar\kappa(\eta)(d\mu)G(\mu,\eta)]$. Note that the conditional probability is well defined since $\mathcal{H}$ is Polish.

When dealing with random systems one usually has to exclude exceptional sets of the disorder from the analysis. These exceptional sets, which may depend on the systems size, should be small enough to be ignored for most purposes. As we will see in our concrete examples this question has to be handled with care; therefore we would like to state an approximation lemma, which shows how exceptional sets of realizations affect the above definitions.

Let us assume that we are given two random sequences $\mu_N(\eta),\mu'_N(\eta)$ of states that become 'close' for most η. We will consider sequences of 'good' sets of realizations $\mathcal{H}(N)\subset\mathcal{H}$; an important role will then be played by the approximation for all η in the set $\underline{\mathcal{H}}:=\liminf_{N\uparrow\infty}\mathcal{H}(N)=\{\eta\in\mathcal{H},\exists N_0:\eta\in\mathcal{H}(N)\ \forall N\geq N_0\}$. This will serve as a relaxation in place of just saying that convergence takes place for η in a full measure set. Then we have

Lemma 1: *Assume that there exist subsets $\mathcal{H}_N\subset\mathcal{H}$ s.t., for all realizations $\eta\in\liminf_{N\uparrow\infty}\mathcal{H}(N)$ we have $\lim_{N\uparrow\infty}d(\mu_N(\eta),\mu'_N(\eta))=0$. Then*

(i) For $\eta\in\liminf_{N\uparrow\infty}\mathcal{H}(N)$ the set of weak cluster points coincide

$$\mathcal{CP}(\mu_N(\eta),\,n=1,2,\dots)=\mathcal{CP}(\mu'_N(\eta),\,n=1,2,\dots) \tag{2.4}$$

(ii) For all $\eta\in\mathcal{H}':=\left\{\eta,\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N}1_{\eta\in\mathcal{H}(n)^c}=0\right\}$ we have

$$\lim_{N\uparrow\infty}\left(\kappa_N(\eta)(F)-\kappa'_N(\eta)(F)\right)=0 \tag{2.5}$$

for all bounded continuous F on $\mathcal{P}(\Omega)$.

(iii) Assume that $\lim_{N\uparrow\infty}I\!P[\mathcal{H}(N)]=1$. Then, for any bounded continuous function $G:\mathcal{P}(\Omega)\times\mathcal{H}\mapsto I\!R$

$$\lim_{N\uparrow\infty}\left(I\!E\left[G(\mu_N(\eta),\eta)\right]-I\!E\left[G(\mu'_N(\eta),\eta)\right]\right)=0 \tag{2.6}$$

Proof: (i) is obvious. To prove (ii), define

$$S_N := \frac{1}{N} \sum_{1 \le n \le N} 1_{\eta \in \mathcal{H}_2^c(n)} \tag{2.7}$$

For any bounded continuous function $\mu \mapsto F(\mu)$ we have

$$\begin{aligned} &\frac{1}{N} \sum_{1 \le n \le N} \left(F(\mu_n(\eta)) - F(\mu'_n(\eta))\right) \\ &= \frac{1}{N} \sum_{1 \le n \le N} \left(F(\mu_n(\eta)) - F(\mu'_n(\eta))\right) 1_{\eta \in \mathcal{H}(n)} + R_N \end{aligned} \tag{2.8}$$

According to its definition we have on $\mathcal{H}'$ that $|R_N| \le \|F\|_\infty S_N \to 0$. Since the first term is a Cesaro sum it sufficies to show that

$$\left(F(\mu_n(\eta)) - F(\mu'_n(\eta))\right) 1_{\eta \in \mathcal{H}(n)} \to 0 \tag{2.9}$$

with $n \uparrow \infty$. But notice that a continuous F is in fact already uniformly continous, due to the compactness of $\mathcal{P}(\Omega)$. Therefore (10) follows directly from the assumption, for both cases that η is an element of $\liminf_{N\uparrow\infty} \mathcal{H}(N)$ or that it is not.

To prove (iii), we split off the exceptional set $\mathcal{H}^c(N)$ to write the l.h.s. of (2.6) as

$$I\!E\left[G(\mu_N(\eta),\eta)\right] - I\!E\left[G(\mu'_N(\eta),\eta)\right] = I\!E\left[\left(G(\mu_N(\eta),\eta) - G(\mu'_N(\eta),\eta)\right) 1_{\eta \in \mathcal{H}(N)}\right] + R_N \tag{2.10}$$

where $|R_N| \le 2\|G\|_\infty I\!P \mathcal{H}_2^c(N) \to 0$. Now, for fixed η, $\mu \mapsto G(\mu,\eta)$ is a uniformly continuous function in μ (due to compactness). Therefor the convergence for fixed η of the expression under the expectation follows directly from the assumption. Using dominated convergence this proves the claim. $\diamondsuit$

Remark: The set $\mathcal{H}'$ is potentially (and sometimes in reality) a bit bigger than the set $\liminf_{N\uparrow\infty} \mathcal{H}(N)$. Our discussion of the CWRFIM will provide an example where, for a natural choice of sets $\mathcal{H}(N)$, the first is a full measure set but not the second. Of course, if $\mathcal{H}(N)$ can be taken as a full measure set which is independent of N, we have $\mathcal{H}(N) = \mathcal{H}'$ and the convergence in (ii) takes place a.s.

3. Mean Field models with quadratic interaction

In this chapter we discuss the models of the above class. We fix approximation criteria (see propositions 1,2) that allow for the computation of the metastates in terms of the relative weights the 'Hubbard-Stratonovich' measure puts on small balls around its concentration set. The models we will consider are of the following type. (See also [BG2], Chapter 2).

The spins $\sigma = (\sigma_i)_{i=1,2,\dots} \in \Omega = S^{\mathbb{N}}$ have an a priori distribution according to a product measure

$$\mu^0(\eta)[\sigma = \omega] = \prod_{i=1}^{N} \mu_i^0(\eta_i)[\sigma_i = \omega_i] \tag{3.1}$$

Here we allow the measures $\mu_i^0(\eta_i)$ to depend on a random variable η_i, $i \in \mathbb{N}$; this enables us to include random field type models. These 'random fields' η_i shall be sitewise i.i.d. Assume that we are given a bounded continuous map

$$(\sigma_1, \eta_1) \mapsto m(\sigma_1, \eta_1) \tag{3.2}$$

taking values in $\mathbb{R}^M$. Then the *order parameter* $\bar{m}_N$ is defined by the empirical average

$$\bar{m}_N(\sigma, \eta) := \frac{1}{N} \sum_{i=1}^{N} m(\sigma_i, \eta_i) \tag{3.3}$$

We consider the Curie Weiss Hamiltonian given by the square of the 2-Norm of the order parameter

$$E_N(\sigma, \eta) := -\frac{N}{2} \bar{m}_N(\sigma, \eta)^2 \equiv -\frac{N}{2} \sum_{\nu=1}^{M} \|\bar{m}_N^\nu(\sigma, \eta)\|_2^2 \tag{3.4}$$

The associated finite volume Gibbs measures are then

$$\mu_N(\eta)[\sigma = \omega] := \frac{\exp\left(-\beta E_N(\omega, \eta)\right)}{Norm.} \mu^0(\eta)[\sigma = \omega] \tag{3.5}$$

We write

$$\bar{\mu}_N(\eta)[\,\cdot\,] := \mu_N\left[\bar{m}_N(\sigma, \eta) \in \cdot\right] \tag{3.6}$$

for the associated image measures on the order parameter. Examples for these models are

(a) The ordinary Curie Weiss Ising ferromagnet: $\sigma_i \in \{-1, 1\}$, $m(\sigma_1, \eta_1) = \sigma_1$, $\mu_i(\eta_i)[\sigma_i = \pm 1] = \frac{1}{2}$ for all i. The choice of random a priori measures according to $\mu_i(\eta_i)[\sigma_i = \pm 1] = \frac{e^{\pm \beta \eta_i}}{2\cosh(\beta \eta_i)}$ gives our first example from the introduction, the CWRFIM.

(b) The Curie Weiss q-state Potts model: $\sigma_i \in \{1, \dots, q\}$, $\left(m^p(\sigma_1, \eta_1)\right)_{p=1,\dots,q} = \left(1_{\sigma_1 = p}\right)_{p=1,\dots,q}$

(c) The Hopfield model: $\sigma_i \in \{-1, 1\}$ with symmetric Bernoulli a priori measures. For traditional reasons we call the random variables in this case ξ instead of η. $(\xi_i^\mu)_{i=1,2,\dots;\mu=1,\dots,M} \equiv (\xi_i)_{i=1,2,\dots}$ are i.i.d. (for different i, μ) with $\mathbb{P}[\xi_i^\mu = \pm 1] = \frac{1}{2}$. The order parameter is defined by $m(\sigma_1, \xi_1) = \sigma_1 \xi_1 \in \{1, -1\}^M$. The empirical mean $\bar{m}_N(\sigma, \xi)$ is then called the *overlap vector*.

Our restriction to quadratic Hamiltonians is convenient because it makes it possible to use the well known trick of the Hubbard-Stratonovich transformation. Let us recall it here for convenience of the reader and to fix notations: One introduces an auxiliary M-dimensional Gaussian integral to write for fixed $\omega = (\omega_1, \dots \omega_N) \in \Omega_{\{1,\dots,N\}}$

$$
\begin{aligned}
\mu_N(\eta)[\sigma = \omega] &= \frac{\int_{I\!R^M} dm \exp\left(-\frac{\beta N m^2}{2} + \beta N m \cdot \bar{m}_N(\omega,\eta)\right) \mu_N^0(\eta)[\sigma = \omega]}{Norm.'}\\
&= \frac{1}{Norm.'} \int_{I\!R^M} dm \exp\left\{-\beta N \left[\frac{m^2}{2} - \frac{1}{\beta N}\log\left(\int \mu_0(\eta)(d\sigma')\exp\left(\beta N m \cdot \bar{m}_N(\sigma',\eta)\right)\right)\right]\right\}\\
&\quad\times \frac{\exp\left(\beta N m \cdot \bar{m}_N(\omega,\eta)\right)}{\int \mu_0(\eta)(d\sigma')\exp\left(\beta N m \cdot \bar{m}_N(\sigma',\eta)\right)}\mu_N^0(\eta)[\sigma = \omega]
\end{aligned}
\tag{3.7}
$$

Now, for fixed m, the second line of the r.h.s. constitutes a probability measure for the variable σ. The variable m is integrated according to the measure that can be read off from the first line of the r.h.s.

Thus one has the following 'factorization formula' that will be the starting point for our analysis

$$
\mu_N(\eta)[\sigma = \omega] = \int_{I\!R^M} \tilde{\mu}_N(\eta)(dm)\mu_N^0(m,\eta)[\sigma = \omega] \tag{3.8}
$$

Here $\mu_N^0(t,\eta)[\sigma = \omega]$ is a product measure over independent spins obtained by 'tilting with the external field' t; that is

$$
\mu_N^0(t,\eta)[\sigma = \omega] = \prod_{i=1}^N \mu_i^0(t,\eta_i)[\sigma_i = \omega_i] \tag{3.9}
$$

where

$$
\mu_i^0(t,\eta_i)[\sigma_i = \omega_i] = \frac{\exp\left(\beta t \cdot m(\omega_i,\eta_i)\right)}{\exp\left(\beta L(t,\eta_i)\right)}\mu_i^0(\eta_i)[\sigma_i = \omega_i] \tag{3.10}
$$

with the associated logarithmic moment generating function

$$
L(t,\eta_i) = \frac{1}{\beta}\log\int \mu_i^0(\eta_i)(d\sigma_i)\exp\left(\beta t \cdot m(\sigma_i,\eta_i)\right) \tag{3.11}
$$

We will write $\mu_\infty^0(t,\eta)$ for the infinite product measure. The 'Hubbard-Stratonovich measures' $\tilde{\mu}_N(\eta)$ are given by

$$
\tilde{\mu}_N(\eta)(dm) := \frac{\exp\left(-\beta N \Phi_N(m,\eta)\right) dm}{\int_{I\!R} dm' \exp\left(-\beta N \Phi_N(m',\eta)\right)} \tag{3.12}
$$

with the function

$$
\Phi_N(m,\eta) := \frac{m^2}{2} - \frac{1}{N}\sum_{1\le i\le N} L(m,\eta_i) \tag{3.13}
$$

(dm means of course integration w.r.t. Lebesgue measure.) Note that $\tilde{\mu}_N(\eta)$ is nothing but the convolution of $\bar{\mu}_N(\eta)$ with a M-dimensional Gaussian Normal variable with covariance matrix $\sigma^2 1 = \frac{1}{\beta N}1$.

It is essential about mean field models that the measures $\tilde{\mu}_N(\eta)$ (and related $\bar{\mu}_N(\eta)$) have exponential concentration properties when $N \uparrow \infty$. The following results, reducing the question of the structure of the phase diagram to *averaged* quantities, are known applications of large deviation techniques ([DS],[DZ],[El]). Define

$$L^*(m) := \inf_t (t \cdot m - I\!\!E L(t, \eta_1)) \tag{3.14}$$

and

$$I(m) := \frac{m^2}{2} + L^*(m) - \inf_{m'} \left(\frac{m'^2}{2} + L^*(m') \right) \tag{3.15}$$

Then there exists a full measure set of η's s.t. a) the measures $\bar{\mu}_N(\eta)$ obey a large deviation principle with the deterministic rate function $I(m)$. b) Any weak limit point of $\mu_N(\eta)$ is of the form

$$\int_{\mathcal{M}} p(dm) \mu^0_\infty(m, \eta) \tag{3.16}$$

where

$$\mathcal{M} := \{m, I(m) = 0\} = \{m, I\!\!E[\Phi_N(m, \eta)] = \min_{m'} I\!\!E[\Phi_N(m', \eta)]\} \tag{3.17}$$

is the concentration set of the measure $\bar{\mu}_N(\eta)$. For a proof see Theorem 5 in [Co] (for the case of nonrandom a priori measures). (3.16) shows that the role of pure infinite volume states is played by the product measures $\mu^0_\infty(m, \eta)$ for m in the cluster set $\mathcal{M}$. [1] Now, for our study, we have to describe in more precision the *finite volume* version of (3.16) in which the random competition among the elements in the cluster set $\mathcal{M}$ manifests itself. For that purpose we need the relative weights that are put by the measure $\tilde{\mu}_N$ close to the elements of the cluster set $m \in M$. Thus we have to go beyond the large deviations on the volume order; we have to look at a scale where the random fluctuations become important.

Let us assume that $\mathcal{M} \subset I\!\!R^M$ is a finite set. In fact, we want to replace (3.16) by

$$\mu_N(\eta) \approx \sum_{m \in \mathcal{M}} p_N^m(\eta) \mu^0_\infty(m, \eta) \tag{3.18}$$

In view of the factorization formula (3.8) we look at the probability vector $p_N(\eta) := (p_N^m(\eta))_{m \in \mathcal{M}}$ as an approximation of the Hubbard-Stratonovich measure $\tilde{\mu}_N(\eta)$. Since this approximation shall be sufficient for the metastates, we are looking for natural conditions that imply the assumption of Lemma 1. Denote by $B_\rho(m)$ the Euclidian ball centered at m with radius ρ. Let us thus make the following

---

[1] Typically, by adding 'magnetic field terms' to the Hamiltonian, one can select one of these to survive as limit point of the modified $\mu_N(\eta)$.

Definition 1: Assume that we are given subsets $\mathcal{H}(N)\subset\mathcal{H}$. We use the abbreviation $\underline{\mathcal{H}} := \liminf_{N\uparrow\infty}\mathcal{H}(N)$. We say that *$\tilde{\mu}_N(\eta)$ becomes close to the probability vector $(p_N^m(\eta))_{m\in\mathcal{M}}$ along the regular sets $\mathcal{H}(N)$* (in short: *they have the property CR(ρ_N)*) if, for all $\eta\in\underline{\mathcal{H}}$, for all $m\in\mathcal{M}$,

$$\lim_{N\uparrow\infty}\left(\tilde{\mu}_N(\eta)\left[B_{\rho_N}(m)\right]-p_N^m(\eta)\right)=0 \tag{3.19}$$

for a decreasing sequence of radii $\rho_N\downarrow 0$. If (3.19) is true for all sufficiently small ρ (independent of N), we say that *they have the property CR(ρ)*.

The reason for this definition as that we have

Lemma 2: *Assume property CR(ρ_N) and define $\mu'_N(\eta) := \sum_{m\in\mathcal{M}} p_N^m(\eta)\mu^0_\infty(m,\eta)$. Then, for all $\eta\in\underline{\mathcal{H}}$ we have $\lim_{N\uparrow\infty} d(\mu_N(\eta),\mu'_N(\eta))=0$.*

Remark: From the fact that $(p_N^m(\eta))_{m\in\mathcal{M}}$ is a probability vector follows in particular that, for all $\eta\in\underline{\mathcal{H}}$,

$$\lim_{N\uparrow\infty}\tilde{\mu}_N(\eta)\left[\left(\cup_{m\in\mathcal{M}}B_\rho(m)\right)^c\right]=0 \tag{3.20}$$

which is just the usual definition of $\mathcal{M}$ being the cluster set of $\tilde{\mu}_N(\eta)$ (see [LPS]).

Remark: Note that CR(ρ_N) for some unspecified ρ_N is implied by CR(ρ). (Put $a_{NK} := \tilde{\mu}_N(\eta)\left[B_{\rho_K}(m)\right]-p_N^m(\eta)$, for some decreasing sequence $\rho_K\downarrow 0$, and use the elementary fact: For each double sequence a_{NK} s.t. $\lim_{N\uparrow\infty}a_{NK}=0$ for fixed K one may find a subsequence $K_N\uparrow\infty$ s.t. $\lim_{N\uparrow\infty}a_{NK_N}=0$.)

Remark: The property CR(ρ) is equivalent with the property $\bar{\text{CR}}(\rho)$, by which we understand, that in the above definition the measures $\tilde{\mu}_N(\eta)$ are replaced with the measures on the order parameter, $\bar{\mu}_N(\eta)$. To see this, note that from their relation as convolutions follows that, for $m\in\mathcal{M}$,

$$\tilde{\mu}_N(\eta)\left[B_\rho(m)\right]\le\bar{\mu}_N(\eta)\left[B_{2\rho}(m)\right]+I\!P\left[\left|\frac{G}{\beta N}\right|>\rho\right] \tag{3.21}$$

with a standard normal variable G. From that we have, for $\eta\in\underline{\mathcal{H}}$,

$$\lim_{N\uparrow\infty}\left(\tilde{\mu}_N(\eta)\left[B_\rho(m)\right]-p_N^m(\eta)\right)\le\lim_{N\uparrow\infty}\left(\bar{\mu}_N(\eta)\left[B_{2\rho}(m)\right]-p_N^m(\eta)\right) \tag{3.22}$$

Similarly we can obtain the lower bound

$$\lim_{N\uparrow\infty}\left(\tilde{\mu}_N(\eta)\left[B_\rho(m)\right]-p_N^m(\eta)\right)\ge\lim_{N\uparrow\infty}\left(\bar{\mu}_N(\eta)\left[B_{\frac{\rho}{2}}(m)\right]-p_N^m(\eta)\right) \tag{3.23}$$

which proves the claim. $\diamondsuit$

We come to the

Proof of Lemma 2: Take $\eta \in \underline{\mathcal{H}}$. We only have to check convergence on a local event of the form $A := \{\sigma_\Lambda = \omega_\Lambda\}$ with fixed ω_Λ. Then, using the factorization formula (3.8), we have

$$\begin{aligned}\left|\mu_N(\eta)[A] - \sum_{m\in\mathcal{M}} p_N^m(\eta)\mu_\infty^0(m,\eta)[A]\right| &\le \tilde\mu_N(\eta)\left[(\cup_{m\in\mathcal{M}} B_{\rho_N}(m))^c\right]\\ + \quad \sum_{m\in\mathcal{M}} &\left|\int_{B_{\rho_N}(m)} \tilde\mu_N(\eta)(d\tilde m)\mu_\infty^0(\tilde m,\eta)[A] - p_N^m(\eta)\mu_\infty^0(m,\eta)[A]\right|\end{aligned} \tag{3.24}$$

where the first term on the r.h.s. vanishes under the N-limit (see first remark). We pick one m in the sum and write

$$\begin{aligned}&\left|\int_{B_{\rho_N}(m)} \tilde\mu_N(\eta)(d\tilde m)\mu_\infty^0(\tilde m,\eta)[A] - p_N^m(\eta)\mu_\infty^0(m,\eta)[A]\right|\\ &\le \int_{B_{\rho_N}(m)} \tilde\mu_N(\eta)(d\tilde m)\left|\mu_\infty^0(\tilde m,\eta)[A] - \mu_\infty^0(m,\eta)[A]\right|\\ &\quad + \left|\tilde\mu_N(\eta)\left[B_{\rho_N}(m)\right] - p_N^m(\eta)\right|\mu_\infty^0(m,\eta)[A]\end{aligned} \tag{3.25}$$

The first term goes to zero with $\rho_N \downarrow 0$, due to the continuity of the function

$$\tilde m \mapsto \mu_\infty^0(\tilde m,\eta)[A] \tag{3.26}$$

(In fact, it is $\mathcal{C}^\infty$ everywhere; all derivatives exist for all $\tilde m \in I\!R^M$, due to the assumed boundedness of the order parameter.) The second term goes to zero according to the assumption (3.19). $\diamondsuit$

Putting the pieces from the Lemmata 1 and 2 together, we immediately obtain the following approximation result that we fix as

Proposition 1: *Suppose that we are given a quadratic random mean field model of the above type whose Hubbard-Stratonovich measures $\tilde\mu_N(\eta)$ obey the approximation property $CR(\rho_N)$ with probability vector $\left(p_N^m(\eta)\right)_{m\in\mathcal{M}}$. Then*

(i) For all $\eta \in \underline{\mathcal{H}}$ we have for the set of weak cluster points in $\mathcal{P}(\Omega)$

$$\mathcal{CP}\left(\mu_N(\eta)\ , N=1,2,\dots\right) = \mathcal{CP}\left(\sum_{m\in\mathcal{M}} p_N^m(\eta)\mu_\infty^0(m,\eta)\ , N=1,2,\dots\right) \tag{3.27}$$

(ii) Define the metastate

$$\tilde\kappa_N(\eta) := \frac{1}{N}\sum_{n=1}^N \delta_{\sum_{m\in\mathcal{M}} p_N^m(\eta)\mu_\infty^0(m,\eta)} \tag{3.28}$$

Then, for all $\eta \in \mathcal{H}' = \left\{\eta, \lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^N 1_{\eta\in\mathcal{H}(n)^c} = 0\right\}$ we have

$$\lim_{N\uparrow\infty}\left(\int \kappa_N(\eta)(d\mu)F(\mu) - \int \tilde\kappa_N(\eta)(d\mu)F(\mu)\right) = 0 \tag{3.29}$$

for all bounded continuous F on $\mathcal{P}(\Omega)$.

(iii) *Assume that* $\lim_{N\uparrow\infty} I\!P[\mathcal{H}(N)] = 1$. *Then, for any bounded continuous function* $G : \mathcal{P}(\Omega) \times \mathcal{H} \mapsto I\!R$

$$\lim_{N\uparrow\infty} \left(I\!E\left[G\left(\mu_N(\eta),\eta\right)\right] - I\!E\left[G\left(\sum_{m\in\mathcal{M}} p_N^m(\eta)\mu_\infty^0(m,\eta),\eta\right)\right]\right) = 0 \tag{3.30}$$

Remark: Again a word of care about the difference of $\underline{\mathcal{H}}$ and $\mathcal{H}'$: The CWRFIM will give an example where, due to this difference, the set of cluster points becomes a.s. larger than the set of measures the metastate will be asympotically supported in (See Chapter 4, Theorem 1').

Let us exploit another piece of information that we expect to hold in these models. Due to the permutation symmetry in mean field models the weights should behave asymptotically in the same way if the random variables in a *finite* volume are changed. This will be easy to verify in our examples, but we refrain from a general investigation here. So, we take this as an assumption and look for the consequence on the distribution of $\kappa_N(\eta)$. Due to the fact that we check convergence of $\kappa_N(\eta)(F)$ with *local F's*, the weights will then become *asymptotically independent* from the random variables the function F feels. Let us use the notation $\|p-p'\|$ for two weights p, p', viewed as elements in $I\!R^M$, for any norm on $I\!R^M$.[1]

The precise consequence of this phenomenon for the distribution of the empirical and for the conditioned metastate is

Proposition 2: *Suppose, in addition to the assumption of proposition 1, that for all $\eta \in \underline{\mathcal{H}}$, for all finite $V \subset I\!N$,*

$$\lim_{N\uparrow\infty} \sup_{\tilde\eta_V} \|p_N(\eta) - p_N(\eta + \tilde\eta_V)\| = 0 \tag{3.31}$$

where $\tilde\eta_V$ is a local perturbation in the finite volume V s.t. $\eta_V + \tilde\eta_V$ lies in the support of the distribution $I\!P$. Let η' denote a copy of disorder variables, independent of η.

(i) *If $I\!P[\mathcal{H}'] = 1$, we have for the empirical metastate*

$$\lim_{N\uparrow\infty} \int \kappa_N(\eta)(d\mu)F(\mu) =^{law} \lim_{N\uparrow\infty} \frac{1}{N}\sum_{n=1}^{N} F\left(\sum_{m\in\mathcal{M}} p_N^m(\eta')\mu_\infty^0(m,\eta)\right) \tag{3.32}$$

for all bounded continuous F on $\mathcal{P}(\Omega)$, whenever the limit on the r.h.s. exists.

(ii) *If $\lim_{N\uparrow\infty} I\!P[\mathcal{H}(N)] = 1$, we have for the conditioned metastate*

$$\int \bar\kappa(\eta)(d\mu)F(\mu) = \lim_{N\uparrow\infty} \int I\!P(d\eta')F\left(\sum_{m\in\mathcal{M}} p_N^m(\eta')\mu_\infty^0(m,\eta)\right) \tag{3.33}$$

[1] Due to the finiteness of M, the choice of the norm doesn't matter; if we allowed M to increase with N, this would become an important point.

for all bounded continuous F on $\mathcal{P}(\Omega)$, whenever the limit on the r.h.s. exists.

Proof: We may restrict to local functions F of the form (2.2). To prove (i) it suffices to show that, given F, there exist versions η_1, η_2, of disorder variables, mutually independent, s.t. for all $\eta \in \underline{\mathcal{H}}$ we have the pointwise limit

$$\lim_{n\uparrow\infty} \left(F\left(\sum_{m\in\mathcal{M}} p_n^m(\eta)\mu_\infty^0(m,\eta)\right) - F\left(\sum_{m\in\mathcal{M}} p_n^m(\eta_1)\mu_\infty^0(m,\eta_2)\right)\right) = 0 \tag{3.34}$$

But note that such a function can be written in the form

$$F\left(\sum_{m\in\mathcal{M}} p_n^m(\eta)\mu_\infty^0(m,\eta)\right) = \hat{F}\left(\left(\sum_{m\in\mathcal{M}} p_n^m(\eta)\mu_\infty^0(m,\eta)(f_j)\right)_{j=1,\dots,l}\right) =: \tilde{F}\left(p_n(\eta), \eta_J\right) \tag{3.35}$$

Due to the $\mu_\infty^0(m,\eta)$ being product measures with local dependence on the randomness, the η-dependence of F other than through $p_n(\eta)$ itself remains *local*; the finite support J of η_J depends of course on the special choice of the functions f_j.

Now, define the variable η_1 to coincide with η on J^c and to coincide with an independent copy on J. Define η_2 to coincide with η on J and to coincide with an independent copy on J^c. Since $\tilde{F}$ is a uniformly continuous function on the compact space of probability vectors (3.34) follows from the assumption (3.31).

The same type of argument proves (ii). $\diamondsuit$

Let us comment on the relations between the various objects we have obtained and the picture that arises from the above propositions, assuming the approximation properties (3.19) and (3.31). The full information on the level of metastates is contained in the object $\tilde{\kappa}_N(\eta)$ (3.28). It is centered on the infinite volume Gibbs states and contains the asymptotic form of the weights in the extremal decomposition. The weights will depend on the overall information of the random variables; therefore they will be asymptotically independent from the variables in a fixed finite volume. But, a local observable feels the underlying randomness only locally. Thus, for the limit *of the distribution* of the empirical metastate, the weights can be replaced with an independent copy, giving rise to an 'additional randomness'. The limiting distribution of $\tilde{\kappa}_N(\eta)$ contains information about the asymptotic behavior along a path $N \mapsto \mu_N(\eta)$. On the other hand, the conditioned metastate contains no path properties at all: The weights, replaced with independent copies with the same distribution are integrated out. In that case, the whole size dependence is averaged 'over infinity'. Its interpretation, suggested by the asymptotic independence, is then: Having no particular knowledge of the given realization of the disorder globally, the conditioned metastate gives the weights with one expects to find a specific mixture of states. This same metastate could be constructed by 'thinning out' the sequence of volumes which occur in the empirical metastate in a nonrandom way, as has been shown for lattice systems in [N].

4. The Curie Weiss Random Field Ising Model in the 2 phase region

In this chapter we prove Theorem 1 for our first example, the CWRFIM, and the fixed realization results of Theorem 1'. By this we provide an easy example of the mean field picture of the last chapter. We will also see in this example that the set of fixed realization cluster can be strictly larger, almost surely, than the support of all the metastates.

In the CWRFIM the logarithmic moment generating function of the order parameter (3.11) becomes

$$L(t,\eta_i) = \frac{1}{\beta}\log\cosh(\beta(t+\eta_i)) \tag{4.1}$$

Due to our assumption that $\eta_i = \pm\epsilon$ takes only two values it can be written in the form

$$L(t,\eta_i) = L_+(t) + L_-(t)\frac{\eta_i}{\epsilon} \tag{4.2}$$

where

$$\begin{aligned} L_+(t) &:= \frac{1}{2\beta}\left(\log\cosh(\beta(t+\epsilon)) + \log\cosh(\beta(t-\epsilon))\right) \\ L_-(t) &:= \frac{1}{2\beta}\left(\log\cosh(\beta(t+\epsilon)) - \log\cosh(\beta(t-\epsilon))\right) \end{aligned} \tag{4.3}$$

Thus the function $\Phi_N(m,\eta)$ becomes

$$\Phi_N(m,\eta) = \frac{m^2}{2} - L_+(m) - L_-(m)\frac{W_N}{N} \tag{4.4}$$

where the dependence on the randomness on only through the random walk

$$W_N := \sum_{1\le i\le N}\frac{\eta_i}{\epsilon} \tag{4.5}$$

This will make the analysis particularly easy, in that it reduces questions on the metastates to questions about the walk W_N.

As said before in Chapter 3, the structure of the phase diagram is determined by the averaged function $\Phi^0_N(m) = \frac{m^2}{2} - L_+(m)$ which has been analysed in detail (see [SW],[APZ]): For 'large magnetic fields' $\epsilon > \frac{1}{2}$, it has only one global quadratic minimum at $m = 0$. For $0 \le \epsilon \le \frac{1}{2}$ there exists a critical inverse temperature $\beta_c(\epsilon)$ s.t. for $\beta > \beta_c(\epsilon)$ the system has two symmetric global quadratic minima at positions $\pm m^* \equiv \pm m^*(\beta,\epsilon)$; for $\beta < \beta_c(\epsilon)$ the system has one global quadratic minimum at $m = 0$. We assume for the rest of this chapter that we are in this two phase region.[1]

The results about the metastate are now easy to understand heuristically: Define $\mu_\infty^\pm(\eta) := \mu_\infty^0(\pm m^*, \eta)$. Let us just replace the integral over m in the definition of $\tilde{\mu}_N(\eta)$ by two delta functions at $\pm m^*$ with weights determined by the values of $\Phi_N(\pm m^*, \eta)$. Let us thus introduce the weights

$$p_N(W_N) := \frac{e^{c(\beta)W_N}}{e^{c(\beta)W_N} + e^{-c(\beta)W_N}} \tag{4.6}$$

with $c(\beta) = \beta L_-(m^*)$. Heuristically we have then

$$\mu_N(\eta) \approx p(W_N)\mu_\infty^+(\eta) + (1 - p(W_N))\mu_\infty^-(\eta) \tag{4.7}$$

Now, the argument in the exponent of $p(W_N)$, $W_N \sim N^{\frac{1}{2}}$, moves on a scale increasing with N. Thus, for the empirical metastate, we might even use the approximation $p(W_N) \approx 1_{W_N > 0}$. Let us thus define

$$n_N(\eta) := \frac{1}{N}\#\{1 \le n \le N | W_n > 0\} \tag{4.8}$$

Then, for a continuous function F on $\mathcal{P}(\Omega)$ we would have

$$\frac{1}{N}\sum_{1\le n\le N} F\left(\mu_n(\eta)\right) \approx n_N(\eta)F(\mu_\infty^+(\eta)) + (1 - n_N(\eta))F(\mu_\infty^-(\eta)) \tag{4.9}$$

which explains the results for the empirical metatate. Denote, following the notation of the last chapter,

$$\tilde{\kappa}_N(\eta) := n_N(\eta)\delta_{\mu^+(\eta)} + (1 - n_N(\eta))\delta_{\mu^-(\eta)} \tag{4.10}$$

Then the precise results are given by Theorem 1 and

Theorem 1':

(i) For all η in a full measure set, the set of weak cluster points equals

$$\begin{aligned}&\mathcal{CP}\{\mu_N(\eta),\, N = 1, 2, \ldots\}\\ &= \left\{q\mu^+(\eta) + (1 - q)\mu^+(\eta),\ \frac{1}{q} = 1 + \exp(-2c(\beta)z),\, z \in \mathbb{Z} \cup \{+\infty\} \cup \{-\infty\}\right\}\end{aligned} \tag{4.11}$$

(ii) For all η in a full measure set, for any continuous function $F : \mathcal{P}(\Omega) \mapsto I\!R$ the empirical metastate is approximated by

$$\lim_{N\uparrow\infty}\left(\int \kappa_N(\eta)(d\mu)F(\mu) - \int \tilde{\kappa}_N(\eta)(d\mu)F(\mu)\right) = 0 \tag{4.12}$$

[1] At the phase transition line itself there exist two regions: For small ϵ there is a unique global quartic minimum at $m = 0$, as for the usual CW ferromagnet; for large ϵ there are three global quadratic minima. These two line segments are seperated by a tricritical point, where there is one global sixth order minimum.

(iii) For all η in a full measure set the conditioned metastate exists and equals

$$\bar{\kappa}(\eta) = \frac{1}{2}\delta_{\mu^+(\eta)} + \frac{1}{2}\delta_{\mu^-(\eta)} \tag{4.13}$$

Remark: Note explicitely, that the conditioned metastate contains only the equal weight distribution on $\{-\frac{1}{2}, \frac{1}{2}\}$, which is obtained by averaging over the variable n_∞ of Theorem 1. The information it contains at all, it thus that, for large N, the system will be in one of the pure phases.

The set of cluster points has also been observed by [APZ]. We would like to point our here that, a.s., it is strictly bigger than the support of the metastates. The special structure is of course due to the discrete nature of the distribution of the random fields; if their distribution were continuous, we would expect to get in fact all mixtures.

The proof is of course an application of the general propositions 1 and 2 plus the model dependent estimates of the laplace asymptotics for the measure $\tilde{\mu}_N(\eta)$. To this end we will now introduce two sorts of 'regular sets' of realizations of the disorder. One is

$$\mathcal{H}_1(N) := \left\{\eta : |W_N(\eta)| \leq N^{\frac{1+\delta}{2}}\right\} \tag{4.14}$$

with some $0 < \delta < \frac{1}{2}$. We consider balls around the minima $\pm m^*$ with radii

$$\rho_N := N^{-\frac{1}{4}+\frac{\delta}{2}} \tag{4.15}$$

Then an estimation of the occuring integrals gives

Proposition 3: *There exists a nonrandom $N_0 = N_0(\beta, \epsilon)$ s.t. for all $N \geq N_0$ for all $\eta \in \mathcal{H}_1(N)$*

$$\tilde{\mu}_N\left[B_{\rho_N}(m^*) \cup B_{\rho_N}(-m^*)\right] \geq 1 - \exp\left(-const\,(\beta,\epsilon) N^{\frac{1}{2}+\delta}\right) \tag{4.16}$$

and

$$\left|\log \frac{\tilde{\mu}_N\left[B_{\rho_N}(m^*)\right]}{\tilde{\mu}_N\left[B_{\rho_N}(-m^*)\right]} - 2c(\beta) W_N\right| \leq Const\,(\beta,\epsilon) N^{-\frac{1}{4}+\frac{\delta}{2}} \tag{4.17}$$

Remark: The proposition shows that outside exceptional sets one has a fairly explicite control about the cluster properties of $\tilde{\mu}_N(\eta)$, including the relative weights. We only remark that it is easy to see same bounds hold for the measure $\bar{\mu}_N$ (with a possible degradation of $const\,(\beta,\epsilon)$ and N_0).

We will postpone the proof to the end of this chapter.

Let us also introduce the smaller regular sets $\mathcal{H}_2(N)$ by imposing as a second condition that the $|W_N(\eta)|$ be not too small:

$$\mathcal{H}_2(N) := \left\{\eta : |W_N(\eta)| \leq N^{\frac{1+\delta}{2}} \quad \text{and} \quad |W_N(\eta)| \geq N^{\tilde{\delta}}\right\} \tag{4.18}$$

for $0 < \tilde{\delta} < \frac{1}{2}$. Denote, following our usual notation,

$$\begin{aligned} \mathcal{H}'_{1,2} &:= \left\{ \lim_{N\uparrow\infty} \frac{1}{N} \sum_{n=1}^{N} 1_{\eta \in \mathcal{H}_{1,2}(n)^c} = 0 \right\} \\ \underline{\mathcal{H}}_{1,2} &:= \liminf_{N\uparrow\infty} \mathcal{H}_{1,2}(N) \end{aligned} \tag{4.19}$$

Then we have

Lemma 3:

(i) $I\!P[\mathcal{H}'_2] = 1,\ I\!P[\underline{\mathcal{H}}_2] = 0$

(ii) $I\!P[\mathcal{H}'_1] = I\!P[\underline{\mathcal{H}}_1] = 1$

Proof: To prove the first claim in (i) we must show that

$$S_N := \frac{1}{N} \sum_{1 \leq n \leq N} 1_{W_n \in B_n} \to 0 \tag{4.20}$$

a.s. where $B_n = \left\{x \in I\!R : |x| \geq N^{\frac{1+\delta}{2}} \quad \text{or} \quad |x| \leq N^{\tilde{\delta}}\right\}$. S_N is nothing but the mean time of the walk spent in the 'bad regions' B_n.

Note that $S_n \leq 2S_{2^{k+1}}$ for $2^k \leq n \leq 2^{k+1}$. Therefor it suffices to show that $S_{2^k} \to 0$ a.s. with $k \uparrow \infty$. By Borel-Cantelli it suffices to show that, for any (rational) ϵ,

$$\sum_{k=1}^{\infty} I\!P\left[S_{2^k} > \epsilon\right] < \infty \tag{4.21}$$

But this follows simply from the Chebycheff inequality since

$$I\!P\left[S_N > \epsilon\right] \leq \frac{I\!ES_N}{\epsilon} = \frac{1}{\epsilon N} \sum_{1 \leq n \leq N} I\!P\left[W_n \in B_n\right] \leq \frac{Const}{\epsilon} N^{-\frac{1}{2}+\tilde{\delta}} \tag{4.22}$$

where we have used that, by standard estimates, $I\!P\left[W_n \in B_n\right] \leq Const \left(N^{-\frac{1}{2}+\tilde{\delta}} + e^{-const\, N^{\delta}}\right)$

The second claim in (i) follows from the recurrence of the random walk. (ii) follows from the law of iterated logarithm. $\diamondsuit$

(i) shows that we really need to distinguish between the sets $\mathcal{H}'_2$ and $\underline{\mathcal{H}}_2$. With these preparations we come to the

Proof of Theorem 1 and 1': It is easy to check that from the estimates in proposition 3 follows property CR(ρ_N) along the sets $\mathcal{H}_1(N)$ with the weights defined by (4.6). To show Theorem 1'(i), we note that it follows from proposition 1 (ii) that the cluster points are described in terms of the cluster points of the weights (4.6), for all η in the full measure set $\underline{\mathcal{H}}_1$. But, due to the recurrence of the walk, these are of the form as in written in (4.11), a.s.

To prove the rest of the statements, we use the different, 'trivial' weights

$$\begin{aligned} p_N^{m^*}(\eta) &= 1_{W_N>0} \\ p_N^{-m^*}(\eta) &= 1_{W_N\leq 0} \end{aligned} \tag{4.23}$$

For Theorem 1'(ii), note that from proposition 3 also follows property CR(ρ_N) along the smaller sets $\mathcal{H}_2(N)$ for the weights (4.23). This is a simple consequence of the imposed minimum size of $|W_N|$. Thus, Theorem 1'(ii) follows from proposition 1(ii) with the full measure set $\mathcal{H}_2'$.

To prove Theorem 1'(iii) and Theorem 1 note that we have for $\eta \in \underline{\mathcal{H}}_2$, because of the minimum size of $|W_N|$,

$$\lim_{N\uparrow\infty} \sup_{\tilde{\eta}_V}(1_{\sum_{i=1}^N \eta_i>0} - 1_{\sum_{i=1}^N \eta_i+\sum_{i\in V}\tilde{\eta}_i)>0}) = 0 \tag{4.24}$$

Note further, that $\lim_{N\uparrow\infty} I\!P[\mathcal{H}_2(N)] = 1$ (as has been seen in the proof of Lemma 3). Thus, Theorem 1' (iii) follows from proposition 2 (ii).

To obtain Theorem 1, remark that, according to proposition 2 (i), the distributional limit is given by the expression

$$\lim_{N\uparrow\infty} \frac{1}{N}\sum_{n=1}^N F(\mu_n(\eta)) =^{law} \lim_{N\uparrow\infty} \left(n_N F\left(\mu_\infty^+(\eta)\right) + (1-n_N)F\left(\mu_\infty^-(\eta)\right)\right) \tag{4.25}$$

where now n_N are random variables with distribution as in (4.8), but independent of η. Now, it is a well known result from elementary fluctuation theory (see e.g. [Fe]) that the n_N converge in distribution to a variable n_∞ which is distributed according to the arcsin-law (1.6). (And *not* to the equidistribution on $\{\frac{1}{2}, -\frac{1}{2}\}$!) This concludes the proof.$\diamondsuit$

Let us finally give the proof of proposition (3). The type of estimates used here are standard; we apply parts of what was used in [BG1] in a far more complicated situation. However, we include these computations here since they are prototypical for random mean field models.

Thus, let $m^* > 0$ is the largest solution of the mean field equation $m = L_+'(m)$. We define $R_\rho := (B_\rho(m^*) \cup B_\rho(-m^*))^c$. We will have to estimate the corresponding integrals

$$\begin{aligned} I_\rho^\pm &:= \int_{B_\rho(\pm m^*)} dm \exp\left(-\beta N\left(\Phi_N(m) - \Phi^0(m^*)\right)\right) \\ J_\rho &:= \int_{R_\rho} dm \exp\left(-\beta N\left(\Phi_N(m) - \Phi^0(m^*)\right)\right) \end{aligned} \tag{4.26}$$

where we have dropped the η in our notation. To prove the proposition we show that there exist $N_0 = N_0(\beta,\epsilon)$ and $const\,(\beta,\epsilon) > 0$ s.t. for all $N \geq N_0$ and for all $\eta \in \mathcal{H}_1(N)$

$$\frac{J_{\rho_N}}{I^{\pm}_{\rho_N}} \leq \exp\left(-const\,(\beta,\epsilon) N^{\frac{1}{2}+\delta}\right) \tag{4.27}$$

and

$$\frac{I^{\pm}_{\rho_N}}{I^{\mp}_{\rho_N}} \exp\left(\mp 2\beta L_-(m^*) W_N\right) \geq 1 - const\,(\beta,\epsilon) N^{-\frac{1}{4}+\frac{\delta}{2}} \tag{4.28}$$

Before we start, we remark for later use that the higher derivatives of $L_\pm$ vanish at infinity:

$$\begin{aligned} \lim_{|m|\uparrow\infty} \left| \left(\frac{\partial}{\partial m}\right)^k L_+(m) \right| &= 0 \quad, k \geq 2 \\ \lim_{|m|\uparrow\infty} \left| \left(\frac{\partial}{\partial m}\right)^k L_-(m) \right| &= 0 \quad, k \geq 1 \end{aligned} \tag{4.29}$$

and are therefore uniformly bounded. We write $m = \pm m^* + v$ and treat the two cases $\pm$ at the same time. Then we have for $|v| \leq \rho$, using the symmetry properties of the functions and of their derivatives,

$$\begin{aligned} &\Phi_N(\pm m^* + v) - \Phi^0(m^*) \pm \frac{W_N}{N} L_-(m^*) \\ &= \frac{\Phi^{0''}(m^* + \theta v)}{2} v^2 - \frac{W_N}{N} L_-'(m^*) v - \frac{W_N}{N} \frac{L_-''(\pm m^* + \theta' v)}{2} v^2 \end{aligned} \tag{4.30}$$

with some $0 \leq \theta, \theta' \leq 1$. Thus, on $|v| \leq \rho$,

$$\Phi_N(\pm m^* + v) - \Phi^0(m^*) \pm \frac{W_N}{N} L_-(m^*) \leq \frac{b_+}{2} v^2 - zv \tag{4.31}$$

with

$$z := \frac{W_N}{N} L_-'(m^*) \tag{4.32}$$

and

$$b_+ := b_+(\rho) := \sup_{v, |v| \leq \rho} \Phi^{0''}(m^* + v) + \left|\frac{W_N}{N}\right| \sup_{v, |v| \leq \rho} \left| L_-''(m^* + v) \right| \tag{4.33}$$

Similarly we have

$$\Phi_N(\pm m^* + v) - \Phi^0(m^*) \pm \frac{W_N}{N} L_-(m^*) \geq \frac{b_-}{2} v^2 - zv \tag{4.34}$$

with

$$b_- := b_-(\rho) := \inf_{v, |v| \leq \rho} \Phi^{0''}(m^* + v) - \left|\frac{W_N}{N}\right| \sup_{v, |v| \leq \rho} \left| L_-''(m^* + v) \right| \tag{4.35}$$

Lemma 4: Denote $P(x) = I\!\!P[G \geq x]$ for a standard Normal G. If $a \in [-\gamma, \gamma]$, $\gamma > 0$

$$\int_{|x| \geq \gamma} e^{-\frac{x^2}{2} + ax} \frac{dx}{\sqrt{2\pi}} = e^{\frac{a^2}{2}} \left(P(\gamma - a) + P(-\gamma - a)\right) \leq e^{-\frac{\gamma^2}{2} + a\gamma} + e^{-\frac{\gamma^2}{2} - a\gamma} \tag{4.36}$$

Proof: From the well known estimate $P(x) \leq \exp(-x^2/2)$.$\diamondsuit$

This gives, for $\rho \geq 4|z|/b_+$,

$$\int_{|v|\geq\rho} e^{-\beta N\left(\frac{b_+}{2}v^2 - zv\right)} dv \leq 2\sqrt{\frac{2\pi}{\beta N b_+}} e^{-\beta N\left(\frac{b_+}{2}\rho^2 - |z|\rho\right)} \leq \sqrt{\frac{8\pi}{\beta N b_+}} \exp\left(-\frac{\beta N b_+ \rho^2}{4}\right) \tag{4.37}$$

With

$$\int_{I\!R} e^{-\beta N\left(\frac{b_+}{2}v^2 - zv\right)} dv = \sqrt{\frac{2\pi}{\beta N b_+}} \exp\left(\frac{z^2 \beta N}{2b_+}\right) \tag{4.38}$$

we obtain from this

$$I_\rho^\pm \geq \exp\left(\pm\beta L_-(m^*)W_N\right) \sqrt{\frac{2\pi}{\beta N b_+}} \left(\exp\left(\frac{z^2\beta N}{2b_+}\right) - 2\exp\left(-\frac{\beta N b_+ \rho^2}{4}\right)\right) \tag{4.39}$$

For the upper bound we simply write

$$\begin{aligned} I_\rho^\pm &\leq \exp\left(\pm\beta L_-(m^*)W_N\right) \int_{I\!R} e^{-\beta N\left(\frac{b_-}{2}v^2 - zv\right)} dv \\ &= \exp\left(\pm\beta L_-(m^*)W_N\right) \sqrt{\frac{2\pi}{\beta N b_-}} \exp\left(\frac{z^2\beta N}{2b_-}\right) \end{aligned} \tag{4.40}$$

Next we estimate the integral over the outer region. We use the following rough estimate.

Lemma 5: *For each ϵ, β in the two phase region there exists a constant $\hat{c}(\beta,\epsilon)$ s.t. for all $v \geq -m^*$*

$$\begin{aligned} &\Phi^0(m^* + v) - \Phi^0(m^*) \geq \hat{c}(\beta,\epsilon)v^2 \\ &\sup_{m\in I\!R} |L_-(m)| =: c_2(\beta,\epsilon) < \infty \end{aligned} \tag{4.41}$$

Proof: The first claim states that Φ^0 is bounded below by a parabol on $I\!R_\geq$. It can be chosen to coincide with Φ at the points $m = 0$ and m^* (where the absolute minimum is attained.) The proof is elementary. To prove the second claim it suffices to verify that $\lim_{m\to\pm\infty} |L_-(m)| < \infty$ which is again elementary. $\diamondsuit$

From this we have

$$\begin{aligned} J_\rho &:= \int_{R_\rho} dm \exp\left(-\beta N\left(\Phi_N(m) - \Phi^0(m^*)\right)\right) \\ &\leq 2\exp\left(c_2(\beta,\epsilon)|W_N|\right) \int_{|v|\geq\rho} dv \exp\left(-\beta N \hat{c}(\beta,\epsilon)v^2\right) \\ &\leq 2\exp\left(c_2(\beta,\epsilon)|W_N| - \beta N\hat{c}(\beta,\epsilon)\rho^2\right) \end{aligned} \tag{4.42}$$

Thus, on $\mathcal{H}_1(N)$,

$$J_\rho \leq Const \exp\left(Const\,(\beta,\epsilon)N^{\frac{1+\delta}{2}} - const\,(\beta,\epsilon)N^{\frac{1}{2}+\delta}\right) \tag{4.43}$$

The choice of ρ_N was made to make the last estimate hold.

Since Φ^0 has bounded third derivatives we have further

$$|\sup_{v,|v|\leq\rho} \Phi^{0''}(\pm m^* + v) - \Phi^{0''}(m^*)| \leq Const\,(\beta,\epsilon)\rho \tag{4.44}$$

Thus, on $\mathcal{H}_1(N)$,

$$|b_+(\rho_N) - \Phi^{0''}(m^*)| \leq Const\,(\beta,\epsilon) N^{-\frac{1}{4}+\frac{\delta}{2}} \tag{4.45}$$

We have from these estimates

$$\frac{J_{\rho_N}}{I^{\pm}_{\rho_N}} \leq Const' \exp\left(Const'(\beta,\epsilon) N^{\frac{1+\delta}{2}} - const\,(\beta,\epsilon) N^{\frac{1}{2}+\delta}\right) \tag{4.46}$$

and

$$\begin{aligned}&\frac{I^{\pm}_{\rho_N}}{I^{\mp}_{\rho_N}} \exp\left(\mp 2\beta L_-(m^*) W_N\right) \geq \sqrt{\frac{b_+(\rho_N)}{b_-(\rho_N)}} \left(1 - 2e^{-\frac{\beta N b_+(\rho_N)\rho_N^2}{4}}\right)\\ &\geq 1 - const\,(\beta,\epsilon)\rho_N = 1 - const\,(\beta,\epsilon) N^{-\frac{1}{4}+\frac{\delta}{2}}\end{aligned} \tag{4.47}$$

from which the claim follows for large enough N.$\diamondsuit$

5. The Hopfield model below the critical temperature

The logarithmic moment generating function of the order parameter is

$$L(t,\xi_i) = \frac{1}{\beta} \log\cosh(\beta t \cdot \xi_i)) \tag{5.1}$$

The structure of the phase diagram is determined by the averaged function $\Phi^0_N(m) = \frac{m^2}{2} - I\!\!E L(m,\xi_1)$. For $\beta > 1$ there exist precisely $2M$ global minima at positions sm^*a^ν, $s = \pm 1$, a^ν being the νth unity vector of $I\!\!R^M$. These are solutions of the averaged mean field equation

$$I\!\!E\left[\xi_1 \tanh(m \cdot \xi_1)\right] = m \tag{5.2}$$

m^* is the largest solution of the ordinary Curie Weiss equation $m = \tanh \beta m$. The M symmetric mixtures of the above product measures

$$\mu^\nu_\infty(\xi) := \frac{1}{2}\left(\mu^0_\infty(m^*a^\nu,\xi) + \mu^0_\infty(-m^*a^\nu,\xi)\right) \tag{5.3}$$

are called 'Mattis states'. They always come in pairs due to the $\pm$ symmetry of the model. For more precise information on the Hopfield model, also in the case where the number of patterns is allowed to go to infinity, see [BGP],[BG1],[BG2].

An important role will be played now by the $M \times M$ matrix $b_N(\xi)$, defined by

$$b_N^{\mu\nu}(\xi) := \sum_{i=1}^{N} (\xi_i^\mu \xi_i^\nu - \delta^{\mu\nu}) \tag{5.4}$$

b_N is symmetric and has vanishing diagonal; note that different elements are uncorrelated (unless prescribed by symmetry) but *not* independent. b_N will describe the random symmetry breaking between the Mattis states in finite volume. Thus, the role that has been played by the random walk $N \mapsto W_N$ in the CWRFIM will now be played by the multidimensional random walk $N \to b_N$.

The asymptotic form of the weights in the extremal decomposition is then given as follows. Let us denote by $\mathcal{A}$ the $\frac{M(M-1)}{2}$ dimensional vector space of $M \times M$ symmetric matrices with vanishing diagonal. Let us denote by $\mathcal{S} = \{(p^\mu)_{\mu=1,\dots,M}\}$ the simplex of M-dimensional probability vectors.

Let us now define the map $p : \mathcal{A} \to \mathcal{S}$ given by

$$p^\nu(V) := \frac{\tilde{p}^\nu(V)}{\sum_{\mu=1}^{M} \tilde{p}^\mu(V)} \quad \text{where} \quad \tilde{p}^\nu(V) := \exp\left(c(\beta)(V^2)^{\nu\nu}\right) \tag{5.5}$$

with

$$c(\beta) = \frac{\beta m^*}{2\left(1 - \beta(1-m^*)^2\right)} \tag{5.6}$$

To obtain the weights in Theorem 2(1.7) from (5.5) take $M(M-1)/2$ independent onedimensional Brownian motions $W_t^{\mu\nu}$ for $\mu < \nu$; we set $W_t^{\nu\mu} := W_t^{\mu\nu}$ and $W_t^{\mu\mu} := 0$ to obtain a Brownian motion $W_t = (W_t^{\mu\nu})_{1\le\mu,\nu\le M}$ with values in $\mathcal{A}$.

With this definition we have the approximate formula

$$\mu_N(\xi) \approx \sum_{1\le\nu\le M} p^\nu(N^{-\frac{1}{2}} b_N(\xi))\mu_\infty^\nu(\xi) \tag{5.7}$$

Note that (not only $M = 1$ but also) $M = 2$ is a trivial case: For $M = 2$ we have $p^{(1)}(V) \equiv p^{(2)}(V) \equiv \frac{1}{2}$, for all $V \in \mathcal{A}$. Nontrivial size dependence in the Hopfield model occurs only if $M \geq 3$.

We remark that the occurence of the matrix $N^{-\frac{1}{2}} b_N(\xi)$ in the weights can be easily understood: In fact, its diagonal elements describe the energy difference between the M pairs of groundstates $\sigma = \pm\xi^\mu$, since $E_N(\sigma = \xi^\mu, \xi) = \frac{1}{2N}\left(b_N^2(\xi)\right)^{\mu\mu} + \frac{N}{2}$. For finite temperature the formula (5.7) can then be understood if one performs a perturbational calculation for the depth of the minima of the random function $m \mapsto \Phi_N(m,\xi)$, thereby considering the deviation from its mean value as a perturbation. Precise estimates (analogues of proposition 3 for the CWRFIM) that allow for the application of proposition 1 and 2 have in fact been done in a different context, so that we need not repeat their proofs here; they can be readily read off from [Gen], where central limit behavior for the measures $\bar{\mu}_N$ around the randomly shifted minima of the function $\Phi_N(m,\xi)$ was proved.

It is important to note that, while in the CWRFIM the arguments in the exponents of the weights were moving on a scale $\sim N^{\frac{1}{2}}$, now the normalization of the central limit theorem is taken. This was the reason for favoring the extemal states in the first case. In the Hopfield model, the weights will remain spread over all mixtures when $N \uparrow \infty$.

To state the results precisely we introduce the following objects. Following old notations we set

$$\tilde{\kappa}_N(\xi) = \frac{1}{N}\sum_{n=1}^{N} \delta_{\sum_{\nu=1}^{M} p^\nu\left(\frac{b_n(\xi)}{\sqrt{n}}\right)\mu_\infty^\nu(\xi)} \tag{5.8}$$

It is possible to get an even nicer form: We find it instructive to introduce also a metastate that differs from the above by strong approximation of $b_N(\xi)$ by a Gaussian process of particularly simple form. To do so, we apply the powerful strong invariance principle for partial sum processes for $I\!R^k$-valued independend random variables, whose proof can be found in a general context in [Rio]. It states that a sequence of Gaussian random variables can be constructed on a *common* probability space having the same $k \times k$ covariance matrix that approximates the partial sum process for a.e. realization.

In our case, from [Rio], page 1712, Cor. 4 follows that there exist onedimensional random variables $\gamma_n^{\mu\nu} = \gamma_n^{\nu\mu}$ for $\nu \neq \mu$, $\gamma_n^{\mu\mu} \equiv 0$, on a *common* probability space with ξ s.t.:

(i) $\gamma = (\gamma_n^{\mu\nu})_{1\leq\mu\neq\nu\leq M; n=1,2,\ldots}$ are i.i.d. Normal Gaussians (for different $\{\mu,\nu\}$ and n)

(ii)
$$\sup_{N=1,2,\ldots} \|b_N^{\mu\nu} - g_N^{\mu\nu}\| = \mathcal{O}(\log N) \tag{5.9}$$

a.s., where

$$g_N^{\mu\nu} = \sum_{n=1}^{N} \gamma_n^{\mu\nu} \tag{5.10}$$

Then we put

$$\hat{\kappa}_N(\gamma,\xi) := \frac{1}{N}\sum_{n=1}^{N} \delta_{\sum_\nu p^\nu\left(\frac{g_N}{\sqrt{n}}\right)\mu_\infty^\nu(\xi)} \tag{5.11}$$

Remark: Note that the matrix elements of g_N have the advantage not only of being Gaussian but also *independent* (unless prescribed by the symmetry of the matrix) which was not true for the matrices b_N. Thus, they form a $M(M-1)/2$ dimensional random walk with Standard Gaussian increments.

With these definitions, the analogue of Theorems 1,1' are Theorem 2 and

Theorem 2':

(i) For all ξ in a full measure set, the set of weak cluster points equals

$$\mathcal{CP}\{\mu_N(\xi),\ N=1,2,\dots\} = \left\{ \sum_{1\leq\nu\leq M} q^\nu \mu^\nu_\infty(\xi),\ (q^\nu)_{\nu=1,\dots,M} \in \mathcal{S}' \right\} \tag{5.12}$$

where $\mathcal{S}' = \{(\frac{1}{2},\frac{1}{2})\}$ for $M=2$ and $\mathcal{S}'=\mathcal{S}$ for $M\geq 3$.

(ii) For all ξ in a full measure set, for any continuous function $F:\mathcal{P}(\Omega)\mapsto I\!R$ the empirical metastate is approximated by

$$\lim_{N\uparrow\infty}\left(\int \kappa_N(\xi)(d\mu)F(\mu) - \int \tilde\kappa_N(\xi)(d\mu)F(\mu)\right) = 0 \tag{5.13}$$

(iii) A.s., for any continuous function $F:\mathcal{P}(\Omega)\mapsto I\!R$ the empirical metastate is approximated by

$$\lim_{N\uparrow\infty}\left(\int \kappa_N(\xi)(d\mu)F(\mu) - \int \hat\kappa_N(\gamma,\xi)(d\mu)F(\mu)\right) = 0 \tag{5.14}$$

(iv) For all ξ in a full measure set the conditioned metastate exists and equals

$$\bar\kappa(\xi)(F) = I\!E_g F\left(\sum_{\nu=1}^M p^\nu(g)\,\mu^\nu_\infty(\xi)\right) \tag{5.15}$$

where g is a Normal Gaussian in $\mathcal{A}$.

In the course of the proof we will have to compare the map $p(V)$ at different arguments in the noncompact space $\mathcal{A}$. To be able to do so, we need some information about the continuity of $V\mapsto p(V)$. We have

Lemma 6: *Define the norm*

$$\|V\|^2_{ss} := \sup_\mu \sum_\nu (V^{\nu\mu})^2 \tag{5.16}$$

Then

$$\|p(V)-p(V')\|_1 \leq 4c(\beta)\left(\|V\|_{ss} + \|V-V'\|_{ss}\right)\|V-V'\|_{ss} \tag{5.17}$$

Proof: Writing $V_{\alpha\beta}=V_{\beta\alpha}$ we view $p(V)$ as a function of the $M(M-1)/2$ variables $V^{\alpha\beta}$ for $\alpha<\beta$. Then the Taylor formula gives

$$p^\nu(V') - p^\nu(V) = \sum_{\alpha<\beta} \frac{\partial p^\nu}{\partial V^{\alpha\beta}}(\tilde V)(V'-V)^{\alpha\beta} \tag{5.18}$$

where $\tilde{V} = V + \theta(V' - V)$. It is easy to compute that

$$\frac{\partial p^\nu}{\partial V^{\alpha\beta}} = p^\nu(1-p^\nu)\frac{\partial \log \tilde{p}^\nu}{\partial V^{\alpha\beta}} - (p^\nu)^2 \frac{1}{\tilde{p}^\nu} \sum_{\rho,\rho\neq\nu} \tilde{p}^\rho \frac{\partial \log \tilde{p}^\rho}{\partial V^{\alpha\beta}} \tag{5.19}$$

Now

$$\frac{\partial \log \tilde{p}^\rho}{\partial V^{\alpha\beta}} = 2cV^{\alpha\beta}\left(\delta_{\alpha\rho} + \delta_{\beta\rho}\right) \tag{5.20}$$

where we write $c \equiv c(\beta)$. Therefore

$$\sum_{\alpha<\beta} \frac{\partial \log \tilde{p}^\rho}{\partial V^{\alpha\beta}}(\tilde{V})(V' - V)^{\alpha\beta} = 2c\left(\tilde{V}(V - V')\right)^{\rho\rho} \tag{5.21}$$

Then

$$\begin{aligned}
|p^\nu(V') - p^\nu(V)| &= 2c\left|p^\nu(1-p^\nu)\left(\tilde{V}(V-V')\right)^{\nu\nu} - (p^\nu)^2\frac{1}{\tilde{p}^\nu}\sum_{\rho,\rho\neq\nu}\tilde{p}^\rho\left(\tilde{V}(V-V')\right)^{\rho\rho}\right| \\
&\leq 2c\left(p^\nu(1-p^\nu) + (p^\nu)^2\frac{1}{\tilde{p}^\nu}\sum_{\rho,\rho\neq\nu}\tilde{p}^\rho\right)\sup_\lambda\left|\left(\tilde{V}(V-V')\right)^{\lambda\lambda}\right| \\
&= 4cp^\nu(1-p^\nu)\sup_\lambda\left|\left(\tilde{V}(V-V')\right)^{\lambda\lambda}\right|
\end{aligned} \tag{5.22}$$

where all $p, \tilde{p}$'s are taken at the argument $\tilde{V}$.

Note that

$$\sup_\lambda\left|\left(\tilde{V}(V-V')\right)^{\lambda\lambda}\right| \leq \|\tilde{V}\|_{ss}\|V-V'\|_{ss} \leq \left(\|V\|_{ss} + \|V-V'\|_{ss}\right)\|V-V'\|_{ss} \tag{5.23}$$

Summing over ν gives the lemma.$\diamondsuit$

Finally we come to the

Proof of Theorem 2 and 2': From [Gen], proposition 1.3. immediately follows that for any $0 < \delta < \frac{1}{2}$, $\rho < \frac{m^*}{2}$, $s = \pm 1$,

$$\bar{\mu}_N(\xi)\left[B_\rho(sm^*a^\nu)\right] = \frac{\tilde{p}^\nu\left(\frac{b_N(\xi)}{\sqrt{N}}\right)(1 + \mathcal{O}(N^{-\delta}))}{\sum_{\mu=1}^M \tilde{p}^\mu\left(\frac{b_N(\xi)}{\sqrt{N}}\right)(1 + \mathcal{O}(N^{-\delta}))} \tag{5.24}$$

$\mathcal{O}(N^{-\delta})$ is here *nonuniform* in ξ.[1]

[1] It means precisely that for a.e. ξ there exist $N_0(\xi)$ and $Const\,(\xi)$, s.t. for all $N \geq N_0(\xi)$ the term is bounded by $Const\,(\xi)N^{-\delta}$.

We have to use information on the minimum and maximum size of $\frac{b_N(\xi)}{\sqrt{N}}$. In fact, from the Law of Iterated Logarithm for partial sums of $I\!R^k$-valued random variables (see for this statement, which is true more generally in Banach spaces, e.g. [LT], Theorem 8.2) we have

$$\left\| \frac{b_N(\xi)}{\sqrt{N}} \right\| \leq Const \sqrt{\ln \ln N} \tag{5.25}$$

a.s. for $N \geq N_0(\xi)$ sufficiently large (with some arbitrary matrix norm.) This gives

$$\tilde{p}\left(\frac{b_N(\xi)}{\sqrt{N}}\right) \leq (\ln N)^K \tag{5.26}$$

with some constants $K = K(\beta)$, for N sufficiently large.

It is easy to see with this information that from (5.24) follows that

$$\lim_{N\uparrow\infty} \left(\bar{\mu}_N(\xi)\left[B_\rho(sm^* a^\nu)\right] - \frac{\tilde{p}^\nu\left(\frac{b_N(\xi)}{\sqrt{N}}\right)}{\sum_{\mu=1}^M \tilde{p}^\mu\left(\frac{b_N(\xi)}{\sqrt{N}}\right)} \right) = 0 \tag{5.27}$$

This, in the language of Chapter 3, is property CR(ρ) along a sequence of N-independent exceptional sets $\mathcal{H}(N) \equiv \mathcal{H}'$ for the fixed full measure set $\mathcal{H}'$ where the assumptions necessary for the above estimates hold. Now we apply our general reasoning. From the third remark after Lemma 2 in Chapter 3 follows that this implies CR(ρ) for $\tilde{\mu}_N$. (In fact, technically, it is typically proven before!) Due to the second remark after Lemma 2 we have then CR(ρ_N) which suffices for all our needs. Note further, that because of the N-independence of $\mathcal{H}(N) = \mathcal{H}'$ we don't have to worry about exceptional sets any more when applying any of the propositions 1 or 2.

Thus, Theorem 2'(ii) follows from proposition 1(ii).

Theorem 2'(iii) follows from proposition 1(ii) and the following fact: Property CR(ρ) with the probability vector $p\left(\frac{b_N}{\sqrt{N}}\right))$ implies the property CR(ρ) with the probability vector $p(\frac{g_N}{\sqrt{N}})$. To show the latter it suffices to show that, a.s.

$$\lim_{N\uparrow\infty} \left\| p\left(\frac{b_N}{\sqrt{N}}\right) - p\left(\frac{g_N}{\sqrt{N}}\right) \right\|_1 = 0 \tag{5.28}$$

But Lemma 6 implies

$$\left\| p\left(\frac{b_N}{\sqrt{N}}\right) - p\left(\frac{g_N}{\sqrt{N}}\right) \right\|_1 \leq \frac{4c(\beta)}{N} \left(\|b_N\|_{ss} + \|b_N - g_N\|_{ss}\right) \|b_N - g_N\|_{ss} \tag{5.29}$$

Using now the law of iterated logarithm (5.25) and the strong approximation property (5.9) for $\|b_N - g_N\|_{ss}$ the desired estimate (5.28) follows.

To prove Theorem 2'(iv) and Theorem 2, let us first note the finite volume perturbation property, necessary for proposition 2: It is clear that, for fixed finite volume V, $\sup_{\xi_V} \|b_N(\xi) - b_N(\xi+\xi_V)\| \leq Const\,(V)$. Then, we have from Lemma 6

$$\lim_{N\uparrow\infty} \sup_{\xi_V} \left\| p\left(\frac{b_N(\xi)}{\sqrt{N}}\right) - p\left(\frac{b_N(\xi+\xi_V)}{\sqrt{N}}\right)\right\|_1 \leq \frac{4c(\beta)}{N}\left(\|b_N(\xi)\|_{ss} + Const\,(V)\right) Const\,(V) \tag{5.30}$$

Using (5.25) the r.h.s. goes to zero for almost all η.

Let us now denote by ξ' an independent copy of ξ. Note that we have the two approximation properties given by proposition 2(i) and (ii). Then we construct, as above, a strongly approximating process g', but this time for ξ', such that it is independent of ξ. It follows that

$$F\left(\sum_{\nu=1}^{M} p^\nu\left(\frac{b_N(\xi')}{\sqrt{N}}\right)\mu_\infty^\nu(\xi)\right) - F\left(\sum_{\nu=1}^{M} p^\nu\left(\frac{g'_N}{\sqrt{N}}\right)\mu_\infty^\nu(\xi)\right) \to 0 \tag{5.31}$$

a.s., for bounded continuous F, with $N\uparrow\infty$. Putting this together with proposition 2(ii), we obtain directly Theorem 2'(iv). For Theorem 2 we get from proposion 2(i)

$$\lim_{N\uparrow\infty}\int \kappa_N(\xi)(d\mu)F(\mu) =^{law} \lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N} F\left(p^\nu\left(\frac{g'_n}{\sqrt{n}}\right)\mu_\infty^\nu(\xi)\right) \tag{5.32}$$

Since we are only interested in distributions, we replace $\frac{g'_n}{\sqrt{n}}$ by $\frac{W_{t_n}}{\sqrt{t_n}}$ with $t_n = \frac{n}{N}$ where W_t is a Brownian motion. But then (5.32) is nothing but a Riemann sum for the continuous function $t \mapsto F\left(p^\nu\left(\frac{W_t}{\sqrt{t}}\right)\mu_\infty^\nu(\xi)\right)$. Thus it converges for almost all realizations of W_t to the corresponding integral with $N\uparrow\infty$. But, from this follows that the distribution of (5.32) is the same as that of (1.7) which proves Theorem 2.

To prove the result about the cluster points, Theorem 1'(i), it suffices to consider the cluster points of the weights $p\left(\frac{b_N}{\sqrt{N}}\right)$, $N = 1, 2, \ldots$. Now we use the following

Lemma 7: *Let X_i, $i = 1, 2, \ldots$ be a sequence of i.i.d. k-dimensional Normal Gaussians. Then, a.s., the set of the cluster points of the sequence $\frac{1}{\sqrt{N}}\sum_{i=1}^{N} X_i$, $N = 1, 2, \ldots$ equals all of $I\!R^k$.*

The proof is not difficult: Given a neighborhood of a rational point in $I\!R^k$ it is easy to construct a sparse subsequence that hits it infinitely often with probability one. We don't give the details here.

But from that we have in particular $\mathcal{CP}\left(\frac{b_N}{\sqrt{N}}, N = 1, 2, \ldots\right) = \mathcal{A}$, a.s. This implies Theorem 1'(i) by continuity of p and

Lemma 8: *$\overline{p(\mathcal{A})}$ equals all of $\mathcal{S}$ for $M \geq 3$.*

Proof: It suffices to show that, given any vector $l = (l_\mu)_{\mu=1,\dots,M} \in I\!R^M$, there exist a real number b and a matrix $V \in \mathcal{A}$, s.t.

$$l_\mu + b = (V^2)^{\mu\mu}, \quad \mu = 1, \dots, M \tag{5.33}$$

The difficulty about this linear system of equations for the $M(M-1)/2$ quantities $(V^{\mu\nu})^2$ is that it fails to give *nonnegative* solutions for arbitrary choices of l and b. Thus the freedom in the choice of b is really necessary. As an ansatz we consider a matrix of the type

$$\begin{aligned} &V^{12} = V^{21} = \sqrt{\frac{\lambda_1}{2}}, \quad V^{13} = V^{31} = \sqrt{\frac{\lambda_2}{2}}, \quad V^{23} = V^{32} = \sqrt{\frac{\lambda_3}{2}}, \\ &V^{\mu-1,\mu} = V^{\mu,\mu-1} = \sqrt{\lambda_\mu}, \quad \mu = 4, \dots, M, \\ &V^{\mu\nu} = V^{\nu\mu} = 0 \quad \text{otherwise} \end{aligned} \tag{5.34}$$

with $\lambda_\mu \geq 0$, where the condition in the second line is empty for $M = 3$. It turns out then that the solution of (5.33) with $b = 0$ has the general form

$$\begin{aligned} \lambda_1 &= l_1 + l_2 - l_3 \;+\; \big(l_4 - l_5 + l_6 - l_7 \pm \dots + (-1)^M l_M\big) \\ \lambda_2 &= l_1 - l_2 + l_3 \;-\; \big(l_4 - l_5 + l_6 - l_7 \pm \dots + (-1)^M l_M\big) \\ \lambda_3 &= -l_1 + l_2 + l_3 \;-\; \big(l_4 - l_5 + l_6 - l_7 \pm \dots + (-1)^M l_M\big) \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} \lambda_4 &= l_4 - l_5 + l_6 - l_7 \pm \dots + (-1)^M l_M \\ \lambda_5 &= l_5 - l_6 + l_7 \pm \dots + (-1)^{M+1} l_M \\ \lambda_6 &= l_6 - l_7 + l_8 \pm \dots + (-1)^M l_M \\ &\dots \\ \lambda_M &= l_M \end{aligned} \tag{5.36}$$

It suffices to prove the statement for l's in the special form $l_3 \geq l_1 \geq l_2$ and $(l_2 \geq)\; l_4 \geq l_5 \geq \dots \geq l_M \geq 0$. But, using this order relation, it follows for the solution of (5.33) with $b = 0$ that $\lambda_\mu \geq 0$ for all $2 \leq \mu \leq M$, whereas λ_1 can be possibly negative. But note that for the solution of (5.33) with $\lambda_\mu \equiv 0$ and $b > 0$, we have $\lambda_1 = b > 0$ for M odd (resp. $\lambda_1 = 2b > 0$ for M even), $\lambda_\mu \geq 0$ for $2 \leq \mu \leq M$. Thus, by adding a sufficiently large $b > 0$ to the fixed l_μ's one can always force the corresponding λ_1 to become positive without destroying the positivity of the other λ_μ's. This proves the claim.$\diamondsuit$

$\diamondsuit\diamondsuit$

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