On the asymptotic analysis of singularly perturbed systems with sliding mode

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Abstract

In this paper we study singularly perturbed systems with discontinuity surfaces. This means that we have a system of ordinary differential equations with a small parameter and a piecewise smooth vector field. The state where the trajectory moves on the discontinuity surface is called *sliding mode*. We present an asymptotic representation for trajectories with temporary sliding and apply the result to stick-slip vibrations.

1 Introduction

There are many applications of the theory of ordinary differential equations with discontinuitiy surface. Often this surface can be described by a scalar equation $s = 0$, for example if there is arising the nonsmooth function $\operatorname{sgn}(s)$ on the righthand side of the differential system. This case represents **oscillations with dry friction** as well as **relay control systems** which are systems with switching devices.

We start our paper with explaining the term *singularly perturbed system with sliding mode.*

1.1 Sliding modes

In early regulators the control has often been of relay type because of their simple implementation.
In control theory the term for the motion on discontinuity surfaces is *sliding mode* motion. From the geometrical viewpoint the trajectory slides on the surface. [1]
In practice, the sliding mode motion is characterized by a high-frequency switching of the relay. LEVANT [Le93] calls it *real sliding.* There is no *ideal sliding* in real relay control systems. Therefore perfect sliding is only an approximation of the real behaviour of such systems. In dry friction systems we have a different situation. There exists (almost) perfect sticking which means that there is ideal sliding.

The so-called *equivalent control method* is characterized by the following procedure (cf. [Utk92, Part I.]): Let's consider the system

$$\begin{aligned} \dot{x} &= f(x,t,u) \\ u_i(x,t) &= \begin{cases} u_i^+(x,t) \text{ if } s_i(x) > 0 \\ u_i^-(x,t) \text{ if } s_i(x) < 0 \end{cases} \quad i = 1, ..., m \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n,\ t \in \mathbb{R}, u, s \in \mathbb{R}^m$. The vector u represents the control by m relays. Whenever a sliding mode appears, the velocity vector in the state space lies on the tangential plane of one or several discontinuity surfaces. The existence of a sliding mode motion means that there is a continuous control function u and a time interval $[t_1, t_2]$ such that for a trajectory starting at time t_1 on the manifold $\{s(x) = 0\} \subset \mathbb{R}^n$ the vector $s(x)$ has a zero time derivative along system (1) on $[t_1, t_2]$:

$$\dot{s} = \frac{\partial s}{\partial x} f(x,t,u) = 0 \ \ \forall t \in [t_1, t_2] \ . \tag{2}$$

A domain $D_{sli} \subset \mathbb{R}^n$ which satisfies the property that all points $d \in D_{sli}$ are starting points of a sliding motion is called *sliding domain* of (1).

[1]In mechanical systems with dry friction this sliding on the surface corresponds to the sticking of two bodies ! If you are not in the sticking region, from the mechanical point of view you have slipping of the bodies. This is maybe confusing.

If there is a unique continuous solution $u = u_{eq}(x,t)$ of the algebraic equation (2) and $u_{eq,i}$ is between u_i^- and u_i^+ then substitute this solution $u_{eq}(x,t)$, referred to as *equivalent control*, in system (1) for u:

$$\dot{x} = f(x, t, u_{eq}(x,t)) \ . \tag{3}$$

This equation is called *sliding mode equation.*
The equivalent control method implies a replacement of the undefined discontinuous control on the discontinuity surface with a continuous control. From the geometric viewpoint, in the case $m = 1$ one should vary the scalar control from u^- to u^+, plot the vector $f(t,x,u)$ and find the intersection point of the vector and the tangential plane at (x,t). If $m = 1$ and f depends linearly on u this vector coincides with the corresponding vector f_0 of Filippov type [Fil88, § 4] . Otherwise the two vectors may be not even colinear. There is no unambigious answer to the question which equation is better for the description of the motion.
There is an important special case regarding system (1) chracterized by the property $x = (x_1, ..., x_{n-m}, s_1, ..., s_m)^T$. In this case we can separate the differential equation in the following way:

$$\dot{y} = f_1(y,s,t,u), \quad \dot{s} = f_2(y,s,t,u)$$

where $y = (x_1, ..., x_{n-m})^T$. If there is a unique solution $u = u_{eq}(y,t)$ of the algebraic equation

$$f_2(y, 0, t, u) = 0$$

this case is called *first order sliding.*
If f_2 does not depend on u then the conditions

$$s = \dot{s} = 0$$

are not sufficient for determining u_{eq}. Here we have to consider

$$\ddot{s} = \frac{d}{dt}\dot{s} = \frac{d}{dt} f_2(y,s,t)$$

Hence, the algebraic equation which determines the equivalent control is

$$\left(\frac{\partial f_2}{\partial y} f_1 + \frac{\partial f_2}{\partial s} f_2 + \frac{\partial f_2}{\partial t}\right)\Big|_{(y,s=0,t,u)} = 0 \ .$$

If there is a unique solution u_{eq} of this equation we have *second order sliding* (cf. [BF92]). The case of arbitrary sliding manifolds of first or higher order is studied in [Le93] and [FL94].

The stability of sliding domains is not considered in this paper. A Lyapunov type Theorem is stated in [Utk92, p.45ff.].

1.2 Discontinuous singularly perturbed systems

One of the major obstacles in the use of efficient tools for analyzing dynamical systems is a high dimension. If there is a small parameter, in many cases high dimensional systems may be reduced to a system of lower dimension by separating them into fast and slow components and substituting the fast system by an algebraic system by neglecting the small parameter in a certain way. This can be achieved in a rightful way using the singular perturbation approach [VBK95] which can be applied to systems of the form

$$\mu\frac{dz}{dt} = F(z,y,t), \quad \frac{dy}{dt} = f(z,y,t), \quad z \in \mathbb{R}^m, \ y \in \mathbb{R}^n \tag{4}$$

which satisfy certain conditions.

In the discontinuous case there is a theorem on the passage from the reduced system to the full system [Utk92, ch.5]. But the result is only valid for a system with linear fast motion equation and a relay control added linearly only to the slow motion equation. And there is only given an approximation of the full system solution to the zeroth order with respect to the small parameter.
In the following sections these restrictions are not supposed. But otherwise the consideration is restricted to singularly perturbed systems with first order sliding mode.

2 Asymptotic representation of trajectories

For smooth singularly perturbed systems there exists not only a result for the 0th order approximation of the full system solution with respect to the small parameter [TVS85, ch.7] but an algorithm for the construction of an approximation of any order [VBK95, ch.1].

For proving existence and uniqueness of the solution of **singularly perturbed relay control systems** as well as their asymptotic behaviour two lemmas will be stated.

At first we define:

$$K^q(r) := \{\chi \in \mathbb{R}^q \mid \ |\chi| \leq r, r > 0\} \ .$$

Consider the system

$$\mu \frac{da}{dt} = F(a,b,t), \ \frac{db}{dt} = f(a,b,t) \tag{5}$$

where $(a,b,t) \in \bar{G} = K^m(r_1) \times K^n(r_2) \times [\tilde{t}, T]$, $T > \ \max\,(t_0, t_0 + t(\mu))$, $\mu \in (0, \mu_e]$ and G is a domain, together with two different initial conditions

$$a(t_0,\mu) = a^0, \ b(t_0,\mu) = b^0, \tag{6}$$

$$a(t_0 + t(\mu),\mu) = a^0 + z(\mu), \ b(t_0 + t(\mu),\mu) = b^0 + y(\mu), \tag{7}$$

$(a^0, b^0, t_0) \in G$. We assume the following hypotheses:

(H_0) $\mu_e > 0$, $\tilde{t} := \min(t_0, \min_{\mu \in [0,\mu_e]}(t_0 + t(\mu))) \in \mathbb{R}$, $t(\mu)$ continuous.

(H_1) $F, f \in C^{k+2}(G)$, $k \geq 0$.

(H_2) For $\mu = 0$ there is an isolated solution $a_0 = \varphi(b_0, t)$, φ smooth, of $F(a,b,t) = 0$, $(b_0, t) \in K^n(r_2) \times [\tilde{t}, T]$ with the same smoothness as F.

(H_3) The slow motion system

$$\frac{db_0}{dt} = f(\varphi(b_0,t), b_0, t), \quad b_0(t_0) = b^0 \tag{8}$$

has a unique solution $b_0(t)$ for $t \in [\tilde{t}, T]$, and $(\varphi(b_0(t),t), b_0(t), t) \in \bar{G}$.

(H_4) There is a $\gamma > 0$ such that for all $t \in [\tilde{t}, T]$ the eigenvalues $\lambda_i(t)$ of

$$\frac{\partial F}{\partial a}(\varphi(b_0(t),t), b_0(t), t)$$

satisfy

$$\mathrm{Re}\ \lambda_i(t) \leq -\gamma \ .$$

for all $i \in \{1, .., m\}$.

(H_5) Let a^0 be in the interior of the domain of attraction of the asymptotically stable equilibrium point $\varphi(b^0, 0)$ of the associated system

$$\frac{d\tilde{a}}{d\tau} = F(\tilde{a}, b^0, t),\ \tilde{a}(t_0) = a^0$$

(H_6) $t(\mu) = \mathcal{O}(\mu^{k+2}),\ z(\mu) = \mathcal{O}(\mu^{k+1}),\ y(\mu) = \mathcal{O}(\mu^{k+1})$

Lemma 1. Assume hypotheses (H_0) to (H_6) are satisfied. Let

$$L(t, \mu) = (A(t, \mu), B(t, \mu))$$

$$N(t, \mu) = (\alpha(t, \mu), \beta(t, \mu))$$

be solutions of system (5) with initial conditions (6) resp. (7).

Then for sufficiently small μ_e $L(t, \mu), N(t, \mu)$ are unique solutions of (5) on $[\tilde{t}, T]$, and it holds

$$|L_k(t, \mu) - N(t, \mu)| = \mathcal{O}(\mu^{k+1})$$

where

$$L_k(t, \mu) = \sum_{i=0}^{k} \mu^i(\bar{c}_i(t) + \Pi_i c(\tau))\ .$$

Proof. It's known [VBK95] that $L_k(t, \mu)$ is the asymptotic representation of L, and there exist $\kappa, \delta > 0$ such that for all $\tau \in [t_0/\mu, T/\mu]$

$$|\Pi_i c(\tau)| < \kappa e^{-\delta\tau},\ \ i = 0, ..., k\ .$$

Existence and uniqueness of the solution L follows from the Tychonov theorem [TVS85, p.191],[VBK95] which can be applied because of the assumptions (H_1) to (H_5). From the continuous dependence of the solution on initial conditions we get the same properties for the solution N, too.
Without restricting the generality we assume that $\tilde{t} = t_0$.
For all $t \in [0, t(\mu)]$ we get

$$||A(t_0 + t, \mu) - a^0|| = ||\int_{t_0}^{t_0+t} F(A(\xi, \mu), B(\xi, \mu), \xi)d\xi/\mu|| \leq K_1 t(\mu)/\mu = \mathcal{O}(\mu^{k+1})$$

$$||B(t_0 + t, \mu) - b^0|| = ||\int_{t_0}^{t_0+t} f(A(\xi, \mu), B(\xi, \mu), \xi)d\xi|| \leq K_2 t(\mu) = \mathcal{O}(\mu^{k+2})$$

where $K_1 = \sup_{(a,b,t)\in\bar{G}} ||F(a, b, t)||$, $K_2 = \sup_{(a,b,t)\in\bar{G}} ||f(a, b, t)||$.
From (H_1) we know that for sufficiently small μ there is a unique solution to initial condition (7) on the interval $[t_0, t_0 + 2t(\mu)]$. It holds due to (H_6)

$$t(\mu) = \mathcal{O}(\mu^{k+2}),\ t = \mathcal{O}(\mu^{k+2}), t(\mu) - t = \mathcal{O}(\mu^{k+2})\ .$$

Expanding α around $t_0 + t(\mu)$ we get

$$\alpha(t_0+t,\mu) = \alpha(t_0+t(\mu),\mu) + \dot{\alpha}(t_0+t(\mu),\mu)(t-t(\mu)) + \ddot{\alpha}(t_0+t(\mu),\mu)(t-t(\mu))^2 + \ldots =$$

$$a^0 + \mathcal{O}(\mu^{k+1}) + F(a^0+z(\mu), b^0+y(\mu), t_0)(t-t(\mu))/\mu + \mathcal{O}(\mu^{2k+1}) \ .$$

With

$$F(a^0+z(\mu), b^0+y(\mu), t_0) = F(a^0, b^0, t_0) + \mathcal{O}(\|(z(\mu), y(\mu), t(\mu))^T\|) = F(a^0, b^0, t_0) + \mathcal{O}(\mu^{k+1})$$

we get

$$\alpha(t_0+t,\mu) = a^0 + \mathcal{O}(\mu^{k+1}) \ .$$

Furthermore, expanding the slow part β around $t_0 + t(\mu)$ we get

$$\beta(t_0+t,\mu) = \beta(t_0+t(\mu),\mu) + \dot{\beta}(t_0+t(\mu),\mu)(t-t(\mu)) + \ddot{\beta}(t_0+t(\mu),\mu)(t-t(\mu))^2 + \ldots =$$

$$= \hat{\beta}(t_0+t(\mu),\mu) + \mathcal{O}(\mu^{k+1}) \ .$$

Therefore, for all $t \in [0, t(\mu)]$ we get

$$L(t_0+t,\mu)) = N(t_0+t,\mu) + \mathcal{O}(\mu^{k+1}) \ .$$

Hence, on $[t_0, t_0+t(\mu)]$ the asymptotical representations of L and N are the same up to the order k. Because the difference of the initial conditions (7) and $(a(t_0+t(\mu),\mu), b(t_0+t(\mu),\mu)) = L(t_0+t(\mu),\mu)$ is $\mathcal{O}(\mu^{k+1})$ with the Vasileva Theorem [VBK95, p.26] we get the same asymptotic representation of both solutions up to the order k on $[t_0+t(\mu), T]$: Consider L_k, the k-th order asymptotic representation of L. Since all functions $\bar{c}_i$ and $\Pi_i c$ are solutions of smooth differential equations and the initial values at time $t_0+t(\mu)$ due to these solutions are the same as in the asymptotic representation N_k of N, we get the uniqueness of these initial value problem solutions. It follows that for all $t \in [t_0+t(\mu), T]$

$$N_k(t,\mu) = L_k(t,\mu) \ .$$

Altogether the assertion is proven because for all $t \in [\tilde{t}, T]$

$$|N(t,\mu) - L_k(t,\mu)| \le |N(t,\mu) - N_k(t,\mu)| + |N_k(t,\mu) - L_k(t,\mu)| = \mathcal{O}(\mu^{k+1})$$

■

Remarks.

1. The exponentially fast decreasing function $\Pi c(\tau)$ which is approximated by the series of $\Pi_i c(\tau)$, is called *boundary layer function.*

2. A smooth dependence of the righthand side (F, f) of the small parameter μ does not change the result because we use the technique of series expansion with respect to μ.

3. If initial time and initial values depend smoothly on mu i.e. $t_0(\mu) = t_0 + \mathcal{O}(\mu)$, $b^0(\mu) = b^0 + \mathcal{O}(\mu)$ we can reduce this situation to the case $t_0(\mu) = 0$, $b^0(\mu) = b^0$ by the following transformation:

$$s = t - t_0(\mu), \quad \eta = b + b^0 - b^0(\mu) .$$

   This is an important fact because without this transformation initial time and initial value of the full and the reduced system are not identical which is a necessary condition in the theorems of Tychonov and Vasileva.

The second lemma is similar to the first but there is an important difference: The initial points are assumed to be near to the fast variable solution φ of the reduced system. This additional condition yields the possibility to weaken the condition regarding $t(\mu)$.

Now consider again system (5) on G, but with two other initial conditions

$$a(t_0, \mu) = \varphi(b^0, t_0) \ , b(t_0, \mu) = b^0, \tag{9}$$

$$a(t_0 + t(\mu), \mu) = \varphi(b^0, t_0) + \hat{z}(\mu) \ , b(t_0 + t(\mu), \mu) = b^0 + \hat{y}(\mu), \tag{10}$$

$(\varphi(b^0, t_0) \ , b^0, t_0) \in G$. We assume the following hypotheses:

(A_1) $F, f \in C^{k+1}(G)$, $k \geq 0$.

(A_2) For $\mu = 0$ there is an isolated solution $\hat{a}_0 = \varphi(\hat{b}_0, t)$, of $F(a, b, t) = 0$, $(\hat{b}_0, t) \in K^n(r_2) \times [t_0, T]$ with the same smoothness as F.

(A_3) The slow motion system

$$\frac{d\hat{b}_0}{dt} = f(\varphi(\hat{b}_0, t), \hat{b}_0, t), \quad \hat{b}_0(t_0) = b^0 \tag{11}$$

has a unique solution $\hat{b}_0(t)$ for $t \in [\tilde{t}, T]$, and $(\varphi(\hat{b}_0(t), t), \hat{b}_0(t), t) \in \bar{G}$.

(A_4) There is a $\hat{\gamma} > 0$ such that for all $t \in [\tilde{t}, T]$ the eigenvalues $\hat{\lambda}_i(t)$ of

$$\frac{\partial F}{\partial a}(\varphi(\hat{b_0}(t), t), \hat{b_0}(t), t)$$

satisfy

$$\text{Re } \hat{\lambda}_i(t) \leq -\hat{\gamma} \; .$$

for all $i \in \{1, .., m\}$.

(A_5) $t(\cdot), \hat{z}(\cdot), \hat{y}(\cdot)$ smooth,
$t(\mu) = \mathcal{O}(\mu^{k+1}),\ \hat{z}(\mu) = \mathcal{O}(\mu^{k+1}), \hat{y}(\mu) = \mathcal{O}(\mu^{k+1})$.

Lemma 2. Assume hypotheses (A_1) to (A_5) are satisfied.Let

$$M(t, \mu) = (\hat{A}(t, \mu), \hat{B}(t, \mu))$$

$$R(t, \mu) = (\hat{\alpha}(t, \mu), \hat{\beta}(t, \mu))$$

be solutions of system (5) with initial conditions (9) resp. (10).
Then for sufficiently small μ_e $M(t, \mu), R(t, \mu)$ are unique solutions of (5) on $[\tilde{t}, T]$, and it holds

$$|M_k(t, \mu) - R(t, \mu)| = \mathcal{O}(\mu^{k+1}) \; .$$

Proof. Existence and uniqueness of the solutions M and R for $t \in [\tilde{t}, T]$ follows from the Tychonov theorem and the continuous dependence on initial conditions. A condition like (H_5) regarding the associated system is satisfied because of condition (9).

Without restricting the generality we assume that $\tilde{t} = t_0$.
It holds due to (A_5)

$$t(\mu) = \mathcal{O}(\mu^{k+1}),\ t = \mathcal{O}(\mu^{k+1}),\ t(\mu) - t = \mathcal{O}(\mu^{k+1}) \; .$$

With (9) we know that

$$\dot{\hat{A}}(t_0, \mu) = F(\chi^0(\mu), b^0, t_0)/\mu = F(\varphi(b^0, t_0) + \mathcal{O}(\mu), b^0, t_0)/\mu = 0 + \mathcal{O}(1) \; .$$

Let $t \in [0, t(\mu)]$. Together with (A_5) we get

$$\hat{A}(t_0 + t, \mu) = \chi^0(\mu) + \mathcal{O}(1)t + \frac{t^2}{2}\ddot{\hat{A}}(t_0, \mu) + ... = \chi^0(\mu) + \mathcal{O}(\mu^{k+1}) \; . \qquad (12)$$

We know further that

$$||\hat{B}(t_0 + t, \mu) - b^0|| = ||\int_{t_0}^{t_0+t} f(\hat{A}(\xi, \mu), \hat{B}(\xi, \mu), \xi)d\xi|| \leq \hat{K}t(\mu) = \mathcal{O}(\mu^{k+1})$$

where $\hat{K} = \sup_{(a,b,t)\in\bar{G}} ||f(a,b,t)||$.
From (A_1) we know that for sufficiently small μ there is a unique solution to the initial condition (10) on the interval $[t_0, t_0 + 2t(\mu)]$. Expanding $\hat{\alpha}$ around $t_0 + t(\mu)$ we get

$$\hat{\alpha}(t_0 + t, \mu) = \hat{\alpha}(t_0 + t(\mu), \mu) + \dot{\hat{\alpha}}(t_0 + t(\mu), \mu)(t - t(\mu)) + \ddot{\hat{\alpha}}(t_0 + t(\mu), \mu)(t - t(\mu))^2 + ...$$

With

$$\hat{\alpha}(t_0 + t(\mu)) = \chi^0(\mu) + \mathcal{O}(\mu^{k+1})$$

and

$$\dot{\hat{\alpha}}(t_0 + t(\mu), \mu)(t - t(\mu)) = F(\chi^0(\mu) + \hat{z}(\mu), b^0 + \hat{y}(\mu), t_0 + t(\mu))(t - t(\mu))/\mu =$$

$$(F(\chi^0(\mu), b^0, t_0)/\mu + \mathcal{O}(\|(\hat{z}(\mu), \hat{y}(\mu), t(\mu))^T\|))(t - t(\mu))/\mu =$$

$$(F(\varphi(b^0, t_0)) + \mathcal{O}(\mu) + \mathcal{O}(\mu^{k+1}))\mathcal{O}(\mu^k) = \mathcal{O}(\mu^{k+1})$$

we conclude that for all $t \in [0, t(\mu)]$

$$\hat{\alpha}(t_0 + t, \mu) = \varphi(b^0, t_0) + \mathcal{O}(\mu^{k+1}) .$$

Furthermore, expanding the slow part $\hat{\beta}$ around $t_0 + t(\mu)$ we get

$$\hat{\beta}(t_0 + t, \mu) = \hat{\beta}(t_0 + t(\mu), \mu) + \dot{\hat{\beta}}(t_0 + t(\mu), \mu)(t - t(\mu)) + \ddot{\hat{\beta}}(t_0 + t(\mu), \mu)(t - t(\mu))^2 + ...$$

$$= \hat{\beta}(t_0 + t(\mu), \mu) + \mathcal{O}(\mu^{k+1}) = b^0 + \hat{y}(\mu) = \hat{\beta}(t_0, \mu) + \mathcal{O}(\mu^{k+1}) .$$

Therefore, for all $t \in [t_0, t_0 + t(\mu)]$ we get

$$M(t_0 + t, \mu)) = R(t_0 + t, \mu) + \mathcal{O}(\mu^{k+1}) .$$

Hence, on $[t_0, t_0 + t(\mu)]$ the asymptotical representations of M and R are the same up to the order k. Because the difference of the initial conditions (10) and $(a(t_0 + t(\mu), \mu), b(t_0 + t(\mu), \mu)) = M(t_0 + t(\mu), \mu)$ is $\mathcal{O}(\mu^{k+1})$ with Vasileva we get the same asymptotic representation of both solutions up to the order k on $[t_0 + t(\mu), T]$:

$$R_k(t, \mu) = M_k(t, \mu) .$$

Altogether the assertion is proven because for all $t \in [\tilde{t}, T]$

$$|R(t, \mu) - M_k(t, \mu)| \leq |R(t, \mu) - R_k(t, \mu)| + |R_k(t, \mu) - M_k(t, \mu)| = \mathcal{O}(\mu^{k+1}) .$$

■

2.1 Transition into sliding

Consider the initial value problem

$$\begin{aligned} \mu\frac{dz}{dt} &= F(z,y,s,u,t) \\ \frac{dy}{dt} &= f(z,y,s,u,t) \\ \frac{ds}{dt} &= h(z,y,s,u,t) \\ z(t_0)=z^0, \quad y(t_0) &= y^0, \quad s(t_0)=s^0 \end{aligned} \tag{13}$$

where $(z,y,s,t)\in\bar{G}=K^m(r_1)\times K^n(r_2)\times K^1(r_3)\times[t_0,T],\ T>t_0,\ \mu\in(0,\mu_e]$ and G is a domain. Moreover,

$$u=\text{ sgn }(s)\ \forall s\neq 0$$

i.e. there is a discontinuity surface $S^*=\{s=0\}\subset\bar{G}$. On this surface we get the so-called *sliding mode equation* by substituting u by the equivalent control function $u_{eq}\in[-1,1]$ [Utk92, p.37],[Fil88, p.54] (cf. assumption (B_7)).

We assume:

(B_0) $(z_0,y_0,s_0,t_0)\in S^+=\{(z,y,s,t)\in\bar{G}|s>0\}$.

(B_1) $F,f,h\in C^{k+2},\ k\geq 0$.

(B_2) The equation $F(z,y,s,1,t)=0$ has an isolated solution $z=\varphi^+(y,s,t)$, φ^+ smooth, with $(y,s,t)\in K^n(r_2)\times K^1(r_3)\times[t_0,\tilde{T}+\delta]$ where $\tilde{T}<T,\delta>0$.

(B_3) The so-called *reduced system*

$$\begin{aligned} \tfrac{dy}{dt} &= f(\varphi^+(y,s,t),y,s,1,t) \\ \tfrac{ds}{dt} &= h(\varphi^+(y,s,t),y,s,1,t) \\ y(t_0)=y^0 \qquad & s(t_0)=s^0 \end{aligned} \tag{14}$$

has a unique solution $(\bar{y}_0^+(t),\bar{s}_0^+(t),t)$ in $K^n(r_2)\times K^1(r_3)\times[t_0,\tilde{T}+\varepsilon]$ for $t\in[t_0,\tilde{T}]$.

(B_4) There is a $\beta_1>0$ such that for all $t\in[t_0,\tilde{T}+\varepsilon]$ the eigenvalues $\lambda_i(t)$ of

$$\frac{\partial F}{\partial z}(\varphi^+(\bar{y}_0^+(t),\bar{s}_0^+(t),t),\bar{y}_0^+(t),\bar{s}_0^+(t),1,t)$$

satisfy

$$\text{Re }\lambda_i(t)\leq-\beta_1$$

for all $i\in\{1,..,m\}$.

(B_5) z^0 is located in the interior of the domain of attraction of the asymptotically stable equilibium point $\varphi^+(y^0, s^0, t_0)$ of the associated system

$$\frac{d\tilde{z}}{d\tau} = F(\tilde{z}; y^0, s^0, 1, t_0), \ \tilde{z}(t_0) = z^0$$

where $\tau = (t - t_0)/\mu, \ \tau \in [0, \infty)$.

(B_6) For the reduced system there exists a moment $t_0^A = \tilde{T} \in (t_0, T)$ where the trajectory arrives at S^* with

- $\bar{s}_0^+(\theta_0^A) = 0$
- $h(\varphi^+(\bar{y}_0^+(\theta_0^A), 0, \theta_0^A), \bar{y}_0^+(\theta_0^A), 0, 1, \theta_0^A) < 0,$
- $h(\varphi^+(\bar{y}_0^+(\theta_0^A), 0, \theta_0^A), \bar{y}_0^+(\theta_0^A), 0, -1, \theta_0^A) > 0.$

(B_7) The equation

$$h(z, y, 0, u, t) = 0$$

has a unique, smooth solution $u = u_{eq}(z, y, t)$ on a neighbourhood $U_1 \subset K^m(r_1) \times K^n(r_2) \times [t_0, T]$ of $(\varphi^+(\bar{y}_0^+(\theta_0^A), 0, \theta_0^A), \bar{y}_0^+(\theta_0^A), \theta_0^A)$. Therefore, the sliding mode equation with the equivalent control u_{eq} is of the following form:

$$\begin{aligned} \mu\frac{dz^*}{dt} &= F(z^*, y^*, 0, u_{eq}(z^*, y^*, t), t) \ , \\ \frac{dy^*}{dt} &= f(z^*, y^*, 0, u_{eq}(z^*, y^*, t), t) \ . \end{aligned} \tag{15}$$

(B_8) There exists an isolated solution $z = \varphi^*(y^*, t)$, φ^* smooth, to

$$0 = F(z^*, y^*, 0, u_{eq}(z^*, y^*, t), t)$$

for $(y^*, t) \in K^n(r_2) \times [\theta_0^A - \varepsilon, T]$.

(B_9) The reduced system in the sliding regime

$$\frac{dy^*}{dt} = f(\varphi^*(y^*, t), y^*, 0, u_{eq}(\varphi^*(y^*, t), y^*, t), t) \ ,$$

$$y^*(\theta_0^A) = \bar{y}_0^+(\theta_0^A)$$

has a unique solution $y^*(t) \in K^n(r_2)$ on $[\theta_0^A - \varepsilon, T]$ with the following properties:

$$h(\varphi^*(y^*(t), t), y^*(t), 0, 1, t) < 0,$$

$$h(\varphi^*(y^*(t), t), y^*(t), 0, -1, t) > 0.$$

(B_{10}) There is a $\beta_2 > 0$ such that for all $t \in [\theta_0^A - \varepsilon, T]$ the eigenvalues $\lambda_i^*(t)$ of

$$\frac{\partial F}{\partial z}(\varphi^*(y^*(t), t), y^*(t), u_{eq}(\varphi^*(y^*(t), t), y^*(t), t), t)$$

satisfy

$$\text{Re } \lambda_i^*(t) \le -\beta_2$$

for all $i \in \{1, .., m\}$.

(B_{11}) $\varphi^+(\bar{y}_0^+(\theta_0^A), 0, \theta_0^A)$ is located in the interior of the domain of attraction of the equilibrium point $\varphi^*(\bar{y}_0^+(\theta_0^A), \theta_0^A)$ of the system

$$\frac{d\tilde{z}^*}{d\tau} = F(\tilde{z}^*; y^*(\theta_0^A), 0, u_{eq}(\varphi^*(y^*(\theta_0^A)), y^*(\theta_0^A)), \theta_0^A) \tag{16}$$

where $\tau = (t - \theta_0^A)/\mu,\ \tau \in [0, \infty)$.

(B_{12}) For all $\lambda \in [0,1]$ it holds

$$h(\lambda(\varphi^+(\bar{y}_0^+(\theta_0^A), 0, \theta_0^A), \bar{y}_0^+(\theta_0^A), 0, 1, \theta_0^A) + (1-\lambda)(\varphi^*(\bar{y}_0^+(\theta_0^A), \theta_0^A), \bar{y}_0^+(\theta_0^A), 0, 1, \theta_0^A)) < 0,$$

$$h(\lambda(\varphi^+(\bar{y}_0^+(\theta_0^A), 0, \theta_0^A), \bar{y}_0^+(\theta_0^A), 0, -1, \theta_0^A) + (1-\lambda)(\varphi^*(\bar{y}_0^+(\theta_0^A), \theta_0^A), \bar{y}_0^+(\theta_0^A), 0, -1, \theta_0^A)) > 0.$$

Theorem 1. Let system (13) satisfy the conditions (B_1) to (B_{12}). Then there exists a $\mu_e > 0$ such that for all $\mu \in (0, \mu_e)$ there is a unique solution $x(t,\mu) = (z(t,\mu), y(t,\mu), s(t,\mu))$ of the system (13) on $[t_0, T]$, and the following estimation holds:

$$|x(t,\mu) - X_k(t,\mu)| = \mathcal{O}(\mu^{k+1}) \tag{17}$$

where

$$X_k(t,\mu) = \sum_{i=0}^{k} \mu^i (\bar{x}_i(t) + \Pi_i^+ x(\tau) + \Pi_i^* x(\tau_{k+1})) \tag{18}$$

and

$$\tau_{k+1} = \frac{t - \Theta_{k+1}^A}{\mu}$$

where $\Theta_{k+1}^A := \theta_0^A + \mu\theta_1^A + \ldots + \mu^{k+1}\theta_{k+1}^A$ is the $(k+1)$th order approximation of the arrival moment $t^A(\mu)$ of the full system.

Proof of Theorem 1. Our assumptions (B_0) to (B_5) are sufficient to apply the theorems of Tychonov and Vasileva to system (13). Hence, the solution of the IVP (13) $(z(t,\mu), y(t,\mu), s(t,\mu))$ exists uniquely in $K^m(r_1) \times K^n(r_2) \times K^1(r_3)$ for $t \in [t_0, \theta_0^A + \varepsilon]$, and for $\mu \to 0$ $z(t,\mu), y(t,\mu), s(t,\mu))$ tends to $(\varphi^+(\bar{y}_0^+(t), \bar{s}_0^+(t), t), \bar{y}_0^+(t), \bar{s}_0^+(t))$ on $(t_0, \theta_0^A + \varepsilon]$. Together with condition (B_6) we may apply the Implicit Function Theorem to $s(t,\mu)$ since we know that $s(\theta_0^A, 0) = \bar{s}_0^+(\theta_0^A) = 0$. Hence, for the full system there exists a arrival moment $t^A(\mu)$ with $\lim_{\mu\to 0} t^A(\mu) = \theta_0^A$ where the solution reaches the discontinuity surface transversally.
We get the asymptotic representation of our solution by separating two steps: analyzing the solution starting in S^+ until it meets S^*, and analyzing the solution moving on S^*. Due to our assumptions we may apply the Theorem of Vasileva to the solution $x^+(t) = (z^+(t), y^+(t), s^+(t))$ of (13) with $u = 1$ on the time interval $[t_0, \theta_0^A + \varepsilon]$, i.e.

$$\max_{t \in [t_0, \theta_0^A + \varepsilon]} ||x^+(t) - X_k^+(t)|| = \mathcal{O}(\mu^{k+1}).$$

The arrival moment $t^A(\mu)$ depends smoothly on μ. We may write

$$t^A(\mu) = \theta_0^A + \mu\theta_1^A + ... + \mu^{k+1}\theta_{k+1}^A + \mu^{k+2}\xi_{k+2} \tag{19}$$

where $\xi_{k+2} = \frac{1}{(k+2)!}\frac{\partial^{k+2}\theta(\kappa^A)}{\partial\mu^{k+2}}$ with $\kappa^A \in [0,\mu]$. For sufficiently small μ we get $t^A(\mu), \Theta_{k+1} \in U_\varepsilon(\theta_0^A)$. Evidently, $t^A(\mu) - \Theta_{k+1}^A = \mathcal{O}(\mu^{k+2})$. Expanding x^+ as series around θ_0^A, we get:

$$(t^A(\mu)) = x^+(\theta_0^A + h) = x^+(\theta_0^A) + \dot{x}^+(\theta_0^A)h + ... + \frac{x^{+(k+1)}(\theta_0^A)}{(k+1)!}h^{k+1} + \mathcal{O}(\mu^{k+2})$$

where $h = \mu\theta_1 + ... + \mu^{k+1}\theta_{k+1} + \mu^{k+2}\xi_{k+2}$, and

$$x^+(\Theta_{k+1}^A) = x^+(\theta_0^A + h_{k+1}) = x^+(\theta_0^A) + \dot{x}^+(\theta_0^A)h_{k+1} + ... + \frac{x^{+(k+1)}(\theta_0^A)}{(k+1)!}h_{k+1}^{k+1} + \mathcal{O}(\mu^{k+2})$$

where $h_{k+1} = \mu\theta_1 + ... + \mu^{k+1}\theta_{k+1}$. Since $\dot{z}^+ = F/\mu$ and

$$h - h_{k+1} = \mathcal{O}(\mu^{k+2}) \ , \ h^j - h_{k+1}^j = (h - h_{k+1})\mathcal{O}(\mu^{j-1}) \text{ if } j \geq 1$$

it follows that $x^+(t^A(\mu)) - x^+(\Theta_{k+1}^A) = \mathcal{O}(\mu^{k+1})$. Moreover, due to Vasileva, we know that $x^+(\Theta_{k+1}^A) - X_k^+(\Theta_{k+1}^A) = \mathcal{O}(\mu^{k+1})$. This yields

$$x^+(t^A(\mu)) - X_k^+(\Theta_{k+1}^A) = \mathcal{O}(\mu^{k+1}) \ .$$

Now we consider the solution moving on S^*: Because of IFT,(B_6) and (B_7) for the solution $x(t,\mu) = (z(t,\mu), y(t,\mu), s(t,\mu))$ of system (13) with $u = u_{eq}$ it holds not only $s(t^A(\mu),\mu) = s^+(t^A(\mu)) = 0$ but for T sufficiently near to $t^A(\mu)$

$$s(t,\mu) = 0 \ \text{ on } [t^A(\mu), T] \ .$$

Therefore, in the following we consider only solutions z_a^*, y_a^* and z_e^*, y_e^* of (15) to the initial conditions

$$z_e^*(t^A(\mu)) = z^+(t^A(\mu)), y_e^*(t^A(\mu)) = y^+(t^A(\mu)) \tag{20}$$

resp.

$$z_a^*(\Theta_{k+1}^A) = Z_k^+(\Theta_{k+1}^A), y_a^*(\Theta_{k+1}^A) = Y_k^+(\Theta_{k+1}^A) \ . \tag{21}$$

For applying Lemma 1 it is necessary to introduce the following new coordinates:

$$\check{z} := z^* \ ,$$

$$\check{y} := y^* + \bar{b}_0^+ - y^+(t^A(\mu)) \ ,$$
$$\check{t} := t - t^A(\mu)$$

where $\bar{b}_0^+ := \bar{y}_0^+(\theta_0^A)$. Hence, due to (15) we get the following two IVPs:

$$\mu\frac{d\check{z}}{d\check{t}} = F(\check{z},\check{y}-\bar{b}_0^+ + y^+(t^A(\mu)),0,u_{eq}(\check{z},\check{y}-\bar{b}_0^+ + y^+(t^A(\mu)),\check{t}+t^A(\mu)),\check{t}+t^A(\mu))\ , \quad (22)$$
$$\frac{d\check{y}}{d\check{t}} = f(\check{z},\check{y}-\bar{b}_0^+ + y^+(t^A(\mu)),0,u_{eq}(\check{z},\check{y}-\bar{b}_0^+ + y^+(t^A(\mu)),\check{t}+t^A(\mu)),\check{t}+t^A(\mu))\ ,$$

$$\check{z}_e(0) = z^+(0)\ ,\ \check{y}_e(0) = \bar{b}_0^+ \text{ resp.}$$
$$\check{z}_a(-\mu^{k+2}\xi_{k+2}) = Z_k^+(-\mu^{k+2}\xi_{k+2})\ ,\ \check{y}_a(-\mu^{k+2}\xi_{k+2}) = Y_k^+(-\mu^{k+2}\xi_{k+2}) + \bar{b}_0^+ - y^+(0)\ .$$

With $t_0 = 0$ and $t(\mu) = -\mu^{k+2}\xi_{k+2}$ we may apply Lemma 1 to system (22) because all hypotheses including (H_6) are satisfied and the righthand side depends smoothly on μ. Hence, for the k-th order approximation $\check{X}_k = (\check{Z}_k, \check{Y}_k, 0)$ of $\check{x} = (\check{z}, \check{y}, 0)$ it follows that

$$\max_{\check{t}\in[-\mu^{k+2}\xi_{k+2},T-t^A(\mu)]} ||\check{x}(\check{t}) - \check{X}_k(\check{t})|| = \mathcal{O}(\mu^{k+1}).$$

After backward transformation we get

$$\max_{t\in[\Theta^A_{k+1},T]} ||x^*(t) - X_k^*(t)|| = \mathcal{O}(\mu^{k+1})$$

because $||x^*(t) - X_k^*(t)|| = ||\check{x}(\check{t}) - \check{X}_k(\check{t})||$. Remark that $x^*(t) = (z^*(t), y^*(t), 0)$ on $[t^A(\mu), T]$. Finally, glueing together the approximations X_k^+ and X_k^* at $t = \Theta^A_{k+1}$ we get the estimation (17) with

$$x(t,\mu) = \begin{cases} x^+(t) & \text{if } t\in[t_0, t^A(\mu)] \\ x^*(t) & \text{if } t\in[t^A(\mu), T] \end{cases} \quad \text{and} \qquad (23)$$

$$X_k(t,\mu) = \begin{cases} X_k^+(t) & \text{if } t\in[t_0, \Theta^A_{k+1}] \\ X_k^*(t) & \text{if } t\in[\Theta^A_{k+1}, T] \end{cases} \qquad (24)$$

such that

$$\bar{x}_i(t) = \begin{cases} \bar{x}_i^+(t) & \text{if } t\in[t_0, \Theta^A_{k+1}] \\ \bar{x}_i^*(t) & \text{if } t\in[\Theta^A_{k+1}, T] \end{cases} \quad i = 0,..,k\ .$$

■

2.2 Leaving sliding domain

Now consider the initial value problem

$$\begin{aligned} \mu\frac{dz}{dt} &= F(z,y,s,\sigma,u,t) \\ \frac{dy}{dt} &= f(z,y,s,\sigma,u,t) \\ \frac{ds}{dt} &= h_1(z,y,s,\sigma,u,t) \\ \frac{d\sigma}{dt} &= h_2(z,y,s,\sigma,u,t) \\ z(t_0)=z^0,\ y(t_0)=y^0 &,\quad s(t_0)=0,\ \sigma(t_0)=\sigma^0 \end{aligned} \tag{25}$$

where $(z,y,s,\sigma,t)\in\bar{G}=K^m(r_1)\times K^n(r_2)\times K^1(r_3)\times K^1(r_4)\times[t_0,T],\ T>t_0,\ \mu\in(0,\mu_e]$ and G is a domain. Moreover,

$$u=\ \mathrm{sgn}\ (s)\ \forall s\neq 0\ .$$

We assume:

(C_1) $F,f,h_1,h_2\in C^{k+2},\ k\geq 0.$

(C_2) The equation

$$h_1(z,y,0,\sigma,u,t)=0$$

has a unique solution $u=u_{eq}(z,y,\sigma,t)$ on a neighborhood U_2 of S^*. Therefore the sliding mode equation is of the following form:

$$\begin{aligned} \mu\frac{dz^*}{dt} &= F(z^*,y^*,0,\sigma^*,u_{eq}(z^*,y^*,\sigma^*,t),t) \\ \frac{dy^*}{dt} &= f(z^*,y^*,0,\sigma^*,u_{eq}(z^*,y^*,\sigma^*,t),t) \\ \frac{d\sigma^*}{dt} &= h_2(z^*,y^*,0,\sigma^*,u_{eq}(z^*,y^*,\sigma^*,t),t) \\ z^*(t_0)=z^0,\ y^*(t_0)=y^0 &,\quad \sigma^*(t_0)=\sigma^0 \end{aligned} \tag{26}$$

(C_3) Let $(z^0,y^0,s^0,\sigma^0,t_0)\in\{(z,y,s,\sigma,t)\in S^*\mid |u_{eq}(z,y,\sigma,t)|<1\}$.

(C_4) There exists a unique solution $z=\varphi^*(y^*,\sigma^*,t)$, φ^* smooth, to

$$0=F(z^*,y^*,0,\sigma^*,u_{eq},t)$$

for $(y^*,\sigma^*,t)\in\{(y,\sigma,t)\in K^n(r_2)\times K^1(r_4)\times[t_0,\tilde{T}+\varepsilon]\mid |u_{eq}(y,\sigma,t)|\leq 1\}$

(C_5) Define $\bar{u}_{eq}(y,\sigma,t) := u_{eq}(\varphi^*(y,\sigma,t),y,\sigma,t)$. The reduced system in the sliding regime

$$\frac{dy^*}{dt} = f(\varphi^*(y^*,\sigma^*,t),y^*,0,\sigma^*,\bar{u}_{eq}(y^*,\sigma^*,t),t),$$
$$\frac{d\sigma^*}{dt} = h_2(\varphi^*(y^*,\sigma^*,t),y^*,0,\sigma^*,\bar{u}_{eq}(y^*,\sigma^*,t),t), \tag{27}$$
$$y^*(t_0) = y^0,\ \sigma^*(t_0) = \sigma^0$$

has a unique solution $(\bar{y}^*(t),\bar{\sigma}^*(t))$ on $[t_0,\tilde{T}+\varepsilon],\ t_0 < \tilde{T} < T$ with the following properties: For $t \in [t_0,\tilde{T})$

$$h_1(\varphi^*(\bar{y}^*(t),\bar{\sigma}^*(t),t),\bar{y}^*(t),0,\bar{\sigma}^*(t),1,t) < 0,$$

$$h_1(\varphi^*(\bar{y}^*(t),\bar{\sigma}^*(t),t),\bar{y}^*(t),0,\bar{\sigma}^*(t),-1,t) > 0.$$

(C_6) There is a $\hat{\beta}_1 > 0$ such that for all $t \in [t_0,\tilde{T}+\varepsilon]$ the eigenvalues $\lambda_i^*(t)$ of

$$\frac{\partial F}{\partial z}(\varphi^*(\bar{y}^*(t),\bar{\sigma}^*(t),t),\bar{y}^*(t),\bar{\sigma}^*(t),\bar{u}_{eq}(\bar{y}^*(t),\bar{\sigma}^*(t),t),t)$$

satisfy

$$\text{Re }\lambda_i^*(t) \le -\hat{\beta}_1$$

for all $i \in \{1,..,m\}$.

(C_7) z^0 is element of the interior of attraction domain of the (asymptotically stable) rest point $\varphi^*(y^0,\sigma^0,t_0)$ of system

$$\frac{dz^*}{d\tau} = F(z^*;y^0,0,\sigma^0,u_{eq}(z^*,y^0,\sigma^0,t_0),t_0) \tag{28}$$

where $\tau = (t-t_0)/\mu,\ \tau \in [0,\infty)$.

(C_8) Let $\Sigma := \{(z,y,\sigma,t) \in K^m(r_1)\times K^n(r_2)\times K^1(r_4)\times[t_0,\tilde{T}+\varepsilon] \mid u_{eq}(z,y,\sigma,t) = 1\}$. Assume (w.l.o.g) that $\Sigma = K^m(r_1)\times K^n(r_2)\times\{0\}\times[t_0,\tilde{T}+\varepsilon]$.
There is a leaving moment ("**B**reak-away moment") $t_0^B = \tilde{T} \in (t_0,T)$ of system (27) such that

- $(\varphi^*(\bar{y}^*(\theta_0^B),0,\theta_0^B),\bar{y}^*(t_0^A),\bar{\sigma}^*(\theta_0^B),\theta_0^B) \in \Sigma$,
- $h_2(\varphi^*(\bar{y}^*(\theta_0^B),0,\theta_0^B),\bar{y}^*(\theta_0^B),0,0,1,\theta_0^B) > 0$,
- $h_1(\varphi^*(\bar{y}^*(\theta_0^B),0,\theta_0^B),\bar{y}^*(\theta_0^B),0,0,1,\theta_0^B) = 0$,
- $\frac{\partial h_1}{\partial\sigma}(\varphi^*(\bar{y}^*(\theta_0^B),0,\theta_0^B),\bar{y}^*(\theta_0^B),0,0,1,\theta_0^B) > 0$,
- $h_1(\varphi^*(\bar{y}^*(\theta_0^B),0,\theta_0^B),\bar{y}^*(\theta_0^B),0,0,-1,\theta_0^B) > 0$.

(C_9) The equation $F(z,y,s,\sigma,1,t) = 0$ has a unique solution $z = \varphi^+(y,s,\sigma,t)$, φ^+ smooth, with $(y,s,\sigma,t) \in R := K^n(r_2)\times K^1(r_3)\times K^1(r_4)\times[\theta_0^B-\varepsilon,T]$.

(C_{10}) The reduced system

$$\frac{dy}{dt} = f(\varphi^+(y,s,\sigma,t),y,s,\sigma,1,t) \tag{29}$$

$$\frac{ds}{dt} = h_1(\varphi^+(y,s,\sigma,t),y,s,\sigma,1,t) \tag{30}$$

$$\frac{d\sigma}{dt} = h_2(\varphi^+(y,s,\sigma,t),y,s,\sigma,1,t) \tag{31}$$

$$y(\theta_0^B) = \bar{y}^*(\theta_0^B)\,, \qquad s(\theta_0^B) = 0\,,\ \sigma(\theta_0^B) = 0 \tag{32}$$

has a unique solution $(\bar{y}_0^+(t), \bar{s}_0^+(t), \bar{\sigma}_0^+(t), t)$ in R on $[\theta_0^B - \varepsilon, T]$.

(C_{11}) There is a $\gamma < 0$ such that for all $t \in [0, \tilde{T}]$ the eigenvalues $\lambda_i(t)$ of

$$\frac{\partial F}{\partial z}(\varphi^+(\bar{y}_0^+(t), \bar{s}_0^+(t), \bar{\sigma}_0^+(t), t), \bar{y}_0^+(t), \bar{s}_0^+(t), \bar{\sigma}_0^+(t), 1, t)$$

satisfy

$$\mathrm{Re}\ \lambda_i(t) \le \gamma$$

for all $i \in \{1,..,m\}$.

(C_{12}) $\tilde{z}^*(0) = \varphi^*(y^*(\theta_0^B), 0, \theta_0^B)$ is element of the interior of attraction domain of $\varphi^+(y^*(\theta_0^B), 0, 0, \theta_0^B)$ of system

$$\frac{d\tilde{z}}{d\tau} = F(\tilde{z}; y^*(\theta_0^B), 0, 0, 1, \theta_0^B) \tag{33}$$

where $\tau = (t - t_0^A)/\mu,\ \tau \in [0, \infty)$.

Theorem 2. Let system (25) satisfy the conditions (C_1) to (C_{12}). Then there exists a $\mu_e > 0$ such that for all $\mu \in (0, \mu_e)$ there is a unique solution $x(t,\mu) = (z(t,\mu), y(t,\mu))$ of the system (25) on $[t_0, T]$, and the following estimation holds:

$$|x(t,\mu) - X_k(t,\mu)| = \mathcal{O}(\mu^{k+1}) \tag{34}$$

where

$$X_k(t,\mu) = \sum_{i=0}^{k} \mu^i (x_i(t) + \Pi_i^* x(\tau) + \Pi_i^+ x(\tau_k))$$

and

$$\tau_k = \frac{t - \Theta_k^B}{\mu}$$

where $\Theta_k^B := \theta_0^B + \mu\theta_1^B + \dots + \mu^k\theta_k^B$ is the k-th order approximation of the leaving moment $t^B(\mu)$ of the full system. Moreover, $\Pi_0^+ x \equiv 0$.

Proof of Theorem 2. Assumptions (C_1) to (C_7) are sufficient to apply the theorem of Tychonov and Vasileva to system (26). Hence, there is a unique solution $(z^*(t), y^*(t), \sigma^*(t))$ of the IVP (26) in $K^m(r_1) \times K^n(r_2) \times K^1(r_4)$ for $t \in [t_0, t_0^A + \varepsilon]$.

With condition (C_8) and the results of Tychonov and Vasileva we may apply the Implicit Function Theorem to $\psi(t,\mu) = \sigma^*(t)$ since we know that $\psi(t_0^A, 0) = \bar{\sigma}^*(t_0^A) = 0$. Hence for the full sliding mode system for sufficiently small μ there exists a time $t^B(\mu)$ continuous depending on μ where the solution meets Σ, which belongs to the boundary of the sliding domain. Furthermore, at this meeting point we have a transversal intersection of the solution with Σ because from the smoothness of h_1 and h_2 we may conclude

$$\frac{\partial h}{\partial \sigma} h_1(\varphi^*(y^*(\theta_0^B), 0, \theta_0^B), y^*(\theta_0^B), 0, 0, 1, \theta_0^B) > 0 \ .$$

Moreover,

$$h_2(\varphi^*(y^*(\theta_0^B), 0, \theta_0^B), y^*(\theta_0^B), 0, 0, 1, \theta_0^B) > 0 \ ,$$

$$h_1(\varphi^*(y^*(\theta_0^B), 0, \theta_0^B), y^*(\theta_0^B), 0, 0, -1, \theta_0^B) > 0 \ .$$

Hence, $t^B(\mu)$ is the leaving moment, i.e. instead of sliding on S^* $(s = 0)$ for $t \leq t^B(\mu)$ the solution $x(t) = (z(t), y(t), s(t), \sigma(t))$ of system (25) moves in S^+ for some time interval starting at $t^B(\mu)$. The reason for this behaviour is the following: Assume there is a $\delta > 0$ such that on $I_\delta := (t^B(\mu), t^B(\mu) + \delta)$ $s(t) < 0$. We can choose δ such that on I_δ $\dot{s}(t) > 0$ taking into account the conditions regarding h_1. But this means that on I_δ $s(t)$ is increasing starting with $s(t^B(\mu)) = 0$. This is an contradiction to the assumption. Alternatively, assume that for the solution on I_δ $s(t) = 0$. Hence, $\dot{s} = h_1 = 0$ on I_δ. Taking into account the conditions regarding h_1 and h_2 in (C_8) we can choose δ such that on I_δ $h_1 > 0$ for $u = 1$ and $u = -1$. According to the definition of solution we get a contradiction to the assumption because $\dot{x}$ would not be almost everywhere element of the set corresponding to the discontinuous vector field. The conclusion is that the solution must leave S^* into S^+.
The leaving moment $t^B(\mu)$ depends smoothly on μ. We may write

$$t^B(\mu) = \theta_0^B + \mu\theta_1^B + \ldots + \mu^k\theta_k^B + \mu^{k+1}\zeta_{k+1} \tag{35}$$

where $\zeta_{k+1} = \frac{1}{(k+1)!}\frac{\partial^{k+1} t^B(\kappa^B)}{\partial \mu^{k+1}}$ with $\kappa^B \in [0, \mu]$. We define the break-away time approximation

$$\Theta_k^B := \theta_0^B + \mu\theta_1^B + \ldots + \mu^k\theta_k^B \ .$$

For sufficiently small μ we get $t^B(\mu), \Theta_k^B \in U_\varepsilon(\theta_0^B)$. Evidently, $t^B(\mu) - \Theta_k^B = \mathcal{O}(\mu^{k+1})$. Expanding x^* as series around θ_0^B, we get

$$x^*(t^B(\mu)) = x^*(\theta_0^B + h) = x^*(\theta_0^B) + \dot{x}^*(\theta_0^B)h + \ldots + \frac{x^{*(k)}(\theta_0^B)}{k!}h^k + \mathcal{O}(\mu^{k+1})$$

where $h = \mu\theta_1^B + ... + \mu^k\theta_k^B + \mu^{k+1}\zeta_{k+1}$, and

$$x^*(\Theta_k^B) = x^*(\theta_0^B + h_k) = x^*(\theta_0^B) + \dot{x}^*(\theta_0^B)h_k + ... + \frac{x^{*^{(k)}}(\theta_0^B)}{k!}h_k^k + \mathcal{O}(\mu^{k+1})$$

where $h_k = \mu\theta_1^B + ... + \mu^k\theta_k^B$. For estimating the difference of $x^*(t^B(\mu))$ and $x^*(\Theta_k^B)$ we have to consider $\dot{x}^* = (\dot{z}^*, \dot{y}^*, 0, \dot{\sigma}^*)$: $\dot{y}^*$ and $\dot{\sigma}^*$ do not depend on μ but

$$\dot{z}^*(\theta_0^B) = \frac{F(z^*(\theta_0^B), \dot{y}^*(\theta_0^B), 0, \dot{\sigma}^*(\theta_0^B), u_{eq}, t_0^A)}{\mu} .$$

Expanding F around $\mu = 0$ we get

$$\dot{z}^*(\theta_0^B) = \frac{F(\varphi^*(y^*(\theta_0^B), \sigma^*(\theta_0^B), u_{eq}, \theta_0^B), \dot{\bar{y}}^*(\theta_0^B), 0, \dot{\bar{\sigma}}^*(\theta_0^B), u_{eq}, t_0^A) + \mathcal{O}(\mu)}{\mu} = 0 + \mathcal{O}(1) .$$

Since

$$h - h_k = \mathcal{O}(\mu^{k+1}) \text{ and } h^j - h_k^j = (h - h_k)\mathcal{O}(\mu^{j-1}) \text{ if } j \geq 1$$

it follows that $x^*(t^B(\mu)) - x^*(\Theta_k^B) = \mathcal{O}(\mu^{k+1})$. Moreover, due to Vasileva, we know that $x^*(\Theta_k^B) - X_k^*(\Theta_k^B) = \mathcal{O}(\mu^{k+1})$. This yields

$$x^*(t^B(\mu)) - X_k^*(\Theta_k^B) = \mathcal{O}(\mu^{k+1}) .$$

Now we consider the solutions $(z_e^+, y_e^+, s_e^+, \sigma_e^+)$ and $(z_a^+, y_a^+, s_a^+, \sigma_a^+)$ of (25) to the initial conditions

$$z_e^+(t^B(\mu)) = z^*(t^B(\mu)),\ y_e^+(t^B(\mu)) = y^*(t^B(\mu)),\ s_e^+(t^B(\mu)) = 0,\ \sigma_e^+(t^B(\mu)) = 0 \quad (36)$$

resp.

$$z_a^+(\Theta_k^B) = Z_k^*(\Theta_k^B),\ y_a^+(\Theta_k^B) = Y_k^*(\Theta_k^B),\ s_a^+(\Theta_k^B) = 0,\ \sigma_a^+(\Theta_k^B) = 0 . \quad (37)$$

Hence,

$$x_e^+(t^B(\mu)) - x_a^+(\Theta_k^B) = \mathcal{O}(\mu^{k+1}) . \quad (38)$$

From (C_8) we know that

$$u_{eq}(\varphi^*(y^*(\theta_0^B), 0, \theta_0^B), y^*(\theta_0^B), 0, \theta_0^B) = 1 .$$

Together with (C_4) it follows that

$$F(\varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B), \bar{y}^*(\theta_0^B), 0, 0, u_{eq}(\varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B), \bar{y}^*(\theta_0^B), 0, \theta_0^B), \theta_0^B) =$$

$$= F(\varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B), \bar{y}^*(\theta_0^B), 0, 0, 1, \theta_0^B) = 0 .$$

And by (C_9) and (C_{10}) it holds that

$$F(\varphi^+(\bar{y}^*(\theta_0^B), 0, 0, \theta_0^B), \bar{y}^*(\theta_0^B), 0, 0, 1, \theta_0^B) = 0 .$$

From the last two equations and the uniqueness of φ^* and φ^+ it follows that

$$\varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B) = \varphi^+(\bar{y}^*(\theta_0^B), 0, 0, \theta_0^B) \ . \tag{39}$$

Furthermore, it is well known that

$$z_e^+(t^B(\mu)) = z^*(t^B(\mu)) = \varphi^*(\bar{y}^*(t^B(\mu)), 0, t^B(\mu)) + \mathcal{O}(\mu) \ .$$

With $\varphi^*(\bar{y}^*(t^B(\mu)), 0, t^B(\mu)) = \varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B) + \mathcal{O}(\mu)$ and (39) we get

$$z_e^+(t^B(\mu)) = \varphi^+(\bar{y}^*(\theta_0^B), 0, 0, \theta_0^B) + \mathcal{O}(\mu) \ . \tag{40}$$

For applying Lemma 2 it is necessary to introduce new coordinates:

$$\begin{aligned} \check{z} &:= z^+ + \varphi^+(\bar{y}^*(\theta_0^B), 0, 0, \theta_0^B) - z^*(t^B(\mu)) \ , \\ \check{y} &:= y^+ + \bar{y}^*(\theta_0^B) - y^*(t^B(\mu)) \ , \\ \check{s} &:= s \ , \\ \check{\sigma} &:= \sigma \ , \\ \check{t} &:= t - t^B(\mu) \ . \end{aligned}$$

Hence, corresponding to (25),(36) and (37) we get the following two IVPs:

$$\begin{aligned} \mu\frac{d\check{z}}{d\check{t}} &= F(\check{z} - \varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B) + z^*(t^B(\mu)), \check{y} - \bar{y}^*(\theta_0^B) + y^*(t^B(\mu)), \check{s}, \check{\sigma}, 1, \check{t} + t_A(\mu)) \ , \\ \frac{d\check{y}}{d\check{t}} &= f(\check{z} - \varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B) + z^*(t^B(\mu)), \check{y} - \bar{y}^*(\theta_0^B) + y^*(t^B(\mu)), \check{s}, \check{\sigma}, 1, \check{t} + t_A(\mu)) \ , \\ \frac{d\check{s}}{d\check{t}} &= h_1(\check{z} - \varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B) + z^*(t^B(\mu)), \check{y} - \bar{y}^*(\theta_0^B) + y^*(t^B(\mu)), \check{s}, \check{\sigma}, 1, \check{t} + t_A(\mu)) \ , \\ \frac{d\check{\sigma}}{d\check{t}} &= h_2(\check{z} - \varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B) + z^*(t^B(\mu)), \check{y} - \bar{y}^*(\theta_0^B) + y^*(t^B(\mu)), \check{s}, \check{\sigma}, 1, \check{t} + t_A(\mu)) \ , \end{aligned} \tag{41}$$

$$\check{z}_e(0) = \varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B), \ \check{y}_e(0) = \bar{y}^*(\theta_0^B) \ , \check{s}_e(0) = 0, \ \check{\sigma}_e(0) = 0 \tag{42}$$

resp.

$$\begin{aligned} \check{z}_a(-\mu^{k+1}\zeta_{k+1}) &= Z_k^*(-\mu^{k+1}\zeta_{k+1}) + \varphi^*(\bar{y}^*(\theta_0^B), 0, \theta_0^B) - z^*(0) \ , \\ \check{y}_a(-\mu^{k+1}\zeta_{k+1}) &= Y_k^*(-\mu^{k+1}\zeta_{k+1}) + \bar{y}^*(\theta_0^B) - y^*(0) \ , \\ \check{s}_a(-\mu^{k+1}\zeta_{k+1}) &= 0, \ \check{\sigma}_a(-\mu^{k+1}\zeta_{k+1}) = 0 \ . \end{aligned} \tag{43}$$

Remark the properties (38) and (40). With $t_0 = 0$ and $t(\mu) = -\mu^{k+1}\zeta_{k+1}$ we may apply Lemma 2 to system (41) because all hypotheses including (A_5) are satisfied and the righthand side depends smoothly on μ. Hence, for the k-th order approximation $\check{X}_k = (\check{Z}_k, \check{Y}_k, \check{S}_k, \check{\Sigma}_k)$ of $\check{x} = (\check{z}, \check{y}, \check{s}, \check{\sigma})$ it follows that

$$\max_{\check{t}\in[-\mu^{k+1}\zeta_{k+1}, T - t^B(\mu)]} ||\check{x}(\check{t}) - \check{X}_k(\check{t})|| = \mathcal{O}(\mu^{k+1}).$$

From the derivation of the asymptotic algorithm in [VBK95] we know that $\Pi_0^+(\check{y}, \check{s}, \check{\sigma}) \equiv 0$. Moreover it follows that $\Pi_0^+\check{z}$ satisfies a linear homogenuous variational equation with initial condition $\Pi_0^+\check{z}(\check{\tau}=0)=\check{z}^0-\varphi^+(\bar{y}^*(\theta_0^B),0,0,\theta_0^B)$. Condition (42) yields $\check{z}^0=\varphi^*(\bar{y}^*(\theta_0^B),0,\theta_0^B)$. Hence, by (39) we get $\Pi_0^+\check{z}(\tau=0)=0$ such that

$$\Pi_0^+\check{z}(\check{\tau})=0\ \ \forall\check{\tau}\ . \tag{44}$$

After backward transformation we get

$$\max_{t\in[\Theta_k^B,T]}||x^+(t)-X_k^+(t)||=\mathcal{O}(\mu^{k+1})$$

because $||x^+(t)-X_k^+(t)||=||\check{x}(\check{t})-\check{X}_k(\check{t})||$. Finally, glueing together the approximations X_k^* and X_k^+ at $t=t_k^B$ we get the estimation (34) with

$$x(t,\mu)=\begin{cases} x^*(t) & \text{if} \quad t\in[t_0,t^B(\mu)] \\ x^+(t) & \text{if} \quad t\in[t^B(\mu),T]\end{cases} \quad \text{and}$$

$$X_k(t,\mu)=\begin{cases} X_k^*(t) & \text{if} \quad t\in[t_0,\Theta_k^B] \\ X_k^+(t) & \text{if} \quad t\in[\Theta_k^B,T]\end{cases}$$

such that

$$\bar{x}_i(t)=\begin{cases} \bar{x}_i^*(t) & \text{if} \quad t\in[t_0,\Theta_k^B] \\ \bar{x}_i^+(t) & \text{if} \quad t\in[\Theta_k^B,T]\end{cases} \quad i=0,..,k\ .$$

From the coordinate change it follows that

$$x^+(t)=\check{x}(t-t^B(\mu))+x_{shift}$$

where

$$x_{shift}=(\varphi^+(\bar{y}^*(\theta_0^B),0,0,\theta_0^B)-z^*(t^B(\mu)),\bar{y}^*(\theta_0^B)-y^*(t^B(\mu)),0,0)^T$$

such that

$$X_k^+(t)=\check{X}_k(t-t^B(\mu))+x_{shift}\ .$$

Especially, because of (44) it follows that

$$X_0^+(t)=\bar{x}_0^+(t)+\Pi_0^+x^+(t)=\check{x}_0(t-t^B(\mu))+x_{shift}\ .$$

This means that the 0th order approximation X_0^+ of x^+ possesses no boundary layer part. Hence, we may conclude that not only $\Pi_0^+(y^+,s^+,\sigma^+)\equiv 0$ but $\Pi_0^+z^+\equiv 0$. Therefore,

$$\Pi_0^+x\equiv 0\ .$$

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3 Application: Coupled Oscillators with dry friction

We consider two pendula which are coupled by a spiral spring. One of them is in contact with a uniformly rotating disk. Both pendula have the same distance $l := l_1 = l_2$ of the centre of gravity from the axis of rotation. In angle coordinates we get the following equation of motion for this 2DOF system:

$$\begin{aligned} m_1 l^2 \ddot{\varphi}_1 &= -k(\varphi_1 - \varphi_2) - m_1 g \sin\alpha \sin\varphi_1 l - a(\dot{\varphi}_1 - \Omega) + F_R l \\ m_2 l^2 \ddot{\varphi}_2 &= \ k(\varphi_1 - \varphi_2) - m_2 g \sin\alpha \sin\varphi_2 l - b(\dot{\varphi}_2 - \Omega) \end{aligned} \tag{45}$$

where m_1, m_2 are the masses of the two pendula, k the hardness of the spring, g the gravity constant, α the clination angle of the disk and a, b constants due to linear damping. F_R represents the friction force with

$$F_R = -\mu(\dot{\varphi}_1) m_1 g \cos\alpha \ \mathrm{sgn}\ (\dot{\varphi}_1 - \Omega)$$

where

$$\mu(\dot{\varphi}_1) = \mu_0(1 - c \arctan(\kappa|\dot{\varphi}_1 - \Omega|)) \ .$$

Assuming that $\varepsilon := m_2 \ll m_1$ and setting $y_1 := \varphi_1, y_2 := \varphi_2, s := \dot{\varphi}_1 - \Omega, z := \dot{\varphi}_2$ we get

$$\begin{aligned} \varepsilon \dot{z} &= \tfrac{k}{l^2}(y_1 - y_2) - \varepsilon g \sin\alpha \sin y_2 - \tfrac{b}{l^2}(z - \Omega) \\ \dot{y}_1 &= s + \Omega \\ \dot{y}_2 &= z \\ \dot{s} &= -\tfrac{k}{m_1 l^2}(y_1 - y_2) - \tfrac{g}{l} \sin\alpha \sin y_1 - \tfrac{F_R}{m_1 l} - \tfrac{a}{m_1 l^2} s \ . \end{aligned} \tag{46}$$

To apply Theorem 2 we do one further transformation

$$\sigma := y_2 - y_1 - \frac{m_1 g l}{k}(\mu_0 \cos\alpha + \sin\alpha \sin y_1) \ .$$

Hence, we consider the following **nondegenerate system**:

$$\begin{aligned} \varepsilon \dot{z} &= -\tfrac{k}{l^2}(\sigma + \tfrac{m_1 g l}{k}(\mu_0 \cos\alpha + \sin\alpha \sin y_1)) - \tfrac{b}{l^2}(z - \Omega) \\ &\quad -\varepsilon \tfrac{g}{l} \sin\alpha \sin(\sigma + y_1 + \tfrac{m_1 g l}{k}(\mu_0 \cos\alpha + \sin\alpha \sin y_1)) &&= F(z, y_1, \sigma) \\ \dot{y}_1 &= s + \Omega &&= f(s) \\ \dot{s} &= \tfrac{k}{m_1 l^2}\sigma + \tfrac{g}{l} \cos\alpha \mu_0 (1 - (1 - c \arctan(\kappa|s|))u) - \tfrac{a}{m_1 l^2} s &&= h_1(s, \sigma, u) \\ \dot{\sigma} &= z - (s + \Omega)(1 + \tfrac{m_1 g l \sin\alpha}{k} \cos y_1) &&= h_2(z, y_1, s) \end{aligned} \tag{47}$$

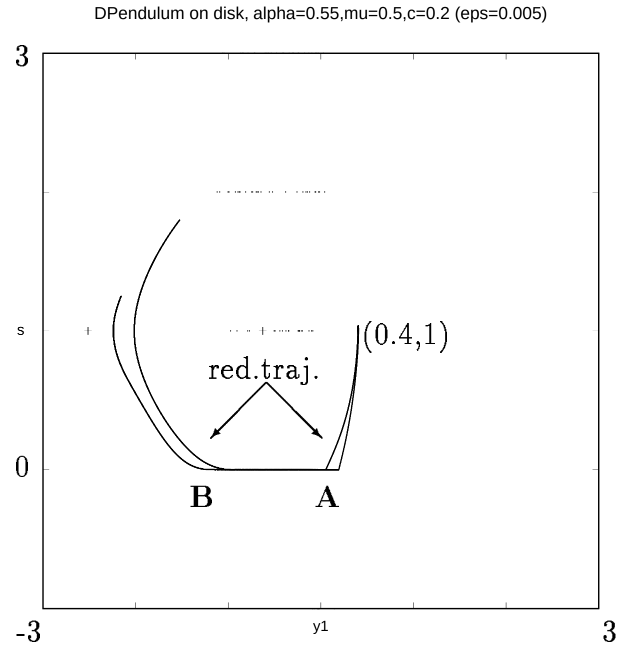

Figure 1: Comparison between full system and reduced system trajectory

In S^+ with $z = \Omega - \frac{k}{b}(\sigma + \frac{m_1 gl}{k}(\mu_0 \cos\alpha + \sin\alpha \sin y_1)) = \varphi^+(y_1, \sigma)$ we get the **reduced system**

$$\begin{aligned} \dot{y}_1 &= f(s) = s + \Omega \\ \dot{s} &= h_1(s, \sigma, 1) = \frac{k}{m_1 l^2}\sigma + \frac{g}{l}\cos\alpha\mu_0(1 - (1 - c\arctan(\kappa|s|))) - \frac{a}{m_1 l^2}s \\ \dot{\sigma} &= h_2(\varphi^+(y_1, \sigma), y_1, s) = \Omega - \frac{k}{b}(\sigma + \frac{m_1 gl}{k}(\mu_0\cos\alpha + \sin\alpha\sin y_1)) \\ &\quad -(s + \Omega)(1 + \frac{m_1 gl\sin\alpha}{k}\cos y_1) \ . \end{aligned} \tag{48}$$

With $u_{eq}(\sigma) = 1 + \frac{k}{m_1 gl\cos\alpha\mu_0}\sigma$ we get the **sliding mode system**

$$\begin{aligned} \varepsilon\dot{z} &= F(z, y_1, \sigma) \\ \dot{y}_1 &= f(0) \\ \dot{\sigma} &= h_2(z, y_1, 0) \ . \end{aligned} \tag{49}$$

Hence, with $z = \varphi^*(y_1, \sigma) = \varphi^+(y_1, \sigma)$ the **reduced sliding mode system** reads as follows:

$$\begin{aligned} &\dot{y}_1 = f(0) = \Omega \\ &\dot{\sigma} = h_2(\varphi^*(y_1, \sigma), y_1, 0) = -\tfrac{k}{b}(\sigma + \tfrac{m_1 gl}{k}(\mu_0\cos\alpha + \sin\alpha\sin y_1)) - \tfrac{m_1 gl\Omega}{k}\cos y_1 \ . \end{aligned} \tag{50}$$

Remarks.
It is easy to verify that the conditions (B_0) to (B_5) and (B_7) to (B_{12}) of Theorem 1 are satisfied. Also it's easy to compute that conditions (C_1) to (C_7) and (C_9) to (C_{12}) of Theorem 2 are satisfied.
The existence and size of the sliding domain depends on the parameters of the oscillator system. Therefore, the transition conditions (B_6) and (C_8) are only satisfied if the parameters belong to an appropriate region of parameter space and the initial value is carefully choosen. An example is displayed in figure 1 with initial values

$$z(0) = 0, y_1(0) = 0.4, s(0) = 1, \sigma(0) = 0 \ .$$

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