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Evaluation of moment Lyapunov exponents for second order linear autonomous SDE

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Abstract

Deterministic methods for evaluation of moment Lyapunov exponents are derived for two-dimensional systems with non-degenerate noise.

Key words: Lyapunov exponents, moment Lyapunov exponents, stability index

1. Introduction

Various characteristics of asymptotic behavior (under $t \to \infty$) of solutions of linear autonomous systems of stochastic differential equations (SDE) such as Lyapunov exponents, moment Lyapunov exponents, stability index, rotation numbers, and some others are derived and studied in [1-6] (see also references therein). It is possible to compare the values of the characteristics with the values of eigenvalues for linear deterministic systems of differential equations with constant coefficients. Meanwhile there are only a few results permitting to find them. Moreover, almost all of them concern the Lyapunov exponent. Among them there is an explicit formula for Lyapunov exponent for two-dimensional systems (see [2]). A method using numerical integration of SDE is proposed in [7]. In the papers [8,9] on systems with small diffusion asymptotic expansions of Lyapunov exponent in powers of small parameter in two-dimensional case are given.

In our paper deterministic methods are derived for evaluation of such characteristics. One of the suggested methods reduces the evaluation of moment Lyapunov exponent to finding a root of an equation with a smooth convex monotone function by the Newton method. The Newton method is known to require a little number of iterations even for reaching high accuracy. The computational efforts for realization of each iteration consist in solving of the linear boundary value problem for the second order ordinary differential equation. Such a problem is profoundly investigated in numerical respect and has a number of good algorithms for its solution. Thus the evaluation of moment Lyapunov exponents becomes reliable and effective matter. The same also concerns other characteristics. In Section 7 some probabilistic representations connected with moment Lyapunov exponents are brought. They are not used directly in numerical respect here because in two-dimensional case the obtained deterministic methods are undoubtedly preferable. But due to the fact that the values of such representations can be calculated with any precision they become very useful in numerical tests connected with numerical integration of SDE and a Monte-Carlo technique. The last Section deals with the problem of maximization of stability index.

2. Preliminary

Consider the second order Ito's linear autonomous system of SDE

$$dX = A_0X\ dt\ +\ \sum_{i=1}^{m} A_iX\ dw_i(t) \tag{2.1}$$

where A_0, A_1,..., A_m are real $2\times2-$matrices, and $w_i(t)$ are independent standard scalar Wiener processes.

Following [1,2], consider the new processes $\rho(t)$ and $\Phi(t)$ where

$$\rho(t) = \ln \mid X(t) \mid$$

and $\Phi(t)$ is defined by relations

$$\frac{X_1(t)}{\mid X(t) \mid} = \cos\Phi(t),\ \frac{X_2(t)}{\mid X(t) \mid} = \sin\Phi(t),\ 0 \le \Phi(t) < 2\pi$$

Introduce the vectors

$$\Lambda(\varphi) = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix},\ \bar{\Lambda}(\varphi) = \begin{bmatrix} \sin\varphi \\ -\cos\varphi \end{bmatrix},$$

the matrix

$$A(\varphi) = \sum_{i=1}^{m} A_i\Lambda(\varphi)\Lambda^{\mathsf{T}}(\varphi)A_i^{\mathsf{T}}$$

and the functions

$$\alpha_i(\varphi) = \Lambda^{\mathsf{T}}(\varphi)A_i\Lambda(\varphi),\ \beta_i(\varphi) = \bar{\Lambda}^{\mathsf{T}}(\varphi)A_i\Lambda(\varphi),\ i = 0,...,m,$$

$$Q(\varphi) = \alpha_0(\varphi) + \frac{1}{2}TrA(\varphi) - \Lambda^{\mathsf{T}}(\varphi)A(\varphi)\Lambda(\varphi) = \alpha_0(\varphi) + \frac{1}{2}TrA(\varphi) - \sum_{i=1}^{m}\alpha_i^2(\varphi),$$

$$k^2(\varphi) = \bar{\Lambda}^{\mathsf{T}}(\varphi)A(\varphi)\bar{\Lambda}(\varphi) = \sum_{i=1}^{m}\beta_i^2(\varphi)$$

We shall suppose throughout what follows that

$$k^2(\varphi) > 0,\ 0 \le \varphi < 2\pi \tag{2.2}$$

Note that all introduced functions and a matrix $A(\varphi)$ are $\pi-$periodic.

Applying Ito's formula (see [1,2]) we have

$$d\Phi(t) = (-\beta_0(\Phi) + \sum_{i=1}^{m}\alpha_i(\Phi)\beta_i(\Phi))dt - \sum_{i=1}^{m}\beta_i(\Phi)dw_i(t), \tag{2.3}$$

$$d\rho(t) = Q(\Phi)dt + \sum_{i=1}^{m}\alpha_i(\Phi)dw_i(t) \tag{2.4}$$

Thanks to (2.2) the Markov process $\Phi(t)$ is ergodic on the unit circle. If $\mu(\varphi)$ is the invariant measure of the process $\Phi(t)$ then for any $x \neq 0$ the following limit exists a.s.

$$\lim_{t\to\infty}\frac{\rho(t)}{t} = \lim_{t\to\infty}\frac{1}{t}\ln \mid X^x(t) \mid = \int_0^{2\pi} Q(\varphi)d\mu(\varphi) := \lambda \tag{2.5}$$

This limit λ is called the Lyapunov exponent of the system (2.1).

It follows from (2.4) for $p \in \mathbf{R}$, $x \neq 0$

$$| X^x(t) |^p = \exp\left\{p(\int_0^t Q(\Phi^\varphi(s))ds + \int_0^t \sum_{i=1}^m \alpha_i(\Phi^\varphi(s))dw_i(s))\right\} \tag{2.6}$$

where $\varphi = \Phi^\varphi(0)$ corresponds to $x = X^x(0)$ by equality $\Lambda(\varphi) = x / | x |$.

The formula

$$T_t(p)f(\varphi) = \mathbf{E}f(\Phi^\varphi(t)) \exp\left\{p(\int_0^t Q(\Phi^\varphi(s))ds + \int_0^t \sum_{i=1}^m \alpha_i(\Phi^\varphi(s))dw_i(s))\right\} \tag{2.7}$$

defines under any $p \in \mathbf{R}$ the strongly continuous semigroup of positive operators on the space of continuous π-periodic functions $f(\varphi)$. The infinitesimal operator $L(p)$ of this semigroup has the form

$$L(p)f(\varphi) = \frac{1}{2}k^2(\varphi)f''(\varphi) + b(\varphi;p)f'(\varphi) + c(\varphi;p)f(\varphi) \tag{2.8}$$

where

$$b(\varphi;p) = -\beta_0(\varphi) + (1-p)\sum_{i=1}^m \alpha_i(\varphi)\beta_i(\varphi),$$

$$c(\varphi;p) = pQ(\varphi) + \frac{1}{2}p^2\sum_{i=1}^m \alpha_i^2(\varphi).$$

In [3-5] the concept of moment Lyapunov exponents for linear autonomous SDE was introduced. It turns out that under assumption (2.2) the following limit

$$g(p) := \lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E} | X^x(t) |^p, \ p \in \mathbf{R} \tag{2.9}$$

exists and is independent of x, $x \neq 0$. The limit $g(p)$ is a convex analytic function of $p \in \mathbf{R}$, $g(0) = 0$, $g(p)/p$ is increasing, and

$$g'(0) = \lim_{p\to 0} \frac{g(p)}{p} := \lambda \tag{2.10}$$

where λ is the Lyapunov exponent defined in (2.5). Moreover, for any $p \in \mathbf{R}$ the number $g(p)$ is the eigenvalue of $L(p)$ which strictly dominates the real part of any point from the rest part of the spectrum of $L(p)$. This eigenvalue is simple and has a strictly positive π-periodic eigenfunction $f(\varphi;p)$:

$$L(p)f(\varphi;p) = g(p)f(\varphi;p) \tag{2.11}$$

Let us note that if the matrix A_0 in (2.1) is replaced with $A_0 + aI$, where a is a scalar and I is the identity matrix, then $g(p)$ is replaced with $g(p) + ap$, and the new Lyapunov exponent is equal to $\lambda + a$.

If $\lambda < 0$ then the trivial solution of the system (2.1) is a.s. asymptotically stable. It is well known (see, for instance, [2]) and follows from (2.10) that in this case $g(p) < 0$ for all sufficiently small positive p, i.e., the solution $X = 0$ of (2.1) is p-stable for such p. It is shown in [4] that $g(p) \to +\infty$ for $p \to +\infty$ unless there exists a non singular matrix G such that GA_iG^{-1}, $i = 1, ..., m$, are skew-symmetric matrices. If $g(p) \to +\infty$ as $p \to +\infty$ then the equation

$$g(p) = 0 \tag{2.12}$$

has the unique positive root γ_0. It is clear that the solution $X = 0$ of (2.1) is p-stable for $0 < p < \gamma_0$ and p-unstable for $p > \gamma_0$. The root γ_0 is connected with the asymptotic behavior of [6]

$$V_\delta(x) := \mathbf{P}\{\sup_{t\geq 0} \mid X^x(t) \mid > \delta\}, \; |x|/\delta \to 0$$

It turns out that for some $K > 0$ and for all $\delta > 0$ and $|x| < \delta$

$$\frac{1}{K}(|x|/\delta)^{\gamma_0} \leq V_\delta(x) \leq K(|x|/\delta)^{\gamma_0} \tag{2.13}$$

It follows from (2.13) that the probability of exit from the ball $|x| < \delta$ has the order $|x|^{\gamma_0}$ for $x \to 0$ for any $\delta > 0$ if (2.1) is stable and γ_0 is the positive root of (2.12). The number γ_0 is called the stability index of the system (2.1). Analogous results are valid in the unstable case (see [6]), where the equation (2.12) has a unique negative root γ_0.

The evaluation of such quantities as $g''(0)$ or $\inf\{g(p) : p \in \mathbf{R}\}$ is also of great interest. The first of them is connected with the rate at which the almost-sure limit $\lambda = \lim_{t\to\infty}(1/t)\ln \mid X^x(t) \mid$ is achieved, and the second is closely related to estimates of $\mathbf{P}\{\mid X^x(t) \mid \geq R\}$ and $\mathbf{P}\{\sup_{s>t} \mid X^x(s) \mid \geq R\}$ as $t \to \infty$ (see [5]).

Remark. For definiteness we consider here linear systems. But all the results of the paper are also correct for a second order nonlinear autonomous SDE

$$dX = f_0(X)dt + \sum_{i=1}^{m} f_i(X)dw_i(t) \tag{2.14}$$

of homogeneous type. More precisely, the vector field $f_0 = (f_0^1, f_0^2)^\top$ is required to be homogeneous of degree one, i.e., $f_0(cx) = cf_0(x)$ for all $c \in \mathbf{R}$, the vector fields $f = (f_i^1, f_i^2)^\top$, $i = 1, ...m$, are required to be positive homogeneous of degree one, i.e., $f_i(cx) = cf_i(x)$ for all $c > 0$, and for any $i = 1, ..., m$ each pair of functions f_i^j, $j = 1, 2$, is required to be even or odd, i.e., or $f_i^j(-x) = f_i^j(x)$, $j = 1, 2$, or $f_i^j(-x) = -f_i^j(x)$, $j = 1, 2$. Besides we suppose that (2.14) is the system with non degenerate noises, i.e., the coefficient $k^2(\varphi)$ for the system (2.14) (see below) is strictly positive.

Indeed, introduce the functions

$$\alpha_{ij}(\varphi) = f_i^j(\cos\varphi, \; \sin\varphi) = f_i^j(\frac{x}{|x|}) = \frac{1}{|x|}f_i^j(x), \; i = 1, ...m; \; j = 1, 2$$

If we set

$$\alpha_i(\varphi) = \alpha_{i1}(\varphi)\cos\varphi + \alpha_{i2}(\varphi)\sin\varphi, \; i = 0, 1, ...m,$$

$$\beta_i(\varphi) = \alpha_{i1}(\varphi)\sin\varphi - \alpha_{i2}(\varphi)\cos\varphi, \; i = 0, 1, ...m,$$

$$Q(\varphi) = \alpha_0(\varphi) + \frac{1}{2}\cos 2\varphi \cdot \sum_{i=1}^{m}(\alpha_{i2}^2(\varphi) - \alpha_{i1}^2(\varphi)) - \sin 2\varphi \cdot \sum_{i=1}^{m}\alpha_{i1}(\varphi)\cdot\alpha_{i2}(\varphi)$$

and suppose that

$$k^2(\varphi) = \sum_{i=1}^{m}\beta_i^2(\varphi) = \sum_{i=1}^{m}(\alpha_{i2}(\varphi)\cos\varphi - \alpha_{i1}(\varphi)\sin\varphi)^2 > 0, \; 0 \leq \varphi < 2\pi$$

then all the relations (2.3)-(2.13) are fulfilled. It will appear by what follows that our results concerning the equation (2.12) use only the strict positiveness of $k^2(\varphi)$ and the $\pi-$periodicity of the functions $k^2(\varphi)$, $b(\varphi;p)$, $c(\varphi;p)$. But under hypothesis made above the $\pi-$periodicity of the mentioned functions also takes place in the case of the system (2.14).

Example 2.1. Consider the system (2.1) where

$$A_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}, \; i = 0, 1, ..., m$$

We have

$$A(\varphi) = \sum_{i=1}^{m} A_i \Lambda(\varphi) \Lambda^{\mathsf{T}}(\varphi) A_i^{\mathsf{T}} =$$

$$\sum_{i=1}^{m} \begin{bmatrix} (a_i \cos\varphi + b_i \sin\varphi)^2 & (a_i \cos\varphi + b_i \sin\varphi)(-b_i \cos\varphi + a_i \sin\varphi) \\ (a_i \cos\varphi + b_i \sin\varphi)(-b_i \cos\varphi + a_i \sin\varphi) & (-b_i \cos\varphi + a_i \sin\varphi)^2 \end{bmatrix},$$

$$TrA(\varphi) = \sum_{i=1}^{m}(a_i^2 + b_i^2), \; \Lambda^{\mathsf{T}}(\varphi) A(\varphi) \Lambda(\varphi) = \sum_{i=1}^{m} a_i^2,$$

$$\alpha_i(\varphi) = a_i, \; \beta_i(\varphi) = b_i, \; k^2(\varphi) = \sum_{i=1}^{m} b_i^2, \; Q(\varphi) = a_0 + \frac{1}{2}\sum_{i=1}^{m}(b_i^2 - a_i^2),$$

$$b(\varphi;p) = -b_0 + (1-p)\sum_{i=1}^{m} a_i b_i, \; c(\varphi;p) = p(a_0 + \frac{1}{2}\sum_{i=1}^{m}(b_i^2 - a_i^2)) + \frac{1}{2}p^2 \sum_{i=1}^{m} a_i^2$$

and the equation (2.11) acquires the form

$$\frac{1}{2}\sum_{i=1}^{m} b_i^2 \cdot f''(\varphi;p) + (-b_0 + (1-p)\sum_{i=1}^{m} a_i b_i) \cdot f'(\varphi;p)$$

$$+ (p(a_0 + \frac{1}{2}\sum_{i=1}^{m}(b_i^2 - a_i^2)) + \frac{1}{2}p^2 \sum_{i=1}^{m} a_i^2) \cdot f(\varphi;p) = g(p) f(\varphi;p) \qquad (2.15)$$

From here

$$g(p) = p(a_0 + \frac{1}{2}\sum_{i=1}^{m}(b_i^2 - a_i^2)) + \frac{1}{2}p^2 \sum_{i=1}^{m} a_i^2, \; f(\varphi;p) \equiv 1 \qquad (2.16)$$

If $\sum_{i=1}^{m} a_i^2 \neq 0$, $\lambda = g'(0) = a_0 + \frac{1}{2}\sum_{i=1}^{m}(b_i^2 - a_i^2) < 0$ then the stability index γ_0 is equal to

$$\gamma_0 = -\frac{2a_0 + \sum_{i=1}^{m}(b_i^2 - a_i^2)}{\sum_{i=1}^{m} a_i^2} \qquad (2.17)$$

3. The equation for $g(p)$

It is not difficult to obtain for every fixed $p \in \mathbf{R}$ a transcendental equation such that $g(p)$ is one of its roots. Indeed, consider the second order ordinary differential equation

$$L(p)f - \nu f = 0 \qquad (3.1)$$

where $\nu \in \mathbf{R}$. Let $f_1(\varphi; p, \nu)$, $f_2(\varphi; p, \nu)$ be a fundamental system of solutions for (3.1) and $f = C_1 f_1 + C_2 f_2$ be the general solution. For π−periodic f we have

$$C_1 f_1(0; p, \nu) + C_2 f_2(0; p, \nu) = C_1 f_1(\pi; p, \nu) + C_2 f_2(\pi; p, \nu),$$

$$C_1 f_1'(0; p, \nu) + C_2 f_2'(0; p, \nu) = C_1 f_1'(\pi; p, \nu) + C_2 f_2'(\pi; p, \nu)$$

If f is non-trivial for some ν then for such ν

$$D(p,\nu) := \det \begin{bmatrix} f_1(0; p, \nu) - f_1(\pi; p, \nu) & f_2(0; p, \nu) - f_2(\pi; p, \nu) \\ f_1'(0; p, \nu) - f_1'(\pi; p, \nu) & f_2'(0; p, \nu) - f_2'(\pi; p, \nu) \end{bmatrix} = 0. \tag{3.2}$$

If p is fixed then (3.2) is an equation with respect to ν and $g(p)$ is a root of this equation. For accurate solution of the equation (3.2) it is necessary to study properties of the function $D(p, \nu)$ thoroughly what seems to be fairly difficult task. Below another equation for $g(p)$ is derived.

Consider the boundary value problem on $[-\pi, \pi]$

$$L(p)y - \nu y = 0, \tag{3.3}$$

$$y(-\pi; p, \nu) = 1, \; y(\pi; p, \nu) = 1 \tag{3.4}$$

Let $\nu_0 = \nu_0(p)$ be maximal eigenvalue for Sturm-Liouville's problem

$$L(p)y - \nu y = 0, \; y(-\pi; p) = y(\pi; p) = 0 \tag{3.5}$$

We note that $\nu_0(p) < \max_{0 \le \varphi \le \pi} c(\varphi; p)$. For all $\nu > \nu_0$ solutions of the equation (3.3) are non oscillating on $[-\pi, \pi]$, and therefore the solution $y(\varphi; p, \nu)$ of the problem (3.3)-(3.4) exists and is unique. It can be found in the following way. Let $y_1(\varphi; p, \nu)$, $y_2(\varphi; p, \nu)$ be the solutions of (3.3) with initial data

$$y_1(-\pi; p, \nu) = 0, \; y_1'(-\pi; p, \nu) = 1,$$

$$y_2(\pi; p, \nu) = 0, \; y_2'(\pi; p, \nu) = -1$$

It is clear (of course, we suppose $\nu > \nu_0$) that $y_1(\varphi; p, \nu) > 0$ on $(-\pi, \pi]$ and $y_2(\varphi; p, \nu) > 0$ on $[-\pi, \pi)$. Let us note in passing that if $y_1(\varphi; p, \nu) > 0$ on $(-\pi, \pi]$ or $y_2(\varphi; p, \nu) > 0$ on $[-\pi, \pi)$ for some ν then $\nu > \nu_0$.

The solution $y(\varphi; p, \nu)$ of (3.3)-(3.4) is evidently expressed in the form

$$y(\varphi; p, \nu) = \frac{y_1(\varphi; p, \nu)}{y_1(\pi; p, \nu)} + \frac{y_2(\varphi; p, \nu)}{y_2(-\pi; p, \nu)} \tag{3.6}$$

Lemma 3.1. *The function $y(\varphi; p, \nu)$ for any $-\pi < \varphi < \pi$ and $p \in \mathbf{R}$ is strongly monotonically decreasing and convex function with respect to ν for $\nu > \nu_0(p)$.*

Proof. For the derivative $\dfrac{\partial y}{\partial \nu}(\varphi; p, \nu)$ we have

$$L(p)\frac{\partial y}{\partial \nu} - \nu \frac{\partial y}{\partial \nu} = y(\varphi; p, \nu), \tag{3.7}$$

$$\frac{\partial y}{\partial \nu}(-\pi; p, \nu) = 0, \; \frac{\partial y}{\partial \nu}(\pi; p, \nu) = 0 \tag{3.8}$$

using (3.3)-(3.4).

But $y_1(\varphi;p,\nu)$, $y_2(\varphi;p,\nu)$ is the fundamental system of solutions for the homogeneous equation which corresponds to (3.7). Introduce the Wronskian

$$W(\varphi;p,\nu) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

Since $y_1(-\pi;p,\nu) = 0$, $y_1'(-\pi;p,\nu) = 1$, $y_2(-\pi;p,\nu) > 0$ then $W(-\pi;p,\nu) < 0$ and hence $W(\varphi;p,\nu) < 0$ for $-\pi \le \varphi \le \pi$. The solution of the problem (3.7)-(3.8) can be expressed in the form

$$\frac{1}{2}\frac{\partial y}{\partial \nu}(\varphi;p,\nu) = y_1(\varphi;p,\nu) \cdot \int_{\varphi}^{\pi} \frac{y_2(\theta;p,\nu) \cdot y(\theta;p,\nu)}{k^2(\theta) \cdot W(\theta;p,\nu)} d\theta$$

$$+ y_2(\varphi;p,\nu) \cdot \int_{-\pi}^{\varphi} \frac{y_1(\theta;p,\nu) \cdot y(\theta;p,\nu)}{k^2(\theta) \cdot W(\theta;p,\nu)} d\theta \tag{3.9}$$

The derivative $\frac{\partial^2 y}{\partial \nu^2}(\varphi;p,\nu)$ satisfies the following boundary value problem

$$L(p)\frac{\partial^2 y}{\partial \nu^2} - \nu \frac{\partial^2 y}{\partial \nu^2} = 2\frac{\partial y}{\partial \nu}(\varphi;p,\nu), \tag{3.10}$$

$$\frac{\partial^2 y}{\partial \nu^2}(-\pi;p,\nu) = 0, \quad \frac{\partial^2 y}{\partial \nu^2}(\pi;p,\nu) = 0 \tag{3.11}$$

Analogously to (3.9) it can be obtained

$$\frac{1}{4}\frac{\partial^2 y}{\partial \nu^2}(\varphi;p,\nu) = y_1(\varphi;p,\nu) \cdot \int_{\varphi}^{\pi} \frac{y_2(\theta;p,\nu) \cdot \frac{\partial y}{\partial \nu}(\theta;p,\nu)}{k^2(\theta) \cdot W(\theta;p,\nu)} d\theta$$

$$+ y_2(\varphi;p,\nu) \cdot \int_{-\pi}^{\varphi} \frac{y_1(\theta;p,\nu) \cdot \frac{\partial y}{\partial \nu}(\theta;p,\nu)}{k^2(\theta) \cdot W(\theta;p,\nu)} d\theta \tag{3.12}$$

Now the assertion of the lemma easily follows from (3.9) and (3.12). Lemma 3.1 is proved.

The following lemma can be proved.

Lemma 3.2. *For any $-\pi < \varphi < \pi$ and $p \in \mathbf{R}$*

$$\lim_{\nu \downarrow \nu_0(p)} y(\varphi;p,\nu) = \infty, \quad \lim_{\nu \uparrow \infty} y(\varphi;p,\nu) = 0 \tag{3.13}$$

Theorem 3.1. *The eigenvalue $g(p)$ of the problem (2.11) is a root of the equation*

$$y(0;p,\nu) = \frac{y_1(0;p,\nu)}{y_1(\pi;p,\nu)} + \frac{y_2(0;p,\nu)}{y_2(-\pi;p,\nu)} = 1 \tag{3.14}$$

$\nu_0(p) < g(p) < \infty$, and the eigenfunction $f(\varphi;p)$ is equal to

$$f(\varphi;p) = y(\varphi;p,g(p)) = \frac{y_1(\varphi;p,g(p))}{y_1(\pi;p,g(p))} + \frac{y_2(\varphi;p,g(p))}{y_2(-\pi;p,g(p))}, \quad 0 \le \varphi \le \pi \tag{3.15}$$

Proof. Due to Lemmas 3.1-3.3 the root of the equation (3.14) exists and is unique on $(\nu_0(p), \infty)$. Denote this root as $\bar{\nu}$, $\nu_0(p) < \bar{\nu} < \infty$. The function $y(\varphi; p, \bar{\nu})$ on $[-\pi, 0]$ is the solution of the boundary value problem

$$L(p)y - \bar{\nu}y = 0, \;\; y(-\pi) = 1, \;\; y(0) = 1$$

and on $[0, \pi]$ is the solution of the boundary value problem

$$L(p)y - \bar{\nu}y = 0, \;\; y(0) = 1, \;\; y(\pi) = 1$$

Each of these problems has a unique solution because the equation

$$L(p)y - \bar{\nu}y = 0$$

has a positive solution on $[-\pi, \pi]$. Since the coefficients of this equation are π-periodic functions we have

$$y(\varphi; p, \bar{\nu}) = y(\varphi + \pi; p, \bar{\nu}), \;\; -\pi \le \varphi \le 0$$

From here

$$y'(-\pi; p, \bar{\nu}) = y'(0; p, \bar{\nu})$$

i.e., $y(\varphi; p, \bar{\nu})$ is the positive π-periodic function. Hence $\bar{\nu} = g(p)$ and (3.15) is realized. Theorem 3.1 is proved.

4. The evaluation of $g(p)$

Thanks to Theorem 3.1 and Lemmas 3.1-3.3 the problem of evaluating $g(p)$ and $f(\varphi; p)$ is sufficiently simple under any fixed p. For localization of the root $\bar{\nu} = g(p)$ of the equation (3.14) the following fact is useful. If $y_1(\varphi; p, \nu) > 0$ on $(-\pi, \pi]$ then $\nu > \nu_0(p)$, and if $y_1(\varphi; p, \nu)$ takes negative values on $(-\pi, \pi]$ then $\nu < \nu_0(p)$. Since the function $y(0; p, \nu)$ of ν is monotone and convex and the derivative $\frac{\partial y}{\partial \nu}(0; p, \nu)$ can be evaluated comparatively easy, the Newton method is preferable. The Newton method converges for all initial approximations ν_1, which are sufficiently close to $\bar{\nu}$, and for all that $\nu_1 > \nu_0(p)$ for which $y(0; p, \nu_1) > 1$. To realize the Newton method

$$\nu_{k+1} = \nu_k - \frac{y(0; p, \nu_k)}{\frac{\partial y}{\partial \nu}(0; p, \nu_k)} \tag{4.1}$$

one must calculate $y(0; p, \nu_k)$ and $\frac{\partial y}{\partial \nu}(0; p, \nu_k)$ at each step. To use the formula (3.9) for calculation $\frac{\partial y}{\partial \nu}(0; p, \nu_k)$ is non rational. It is much simpler to find the solution $z_0(\varphi; p, \nu_k)$ of the Cauchy problem (see equation (3.7))

$$L(p)z - \nu_k z = y(\varphi; p, \nu_k), \;\; z(-\pi) = 0, \;\; z'(-\pi) = 0 \tag{4.2}$$

and to obtain

$$\frac{\partial y}{\partial \nu}(\varphi; p, \nu_k) == -\frac{z_0(\pi; p, \nu_k)}{y_1(\pi; p, \nu_k)} \cdot y_1(\varphi; p, \nu_k) + z_0(\varphi; p, \nu_k) \tag{4.3}$$

Thus the realization of a single Newton method's step is reduced to solving of three Cauchy problems: two problems for the equation (3.3) and one problem for the equation (4.2). To avoid storing the function $y(\varphi; p, \nu_k)$ it can be recommended to solve the corresponding Cauchy problem for the systems (3.3), (4.2).

Another approach for searching of $y(0; p, \nu_k)$ and $\dfrac{\partial y}{\partial \nu}(0; p, \nu_k)$ consists in solving of the boundary value problems (3.3)-(3.4) and (3.7)-(3.8) by a finite difference method. Such a method is more preferable if a Cauchy problem is not well-posed.

Let the point $(p, g(p))$ be known. Then the point $(p + \Delta p, g(p + \Delta p))$ under sufficiently small Δp can be calculated by the Newton method

$$\nu_{k+1} = \nu_k - \frac{y(0; p + \Delta p, \nu_k)}{\dfrac{\partial y}{\partial \nu}(0; p + \Delta p, \nu_k)}, \quad k = 1, 2, ... \tag{4.4}$$

if as the first approximation for $g(p + \Delta p)$ the value

$$\nu_1 = g(p) \tag{4.5}$$

is taken. It is clear $\nu_k < g(p + \Delta p)$, $k = 2, 3, ...$.

The value ν_2 approximates $g(p + \Delta p)$ to within $\mathcal{O}((\Delta p)^2)$ and ν_k approximates up to $\mathcal{O}((\Delta p)^{2^{k-1}})$.

In this way it is possible to construct numerically the function $g(p)$ on any interval $[0, p]$ (remember $g(0) = 0$).

5. Differential equation for the function $g(p)$ and evaluation of $g'(p)$

According to Theorem 3.1 the function $g(p)$ satisfies the equation

$$y(0; p, g(p)) = 1 \tag{5.1}$$

From here

$$g'(p) = -\frac{\dfrac{\partial y}{\partial p}(0; p, g(p))}{\dfrac{\partial y}{\partial \nu}(0; p, g(p))} \tag{5.2}$$

i.e., $g(p)$ is the solution of the following Cauchy problem for ordinary differential equation

$$\frac{d\nu}{dp} = F(p, \nu), \ \nu(0) = 0 \tag{5.3}$$

where

$$F(p, \nu) = -\frac{\dfrac{\partial y}{\partial p}(0; p, \nu)}{\dfrac{\partial y}{\partial \nu}(0; p, \nu)}. \tag{5.4}$$

It was shown in the previous sections how to evaluate $\dfrac{\partial y}{\partial \nu}(\varphi; p, \nu)$. Consider an evaluation of $\dfrac{\partial y}{\partial p}(\varphi; p, \nu)$. From (3.3)-(3.4) (see also (2.8)) it follows

$$L(p)\frac{\partial y}{\partial p} - \nu\frac{\partial y}{\partial p} = \sum_{i=1}^{m} \alpha_i(\varphi)\beta_i(\varphi) \cdot y'(\varphi; p, \nu) - (Q(\varphi) + p\sum_{i=1}^{m} \alpha_i^2(\varphi)) \cdot y(\varphi; p, \nu), \tag{5.5}$$

$$\frac{\partial y}{\partial p}(-\pi;p,\nu)=0,\ \frac{\partial y}{\partial p}(\pi;p,\nu)=0 \tag{5.6}$$

The boundary value problem (5.5)-(5.6) is uniquely solvable. Let $u_0(\varphi;p,\nu)$ be the solution of the following Cauchy problem

$$L(p)u-\nu u=\sum_{i=1}^{m}\alpha_i(\varphi)\beta_i(\varphi)\cdot y'(\varphi;p,\nu)-(Q(\varphi)+p\sum_{i=1}^{m}\alpha_i^2(\varphi))\cdot y(\varphi;p,\nu),$$

$$u(-\pi)=0,\ u'(-\pi)=0 \tag{5.7}$$

Then evidently

$$\frac{\partial y}{\partial p}(\varphi;p,\nu)=-\frac{u_0(\pi;p,\nu)}{y_1(\pi;p,\nu)}\cdot y_1(\varphi;p,\nu)+u_0(\varphi;p,\nu)$$

Of course, $\frac{\partial y}{\partial p}(\varphi;p,\nu)$ just as $\frac{\partial y}{\partial \nu}(\varphi;p,\nu)$ can also be found by finite difference method.

As far as the function $F(p,\nu)$ can be calculated sufficiently simply, it is possible for searching $g(p)$ to solve the Cauchy problem (5.3) by any Runge-Kutta method of numerical integration. But due to the equation (5.1) the simplest method, i.e., Euler's method, is preferable. Indeed, suppose the point $(p,g(p))$ to be known (maybe $g(p)$ is known approximately). Calculating $F(p,g(p))$ according to (5.4) we find

$$g(p+\Delta p)=g(p)+F(p,g(p))\Delta p:=g_1$$

The convexity of $g(p)$ gives $g_1<g(p+\Delta p)$ and consequently (see Lemma 3.1) $y(0;p+\Delta p,g_1)>1$.

Use the formula (4.4), where the first approximation in the Newton method unlike (4.5) is equal to g_1, i.e.,

$$\nu_{k+1}=\nu_k-\frac{y(0;p+\Delta p,\nu_k)}{\frac{\partial y}{\partial \nu}(0;p+\Delta p,\nu_k)},\ k=1,2,...,\ \nu_1=g_1 \tag{5.8}$$

Here ν_1 differs from $g(p+\Delta p)$ by a quantity of $\mathcal{O}((\Delta p)^2)$. The correction of ν_1 in accordance with (5.8) gives $\nu_2>\nu_1=g_1$ with the error $\mathcal{O}((\Delta p)^4)$, which has the same accuracy as a Runge-Kutta method of the third order. The next approximation $\nu_2<\nu_3<g(p+\Delta p)$ has the error $\mathcal{O}((\Delta p)^8)$, and it is much better than that does the most used Runge-Kutta method of the fourth order. Besides such an approach does not lead to any error accumulation. Taking ν_2 or ν_3 as $g(p+\Delta p)$ we obtain the next point $(p+\Delta p,g(p+\Delta p))$ (of course, approximately) and the procedure can be repeated.

Computational efforts for the construction of $g(p)$ here are the same as in the method suggested in the previous section but the important function $g'(p)$ is simultaneously calculated now.

Having $g'(p)$ it is also very easy to find the stability index γ_0, which is the root of the equation $g(p)=0$, by the Newton method.

For evaluation of $\inf\{g(p):p\in\mathbf{R}\}$ one can solve the equation

$$g'(p)=0$$

which can accurately be solved by using $g''(p)$:

$$g''(p) = -\frac{\frac{\partial^2 y}{\partial p^2}(0;p,g(p)) + 2\frac{\partial^2 y}{\partial p \partial \nu}(0;p,g(p)) \cdot g'(p) + \frac{\partial^2 y}{\partial \nu^2}(0;p,g(p)) \cdot (g'(p))^2}{\frac{\partial y}{\partial \nu}(0;p,g(p))}$$

The derivatives $\frac{\partial^2 y}{\partial p^2}(\varphi;p,g(p))$, $\frac{\partial^2 y}{\partial p \partial \nu}(\varphi;p,g(p))$, $\frac{\partial^2 y}{\partial \nu^2}(\varphi;p,g(p))$ can be found from boundary value problems which are analogous to the problems (3.7)-(3.8) and (5.5)-(5.6).

Let us note in passing that under $p = 0$, $\nu = 0$ the homogeneous equation corresponding to the problems for calculating derivatives has the form

$$\frac{1}{2}k^2(\varphi)u'' + (-\beta_0(\varphi) + \sum_{i=1}^{m} \alpha_i(\varphi)\beta_i(\varphi))u' = 0$$

From here it follows that any derivative $g^{(n)}(0)$, $n = 1, 2, ...$, can be found by quadratures.

6. The second method of evaluation of $g'(p)$

We have (see (2.8) and (2.11))

$$\tilde{L}(p)f - \frac{2g(p)}{k^2(\varphi)}f$$

$$:= f''(\varphi;p) + \frac{2b(\varphi;p)}{k^2(\varphi)}f'(\varphi;p) + \frac{2c(\varphi;p)}{k^2(\varphi)}f(\varphi;p) - \frac{2g(p)}{k^2(\varphi)}f(\varphi;p) = 0, \tag{6.1}$$

$$f(0;p) = f(\pi;p), \; f'(0;p) = f'(\pi;p) \tag{6.2}$$

Therefore

$$\tilde{L}(p)\frac{\partial f}{\partial p} - \frac{2g(p)}{k^2(\varphi)}\frac{\partial f}{\partial p} = \frac{2g'(p)}{k^2(\varphi)} \cdot f(\varphi;p)$$

$$+ \frac{2}{k^2(\varphi)}\sum_{i=1}^{m} \alpha_i(\varphi)\beta_i(\varphi) \cdot f'(\varphi;p) - \frac{2}{k^2(\varphi)}(Q(\varphi) + p\sum_{i=1}^{m} \alpha_i^2(\varphi)) \cdot f(\varphi;p), \tag{6.3}$$

$$\frac{\partial f}{\partial p}(0;p) = \frac{\partial f}{\partial p}(\pi;p), \; \frac{d}{d\varphi}(\frac{\partial f}{\partial p})(0;p) = \frac{d}{d\varphi}(\frac{\partial f}{\partial p})(\pi;p) \tag{6.4}$$

A solution of the problem (6.3)-(6.4) there exists iff the right hand side of the equation (6.3) is orthogonal to the nontrivial solution of the homogeneous conjugate problem

$$\tilde{L}^*(p)z := \frac{d^2 z}{d\varphi^2} - \frac{d}{d\varphi}(\frac{2b(\varphi;p)}{k^2(\varphi)}z) + \frac{2(c(\varphi;p) - g(p))}{k^2(\varphi)}z = 0, \tag{6.5}$$

$$z(0;p) = z(\pi;p), \; z'(0;p) = z'(\pi;p) \tag{6.6}$$

Let $f_1(\varphi;p)$, $f_2(\varphi;p)$ be the solutions of (6.1) with initial data

$$f_1(0;p) = 0, \; f_1'(0;p) = 1,$$

$$f_2(\pi;p) = 0,\ f_2'(\pi;p) = -1$$

and $W(\varphi;p)$ is the Wronskian

$$W(\varphi;p) = \det\begin{bmatrix} f_1(\varphi;p) & f_2(\varphi;p) \\ f_1'(\varphi;p) & f_2'(\varphi;p) \end{bmatrix}$$

It is clear that $W(0;p) = -f_2(0;p) < 0$, $W(\pi;p) = -f_1(\pi,p) < 0$.

Lemma 6.1. *The problem (6.5)-(6.6) has a positive solution which is equal to*

$$z(\varphi;p) = -\frac{f_1(\varphi;p)}{W(\varphi;p)} - \frac{f_2(\varphi;p)}{W(\varphi;p)} \tag{6.7}$$

Proof. Since the problem (6.1)-(6.2) has a nontrivial solution which is equal to

$$f(\varphi;p) = \frac{f_1(\varphi;p)}{f_1(\pi;p)} + \frac{f_2(\varphi;p)}{f_2(\pi;p)} \tag{6.8}$$

then (see (3.2))

$$D(p) := \det\begin{bmatrix} f_1(0;p) - f_1(\pi,p) & f_2(0;p) - f_2(\pi;p) \\ f_1'(0;p) - f_1'(\pi,p) & f_2'(0;p) - f_2'(\pi;p) \end{bmatrix}$$

$$= \det\begin{bmatrix} -f_1(\pi,p) & f_2(0;p) \\ 1 - f_1'(\pi,p) & f_2'(0;p) + 1 \end{bmatrix} = 0 \tag{6.9}$$

It is not difficult to verify directly that if $y(\varphi;p)$ is any solution of the equation (6.1) then $y(\varphi;p)/W(\varphi;p)$ is the solution of the equation (6.5) (remember $W' = -2bW/k^2$). Therefore the function z defined by (6.7) is the solution of (6.5). Clearly $z(0;p) = z(\pi;p) = 1$. The relation $z'(0;p) = z'(\pi;p)$ follows from direct calculation taking account of (6.9). Lemma 6.1 is proved.

Corollary.

$$g'(p) = \frac{R - S}{\int_0^\pi \frac{1}{k^2(\varphi)} \cdot f(\varphi;p) \cdot z(\varphi;p)d\varphi} \tag{6.10}$$

where R and S are equal to

$$R = \int_0^\pi \frac{Q(\varphi) + p\sum_{i=1}^m \alpha_i^2(\varphi)}{k^2(\varphi)} \cdot f(\varphi;p) \cdot z(\varphi;p)d\varphi\ ,$$

$$S = \int_0^\pi \frac{\sum_{i=1}^m \alpha_i(\varphi)\beta_i(\varphi)}{k^2(\varphi)} \cdot f'(\varphi;p) \cdot z(\varphi;p)d\varphi$$

and $f(\varphi;p)$, $z(\varphi;p)$ are from (6.8) and (6.7).

Due to this formula we do not need to solve any additional boundary value problems for calculation of $g'(p)$. But the calculation of integrals in (6.10) is sufficiently labor-consuming problem.

Remark. Of course,

$$f_1(\varphi;p) = y_1(\varphi - \pi;p,g(p)),\ f_2(\varphi;p) = y_2(\varphi;p,g(p)),\ 0 \le \varphi \le \pi,$$

where y_1 and y_2 are from (3.6).

7. Some probabilistic representations

The solution $y(\varphi; p, \nu)$ of the boundary value problem (3.3)-(3.4) has the following well known probabilistic representation for $\nu > \nu_0$

$$y(\varphi; p, \nu) = \mathbf{E} \exp\{-\nu\tau^{\varphi} + p \cdot I(\varphi, \tau^{\varphi})\} \tag{7.1}$$

where τ^{φ} is the time at which the solution $\Phi^{\varphi}(s)$ of the equation (2.3) leaves the interval $(-\pi, \pi)$ and

$$I(\varphi, \tau^{\varphi}) = \int_0^{\tau^{\varphi}} Q(\Phi^{\varphi}(s))ds + \int_0^{\tau^{\varphi}} \sum_{i=1}^{m} \alpha_i(\Phi^{\varphi}(s))dw_i(s)$$

The value $g(p)$ is the very number for which

$$\mathbf{E} \exp\left\{-g(p)\tau^0 + p \cdot I(0, \tau^0)\right\} = 1 \tag{7.2}$$

and for stability index γ_0 the following equality

$$\mathbf{E} \exp\left\{\gamma_0 \cdot I(0, \tau^0)\right\} = 1 \tag{7.3}$$

holds.

Having differentiated (7.2) with respect to p two times we obtain

$$\mathbf{E} \exp\left\{-g(p)\tau^0 + p \cdot I(0, \tau^0)\right\} (-g'(p)\tau^0 + I(0, \tau^0)) = 0\ , \tag{7.4}$$

$$\mathbf{E} \exp\left\{-g(p)\tau^0 + p \cdot I(0, \tau^0)\right\} [(-g'(p)\tau^0 + I(0, \tau^0))^2 - g''(p)\tau^0] = 0 \tag{7.5}$$

Of course, it is possible to differentiate (7.2) further.

Let us prove, for example, (7.4). To this end we establish the possibility of differentiating the expression

$$\mathbf{E}G(p) = \mathbf{E} \exp\left\{-g(p)\tau^0 + p \cdot I(0, \tau^0)\right\}$$

Let $p = \bar{p}$ be fixed. Note that there exists a number $\delta > 0$ such that

$$\mathbf{E} \exp\left\{-\nu\tau^0 + p \cdot I(0, \tau^0)\right\} < \infty \text{ for } \nu > -g(\bar{p}) - 2\delta,\ \bar{p} - 2\delta \le p \le \bar{p} + 2\delta \tag{7.6}$$

Due to mean value theorem

$$\frac{1}{\Delta p}(G(\bar{p} + \Delta p) - G(\bar{p})) = \exp\left\{-g(\tilde{p})\tau^0 + \tilde{p} \cdot I(0, \tau^0)\right\} \cdot (-g'(\tilde{p})\tau^0 + I(0, \tau^0)) \tag{7.7}$$

where $\tilde{p}$ depends on elementary event ω and $\bar{p} - \Delta p \le \tilde{p} \le \bar{p} + \Delta p$. For definiteness let $\Delta p > 0$ and let Δp be so small that $\Delta p < \delta$ and $-g(\tilde{p}) < -g(\bar{p}) + \delta$. Then

$$\exp\left\{-g(\tilde{p})\tau^0 + \tilde{p} \cdot I(0, \tau^0)\right\} \le$$

$$\exp\left\{-(g(\bar{p}) - \delta)\tau^0 + (\bar{p} + \delta) \cdot I(0, \tau^0)\right\} \cdot \chi_{I(0,\tau^0)\ge 0}(\omega) +$$

$$\exp\left\{-(g(\bar{p}) - \delta)\tau^0 + (\bar{p} - \delta) \cdot I(0, \tau^0)\right\} \cdot \chi_{I(0,\tau^0)<0}(\omega) \le$$

$$\exp\left\{-(g(\bar p)-\delta)\tau^0+(\bar p+\delta)\cdot I(0,\tau^0)\right\}+\exp\left\{-(g(\bar p)-\delta)\tau^0+(\bar p-\delta)\cdot I(0,\tau^0)\right\} \tag{7.8}$$

Thanks (7.6) the right-hand side of (7.8) has bounded mathematical expectation. Let $|g'(\tilde p)|\le K$ for $\bar p-\Delta p\le\tilde p\le\bar p+\Delta p$. From (7.7) and (7.8) we have

$$\frac{1}{\Delta p}|G(\bar p+\Delta p)-G(\bar p)|\le\exp\left\{-(g(\bar p)-\delta)\tau^0+(\bar p+\delta)\cdot I(0,\tau^0)\right\}\cdot(K\tau^0+|I(0,\tau^0)|)+$$

$$\exp\left\{-(g(\bar p)-\delta)\tau^0+(\bar p-\delta)\cdot I(0,\tau^0)\right\}\cdot(K\tau^0+|I(0,\tau^0)|) \tag{7.9}$$

It is not difficult to justify the boundedness of $\mathbf{E}(\tau^0)^n$ and $\mathbf{E}|I(0,\tau^0)|^n$ for any positive integer n. Due to (7.6) the functions

$$\exp\left\{(1+\frac{1}{n-1})\cdot[-(g(\bar p)-\delta)\tau^0+(\bar p\pm\delta)\cdot I(0,\tau^0)]\right\}$$

are integrable for sufficiently big n. Thus the summability of the right-hand side of (7.9) follows from Hölder's inequality. Now the differentiability of $\mathbf{E}G(p)$ and formula (7.4) imply from the Lebesgue theorem.

From (7.4)-(7.5) we can write down the formulae for $g'(p)$ and $g''(p)$ and specifically

$$\lambda=g'(0)=\frac{\mathbf{E}\int_0^{\tau^0}Q(\Phi^0(s))ds}{\mathbf{E}\tau^0},$$

$$g''(0)=\frac{\mathbf{E}(-g'(0)\tau^0+I(0,\tau^0))^2}{\mathbf{E}\tau^0}$$

It is possible to use different representations for $y(\varphi;p,\nu)$. For example,

$$y(\varphi;p,\nu)=\mathbf{E}\exp\left\{-\nu\tau^{\varphi;p}+\int_0^{\tau^{\varphi;p}}c(\Phi^\varphi(s;p);p)ds\right\} \tag{7.10}$$

along the solution of the equation

$$d\Phi=b(\Phi;p)dt+k(\Phi)dw(s)$$

where $w(s)$ is a scalar Wiener process.

But the representation (7.1), (2.3) is remarkable in that respect that Φ (and consequently τ) does not depend on p.

In view of such formulae as (7.1) or (7.10) many assertions of Section 3 become easy-to-interpret. For example, the assertion

$$\lim_{\nu\uparrow\infty}y(\varphi;p,\nu)=0$$

from Lemma 3.2 becomes evident after (7.10).

These probabilistic representations, as we can calculate $g(p)$, $g'(p)$, γ_0, and so on with any precision by deterministic methods, are very useful in numerical tests which are connected with numerical integration of SDE and a Monte-Carlo technique.

At last we turn out our attention to the probabilistic meaning of the eigenfunction $f(\varphi;\gamma_0)=y(\varphi;\gamma_0,0)$ of the problem (2.11) which can be easily found after γ_0 (see Sections 3-5). Let $\gamma_0>0$. At first let us find the probability

$$V(x)=\mathbf{P}(X^x(\tau)=1)$$

where $X^x(t)$, $0 < x < 1$, is the solution of the one-dimensional equation

$$dX = aXdt + \sigma X dw(t) \tag{7.11}$$

and τ is the exit time at which $X^x(t)$ leaves the interval $(0,1)$.

The function $V(x)$ satisfies the following linear boundary value problem

$$\frac{1}{2}\sigma^2 x^2 V'' + axV' = 0 \ , \ V(0) = 0 \ , \ V(1) = 1 \tag{7.12}$$

The problem (7.12) has the solution

$$V(x) = x^{-\frac{2a}{\sigma^2}+1} \tag{7.13}$$

As the stability index for the equation (7.11) is equal to $\gamma_0 = -\frac{2a}{\sigma^2} + 1$, we have obtained that the exponent in (7.13) coincides with the stability index.

Such a problem can be treated for $n-$dimensional systems as well. Let us consider the more general problem and for definiteness we restrict ourselves to two-dimensional case. We shall find out

$$V(x) = \mathbf{E}F(X^x(\tau))$$

where $X^x(t)$, $0 < |x| < 1$, is the solution of the equation (2.1), τ is the exit time at which the process $X^x(t)$ leaves the open sphere of radius 1 with center at zero, and F is a twice continuously differentiable function defined on the boundary of the sphere. If $F \equiv 1$ then $V(x) = \mathbf{P}(|X^x(\tau)| = 1)$. The function $V(x)$ satisfies the equation (in the open sphere with deleted center)

$$LV(x) := (A_0 x, \frac{\partial V}{\partial x}) + \frac{1}{2}\sum_{i=1}^{m}(A_i x, \frac{\partial}{\partial x})^2 V = 0 \ , \ |x| < 1, \ x \neq 0 \tag{7.14}$$

and the boundary conditions

$$V(0) = 0, \ V\mid_{|x|=1} = F(x) \tag{7.15}$$

Let us try to find out a solution of the linear boundary value problem (7.14)-(7.15) in the form of separating variables

$$V(x) = |x|^\gamma \cdot F(\frac{x}{|x|}) = |x|^\gamma \cdot f(\varphi) \tag{7.16}$$

where $\gamma > 0$, $f(\varphi) = F(\cos\varphi, \sin\varphi)$ is a strictly positive function.

Let $V(x)$ of the form (7.16) with $\gamma > 0$ be the solution of (7.14)-(7.15). Then (see [3]-[4])

$$L(\gamma)f(\varphi) = |x|^{-\gamma} LV = 0$$

i.e., γ is equal to the stability index γ_0 and $f(\varphi) = f(\varphi;\gamma_0)$ is the strictly positive eigenfunction for the operator $L(\gamma_0)$. On the contrary, if $\gamma = \gamma_0 > 0$ is the stability index and $f(\varphi) = f(\varphi;\gamma_0)$ is the corresponding eigenfunction then $V(x)$ of the form (7.16) is the solution of (7.14)-(7.15).

Now it is not difficult to obtain the following known bounds for $\mathbf{P}(|X^x(\tau)| = 1)$ with the help of $f(\varphi;\gamma_0)$

$$\frac{m}{M} \cdot |x|^{\gamma_0} \leq \mathbf{P}(|X^x(\tau)| = 1) \leq \frac{M}{m} \cdot |x|^{\gamma_0}$$

where

$$m = \min_{0 \leq \varphi < 2\pi} f(\varphi;\gamma_0) \ , \ M = \max_{0 \leq \varphi < 2\pi} f(\varphi;\gamma_0)$$

8. Optimization of parameters

Let us consider a system of SDE with control in two-dimensional case, for instance, of the following type

$$dX = (A_0X + cu)dt + \sum_{i=1}^{m}(A_iX + \sigma_i u)dw_i(t) \tag{8.1}$$

where u is a scalar control and c, σ_i are vectors. If the control u is constructed in the form of linear feedback $u = k_1X_1 + k_2X_2$ then the system (8.1) acquires the form

$$dX = A_0(k)Xdt + \sum_{i=1}^{m} A_i(k)Xdw_i(t) \tag{8.2}$$

In more general situation the parameter $k = (k_1, ...k_n)$ can be taken with n greater than two, matrices $A_i(k)$, $i = 0, ..., m$, can depend on k in an arbitrary (nonlinear) manner, and $k_1, ...k_n$ can satisfy some restrictions.

As performance criterion for choice of k it is natural to take such a criterion which does not depend on the initial state of the process $X(t)$ and which, of course, depends on k. The moment Lyapunov exponent under fixed p and stability index give examples of such criterions.

Consider the problem of maximization of stability index

$$\gamma_0(k) \longrightarrow \max_{k \in G} \tag{8.3}$$

where G is a certain set (bounded or unbounded).

To solve optimization problems the crucial moment is the possibility of evaluating the gradient of an optimized function.

We have

$$g(\gamma_0(k); k) = 0$$

where $g(p; k)$ is the moment Lyapunov exponent for the system (8.2). If k belongs to the interior of G, we can calculate $\partial\gamma_0/\partial k_i$ according to the formula

$$\frac{\partial\gamma_0}{\partial k_i} = -\frac{\frac{\partial g}{\partial k_i}(\gamma_0(k); k)}{\frac{\partial g}{\partial p}(\gamma_0(k); k)} \tag{8.4}$$

The derivatives $\partial g/\partial k_i$ are calculated like $\partial g/\partial p$ what has been done in Section 6.

Of course, the problem (8.3) subject to $A_i(k)$, $i = 0, ..., m$, and g can be very difficult but due to above stated results in many interesting cases it is possible to manage such problems numerically.

Consider an example with explicit calculation of $\partial\gamma_0/\partial k_i$ under some value of the parameter k.

Example 7.1. Consider a system with two scalar controls

$$dX = (A_0X + c_1u_1 + c_2u_2)dt + (A_1X + c_3u_1)dw_1(t) + (A_2X + c_4u_2)dw_2(t) \tag{8.5}$$

where

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad A_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}, \quad i = 0, 1, 2,$$

$$c_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c_3 = \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix}, c_4 = \begin{bmatrix} 0 \\ \sigma_2 \end{bmatrix}$$

Let the only coordinate X_1 be observed and the controls u_1, u_2 be constructed in the form of linear feedback $u_1 = k_1 X_1$, $u_2 = k_2 X_1$. Then the system (8.5) acquires the form

$$dX = A_0(k)X dt + A_1(k)X dw_1(t) + A_2(k)X dw_2(t) \tag{8.6}$$

where $k = (k_1, k_2)$ is the two-dimensional parameter and

$$A_0(k) = \begin{bmatrix} a_0 + k_1 & b_0 \\ -b_0 + k_2 & a_0 \end{bmatrix}, A_1(k) = \begin{bmatrix} a_1 + \sigma_1 k_1 & b_1 \\ -b_1 & a_1 \end{bmatrix}, A_2(k) = \begin{bmatrix} a_2 & b_2 \\ -b_2 + \sigma_2 k_2 & a_2 \end{bmatrix}$$

Calculate

$$A(\varphi; k) = \sum_{i=1}^{2} A_i(k)\Lambda(\varphi)\Lambda^\top(\varphi)A_i^\top(k) = \begin{bmatrix} a_{11}(\varphi; k) & a_{12}(\varphi; k) \\ a_{21}(\varphi; k) & a_{22}(\varphi; k) \end{bmatrix}$$

where

$$a_{11}(\varphi; k) = ((a_1 + \sigma_1 k_1)\cos\varphi + b_1 \sin\varphi)^2 + (a_2 \cos\varphi + b_2 \sin\varphi)^2 ,$$

$$a_{12}(\varphi; k) = a_{21}(\varphi; k) = ((a_1 + \sigma_1 k_1)\cos\varphi + b_1 \sin\varphi) \cdot (-b_1 \cos\varphi + a_1 \sin\varphi)$$
$$+(a_2 \cos\varphi + b_2 \sin\varphi) \cdot ((-b_2 + \sigma_2 k_2)\cos\varphi + a_2 \sin\varphi) ,$$

$$a_{22}(\varphi; k) = (-b_1 \cos\varphi + a_1 \sin\varphi)^2 + ((-b_2 + \sigma_2 k_2)\cos\varphi + a_2 \sin\varphi)^2$$

Further

$$\alpha_0(\varphi; k) = \Lambda^\top(\varphi)A_0(k)\Lambda(\varphi) = a_0 + k_1 \cos^2\varphi + k_2 \cos\varphi \sin\varphi ,$$

$$\alpha_1(\varphi; k) = \Lambda^\top(\varphi)A_1(k)\Lambda(\varphi) = a_1 + \sigma_1 k_1 \cos^2\varphi ,$$

$$\alpha_2(\varphi; k) = \Lambda^\top(\varphi)A_2(k)\Lambda(\varphi) = a_2 + \sigma_2 k_2 \cos\varphi \sin\varphi ,$$

$$\beta_0(\varphi; k) = \bar{\Lambda}^\top(\varphi)A_0(k)\Lambda(\varphi) = b_0 + k_1 \cos\varphi \sin\varphi - k_2 \cos^2\varphi ,$$

$$\beta_1(\varphi; k) = \bar{\Lambda}^\top(\varphi)A_1(k)\Lambda(\varphi) = b_1 + \sigma_1 k_1 \cos\varphi \sin\varphi ,$$

$$\beta_2(\varphi; k) = \bar{\Lambda}^\top(\varphi)A_2(k)\Lambda(\varphi) = b_2 - \sigma_2 k_2 \cos^2\varphi$$

and finally

$$Q(\varphi; k) = \alpha_0(\varphi; k) + \frac{1}{2} Tr A(\varphi; k) - \sum_{i=1}^{2} \alpha_i^2(\varphi; k)$$

$$= a_0 + \frac{1}{2}(b_1^2 + b_2^2 - a_1^2 - a_2^2) + (k_1 - a_1\sigma_1 k_1 - b_2\sigma_2 k_2)\cos^2\varphi$$

$$+(b_1\sigma_1 k_1 + k_2 - a_2\sigma_2 k_2)\cos\varphi \sin\varphi + \frac{1}{2}(-\sigma_1^2 k_1^2 + \sigma_2^2 k_2^2)\cos^2\varphi \cos 2\varphi ,$$

$$k^2(\varphi; k) = \sum_{i=1}^{2} \beta_i^2(\varphi; k) = (b_1 + \sigma_1 k_1 \cos\varphi \sin\varphi)^2 + (b_2 - \sigma_2 k_2 \cos^2\varphi)^2 ,$$

$$b(\varphi; p, k) = -\beta_0(\varphi; k) + (1-p)\sum_{i=1}^{2} \alpha_i(\varphi; k)\beta_i(\varphi; k) = -b_0 - k_1 \cos\varphi \sin\varphi + k_2 \cos^2\varphi$$

$$+(1-p)((a_1 + \sigma_1 k_1 \cos^2\varphi)(b_1 + \sigma_1 k_1 \cos\varphi \sin\varphi) + (a_2 + \sigma_2 k_2 \cos\varphi \sin\varphi)(b_2 - \sigma_2 k_2 \cos^2\varphi)),$$

$$c(\varphi;p,k)=pQ(\varphi;k)+\frac{1}{2}p^2\sum_{i=1}^{2}\alpha_i^2(\varphi;k)$$

$$=pQ(\varphi;k)+\frac{1}{2}p^2((a_1+\sigma_1k_1\cos^2\varphi)^2+(a_2+\sigma_2k_2\cos\varphi\sin\varphi)^2)$$

Write down the equation (2.11)

$$\frac{1}{2}k^2(\varphi;k)f''(\varphi;p,k)+b(\varphi;p,k)f'(\varphi;p,k)+c(\varphi;p,k)f(\varphi;p,k)=g(p,k)f(\varphi;p,k) \quad (8.7)$$

Clearly

$$g(p,0)=p(a_0+\frac{1}{2}(b_1^2+b_2^2-a_1^2-a_2^2))+\frac{1}{2}p^2(a_1^2+a_2^2)\ ,\ f(\varphi;p,0)\equiv 1$$

Denote

$$u_1(\varphi;p)=\frac{\partial f}{\partial k_1}(\varphi;p,0)\ ,\ u_2(\varphi;p)=\frac{\partial f}{\partial k_2}(\varphi;p,0)$$

Differentiating (8.7) with respect to k_1 and setting $k_1=0$, $k_2=0$ leads to the following boundary value problem (remember f is π-periodic)

$$\frac{1}{2}(b_1^2+b_2^2)u_1''+(-b_0+(1-p)(a_1b_1+a_2b_2))u_1'$$

$$+p((1-a_1\sigma_1)\cos^2\varphi+b_1\sigma_1\cos\varphi\sin\varphi)+p^2a_1\sigma_1\cos^2\varphi-\frac{\partial g}{\partial k_1}(p,0)=0 \quad (8.8)$$

$$u_1(0;p)=u_1(\pi;p)\ ,\ u_1'(0;p)=u_1'(\pi;p) \quad (8.9)$$

The corresponding homogeneous conjugate problem to the boundary value problem (8.8)-(8.9) has the solution $z\equiv 1$. Due to Corollary of Lemma 6.1

$$\frac{\partial g}{\partial k_1}(p,0)=\frac{\pi}{2}p(1-a_1\sigma_1+pa_1\sigma_1)$$

It is possible to obtain analogously

$$\frac{\partial g}{\partial k_2}(p,0)=-\frac{\pi}{2}pb_2\sigma_2$$

Taking into account that $g(p,0)$ is equal to $g(p)$ from (2.16) and $\gamma_0(0)$ is equal to γ_0 from (2.17), we obtain

$$\frac{\partial g}{\partial p}(\gamma_0(0),0)=-a_0-\frac{1}{2}(b_1^2+b_2^2-a_1^2-a_2^2)$$

For sufficiently small k_1 and k_2 the formula (8.4) gives (of course, we suppose $\gamma_0<0$)

$$\gamma_0(k)\doteq\gamma_0-\frac{\pi}{a_1^2+a_2^2}(1-a_1\sigma_1+\gamma_0a_1\sigma_1)k_1+\frac{\pi}{a_1^2+a_2^2}b_2\sigma_2k_2 \quad (8.10)$$

Due to (8.10) we can do the first step to increase $\gamma_0(k)$. The next steps are similar but they can be done only numerically. It is interesting to note that increasing of $\gamma_0(k)$ (stability properties are better in a sense) can be accompanied by decreasing of $|\lambda(k)|$ for some values of the coefficients (stability properties are worse in a sense) where the Lyapunov exponent $\lambda(k)$ is equal to

$$\lambda(k)=\frac{\partial g}{\partial p}(0,k)\doteq a_0+\frac{1}{2}(b_1^2+b_2^2-a_1^2-a_2^2)+\frac{\pi}{2}(1-a_1\sigma_1)k_1-\frac{\pi}{2}b_2\sigma_2k_2$$

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References

[1] R.Z.Khasminskii, Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems, Theory Probab. Appl. **12**(1967), pp. 144-147.

[2] R.Z.Khasminskii, Stochastic stability of differential equations, Nauka, Moscow, 1969; English transl., Sijthoff and Noordhoff, Alphen, 1980.

[3] L.Arnold, A formula connecting sample and moment stability of linear stochastic systems, SIAM J.Appl.Math. **44**(1984), pp. 793-802.

[4] L.Arnold, E.Oeljeklaus, and E.Pardoux, Almost sure and moment stability for linear Ito equations, Lyapunov exponents (L.Arnold and V. Wihstutz, eds.), Lecture Notes in Math., vol. 1186, Springer-Verlag, Berlin, 1986, pp. 129-159.

[5] L.Arnold, G.Papanicolaou and V.Wihstutz, Asymptotic analysis of the Lyapunov exponent and rotation of the random oscillator and applications, SIAM J.Appl.Math.,46(1986), pp. 427-450.

[6] P.Baxendale, Moment stability and large deviations for linear stochastic differential equations, Proc. Taniguchi Symp. on Probab. Meth. In Math. Physics, Katata and Kyoto 1985 (N.Ikeda, ed.), Kinokuniya, Tokyo, 1987, pp. 31-54.

[7] D.Talay, Approximation of upper Lyapunov exponents of bilinear stochastic differential equations, SIAM J.Numer.Anal. 28(1991), pp. 1141-1164.

[8] E.I.Auslender and G.N.Milstein, Asymptotic expansion of the Lyapunov index for linear stochastic systems with small noise, J.Appl.Math.Mech., (1982), pp.358-365.

[9] E.Pardoux and V.Wihstutz, Lyapunov exponent and rotation number of two-dimensional linear stochastic systems with small diffusion, SIAM J.Appl.Math. 48(1988), pp. 442-457.

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