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Abstract forced symmetry breaking

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Contents

1 **Introduction** 2

2 **Notation, Assumptions and Set-up** 7

3 **Liapunov-Schmidt Reduction** 8

4 **The Solutions in the Case of Vanishing Symmetry Breaking Parameter** 12

5 **Symmetry Breaking and Locking Cones** 16

6 **Stability from the Reduced Bifurcation Equation** 24

7 **Abstract Forced S^1-Equivariant Equations** 27

A **Appendix** 34

References 36

Abstract

We consider abstract forced symmetry breaking problems of the type

$$F(x,\lambda) = y, \;\; x \approx \mathcal{O}(x_0), \; \lambda \approx \lambda_0, \; y \approx 0.$$

It is supposed that for all λ the maps $F(\,\cdot\,,\lambda)$ are equivariant with respect to representations of a given compact Lie group, that $F(x_0,\lambda_0) = 0$ and, hence, that $F(x,\lambda_0) = 0$ for all elements x of the group orbit $\mathcal{O}(x_0)$ of x_0.

We look for solutions x which bifurcate from the solution family $\mathcal{O}(x_0)$ as λ and y move away from λ_0 and zero, respectively. Especially, we describe the number of different solutions x (for fixed control parameters λ and y), their dynamic stability, their asymptotic behavior for y tending to zero and the structural stability of all these results. Further, generalizations are given to problems of the type $F(x,\lambda) = y(x,\lambda), \;\; x \approx \mathcal{O}(x_0), \; \lambda \approx \lambda_0, \; y(x,\lambda) \approx 0$.

This work is a generalization of results of J. K. Hale, P. Táboas , A. Vanderbauwhede and E. Dancer to such extend that the conclusions are applicable to forced frequency locking problems for rotating and modulated wave solutions of certain $\mathbf{S}^1$-equivariant evolution equations which arise in laser modeling.

1 Introduction

In this paper we consider abstract forced (or "induced") symmetry breaking problems of the type

$$F(x, \lambda) = y, \quad x \approx \mathcal{O}(x_0), \; \lambda \approx \lambda_0, \; y \approx 0. \tag{1.1}$$

In (1.1), F is a smooth mapping such that for all λ the maps $F(\cdot, \lambda)$ are equivariant with respect to representations of a given compact Lie group, that $F(x_0, \lambda_0) = 0$ and, hence, that $F(x, \lambda_0) = 0$ for all elements x of the group orbit $\mathcal{O}(x_0)$ of x_0. We look for solutions to (1.1) which bifurcate from the solution family $\mathcal{O}(x_0)$ as λ and y move away from λ_0 and zero, respectively. Thus, x is the "state parameter", λ is the "internal, symmetry preserving control parameter", and y is the "external, symmetry breaking control parameter".

The aim of this work is to present a simple analytic and geometric strategy for predicting, or engineering, solutions to (1.1) in the case of $\dim \mathcal{O}(x_0) > 0$. The strategy is simple because it is founded on a Liapunov-Schmidt reduction, certain scaling techniques (Hadamard's lemma) and the Implicit Function Theorem, only. For example, we provide a criterion which implies that for a given subspace Λ_* of the space of all internal, symmetry preserving control parameters the following is true: For each nondegenerate (in the sense of Corollary 5.3) external, symmetry breaking control parameter y near zero there exist a $\lambda_* \in \Lambda_*$ near zero and an x near $\mathcal{O}(x_0)$ such that $F(x, \lambda_0 + \lambda_*) = y$. In other words: For each nondegenerate y near zero, it is possible to adjust λ near λ_0, by variing the components in Λ_* only, such that (1.1) gets solvable.

Our results make it possible to determine (under certain assumptions) the number of different solutions x to (1.1) (for fixed control parameters λ and y), their dynamic stability, their asymptotic behavior for y tending to zero and the structural stability of all these results.

In fact, this work is a generalization of results of J. K. HALE and P. TÁBOAS [24, 26, 39], A. VANDERBAUWHEDE [41] and E. DANCER [16] to such extend that these results are applicable to forced frequency locking problems for rotating and modulated wave solutions of certain $\mathbf{S}^1$-equivatiant evolution equations which arise in laser modeling (cf. [32]). For example, in the context of frequency locking of a self-pulsating two section DFB lasers under periodically modulated external signals, the criterion mentioned above yields the following (cf. [32]): For each external signal with small amplitude, with arbitrary modulation frequency near a given one and with arbitrary, nondegenerate modulation profile, it is possible to adjust the laser state, by variing only the two laser currents near given values, such that frequency locking takes place.

The paper is organized as follows.

In Section 2 we introduce some notation and assumptions.

Using an approach of VANDERBAUWHEDE [42] and DANCER [16], in Section 3 we carry out a Liapunov-Schmidt reduction for (1.1) which leads to a smooth bifurcation equation (though we do not suppose the Lie group to act smoothly on the state space). This reduction is "semi-global" in the sense that the control parameters λ and y have to move near points ($\lambda = \lambda_0$ and $y = 0$), but the state parameter x may vary in a neighbourhood of the compact submanifold $\mathcal{O}(x_0)$ in the state space. The main assumptions for this reduction are that the partial derivative $\partial_x F(x_0, \lambda_0)$ is a Fredholm operator and that its kernel is as small as it is possible in the given situation of equivariance (namley $\ker \partial_x F(x_0, \lambda_0)$ is equal to $T_{x_0}\mathcal{O}(x_0)$, the tangential space at $\mathcal{O}(x_0)$ in the point x_0).

In Section 4 we describe the solution behavior of (1.1) in the case of vanishing symmetry breaking control parameter:

$$F(x, \lambda) = 0, \quad x \approx \mathcal{O}(x_0), \ \lambda \approx \lambda_0. \tag{1.2}$$

We show that generically there exists a smooth submanifold $\mathcal{M}$ in the λ-space with $\lambda_0 \in \mathcal{M}$ and tangential space

$$T_{\lambda_0}\mathcal{M} = \{\lambda \in \Lambda : \ \partial_\lambda F(x_0, \lambda_0)\lambda \in \operatorname{im} \partial_x F(x_0, \lambda_0)\}$$

such that (1.2) is solvable iff $\lambda \in \mathcal{M}$. Here we use and generalize results of DANCER [15, 16, 18], who considered the case of $\operatorname{codim}\mathcal{M} = 0$, i.e. the case that (1.2) is solvable for all $\lambda \approx \lambda_0$. We are mainly interested (because of the applications in [34, 32]) in the case that $\operatorname{codim}\mathcal{M} = \dim \mathcal{O}(x_0)$ (that is the largest generically possible codimension of $\mathcal{M}$).

Assuming $\dim \mathcal{O}(x_0) = \dim \Gamma$, in Section 5 we describe solution families of (1.1) which are obtained by a scaling technique. These families are smoothly parametrized by the control parameter (λ, y) belonging to certain open subsets (so-called locking cones) of the (λ, y)-space.

To be more precise, let Λ_2 be a topological complement of $T_{\lambda_0}\mathcal{M}$ in Λ, and let $\hat{\lambda}_2 : T_{\lambda_0}\mathcal{M} \to \Lambda_2$ be a parametrization of $\mathcal{M}$ near λ_0, i.e.

$$\mathcal{M} = \{\lambda_0 + \lambda_1 + \hat{\lambda}_2(\lambda_1) : \ \lambda_1 \in T_{\lambda_0}\mathcal{M}, \ \lambda_1 \approx 0\}. \tag{1.3}$$

Then the scaling, used in Section 5, is

$$\lambda = \lambda_0 + \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \quad y = \epsilon z, \tag{1.4}$$

where $\epsilon \in \mathbb{R}$ and $\lambda_1 \in T_{\lambda_0}\mathcal{M}$ are small, and $\mu \in \Lambda_2$ and $z \in \tilde{X}$ are new scaled control parameters. Each isolated solution $\gamma = \gamma_0$ to the so-called reduced bifurcation equation

$$(I - \tilde{P})[\partial_\lambda F(x_0, \lambda_0)\mu_0 - \tilde{S}(\gamma)^{-1}z_0] = 0 \tag{1.5}$$

generates a family of solutions to (1.1), and the corresponding locking cone is the set of all control parameters (λ, y) of the type (1.4), where ϵ and λ_1 vary near zero, μ near μ_0 and z near z_0. In (1.5), $\tilde{S} : \Gamma \to \mathcal{L}(\tilde{X})$ is one of the Γ-representations mentioned above, and $\tilde{P} \in \mathcal{L}(\tilde{X})$ is a projector onto $\operatorname{im} \partial_x F(x_0, \lambda_0)$ which commutes with $\tilde{S}(\gamma)$ for all γ belonging to the isotropy subgroup of x_0. The reduced bifurcation equation (1.5) is the condition for vanishing of the first order terms of the ϵ-expansion of an equation which is created by inserting (1.4) (with $\mu = \mu_0$ and $z = z_0$) and the ansatz

$$x = S(\gamma \exp(\epsilon a_1 + \epsilon^2 a_2 + \ldots))(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots)$$

into (1.1). Here $S : \Gamma \to \mathcal{L}(X)$ is the other Γ-representation (and the equivariance assumption is $\tilde{S}(\gamma)F(x, \lambda) = F(S(\gamma)x, \lambda)$ for all γ, x and λ).

There exists a remarkable difference between the solution behavior of problem (1.1) and that of problem (1.2): The parameter $\lambda_2 \in \Lambda_2$ is a "state parameter" for (1.2), because (1.2) determines λ_2 to be a function of the "control parameter" λ_1 (which may vary in an open subset of the λ_1-space, cf. (1.3)). But for equation (1.1), λ_2 is a "control parameter" as well as λ_1, because for all $\lambda = (\lambda_1, \lambda_2)$, belonging to the locking cones, (1.1) is solvable.

In Section 6 we present a simple criterion which implies linearized stability (resp. linearized instability) simultaneously for all solutions to (1.1) belonging to the solution family corresponding to a solution $\gamma = \gamma_0$ to (1.5). Essentially, the criterion consists in whether or not all eigenvalues of the linearization of (1.5) with respect to γ in the solution $\gamma = \gamma_0$ have negative real parts. The contents of Section 6 are natural generalisations of results about the so-called principle of reduced stability for bifurcation from isolated solutions (cf. VANDERBAUWHEDE [44] and RECKE [33]).

In Section 7 we assume the Lie group to be $\mathbf{S}^1$. Then the reduced bifurcation equation takes values in a one-dimensional space, and the control parameter λ_2 is one-dimensional. In this case (and under certain generic assumptions) we describe how the solution families may be smoothly continued and what sort of bifurcations occures if the control parameter (λ, y) tends to the boundary of a maximal domain of continuation. Here we use techniques of HALE and TÁBOAS [24, 26] (see also the results about "abstract bifurcation near a closed curve" in [14, Section 11.5] and about "symmetry and bifurcation near families of solutions" in [42, Chapter 8]).

We confine us to forced symmetry breaking problems of type (1.1) by reasons of simplicity, only (and because the applications, we have in mind, are of this type). There exist straightforward generalizations of our results to forced symmetry breaking problems of the more general type

$$F(x, \lambda) = y(x, \lambda), \;\; x \approx \mathcal{O}(x_0), \; \lambda \approx \lambda_0, \; y(x, \lambda) \approx 0.$$

Such generalizations are presented in the Remarks 3.5, 5.10, 6.3 and 7.7.

In [34], which is a direct continuation of the present work, we apply our results on abstract forced symmetry breaking to two problems for parameter depending forced $\mathbf{S}^1$-equivariant ordinary differential equations of the type

$$\dot{\xi}(t) = f(\xi(t), \lambda) - \eta(t). \tag{1.6}$$

In (1.6), we suppose $S(e^{i\gamma})f(\xi, \lambda) = f(S(e^{i\gamma})\xi, \lambda)$ for all γ, ξ and λ, where S is an $\mathbf{S}^1$-representation on the ξ-space.

First, we suppose the unperturbed equation

$$\dot{\xi}(t) = f(\xi(t), \lambda_0) \tag{1.7}$$

to have an orbitally stable rotating wave solution $\xi_0(t) = S(e^{i\alpha_0 t})x_0$, and we describe the frequency locking of this solution to a forcing $\eta(t) = S(e^{i\alpha t})y$ with $\alpha \approx \alpha_0$ and $y \approx 0$. We show that for small forcings (i.e. for small $||y||$) near the rotating wave solution occures a modulated wave solution, which has a modulation frequency near $|\alpha - \alpha_0|$, and the modulation oscillation $\max\{\, ||\xi(t)|| \;:\; t \in \mathbb{R}\,\} - \min\{\, ||\xi(t)|| \;:\; t \in \mathbb{R}\,\}$ of which tends to zero for $||y||$ tending to zero. If the forcing increases then the modulation oscillation increases, too, but the modulation frequency decreases. Moreover, at a certain value of the forcing intensity the modulation frequency vanishes, and the modulated wave solution changes "back" into (generically two) rotating wave solutions, which are close to fixed phase shifts of the "initial" rotating wave solution $\xi_0(t)$ and which have exactly the same frequency as the forcing (saddle node bifurcation of rotating waves). We describe which of them are stable and which are unstable. In this sense, frequency locking of the rotating wave solution of the unperturbed equation (1.7) with a forcing $\eta(t) = S(e^{i\alpha t})y$ of "rotating wave type" occures. If the intensity of the forcing is increased further then a second saddle node bifurcation of the rotating wave solutions (into a modulated wave solution) may occure or not. This depends on whether or not the locking cone is "lop-sided". We describe this bifurcation scenario rigorously and uniformly for all control parameters $\lambda \approx \lambda_0$, $\alpha \approx \alpha_0$ and $y = \epsilon z$ with $\epsilon \in \mathbb{R}$ near zero and $z \in \mathbb{R}^m$ near z_0, where z_0 is a "direction" in $\mathbb{R}^m$ such that the corresponding reduced bifurcation equation has nondegenerate solutions.

And second, we suppose equation (1.7) to have an orbitally stable modulated wave solution $\xi_0(t) = S(e^{i\alpha_0 t})x_0(t)$ with $x_0(t) = x_0(t + \frac{2\pi}{\beta_0})$ for all t, and we describe the quasiperiodic frequency locking of this solution to a forcing $\eta(t) = S(e^{i\alpha t})y(t)$ with $y(t) = y(t + \frac{2\pi}{\beta})$ and $y(t) \approx 0$ for all t, $\alpha \approx \alpha_0$ and $\beta \approx \beta_0$.

The motivation for our investigations comes from problems in semiconductor laser modeling. At present, self-pulsations (i.e. periodic intensity change in the output power with frequencies of tenth of gigahertz, cf., e.g., [31, 37, 9, 47, 46, 8]) and frequency locking of self-pulsations to optically injected modulations (cf. [5, 19, 28, 32]) are topics of intensive experimental and theoretical research. The mathematical models are, as a rule, ordinary differential equations (rate equations for the carrier densities) which are nonlinearly coupled with boundary value problems for dissipative hyperbolic systems of first order partial differential equations ("coupled mode equations" for the complex amplitudes of the electric field). Moreover, the models are equivariant with respect to an $\mathbf{S}^1$-representation on the state space ($e^{i\gamma} \in \mathbf{S}^1$ works trivially on the carrier densities and by multiplication on the complex amplitudes).

By means of the results of the present paper, the forced frequency locking behavior of these models can be described to a great extent by analogy with the description of the forced frequency locking behavior of $\mathbf{S}^1$-equivariant ordinary differential equations (which is presented [34]). The frequencies α and α_0 (resp. β and β_0) are the so-called optical or carrier frequencies (resp. the power frequencies) of the external light signal and the self-pulsation, respectively, and the internal, symmetry preserving control parameter λ describes the internal laser parameter (laser currents, geometric and material parameters, facet reflectivities), for details see [32].

Let us introduce some notation.

All the vector spaces considered in this paper are real.

If X and $\tilde{X}$ are normed vector spaces then $\mathcal{L}(X, \tilde{X})$ is the vector space of all linear bounded operators from X into $\tilde{X}$. Further, we denote $\mathcal{L}(X) := \mathcal{L}(X, \tilde{X})$, and $X^* := \mathcal{L}(X, \mathbb{R})$ is the dual space to X.

For $L \in \mathcal{L}(X, \tilde{X})$ we denote by $\ker L := \{x \in X : \; Lx = 0\}$ and $\operatorname{im} L := \{Lx \in \tilde{X} :$

$x \in X\}$ the kernel and the image of the operator L, respectively.

Partial derivatives will be denoted in a usual manner. For example, if Λ is a further normed vector space and $F : X \times \Lambda \to \tilde{X}$ is a C^1-map then $\partial_x F(x_0, \lambda_0) \in \mathcal{L}(X, \tilde{X})$ denotes the partial derivative of F with respect to $x \in X$ in the point $(x_0, \lambda_0) \in X \times \Lambda$.

Let Γ be a group. A map $S : \Gamma \to \mathcal{L}(X)$ is called a representation of Γ on X if $S(\gamma\delta) = S(\gamma)S(\delta)$ for all γ and δ and if S maps the unit element from Γ onto the identity map in X. For $x \in X$ we denote by $\mathcal{O}(x) := \{S(\gamma)x \in X : \gamma \in \Gamma\}$ and $\Gamma(x) := \{\gamma \in \Gamma : S(\gamma)x = x\}$ the group orbit and the isotropy subgroup of x with respect to the representation S, respectively.

2 Notation, Assumptions and Set-up

Throughout in this paper X and $\tilde{X}$ are fixed Banach spaces, Λ is a normed vector space, $k \geq 2$ is a natural number, $F : X \times \Lambda \to \tilde{X}$ is a C^k-map, and $x_0 \in X$ and $\lambda_0 \in \Lambda$ are points such that

(I) $F(x_0, \lambda_0) = 0$,

(II) $\partial_x F(x_0, \lambda_0)$ is a Fredholm operator from X into $\tilde{X}$.

Further, by Γ we denote a fixed compact Lie group, and $S : \Gamma \to \mathcal{L}(X)$ and $\tilde{S} : \Gamma \to \mathcal{L}(\tilde{X})$ are representations of the group Γ on the spaces X and $\tilde{X}$, respectively, such that

(III) $F(S(\gamma)x, \lambda) = \tilde{S}(\gamma)F(x, \lambda)$ for all $x \in X$, $\lambda \in \Lambda$ and $\gamma \in \Gamma$,

(IV) $\gamma \in \Gamma \longmapsto (S(\gamma)x, \tilde{S}(\gamma)\tilde{x}) \in X \times \tilde{X}$ is continuous for all $x \in X$ and $\tilde{x} \in \tilde{X}$.

Assumptions (I) – (IV) imply that the map $\gamma \in \Gamma \mapsto S(\gamma)x \in X$ is C^k-smooth (cf. [18]). Hence, the group orbit $\mathcal{O}(x_0) := \{S(\gamma)x_0 \in X : \gamma \in \Gamma\}$ is a C^k-submanifold in X, the map $\gamma \in \Gamma \mapsto S(\gamma)x \in \mathcal{O}(x_0)$ is a submersion, and

$$\dim \mathcal{O}(x_0) = \dim \Gamma - \dim \Gamma(x_0) \tag{2.1}$$

(cf. [42]). Here $\Gamma(x_0) := \{\gamma : S(\gamma)x_0 = x_0\}$ is the isotropy subgroup of the point x_0. Moreover, the assumptions (I) – (IV) imply that the tangential space $T_{x_0}\mathcal{O}(x_0)$ at $\mathcal{O}(x_0)$ in x_0 is a subspace of the kernel $\ker \partial_x F(x_0, \lambda_0)$. We assume that this kernel is as small as it is possible in our situation, i.e.

(V) $\ker \partial_x F(x_0, \lambda_0) = T_{x_0}\mathcal{O}(x_0)$.

Finally, from (I) and (III) follows that the subspaces $\ker \partial_x F(x_0, \lambda_0)$ and $\operatorname{im} \partial_x F(x_0, \lambda_0)$ are invariant with respect to $S(\gamma)$ and $\tilde{S}(\gamma)$ with $\gamma \in \Gamma(x_0)$, respectively. Hence (cf. [42]), (II) and (IV) imply that there exist projectors $P \in \mathcal{L}(X)$ and $\tilde{P} \in \mathcal{L}(\tilde{X})$ such that

$$\ker P = \ker \partial_x F(x_0, \lambda_0),\ \operatorname{im} \tilde{P} = \operatorname{im} \partial_x F(x_0, \lambda_0) \tag{2.2}$$

and

$$S(\gamma)P = PS(\gamma),\ \tilde{S}(\gamma)\tilde{P} = \tilde{P}\tilde{S}(\gamma) \text{ for all } \gamma \in \Gamma(x_0). \tag{2.3}$$

In most of the applications there is a natural unique choise of the projectors P and $\tilde{P}$ because it holds

$$\begin{gathered} X \text{ is continuously embedded into } \tilde{X}, \\ \tilde{X} = \ker \partial_x F(x_0, \lambda_0) \oplus \operatorname{im} \partial_x F(x_0, \lambda_0), \\ S(\gamma)x = \tilde{S}(\gamma)x \text{ for all } x \in X. \end{gathered} \tag{2.4}$$

In that case the projectors P and $\tilde{P}$ may be (uniquely) chosen such that, in addition to (2.2) and (2.3), we have

$$\operatorname{im} P = X \cap \operatorname{im} \partial_x F(x_0, \lambda_0),\quad \ker \tilde{P} = \ker \partial_x F(x_0, \lambda_0) \tag{2.5}$$

and, hence, $Px = \tilde{P}x$ for all $x \in X$.

Finally, throughout in this paper Y is a normed vector space such that Y is continuously embedded into $\tilde{X}$, that $\tilde{S}(\gamma)y \in Y$ for all $y \in Y$ and that

(VI) $(\gamma, y) \in \Gamma \times Y \longmapsto \tilde{S}(\gamma)y \in \tilde{X}$ is C^k-smooth.

3 Liapunov-Schmidt Reduction

The following proposition is due to VANDERBAUWHEDE (cf. [41, 42, 43]). It describes a parametrization of a tubular neighbourhood of the group orbit $\mathcal{O}(x_0)$ which is invariant with respect to the action of Γ on X:

Proposition 3.1 *Suppose* (I) – (IV). *Then there exist neighbourhoods $U \subseteq \operatorname{im} P$ of zero and $V \subseteq X$ of $\mathcal{O}(x_0)$ such that the map*

$$(\gamma, u) \in \Gamma \times U \longmapsto S(\gamma)(x_0 + u) \in V \tag{3.1}$$

is surjective. Moreover, for $(\gamma_j, u_j) \in \Gamma \times U$ $(j = 1, 2)$ we have $S(\gamma_1)(x_0+u_1) = S(\gamma_2)(x_0+u_2)$ if and only if $S(\gamma_1)x_0 = S(\gamma_2)x_0$.

Remark 3.2 Later on we will formulate results which are valid in certain neighbourhoods of zero in $\operatorname{im} P$, of $\mathcal{O}(x_0)$ in X etc. As in Proposition 3.1, these neighbourhoods will be denoted by U, V etc., though these "new" neighbourhoods are not the same as in Proposition 3.1 (but, may be, smaller one's).

In what follows in this work we will consider the following abstract forced symmetry breaking problem

$$F(x,\lambda) = y, \quad x \approx \mathcal{O}(x_0), \ \lambda \approx \lambda_0, \ y \approx 0 \tag{OE}$$

(here "OE" stands for "original equation"). This problem, written in the "new coordinates" (3.1), is

$$F(S(\gamma)(x_0 + u), \lambda) = y, \tag{3.2}$$

or, equivalently (cf. (III)),

$$F(x_0 + u, \lambda) = \tilde{S}(\gamma)^{-1} y. \tag{3.3}$$

The following lemma proceeds with a Liapunov-Schmidt reduction for equation (3.3) with $u \in \operatorname{im} P$ near zero, $\lambda \in \Lambda$ near λ_0, $y \in Y$ near zero and arbitrary $\gamma \in \Gamma$. It is similar to [42, Lemma 8.2.10].

Lemma 3.3 *Suppose* (I) – (VI). *Then there exist neighbourhoods* $W \subseteq \Lambda \times Y$ *of* $(\lambda_0, 0)$ *and* $U \subseteq \operatorname{im} P$ *of zero and a* C^k*-map* $\hat{u} : \Gamma \times W \to \operatorname{im} P$ *such that:*

(i) *It holds* $\tilde{P}[F(x_0 + u, \lambda) - \tilde{S}^{-1}(\gamma) y] = 0$, $u \in U$, $(\lambda, y) \in W$ *if and only if* $u = \hat{u}(\gamma, \lambda, y)$.

(ii) *It holds* $\hat{u}(\gamma, \lambda_0, 0) = 0$ *for all* $\gamma \in \Gamma$.

(iii) *It holds* $\hat{u}(\delta\gamma, \lambda, y) = \hat{u}(\gamma, \lambda, \tilde{S}(\delta)^{-1} y)$ *for all* $\gamma, \delta \in \Gamma$ *and* $(\lambda, y) \in W$.

(iv) *It holds* $\hat{u}(\gamma\delta, \lambda, y) = S(\delta)^{-1} \hat{u}(\gamma, \lambda, y)$ *for all* $\gamma \in \Gamma$, $\delta \in \Gamma(x_0)$ *and* $(\lambda, y) \in W$.

Proof The partial derivative of $\tilde{P}[F(x_0 + u, \lambda) - \tilde{S}(\gamma)^{-1} y]$ with respect to u in $u = 0$, $\lambda = \lambda_0$, $y = 0$ (and in an arbitrary γ) equals to the restriction of $\tilde{P}\partial_x F(x_0, \lambda_0) = \partial_x F(x_0, \lambda_0)$ on $\operatorname{im} P$ (cf. (2.2)). But the assumption (II) yields that

$$\partial_x F(x_0, \lambda_0) \text{ is an isomorphism from } \operatorname{im} P \text{ onto } \operatorname{im} \tilde{P}. \tag{3.4}$$

Therefore, the Implicit Function Theorem (together with the compactness of Γ) implies assertions (i) and (ii) of the lemma.

Further, from (III) and (2.3) follows

$$\tilde{P}[F(x_0 + u, \lambda) - \tilde{S}(\gamma\delta)^{-1} y] = \tilde{S}(\delta)^{-1} \tilde{P}[F(x_0 + S(\delta) u, \lambda) - \tilde{S}(\gamma)^{-1} y]$$

for all $\gamma \in \Gamma$, $\delta \in \Gamma(x_0)$ and $(\lambda, y) \in W$. Therefore, assertion (iv) follows from the uniqueness assertion of the Implicit Function Theorem.

A similar argument proves (iii).

■

Let us define a map $G : \Gamma \times W \to \ker P$ by

$$G(\gamma, \lambda, y) := (I - \tilde{P})[F(x_0 + \hat{u}(\gamma, \lambda, y), \lambda) - \tilde{S}(\gamma)^{-1} y]. \tag{3.5}$$

In (3.5), I is the identity in the space $\tilde{X}$. Because of (III), (2.3) and Lemma 3.3 we have for all $\gamma \in \Gamma$ and $(\lambda, y) \in W$

$$G(\gamma, \lambda_0, 0) = 0, \tag{3.6}$$

$$G(\delta\gamma, \lambda, y) = G(\gamma, \lambda, \tilde{S}(\delta) y) \text{ for all } \delta \in \Gamma, \tag{3.7}$$

$$G(\gamma\delta, \lambda, y) = \tilde{S}^{-1}(\delta) G(\gamma, \lambda, y) \text{ for all } \delta \in \Gamma(x_0). \tag{3.8}$$

The correspondence between the solutions of (OE) and of the Liapunov–Schmidt bifurcation equation

$$G(\gamma, \lambda, y) = 0, \quad \gamma \in \Gamma, \ \lambda \approx \lambda_0, \ y \approx 0 \tag{BE}$$

(here "BE" stands for "bifurcation equation") may be described in the following way:

Let (x, λ, y) be a solution to (OE). Then there existes a $\gamma_* \in \Gamma$ such that $x = S(\gamma_*)(x_0 + \hat{u}(\gamma_*, \lambda, y))$ and that (γ, λ, y) is a solution to (BE) iff $\gamma = \gamma_* \delta$ with $\delta \in \Gamma(x_0)$.

And conversely, let (γ, λ, y) be a solution to (BE). Then, for all $\delta \in \Gamma(x_0)$, $(\gamma\delta, \lambda, y)$ is a solution to (BE), too, $x := S(\gamma\delta)(x_0 + \hat{u}(\gamma\delta, \lambda, y))$ does not depend on δ, and (x, λ, y) is a solution to (OE).

Remark 3.4 We do not assume the representations S and $\tilde{S}$ to be smooth (because in many applications they are not smooth). Therefore, in our setting the parametrization (3.1) and the equation (3.2) are not smooth, in general. But we overcome this technical difficulty easily by transforming equation (3.2) into equation (3.3), which is C^k–smooth already (because of assumption (VI)).

If the symmetry breaking parameter does not appear as a right hand side in the equation (OE), such an approach is not possible, in general. Then one can use a result of DANCER [16], who showed that there exists a smooth vector subbundle of the trivial vector bundle $\mathcal{O}(x_0) \times X$ which is invariant with respect to the representation S and which has the property that $(x, v) \mapsto x + v$ (with $x \in \mathcal{O}(x_0)$ and v belonging to the fiber over x) is a smooth parametrization of a tubular neighbourhood of $\mathcal{O}(x_0)$ in X. Another way to deal with such more general forced symmetry breaking problems is described in the following Remark 3.5.

Remark 3.5 There exist staightforward generalizations of Lemma 3.3 (and, hence, of the results of the Sections 4–7 of this paper, which follow from Lemma 3.3) to original equations of the type

$$F(x,\lambda,y)=0, \quad x\approx\mathcal{O}(x_0),\ \lambda\approx\lambda_0,\ y\approx 0 \tag{3.9}$$

with $F(x_0,\lambda_0,0)=0$ and

$$F(S(\gamma)x,\lambda,T(\gamma)y)=\tilde{S}(\gamma)F(x,\lambda,y) \text{ for all } x\in X,\ \lambda\in\Lambda \text{ and } \gamma\in\Gamma, \tag{3.10}$$

where $T:\Gamma\to\mathcal{L}(Y)$ is a Γ–representation on the space Y of the symmerty breaking parameters such that the map $(x,\lambda,y,\gamma)\in X\times\Lambda\times Y\times\Gamma\mapsto F(x,\lambda,T(\gamma)y)\in\tilde{X}$ is C^k–smooth.

Especially, forced symmetry breaking problems

$$F(x,\lambda)=y(x,\lambda), \quad x\approx\mathcal{O}(x_0),\ \lambda\approx\lambda_0,\ y\approx 0$$

are of this type, where the symmetry breaking parameter space Y is a suitable subspace of the space of all C^k–maps $y:X\times\Lambda\longrightarrow\tilde{X}$ such that the map

$$(x,\lambda,y,\gamma)\in X\times\Lambda\times Y\times\Gamma\longmapsto\tilde{S}(\gamma)y(S(\gamma)^{-1}x,\lambda)\in\tilde{X} \tag{3.11}$$

is C^k–smooth. In this case the Γ–representation T on Y has to be defined by

$$[T(y)](x,\lambda):=\tilde{S}(\gamma)y(S(\gamma)^{-1}x,\lambda). \tag{3.12}$$

Let us indicate a typical example for the situation described above.

Let $\tilde{X}$ be the space of all continuous 2π-periodic maps $\tilde{x}:\mathbb{R}\to\mathbb{R}^m$, and let X be the space of all C^1–smooth elements of $\tilde{X}$ (with the usual supremum norms). Let Γ be $\mathbf{S}^1:=\{e^{i\varphi}\in\mathbb{C}:\varphi\in\mathbb{R}\}$,

$$[\tilde{S}(e^{i\varphi})\tilde{x}](t):=\tilde{x}(t+\varphi),$$

and let $S(e^{i\varphi})$ be the restriction of $\tilde{S}(e^{i\varphi})$ on X. Let $\Lambda:=\mathbb{R}^n$ and

$$[F(x,\lambda)](t):=\dot{x}(t)+f(x(t),\lambda)$$

with a C^k–smooth map $f:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m$. Finally, let Y be the space of all superposition operators $y:X\to\tilde{X}$ of the type

$$[y(x)](t):=\tilde{y}(t,x(t))$$

with a C^k–smooth generating map $\tilde{y} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ such that $\tilde{y}(\,\cdot\,, x)$ is 2π-periodic for all $x \in \mathbb{R}^m$ and that $\tilde{y}$ and all its derivatives up to the k-th one $\tilde{y}^{(k)}$ are bounded. In Y we use the norm $\sup\{\|\tilde{y}^{(l)}(t,x)\| : t \in \mathbb{R},\ x \in \mathbb{R}^m,\ l = 0, 1, ..., k\}$, where $\|\cdot\|$ is a norm in $\mathbb{R}^m$. From (3.12) follows

$$[(T(e^{i\varphi})y)(x)](t) = \tilde{y}(t + \varphi, x).$$

Hence, the map (3.11) is C^k–smooth, because the so-called evaluation map $(t, x, y) \in \mathbb{R} \times \mathbb{R}^m \times Y \mapsto \tilde{y}(t, x) \in \mathbb{R}$ is C^k–smooth (cf., e.g., [1, Proposition 2.4.17]).

Analogously, forced symmetry breaking problems for symmetric elliptic boundary value problems on symmetric domains may be formulated in this way. Here one has to use known smoothness properties of superposition operators between Sobolev or Hölder spaces (cf., e.g., [40, 3]).

4 The Solutions in the Case of Vanishing Symmetry Breaking Parameter

In this section we describe the solution behavior of the bifurcation equation (BE) and, hence, of the original equation (OE) in the case of vanishing symmetry breaking parameter y:

$$F(x, \lambda) = 0, \quad x \approx \mathcal{O}(x_0),\ \lambda \approx \lambda_0. \tag{4.1}$$

Because of (3.7), $G(\gamma, \lambda, 0)$ is independent of γ. Hence, it is correct to define

$$G_0(\lambda) := G(\gamma, \lambda, 0). \tag{4.2}$$

Here G_0 is a C^k-map which is defined for all $\lambda \in \Lambda$ with $(\lambda, 0) \in W$ ($W \subseteq \Lambda \times Y$ is the neighbourhood of $(\lambda_0, 0)$ from Lemma 3.3) and which takes values in $\ker \tilde{P}$ ($\tilde{P} \in \mathcal{L}(\tilde{X})$ is the projector introduced in (2.2)), and (3.5) and (4.2) imply

$$G_0(\lambda) = (I - \tilde{P})F(x_0 + \hat{u}(\gamma, \lambda, 0), \lambda). \tag{4.3}$$

Let

$$X_0 := \{x \in X : S(\gamma)x = x \text{ for all } \gamma \in \Gamma(x_0)\},$$
$$\tilde{X}_0 := \{\tilde{x} \in \tilde{X} : \tilde{S}(\gamma)\tilde{x} = \tilde{x} \text{ for all } \gamma \in \Gamma(x_0)\}$$

be the isotropy subspaces corresponding to the isotropy subgroup $\Gamma(x_0)$, respectively. Then, because of (III), $F(\cdot, \lambda)$ maps X_0 into $\tilde{X}_0$ for all λ. Hence,

$$\partial_x F(x_0, \lambda_0)X_0 \subseteq \tilde{X}_0, \quad \partial_\lambda F(x_0, \lambda_0)\Lambda \subseteq \tilde{X}_0. \tag{4.4}$$

Moreover, (3.4) yields that $\partial_x F(x_0, \lambda_0)$ is an isomorphism from $X_0 \cap \text{im} P$ onto $\tilde{X}_0 \cap \text{im} \tilde{P}$, and we get

$$\partial_x F(x_0, \lambda_0) X_0 = \tilde{X}_0 \cap \partial_x F(x_0, \lambda_0) X. \tag{4.5}$$

Thus, $\partial_x F(x_0, \lambda_0) X_0$ is a closed subspace of finite codimension in $\tilde{X}_0$ (cf. (II)), and we denote this codimension by $\text{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0) X_0$. From (4.5) follows

$$\text{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0) X_0 = \dim(\tilde{X}_0 \cap \ker \tilde{P}). \tag{4.6}$$

The following theorem describes the solution behavior of equation (4.1) under the assumption that the subspaces $\partial_x F(x_0, \lambda_0) X_0$ and $\partial_\lambda F(x_0, \lambda_0) \Lambda$ are transversal in $\tilde{X}_0$:

Theorem 4.1 *Suppose* (I) – (V), *and let* Λ_2 *be a closed subspace in* Λ *such that*

(VII) $\dim \Lambda_2 = \text{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0) X_0$,

(VIII) $\tilde{X}_0 = \partial_x F(x_0, \lambda_0) X_0 \oplus \partial_\lambda F(x_0, \lambda_0) \Lambda_2$.

Further, let Λ_1 *be a complement of* Λ_2 *in* Λ, *and let* $\lambda_0 = \lambda_{01} + \lambda_{02}$ *with* $\lambda_{0j} \in \Lambda_j$ $(j = 1, 2)$.

Then there exist neighbourhoods $V \subseteq X$ *of* $\mathcal{O}(x_0)$ *and* $W_j \subseteq \Lambda_j$ *of* λ_{0j} *and* C^k*–maps* $\hat{x}_0 : W_1 \to X_0$ *and* $\hat{\lambda}_2 : W_1 \to \Lambda_2$ *with* $\hat{x}_0(\lambda_{01}) = x_0$ *and* $\hat{\lambda}_2(\lambda_{01}) = \lambda_{02}$ *such that the following is true: It holds* $F(x, \lambda_1 + \lambda_2) = 0$ *with* $x \in V$ *and* $\lambda_j \in \Lambda_j$ *if and only if* $\lambda_2 = \hat{\lambda}_2(\lambda_1)$ *and* $x = S(\gamma)\hat{x}_0(\lambda_1)$ *for some* $\gamma \in \Gamma$.

Proof Because of (3.5), (3.8) and (4.2), we have $G_0(\lambda) \in \tilde{X}_0 \cap \ker \tilde{P}$ for all λ. Let $G_0'(\lambda_0) \in \mathcal{L}(\Lambda; \tilde{X} \cap \ker \tilde{P})$ be the derivative of the map G_0 in the point λ_0. Then (2.2) and (4.3) yield

$$G_0'(\lambda_0) = (I - \tilde{P}) \partial_\lambda F(x_0, \lambda_0). \tag{4.7}$$

Let us show that the restriction of $G_0'(\lambda_0)$ on Λ_2 is injective. Thus, let $G_0'(\lambda_0)\lambda_2 = 0$ with $\lambda_2 \in \Lambda_2$. Then (4.7) yields that $\partial_\lambda F(x_0, \lambda_0)\lambda_2 \in \text{im}\, \tilde{P}$. Hence, (4.4) and (4.5) imply that $\partial_\lambda F(x_0, \lambda_0)\lambda_2 \in \partial_x F(x_0, \lambda_0) X_0$, and from (VIII) follows that $\partial_\lambda F(x_0, \lambda_0)\lambda_2 = 0$. But assumptions (VII) and (VIII) provide, moreover, that $\partial_\lambda F(x_0, \lambda_0)$ is injective on Λ_2. Therefore $\lambda_2 = 0$.

On the other hand, from (4.4), (4.7) and (VII) follows that $G_0'(\lambda_0)\Lambda_2 = \tilde{X}_0 \cap \ker \tilde{P}$. Hence, the restriction of $G_0'(\lambda_0)$ on Λ_2 is an isomorphism from Λ_2 onto $\tilde{X}_0 \cap \ker \tilde{P}$, and the Implicit Function Theorem solves equation $G_0(\lambda) = 0$ for $\lambda \approx \lambda_0$ in form of $\lambda_2 = \hat{\lambda}_2(\lambda_1)$. Thus, the theorem is proved with

$$\hat{x}_0(\lambda_1) := x_0 + \hat{u}(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1), 0). \tag{4.8}$$

Remark that, because of Lemma 3.3(iii) and (iv), the right hand side of (4.8) belongs to X_0 and does not depend on γ.

■

Corollary 4.2 *Suppose the assumptions of Theorem* 4.1 *to be satisfied.*

Then $\Lambda_1 := \{\lambda \in \Lambda : \partial_\lambda F(x_0, \lambda_0)\lambda \in \text{im}\, \partial_\lambda F(x_0, \lambda_0)\}$ *is a closed complement of* Λ_2 *in* Λ. *Moreover, this choise of* Λ_1 *implies that the derivative of the map* $\hat{\lambda}_2$ *in* λ_{01} *vanishes.*

Proof In the proof of Theorem 4.1 we showed that for a parameter $\lambda_2 \in \Lambda_2$ the condition $\partial_\lambda F(x_0, \lambda_0)\lambda_2 \in \text{im}\, \tilde{P}$ implies that $\lambda_2 = 0$. But $\text{im}\, \tilde{P} = \text{im}\, \partial_x F(x_0, \lambda_0)$ (cf. (2.2)). Hence, $\Lambda_1 \cap \Lambda_2 = \{0\}$.

On the other hand, we have

$$\begin{aligned}
\text{codim}\, \Lambda_1 &= \text{codim ker}\, (I - \tilde{P})\, \partial_\lambda F(x_0, \lambda_0) = \dim \text{im}\, (I - \tilde{P})\, \partial_\lambda F(x_0, \lambda_0) = \\
&= \dim\, (I - \tilde{P})\, \partial_\lambda F(x_0, \lambda_0)\Lambda_2 \leq \\
&\leq \dim \partial_\lambda F(x_0, \lambda_0)\Lambda_2 = \text{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0) X_0 = \dim \Lambda_2.
\end{aligned}$$

Here we used that $(I - \tilde{P})\, \partial_\lambda F(x_0, \lambda_0)$ is an isomorphism from Λ_2 onto $\tilde{X}_0 \cap \ker \tilde{P}$ (see the proof of Theorem 4.1) and, hence, that $(I - \tilde{P})\, \partial_\lambda F(x_0, \lambda_0)\Lambda = (I - \tilde{P})\, \partial_\lambda F(x_0, \lambda_0)\Lambda_2$ (cf. (4.4)). So we get $\Lambda = \Lambda_1 \oplus \Lambda_2$.

Finally, (4.7) yields $\Lambda_1 = \ker G_0'(\lambda_0)$. Moreover, if we differentiate in $\lambda_1 = \lambda_{01}$ the identity $(I - \tilde{P})F(\hat{x}_0(\lambda_1), \lambda_1 + \hat{\lambda}_2(\lambda_1)) = 0$, we get $\hat{\lambda}_2'(\lambda_{01})\Lambda_1 \subseteq \ker G_0'(\lambda_0)$. Hence, $\hat{\lambda}_2'(\lambda_{01}) = 0$. ■

Using a more geometrical language, Theorem 4.1 can be formulated as follows:

Suppose (I) – (V), (VII) and (VIII). Then there exist a C^k–submanifold $\mathcal{M}$ in Λ (namley $\mathcal{M} := \{\lambda_1 + \hat{\lambda}_2(\lambda_1) : \lambda_1 \approx \lambda_{01}\}$, here Λ_1 and Λ_2 are closed subspaces of Λ which satisfy the assumptions of Theorem 4.1, $\hat{\lambda}_2$ is the corresponding map, given by Theorem 4.1, and, obviously, $\mathcal{M}$ does not depend on the choise of Λ_1 and Λ_2) with $\lambda_0 \in \mathcal{M}$ and

$$T_{\lambda_0}\mathcal{M} = \{\lambda \in \Lambda : \partial_\lambda F(x_0, \lambda_0)\lambda \in \text{im}\, \partial_\lambda F(x_0, \lambda_0)\} \tag{4.9}$$

$$\text{codim}\mathcal{M} = \text{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0) X_0 \tag{4.10}$$

and a map $\mathcal{F}$ from $\mathcal{M}$ into the orbit space X/Γ such that (4.1) holds iff $\lambda \in \mathcal{M}$ and $\mathcal{O}(x_0) = \mathcal{F}(\lambda)$. All the orbits $\mathcal{F}(\lambda)$ are C^k–diffeomorphic to $\mathcal{O}(x_0)$.

The following lemma states, under the assumption (2.4), a necessary and sufficient condition for a subspace Λ_2 of Λ to satisfy (VII) and a sufficient condition for (VIII):

Lemma 4.3 *Suppose* (I) – (V) *and* (2.4)*, and let* Λ_2 *be a closed subspace of* Λ*. Then the following is true:*

(i) *Condition* (VII) *is satisfied if and only if*

$$\dim \Lambda_2 = \dim[X_0 \cap T_{x_0}\mathcal{O}(x_0)] \tag{4.11}$$

(ii) *Condition* (VIII) *is satisfied if*

$$\partial_\lambda F(x_0, \lambda_0)\Lambda_2 = X_0 \cap T_{x_0}\mathcal{O}(x_0). \tag{4.12}$$

Proof (i) Because of (2.4) we have $\tilde{X}_0 = [X_0 \cap \ker \partial_x F(x_0, \lambda_0)] \oplus [\tilde{X}_0 \cap \text{im } \partial_x F(x_0, \lambda_0)]$. Hence, (V) and (4.5) yield

$$\tilde{X}_0 = [X_0 \cap T_{x_0}\mathcal{O}(x_0)] \oplus \partial_x F(x_0, \lambda_0) X_0. \tag{4.13}$$

Therefore, (4.11) is equivalent to (VII).

(ii) Obviously, (4.12) and (4.13) imply (VIII).

■

Let us mention two in a certain sense "extremal" situations described by Theorem 4.1.

In the first situation the codimension of the submanifold $\mathcal{M}$ of all control parameters $\lambda \in \Lambda$ near λ_0 such that (4.1) is solvable, is as large as it is possible under the assumptions of Theorem 4.1. Because of (2.1), (II), (4.6), (VII) and (4.10) this is the case if

$$\text{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0) X_0 = \dim\Gamma - \dim\Gamma(x_0). \tag{4.14}$$

For example, if Γ_0 consists of the unit element only (and, hence, $X_0 = X$ and $\tilde{X}_0 = \tilde{X}$) then (4.14) is satisfied. If, in addition to the assumptions of Theorem 4.1, condition (2.4) holds, then (4.14) is equivalent to $T_{x_0}\mathcal{O}(x_0) \subseteq X_0$. This condition is fulfilled, for example, if Γ is Abelian.

In the second situation the codimension of $\mathcal{M}$ is as small as it is possible:

$$\text{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0) X_0 = 0. \tag{4.15}$$

In that case (VII) and (VIII) are satisfied with $\Lambda_2 = \{0\}$, and Theorem 4.1 states that for all $\lambda \approx \lambda_0$ there exists exactly one orbit of solutions to (4.1). This is the so-called G-Invariant Implicit Function Theorem of Dancer [15, 16, 18]. If, moreover, (2.4) is fulfilled, then (4.15) is equivalent to $X_0 \cap T_{x_0}\mathcal{O}(x_0) = \{0\}$, the so-called $\mathcal{P}$-property of

DANCER. If it is violated then for generic $\lambda \approx \lambda_0$ no solutions to (4.1) will exist (cf. [17, 18, 21]).

On the other hand, if the assumptions (I) – (V) are satisfied with Hilbert spaces $X = \tilde{X}$, if the representations $S = \tilde{S}$ are unitary and if $F(\cdot, \lambda)$ is a gradient map (for each λ), then, even if the property $\mathcal{P}$ is not fulfilled, we have $G(\gamma, \lambda, 0) = 0$ for all $\gamma \in \Gamma$ and $\lambda \approx \lambda_0$, and, hence, for all $\lambda \approx \lambda_0$ there exists exactly one orbit of solutions $x \approx \mathcal{O}(x_0)$ to (4.1) (cf. [16]).

Remark 4.4 It is easy to see that all the assumptions (I)–(VIII) (and (I.9) and (I.10) from the subsequent Sections 5 and 6, too) remain to be satisfied if one replaces x_0 and λ_0 by $S(\gamma)\hat{x}_0(\lambda_1)$ and $\lambda_1 + \hat{\lambda}_2(\lambda_1)$, respectively, where $\gamma \in \Gamma$ and $\lambda_1 \in W_1$ are arbitrary. Hence, all the results of this paper remain to be valied under such a "change of the starting solution". For example, we have

$$\dim \ker \partial_x F(S(\gamma)\hat{x}_0(\lambda_1), \lambda_1 + \hat{\lambda}_2(\lambda_1)) = \dim \Gamma - \dim \Gamma_0 \tag{4.16}$$

for all $\gamma \in \Gamma$ and $\lambda_1 \in W_1$. Moreover, we will show that our bifurcation results (for example the existence of locking cones, cf. Theorem 5.2) hold not only for each (fixed) such "starting solution", but in a certain sense uniformely with respect to them.

5 Symmetry Breaking and Locking Cones

In this section we suppose the assumptions (I)–(VIII) to be satisfied.

We introduce in the original problem

$$F(x, \lambda) = y, \quad x \approx \mathcal{O}(x_0), \; \lambda \approx \lambda_0, \; y \approx 0 \tag{OE}$$

and in the bifurcation equation

$$G(\gamma, \lambda, y) = 0, \quad \gamma \in \Gamma, \; \lambda \approx \lambda_0, \; y \approx 0 \tag{BE}$$

new control parameters $\epsilon \in \mathbb{R}$, $\lambda_1 \in \Lambda_1$, $\mu \in \Lambda_2$ and $z \in Y$ by scaling the old control parameters $\lambda \in \Lambda$ and $y \in Y$ in the following way:

$$\begin{aligned} &\lambda = \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \quad y = \epsilon z \\ &\epsilon \approx 0, \; \lambda_1 \approx \lambda_{01}, \; (\mu, z) \in \mathcal{S} := \{(\lambda_2, y) \in \Lambda_2 \times Y : \|\lambda_2\|^2 + \|y\|^2 = 1\}. \end{aligned} \tag{5.1}$$

In (5.1), the symbol $\|\cdot\|$ is used for the norms in Λ and Y, respectively, and $\hat{\lambda}_2$ is the map given by Theorem 4.1. The parameters $(\mu, z) \in \mathcal{S}$ are "directions" in the space $\Lambda_2 \times Y$.

Because of Theorem 4.1, $G(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z)$ vanishes for $\epsilon = 0$. Therefore

$$G(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z) = \epsilon H(\gamma, \epsilon, \lambda_1, \mu, z) \tag{5.2}$$

with

$$\begin{aligned} &H(\gamma, \epsilon, \lambda_1, \mu, z) := \\ &:= \int_0^1 [\partial_\lambda G(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1) + t\epsilon\mu, t\epsilon z)\mu + \partial_y G(\gamma, \lambda_1 + \hat{\lambda}_2(\lambda_1) + t\epsilon\mu, t\epsilon z)z]dt. \end{aligned} \tag{5.3}$$

Especially, for $\epsilon = 0$ we have (cf. (3.5))

$$\begin{aligned} &H_0(\gamma, \mu, z) := \\ &:= H(\gamma, 0, \lambda_{01}, \mu, z) = (I - \tilde{P})[\partial_\lambda F(x_0, \lambda_0)\mu - \tilde{S}(\gamma)^{-1} z]. \end{aligned} \tag{5.4}$$

The solutions with $\epsilon \neq 0$ to the problem

$$H(\gamma, \epsilon, \lambda_1, \mu, z) = 0, \ \ \gamma \in \Gamma, \ \epsilon \approx 0, \ \lambda_1 \approx \lambda_{01}, \ (\mu, z) \in \mathcal{S} \tag{SE}$$

(here "SE" stands for "scaled bifurcation equation") correspond, via (5.1), to solutions of (BE) and, hence, to solutions of (OE).

The aim of this section is to look for solutions $\gamma = \gamma_0, \ \epsilon = 0, \ \lambda_1 = \lambda_{01}, \ \mu = \mu_0$ and $z = z_0$ of (SE), i.e. of the problem (cf. (5.4))

$$H_0(\gamma_0, \mu_0, z_0) = 0, \ \ \gamma_0 \in \Gamma, \ (\mu_0, z_0) \in \mathcal{S} \tag{RE}$$

(here "RE" stands for "reduced bifurcation equation"), such that in these solutions the Implicit Function Theorem works with respect to γ. Such solutions to (RE) produce families of solution to (SE) with $\gamma \approx \gamma_0, \ \epsilon \approx 0, \ \lambda_1 \approx \lambda_{01}, \ \mu \approx \mu_0$ and $z \approx z_0$ and, hence, families of solutions to (OE) with control parameters $(\lambda, y) \in \Lambda \times Y$ defined by (5.1) with $\epsilon \approx 0, \ \lambda_1 \approx \lambda_{01}, \ \mu \approx \mu_0$ and $z \approx z_0$. In order to describe the sets of such control parameters (λ, y), we introduce for $\epsilon_0 > 0, \ (\mu_0, z_0) \in \mathcal{S}$ and for neighbourhoods $W \subseteq \Lambda_1 \times \mathcal{S}$ of $(\lambda_{01}, \mu_0, z_0)$ the following notation:

$$\begin{aligned} &K(\epsilon_0, \mu_0, z_0, W) := \\ &:= \{(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z) \in \Lambda \times Y : 0 < |\epsilon| < \epsilon_0, (\lambda_1, \mu, z) \in W\}. \end{aligned} \tag{5.5}$$

Because of the applications we have in mind (cf. Sections 9 and 10 of this paper) we call the sets (5.5) locking cones.

Let (γ_0, μ_0, z_0) be a solution to (RE). The Implicit Function Theorem works in this solution in order to solve (SE) with respect to γ iff the operator

$$\partial_\gamma H_0(\gamma_0, \mu_0, z_0) = -(I - \tilde{P})\frac{d}{d\gamma}\Big[\tilde{S}(\gamma)^{-1} z_0\Big]_{\gamma=\gamma_0} \tag{5.6}$$

(cf. (5.4)) is an isomorphism from the tangential space $T_{\gamma_0}\Gamma$ onto $\ker \tilde{P}$. Obviously, for that condition to be fulfilled it is necessary that the following condition

(IX) $\dim \Gamma(x_0) = 0$,

is satisfied, because it holds $H_0(\gamma_0\delta, \mu_0, z_0) = \tilde{S}^{-1}(\delta)H_0(\gamma_0, \mu_0, z_0)$ for all $\delta \in \Gamma(x_0)$ (cf. (3.8), (5.2) and (5.4)).

Let $\mathcal{A}$ be the Lie algebra of the Lie group Γ, exp: $\mathcal{A} \to \Gamma$ the corresponding exponential map, $n := \dim \Gamma$ and $\{a_1, ..., a_n\}$ a basis in the vector space $\mathcal{A}$. Assumption (IX) implies that the vectors

$$v_j := \frac{d}{dt}[S(\exp(ta_j)x_0)]_{t=0} \tag{5.7}$$

form a basis in $T_{x_0}\mathcal{O}(x_0) = \ker \partial_x F(x_0, \lambda_0)$ (cf. (V)).

Further, let $\partial_x F(x_0, \lambda_0)^* \in \mathcal{L}(\tilde{X}^*, X^*)$ be the adjoint operator to $\partial_x F(x_0, \lambda_0)$. Then (II), (V), (IX) and (2.1) imply that dim ker $\partial_x F(x_0, \lambda_0)^* =$ dim Γ. If, moreover, (2.4) is satisfied then there exists a basis $\{v_1^*, ..., v_n^*\}$ in ker $\partial_x F(x_0, \lambda_0)^*$ such that

$$\langle v_i, v_j^* \rangle = \delta_{ij} \text{ and } \tilde{P}\tilde{x} = \tilde{x} - \sum_{j=1}^{n} < \tilde{x}, v_j^* > v_j \text{ for } \tilde{x} \in \tilde{X}. \tag{5.8}$$

Here $\langle \cdot, \cdot \rangle : \tilde{X} \times \tilde{X}^* \to \mathbb{R}$ is the dual pairing, and δ_{ij} is the Kronecker symbol.

The following lemma states two necessary and sufficient conditions for the operator (5.6) to be an isomorphism:

Lemma 5.1 *Suppose* (I) - (VI) *and* (IX). *Then the following is true:*

(i) The operator (5.6) *is an isomorphism from* $T_{\tilde{S}(\gamma_0)^{-1}z_0}\mathcal{O}(z_0)$ *onto ker* $\tilde{P}$ *if and only if*

$$\tilde{X} = T_{\tilde{S}(\gamma_0)^{-1}z_0}\mathcal{O}(z_0) \oplus \text{im } \partial_x F(x_0, \lambda_0). \tag{5.9}$$

(ii) Suppose (2.4). *Then the operator* (5.6) *is an isomorphism from* $T_{\tilde{S}(\gamma_0)^{-1}z_0}\mathcal{O}(z_0)$ *onto* ker $\tilde{P}$ *if and only if the matrix*

$$\left[\langle \frac{d}{dt}[\tilde{S}(\exp(ta_i)\gamma_0^{-1})z_0]_{t=0}, v_j^* \rangle\right]_{i,j=1}^{n} \tag{5.10}$$

has a non-vanishing determinant.

Proof Because of (II), (V), (IX), (2.1) and (2.2) we have dim Γ = dim ker $\tilde{P}$. Hence, (5.6) is an isomorphism from $T_{\tilde{S}(\gamma_0)^{-1}z_0}\mathcal{O}(z_0)$ onto ker $\tilde{P}$ iff it is injective.

(i) Suppose (5.6) to be injective. Then dim $\Gamma(z_0) = 0$ and, hence, dim $\mathcal{O}(z_0) =$ dim Γ = codim $\partial_x F(x_0, \lambda_0)$ (cf. (II), (V),(IX) and (2.1)). Therefore, for (5.9) it remains to show that

$$T_{\tilde{S}(\gamma_0)^{-1}z_0}\mathcal{O}(z_0) \cap \text{im } \partial_x F(x_0, \lambda_0) = \{0\}. \tag{5.11}$$

Let $\tilde{x}$ be an element of the left hand side of (5.11). Then there exists a $\bar{\gamma} \in T_{\gamma_0}\Gamma$ such that, on the one hand,

$$\tilde{x} = \frac{d}{d\gamma}\Big[\tilde{S}(\gamma)^{-1}z_0\Big]_{\gamma=\gamma_0}\bar{\gamma}$$

and, on the other hand, (5.6) maps $\bar{\gamma}$ into zero. But (5.6) is injective, therefore $\bar{\gamma} = 0$.

Now, conversely, suppose (5.9). Then, as above, $\dim \mathcal{O}(z_0) = \operatorname{codim} \partial_x F(x_0, \lambda_0) = \dim \Gamma$ and, hence, $\dim \Gamma(z_0) = 0$. Therefore, $\frac{d}{d\gamma}[\tilde{S}(\gamma)^{-1}z_0]_{\gamma=\gamma_0}$ is injective, and (5.9) yields that (5.6) is injective, too.

(ii) The map $\gamma \in \Gamma \mapsto -(I - \tilde{P})\tilde{S}(\gamma)^{-1}z_0 \in \ker \tilde{P}$ is a local diffeomorphism in $\gamma = \gamma_0$ iff the map

$$a \in \mathcal{A} \longmapsto -(I - \tilde{P})\tilde{S}(\exp(a)\gamma_0)^{-1}z_0 \in \ker\tilde{P} \tag{5.12}$$

is a local diffeomorphism in $a = 0$. But (5.10) is the matrix representation with respect to the bases $\{a_1, ..., a_n\}$ of $\mathcal{A}$ and $\{v_1, ..., v_n\}$ of $\ker \tilde{P} = T_{x_0}\mathcal{O}(x_0)$ (cf. (2.4)) of the derivative of (5.12) in $a = 0$. ∎

Again, using a more geometrical language, Lemma 5.1(i) can be formulated in the following way:

Suppose (I) – (VI) and (IX). Then (γ_0, μ_0, z_0) is a regular solution to the reduced bifurcation equation (RE) iff the group orbit $\mathcal{O}(z_0)$ intersects the affine subspace $\partial_\lambda F(x_0, \lambda_0)\mu_0 + \operatorname{im} \partial_x F(x_0, \lambda_0)$ in $\tilde{S}(\gamma_0)^{-1}z_0$ transversaly.

The following theorem is the main result of this section. In its formulation we use the maps $\hat{x}_0$ and $\hat{\lambda}_2$, given by Theorem 4.1.

Theorem 5.2 *Suppose* (I) – (IX), *and let* (γ_0, μ_0, z_0) *be a solution to* (RE) *with* (5.9).

Then there exist $\epsilon > 0$, *neighbourhoods* $V \subseteq X$ *of* $S(\gamma_0)x_0$, $W \subseteq \Lambda_1 \times \mathcal{S}$ *of* $(\lambda_{01}, \mu_0, z_0)$, *a* C^{k-1}*-map* $\hat{\gamma} : W \to \Gamma$ *with* $\hat{\gamma}(\lambda_{01}, \mu_0, z_0) = \gamma_0$ *and a* C^k*-map* $\hat{x} : K(\epsilon_0, \mu_0, z_0, W) \to X$ *such that the following is true:*

(i) *It holds* $F(x, \lambda) = y$ *with* $x \in V$ *and* $(\lambda, y) \in K(\epsilon_0, \mu_0, z_0, W)$ *if and only if* $x = \hat{x}(\lambda, y)$.

(ii) Let $(\lambda_1, \mu, z) \in W$ *be fixed. Then* $\hat{x}(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z)$ *tends to* $S(\hat{\gamma}(\lambda_1, \mu, z))\hat{x}_0(\lambda_1)$ *for* $\epsilon \to 0$.

Proof The Implicit Function Theorem yields a relation $\gamma = \check{\gamma}(\epsilon, \lambda_1, \mu, z)$ solving (SE) near the solution $\gamma = \gamma_0$, $\epsilon = 0$, $\lambda_1 = \lambda_{01}$, $\mu = \mu_0$ and $z = z_0$ (especially it hold

$\check{\gamma}(0, \lambda_{01}, \mu_0, z_0) = \gamma_0$). Hence, (i) follows with

$$\begin{aligned} &\hat{x}(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z) := \\ &:= S(\check{\gamma}(\epsilon, \lambda_1, \mu, z))(x_0 + \hat{u}(\check{\gamma}(\epsilon, \lambda_1, \mu, z), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z)). \end{aligned} \tag{5.13}$$

In (5.13), $\hat{u}$ is the map given by Lemma 3.3, and Lemma 3.3(ii) and (4.8) imply assertion (ii) with $\hat{\gamma}(\lambda_1, \mu, z) := \check{\gamma}(0, \lambda_1, \mu, z)$.

Remark that the map H is only C^{k-1}-smooth in arguments with $\epsilon = 0$ (cf. (5.2) and (5.3)), therefore the map $\hat{\gamma}$ is only C^{k-1}-smooth, in general. ■

Suppose (I) – (IX). Then, by means of Theorem 5.2, there exists a straightforward procedure to construct control parameters λ and y such that (OE) is solvable: Just take $(\mu_0, z_0) \in \mathcal{S}$ such that the orbit $\mathcal{O}(z_0)$ intersects the affine subspace $\partial_\lambda F(x_0, \lambda_0)\mu_0 + \partial_x F(x_0, \lambda_0)X$ in at least one point transversaly. Then $\lambda = \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu$ and $y = \epsilon z$ with arbitrary $\epsilon \in \mathbb{R}$ near zero, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near μ_0 and $z \in Y$ near z_0 are parameters of the type being in demand.

In applications, however, often one has to answer more specific questions about the solvability of (OE):

One of such questions, for example, is the following: Given an external, symmetry breaking control parameter $y \in Y$ near zero, do there exist internal, symmetry preserving control parameters $\lambda \in \Lambda$ near λ_0 and state parameters $x \in X$ near $\mathcal{O}(x_0)$ such that $F(x, \lambda) = y$? In other words, is it possible to adjust λ near λ_0 (in a manner, depending on the given y) such that (OE) is solvable?

The answer is "yes" if there exists a $\lambda_2 \in \Lambda_2$ such that $\mathcal{O}(y)$ intersects $\partial_\lambda F(x_0, \lambda_0)\mu_0 + \partial_x F(x_0, \lambda_0)X$ in at least one point transversaly. Moreover, the answer is "no" if none of the affine subspaces $\partial_\lambda F(x_0, \lambda_0)\lambda_2 + \partial_x F(x_0, \lambda_0)X$ (with $\lambda_2 \in \Lambda_2$) intersects $\mathcal{O}(y)$ (cf. Remark 5.4 below). Hence, the answer depends considerably on the number $\dim \Lambda_2 = \operatorname{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0)X_0$ (cf. (VII)). For example, if $\operatorname{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0)X_0 = 0$ (i.e. if the unperturbed equation (4.1) is solvable for all $\lambda \approx \lambda_0$, cf. the discussion after (4.14)) then $\mathcal{O}(y)$ may intersect the subspace $\partial_x F(x_0, \lambda_0)X$ or not, independently on any λ_2. Hence, no adjustment of λ can influence the question of the solvability of (OE) with the given y.

Now suppose, by contrast,

$$\operatorname{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0)X_0 = \dim \Gamma \tag{5.14}$$

(that is the largest codimension of $\partial_x F(x_0, \lambda_0)X_0$ in $\tilde{X}_0$, which is possible under the assumptions (I) – (IX), cf. the discussion after (4.13)). In this case it holds $\ker \tilde{P} \subseteq \tilde{X}_0$, and, hence, for each $\gamma \in \Gamma$ there exists a unique $\lambda_2 \in \Lambda_2$ such that $(I - \tilde{P})(\partial_\lambda F(x_0, \lambda_0)\lambda_2 -$

$\tilde{S}(\gamma)^{-1}y) = 0$ (because $(I - \tilde{P})\partial_\lambda F(x_0, \lambda_0)$ is an isomorphism from Λ_2 onto $\tilde{X}_0 \cap \ker \tilde{P}$, cf. the proof of Theorem 4.1). Thus, in this case for any $y \in Y$ near zero and for any $\gamma \in \Gamma$ there exists a $\lambda_2 \in \Lambda_2$ such that $\mathcal{O}(y)$ intersects $\partial_\lambda F(x_0, \lambda_0)\lambda_2 + \partial_x F(x_0, \lambda_0)X$ in $\tilde{S}(\gamma)^{-1}y$. This intersection is transversal if $(I - \tilde{P})\frac{d}{d\delta}[\tilde{S}(\delta)^{-1}y]_{\delta=\gamma}$ is injective. In other words: If $\mathrm{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0)X_0 = \dim \Gamma$ then, for any $y \in Y$ near zero such that $(I - \tilde{P})\frac{d}{d\delta}[\tilde{S}(\delta)^{-1}y]_{\delta=\gamma}$ is injective for at least one $\gamma \in \Gamma$, there exists a $\lambda \in \Lambda$ such that (OE) is solvable.

Let us consider an even more specific question concerning the solvability of (OE), which arises from applications and which may be answered by means of Theorem 5.2.

Suppose there is given a closed subspace Λ_* in Λ. In the applications, we have in mind, the internal, symmetry preserving parameters in Λ_* are distinguished by the property that it is much easier to vary them then other internal, symmetry preserving parameters (by reasons of the technology of the real system which is modeled by (OE)). In laser modeling, for example, the parameters in Λ_* are the laser currents which are much easier to vary then other laser parameters (as geometric and material parameters or facet reflectivities).

Now, a natural question is the following: Given an $y \in Y$ near zero, do there exist parameters $\lambda_* \in \Lambda_*$ near zero and state parameters $x \in X$ near $\mathcal{O}(x_0)$ such that $F(x, \lambda_0 + \lambda_*) = y$? In other words: Is it possible to adjust λ near λ_0, by variing the components in Λ_* only, such that (OE) with the given y is solvable?

The answer depends, firstly, on whether or not the orbit $\mathcal{O}(y)$ intersects one of the affine subspaces $\partial_\lambda F(x_0, \lambda_0)\lambda_2 + \partial_x F(x_0, \lambda_0)X$ in at least one point transversaly. This question does not depend on the choise of Λ_* and is discussed above. And second, if $\mathcal{O}(y)$ intersects $\partial_\lambda F(x_0, \lambda_0)\lambda_2 + \partial_x F(x_0, \lambda_0)X$ (with a fixed $\lambda_2 \in \Lambda_2$ near zero) in at least one point transversaly, then the answer depends on whether or not the affine subspace $\{(\lambda_0 + \lambda_*, y) \in \Lambda \times Y : \lambda^* \in \Lambda^*\}$ intersects the corresponding locking cone $K(\epsilon_0, \mu_0, z_0, W)$ with

$$\mu_0 := \frac{\lambda_2}{\sqrt{\|\lambda_2\|^2 + \|y\|^2}} \text{ and } z_0 := \frac{y}{\sqrt{\|\lambda_2\|^2 + \|y\|^2}}$$

(cf. (5.5)). This second question is equivalent to the question whether or not there exist $\lambda_* \in \Lambda_*$ near zero and $\lambda_1 \in \Lambda_1$ near λ_{01} such that $\lambda_0 + \lambda_* = \lambda_1 + \hat{\lambda}_2(\lambda_1) + \lambda_2$, and, hence, the answer is "yes" if the affine subspace $\lambda_0 + \Lambda_*$ is transversal to the submanifold $\mathcal{M} = \{\lambda_1 + \hat{\lambda}_2(\lambda_1) : \lambda_1 \approx \lambda_{01}\}$ (cf. (4.9)). Especially, for that it is necessary that $\dim \Lambda_* \geq \mathrm{codim}_{\tilde{X}_0} \partial_x F(x_0, \lambda_0)X_0$ (cf. (4.10)). Summarizing, we get the following

Corollary 5.3 *Suppose* (I) – (IX) *and* (5.14). *Let Λ_* be a closed subspace in Λ which is transversal to the subspace* (4.9).

Then for each $y \in Y$ near zero such that $\tilde{X} = T_{\tilde{y}}\mathcal{O}(y) \oplus \mathrm{im}\, \partial_x F(x_0, \lambda_0)$ for at least one

$\tilde{y} \in \mathcal{O}(y)$ *there exist* $\lambda_* \in \Lambda_*$ *near zero and* $x \in X$ *near* $\mathcal{O}(x_0)$ *with* $F(x, \lambda_0 + \lambda_*) = y$.

Remark 5.4 Let us consider the uniqueness assertion of Theorem 5.2(i) in more detail.

Suppose (I) – (IX), let $(\mu_0, z_0) \in \mathcal{S}$ be fixed, and let $\gamma = \gamma_0$ be a solution to

$$H_0(\gamma, \mu_0, z_0) = 0 \tag{5.15}$$

with (5.9). Then Theorem 5.2 asserts that for $(\lambda, y) \in K(\epsilon_0, \mu_0, z_0, W)$ there exists exactly one solution x near $S(\gamma_0)x_0$ to (OE). Of course, there may exist other solutions x near $\mathcal{O}(x_0)$ to (OE) (with the same control parameter (λ, y)), not close to $S(\gamma_0)x_0$.

Now, suppose that all solutions $\gamma \in \Gamma$ to (5.15) satisfy (5.9). Then the number of these solutions is finite (because Γ is compact), each such solution generates a family of solutions to (OE) (via Theorem 5.2), and we have the following "global" uniqueness assertion:

If $(\lambda, y) \approx (\lambda_0, 0)$ belongs to the intersection of the locking cones corresponding to the solutions to (5.15) (this intersection is a locking cone, i.e. of structure (5.5), again) and if $x \approx \mathcal{O}(x_0)$ is a solution to (OE) with this control parameter (λ, y), then x is a member of one of the families of solutions to (OE) corresponding to the solutions to (5.15). Especially, if (5.15) (with fixed $(\mu_0, z_0) \in \mathcal{S}$) is not solvable then there do not exist solutions to $F(x, \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu) = \epsilon z$ with $x \in X$ near $\mathcal{O}(x_0)$, $\epsilon \in \mathbb{R}$ near zero, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near μ_0 and $z \in Y$ near z_0. Moreover, if (5.15) (with fixed $z_0 \in Y$) is not solvable for each μ_0 such that $(\mu_0, z_0) \in \mathcal{S}$, then there do not exist solutions to $F(x, \lambda) = \epsilon z$ with $x \in X$ near $\mathcal{O}(x_0)$, $\epsilon \in \mathbb{R}$ near zero, $\lambda \in \Lambda$ near λ_0 and $z \in Y$ near z_0.

Remark 5.5 This is a remark on the "continuation assertion" (ii) of Theorem 5.2.

Suppose the assumptions of Theorem 5.2 to be satisfied. Then, if the control parameter $(\lambda, y) \in K(\epsilon_0, \mu_0, z_0, W)$ tends to $(\lambda_1 + \hat{\lambda}_2(\lambda_1), 0)$ (with fixed $(\lambda_1, \mu_0, z_0) \in W$), but not along a straight line $\{(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z) : \epsilon \in \mathbb{R}\}$, the solution $\hat{x}(\lambda, y)$ does not converge, in general (because the right hand side of (5.13) with $\epsilon = 0$ depends on μ and z, in general). In other words, in general it is not possible to continue continuously the family $(\lambda, y) \in K(\epsilon_0, \mu_0, z_0, W) \mapsto \hat{x}(\lambda, y) \in X$ of solutions to (OE) onto the closure of $K(\epsilon_0, \mu_0, z_0, W)$, for example. This phenomenon is well-known in the theory of damping and forcing of second order ordinary differential equations (cf. [26, 39]).

Remark 5.6 Let the assumptions of Theorem 5.2 be satisfied, and let $(\lambda, y) \in$

$K(\epsilon_0, \mu_0, z_0, W)$. Then, because of the uniqueness assertion of Theorem 5.2(i),

$$S(\gamma)\hat{x}(\lambda, y) = \hat{x}(\lambda, \tilde{S}(\gamma)y)$$

for all γ near the unit element. Hence, the map $\gamma \in \Gamma \mapsto S(\gamma)\hat{x}(\lambda, y) \in X$ is C^k-smooth (cf.(VI)). This is a kind of "abstract solution regularity" result for (OE): The representation S does not act smoothly on each element $x \in X$, but on solutions to (OE) it does.

Remark 5.7 If assumption (IX) is not satisfied then, at first glance, the parametrization (3.1) seems to be unsuitable for solving (OE) because we have

$$H(\gamma\delta, \epsilon, \lambda_1, \mu, z) = \tilde{S}(\delta)^{-1} H(\gamma, \epsilon, \lambda_1, \mu, z) \text{ for all } \delta \in \Gamma(x_0)$$

and all γ, ϵ, λ_1, μ and z (cf. (3.8) and (5.2)). Hence, the solutions $\gamma \in \Gamma$of the scaled bifurcation equation (SE) (with fixed control parameters ϵ, λ_1, μ and z) are not isolated, but occure as $\Gamma(x_0)$-orbits. On the other hand, because of the "uniqueness assertion" of Proposition 3.1, each such $\Gamma(x_0)$-orbit of solutions $\gamma \in \Gamma$ to (SE) (with fixed control parameters ϵ, λ_1, μ and z) corresponds to an isolated solution $x \in X$ of (OE) (with control parameters $(\lambda, y) \in \Lambda \times Y$ determined by (5.1)). Therefore, it should be possible to apply the results of Sections 3 and 4 (especially a second Liapunov-Schmidt reduction) in order to get families of $\Gamma(x_0)$-orbits of solutions to (SE) and, hence, families of isolated solutions to (OE). These families would be parametrized by the control parameters of the corresponding equations which have to belong to certain submanifolds in the control parameter spaces of codimension less or equal to dim $\Gamma(x_0)$. This is a topic of future research.

Remark 5.8 From (3.7) and (5.2) follows

$$H(\delta\gamma, \epsilon, \lambda_1, \mu, z) = H(\gamma, \epsilon, \lambda_1, \mu, z) \text{ for all } \delta \in \Gamma(z) \tag{5.16}$$

and for all suitable γ, ϵ, λ_1, μ and z. Therefore, the solutions $\gamma \in \Gamma$ to the scaled bifurcation equation (SE) (with fixed control parameters ϵ, λ_1, μ and z) occure as $\Gamma(z)$-orbits, and, hence, are not isolated, if $\dim \Gamma(z) > 0$. In contrast to the situation, considered in Remark 5.7, such $\Gamma(z)$-orbit of solutions to (SE) corresponds to a $\Gamma(z)$-orbit of, in general, nonisolated solutions $x \in X$ to (OE) (with fixed control parameters λ and y, determined by (5.1)).

Now, suppose $H_0(\gamma_0, \mu_0, z_0) = 0$ and $\dim \Gamma(z_0) > 0$. Then, because of (5.16), (5.9) is not satisfied, and Theorem 5.2 is not applicable. But (SE) with $z \approx z_0$ is a forced symmetry breaking problem, again (the parameter $z - z_0$ breaks the $\Gamma(z_0)$-symmetry). Therefore, in principle one can apply the results of this section to (SE) with $z \approx z_0$ (especially a second Liapunov-Schmidt reduction and a second scaling in order to get a "scaled bifurcation equation for the scaled bifurcation eqution"). Then Theorem 5.2 yields families of isolated solutions to (OE) which are parametrized by control parameters (λ, y) belonging to certain "locking cones of second kind".

Remark 5.9 Theorem 5.2 has the advantage that the left hand side (5.4) of the reduced bifurcation equation (RE) does not depend on the implicit given map $\hat{u}$. Moreover, in certain cases (with $\Lambda_2 = \{0\}$) this left hand side does not depend on the map F (the left hand side of the original problem (OE)) at all, but only on the action of the group Γ on x_0 and on Y. For example, if $X = \tilde{X}$ are Hilbert spaces, if the representations $S = \tilde{S}$ are unitary and if $F(\cdot, \lambda)$ is a gradient map (for each λ), then this left hand side is equal to

$$H_0(\gamma, z) = -(I - P)S(\gamma)^{-1}z,$$

where $I - P$ is the orthoprojector onto $T_{x_0}\mathcal{O}(x_0)$ in X. CHILLINGWORTH, MARSDEN and WAN used this property in their study of the dead load traction problem in three-dimensional elastostatics [13, 45, 11].

Remark 5.10 The generalization of Theorem 5.2 to problems of the type (3.9) with (3.10) (cf. Remark 3.5) is straightforward. In this case one has to use the following more general form of the left hand side of the reduced bifurcation equation

$$H_0(\gamma_0, \mu_0, z_0) = (I - \tilde{P})[\partial_\lambda F(x_0, \lambda_0, 0)\mu_0 + \partial_y F(x_0, \lambda_0, 0)T(\gamma)^{-1}z_0], \tag{5.17}$$

and the matrix (5.10) has to be replaced by the matrix

$$-\left[\langle \partial_y F(x_0, \lambda_0, 0)\frac{d}{dt}[T(\exp(ta_i)\gamma_0^{-1})z_0]_{t=0}, v_j^*\rangle\right]_{i,j=1}^{n}. \tag{5.18}$$

6 Stability from the Reduced Bifurcation Equation

Theorem 5.2 shows that regular solutions to the reduced bifurcation equation (RE) generate families of solutions to the original equation (OE). In this section we show that, moreover, under certain natural assumptions the linearization of (RE) with respect to the state parameter $\gamma \in \Gamma$ in such a solution to (RE) determines the linearized stability of all the corresponding solutions to (OE).

In this section we assume the conditions (I) – (IX) and (2.4) to be satisfied.

As usual, for $L \in \mathcal{L}(X;\tilde{X})$ we denote by $\operatorname{spec} L$ the set of all complex numbers ξ such that the operator $L - \xi J$ is not an isomorphism of the complexification of X onto the complexification of $\tilde{X}$. Here $J \in \mathcal{L}(X;\tilde{X})$ is the embedding operator.

Theorem 6.1 *Suppose* (I) – (IX), (2.4) *and*

(X) $\sup\{\operatorname{Re}\xi : \xi \in \operatorname{spec} \partial_x F(x_0, \lambda_0),\ \xi \neq 0\} < 0.$

Then, if ϵ_0 and W are sufficiently small, for all $(\lambda, y) \in K(\epsilon_0, \mu_0, z_0, W)$ the following is true: If all eigenvalues of the matrix (5.10) have negative real parts (resp. one such eigenvalue has a positive real part) then $\sup\{\operatorname{Re}\xi : \xi \in \operatorname{spec} \partial_x F(\hat{x}(\lambda, y), \lambda)\}$ *is negative (resp. positive).*

Proof Let $P \in \mathcal{L}(X)$ and $\tilde{P} \in \mathcal{L}(\tilde{X})$ be the projectors defined by (2.2) and (2.5), and let $\{v_1, ..., v_n\}$ and $\{v_1^*, ..., v_n^*\}$ (with $n := \dim \Gamma$) be the bases in $\ker \partial_x F(x_0, \lambda_0)$ and $\ker \partial_x F(x_0, \lambda_0)^*$, respectively, which are defined in (5.7) and (5.8). Further, let $\hat{u}$ be the map which is determined by Lemma 3.3, $\hat{x}_0$ and $\widehat{\lambda}_2$ the maps determined by Theorem 4.1 and $\hat{x}$, $\hat{\gamma}$ (resp. $\check{\gamma}$) the maps determined by (resp. in the proof of) Theorem 5.2.

From (III) and (5.13) we get

$$\begin{aligned}&\partial_x F(\hat{x}(\lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu)) S(\check{\gamma}(\epsilon, \lambda_2, \mu, z)) = \\ &= \tilde{S}(\check{\gamma}(\epsilon, \lambda_1, \mu, z)) \partial_x F(x_0 + \hat{u}(\check{\gamma}(\epsilon, \lambda_1, \mu, z)), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z)). \qquad (6.1)\end{aligned}$$

A clever application of the Implicit Function Theorem (cf. VANDERBAUWHEDE [44] and RECKE [33]) yields that there exist families of operators $A(\epsilon, \lambda_1, \mu, z) \in \mathcal{L}(\ker P)$, $B(\epsilon, \lambda_1, \mu, z) \in \mathcal{L}(\operatorname{im} P; \operatorname{im} \tilde{P})$, $C(\epsilon, \lambda_1, \mu, z) \in \mathcal{L}(X)$ and $\tilde{C}(\epsilon, \lambda_1, \mu, z) \in \mathcal{L}(\tilde{X})$, which are C^{k-1}-smoothly parametrized by $\epsilon \in \mathbb{R}$ near zero, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near μ_0 and $z \in Y$ near z_0, such that

$$\begin{aligned} A(0, \lambda_{01}, \mu_0, z_0) &= 0, \\ B(0, \lambda_{01}, \mu_0, z_0) &= \partial_x F(x_0, \lambda_0) \text{ on } \operatorname{im} P, \\ \tilde{C}(0, \lambda_{01}, \mu_0, z_0) &= I; \end{aligned} \qquad (6.2)$$

and that for all suitable ϵ, λ_1, μ and z we have

$$C(\epsilon, \lambda_1, \mu, z) = \tilde{C}(\epsilon, \lambda_1, \mu, z) \text{ on } X \qquad (6.3)$$

and

$$\begin{aligned}&\partial_x F(x_0 + \hat{u}(\check{\gamma}(\epsilon, \lambda_1, \mu, z), \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\mu, \epsilon z)) = \\ &= \tilde{C}(\epsilon, \lambda_1, \mu, z)[A(\epsilon, \lambda_1, \mu, z) \oplus B(\epsilon, \lambda_1, \mu, z)] C(\epsilon, \lambda_1, \mu, z)^{-1}. \qquad (6.4)\end{aligned}$$

In (6.4), $A \oplus B \in \mathcal{L}(X; \tilde{X})$ is the "diagonal" operator, which is defined by $(A \oplus B)(a+b) := Aa + Bb$ for $a \in \ker P$ and $b \in \operatorname{im} P$.

From (IX) and (4.16) follows that the dimension of the kernel of $\partial_x F(\hat{x}_0(\lambda_1), \lambda_1 + \hat{\lambda}(\lambda_1))$ (which is the limit for $\epsilon \to 0$ of the left hand side of (6.4), cf. (4.8) and Theorem 5.2(ii)) is equal to dim Γ. On the other hand, (3.4) and (6.2) imply that $B(\epsilon, \lambda_1, \mu, z)$ is an isomorphism from im P onto im $\tilde{P}$ for $\epsilon \approx 0$, $\lambda_1 \approx \lambda_{01}$, $\mu \approx \mu_0$ and $z \approx z_0$. Hence, $\dim \ker A(0, \lambda_1, \mu, z) = \dim \Gamma$ for such λ_1, μ and z. But $\dim \Gamma$ is the dimension of the space where $A(0, \lambda_1, \mu, z)$ is defined (cf. (2.1), (2.2), (V) and (IX)). Therefore, $A(0, \lambda_1, \mu, z)$ is the zero operator, and we have

$$A(\epsilon, \lambda_1, \mu, z) = \epsilon \bar{A}(\epsilon, \lambda_1, \mu, z), \tag{6.5}$$

where $\bar{A}(\epsilon, \lambda_1, \mu, z) \in \mathcal{L}(\ker P)$ depends C^{k-2}-smoothly (in arguments with $\epsilon \neq 0$ C^{k-1}-smoothly) on ϵ, λ_1, μ and z. Moreover, (6.2) – (6.5) imply

$$\begin{aligned} &\bar{A}(0, \lambda_{01}, \mu_0, z_0) v_j = \\ &\quad = P \frac{d}{d\epsilon} \Big[\partial_x F(x_0 + \hat{u}(\check{\gamma}(\epsilon, \lambda_{01}, \mu_0, z_0), \lambda_0 + \epsilon\mu_0, \epsilon z_0)) v_j \Big]_{\epsilon=0} \end{aligned} \tag{6.6}$$

for all $j = 1, ..., n$.

Let us denote

$$\gamma(\epsilon) := \check{\gamma}(\epsilon, \lambda_{01}, \mu_0, z_0), \quad u(\epsilon) := \hat{u}(\gamma(\epsilon), \lambda_0 + \epsilon\mu_0, \epsilon z_0). \tag{6.7}$$

Theorem 5.2 and (5.13) imply that $F(S(\gamma(\epsilon))(x_0 + u(\epsilon)), \lambda_0 + \epsilon\mu_0) = \epsilon z_0$ for all small $\epsilon \in \mathbb{R}$. Hence, (III) yields

$$F(S(\exp(ta_j))(x_0 + u(\epsilon), \lambda_0 + \epsilon\mu_0) = \epsilon \tilde{S}(\exp(ta_j)) \tilde{S}(\gamma(\epsilon))^{-1} z_0. \tag{6.8}$$

for all small $\epsilon \in \mathbb{R}$ and $j = 1, ..., n$. We differentiate the identity (6.8) with respect to t in $t = 0$ and obtain (using (5.7))

$$\begin{aligned} &\partial_x F(x_0 + u(\epsilon), \lambda_0 + \epsilon\mu_0)(v_j + \frac{d}{dt}[S(\exp(ta_j))u(\epsilon)]_{t=0}) = \\ &\quad = \epsilon \frac{d}{dt} \Big[\tilde{S}(\exp(ta_j)\gamma(\epsilon)^{-1}) z_0 \Big]_{t=0}. \end{aligned} \tag{6.9}$$

Further, we differentiate the identity (6.9) with respect to ϵ in $\epsilon = 0$ and get (using (6.6) and (6.7))

$$\bar{A}(0, \lambda_0, \mu_0, z_0) = (I - \tilde{P}) \frac{d}{dt} \Big[\tilde{S}(\exp(ta_j)\gamma_0^{-1}) z_0 \Big]_{t=0}.$$

Hence, the matrix (5.10) is the matrix representation of the operator $\bar{A}(0, \lambda_{01}, \mu_0, z_0)$ with respect to the basis $\{v_1, ..., v_n\}$ ((cf. 5.7) and (5.8)).

Let us summarize. Denote by M the matrix (5.10). Then spec M equals to the spectrum of $\tilde{A}(0, \lambda_{01}, \mu_0, z_0)$. Hence, (6.5) yields that

$$\text{sgn max}\,\{\text{Re}\,\xi : \ \xi \in \text{spec}\, M\} = \text{sgn max}\,\{\text{Re}\,\xi : \ \xi \in \text{spec}\, A(\epsilon, \lambda_1, \mu, z)\} \tag{6.10}$$

for all small $\epsilon > 0$, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near λ_{02} and $z \in Y$ near z_0. Further, assumption (X), (6.2) and the upper-semicontinuity of spectra (cf. [14, Chapter 14]) provide

$$\sup\,\{\text{Re}\,\xi : \ \xi \in \text{spec}\, B(\epsilon, \lambda_1, \mu, z)\} < 0 \tag{6.11}$$

for all small $\epsilon > 0$, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\mu \in \Lambda_2$ near λ_{02} and $z \in Y$ near z_0. Now, (6.1), (6.4), (6.10) and (6.11) imply the desired result. ■

Remark 6.2 Suppose (I) – (IX) and, for the sake of simplicity, that $X = \tilde{X} = \mathbb{R}^m$.

A solution $x = x_0$ of the equation $F(x, \lambda_0) = 0$, which satisfies condition (X), is usually called linearly orbitally stable. This property implies the so-called asymptotic orbital stability with asymptotic phase of the stationary solution $x = x_0$ of the Γ-equivariant ordinary differential equation

$$\dot{x} = F(x, \lambda_0), \tag{6.12}$$

i.e. each solution $x(t)$ of (6.12) with $x(0) \approx x_0$ exists and stays near x_0 for all times $t \geq 0$, and there exists a $\gamma_0 \in \Gamma$ such that $x(t) \to S(\gamma_0)x_0$ for $t \to \infty$ (cf. [6, 22]).

The solution $x = \hat{x}(\lambda, y)$ of equation $F(x, \lambda) = y$ with $\sup\,\{\text{Re}\,\xi : \ \xi \in \text{spec}\, \partial_x F(x, \lambda)\} < 0$ (resp. > 0) is usually called linearly stable (resp. linearly unstable). As it is well-known, such a solution is an asymptotically stable (resp. unstable) stationary solution of $\dot{x} = F(x, \lambda) - y$.

In the case of dim X = dim $\tilde{X} = \infty$, the things are more difficult, of course (see, e.g., [27, 7] for corresponding criteria for isolated stationary solutions of evolution equations).

Remark 6.3 The generalization of Theorem 5.2 to problems of the type (3.9) with (3.10) (cf. Remarks 3.5 and 5.10) is straightforward. In this case the eigenvalues of the matrix (5.18) determine the linearized stability of the solution families to (3.9) corresponding to solutions to the reduced bifurcation equation with left hand side (5.17).

7 Abstract Forced S^1-Equivariant Equations

Theorem 5.2 describes solution families of the problem

$$F(x, \lambda) = y, \ \ x \approx \mathcal{O}(x_0), \ \lambda \approx \lambda_0, \ y \approx 0 \tag{OE}$$

which are smoothly parametrized by the control parameter (λ, y) belonging to the locking cone $K(\epsilon_0, \mu_0, z_0, W) \subset \Lambda \times Y$. But Theorem 5.2 does not state any assertion about the questions whether or not these families have a smooth continuation outside of $K(\epsilon_0, \mu_0, z_0, W)$ (with the exception of the assertion of the impossibility of continuous continuation onto the points $(\lambda, y) = (\lambda_1 + \hat{\lambda}_2(\lambda_1), 0)$, cf. Remark 5.5), whether or not there exists a maximal domain of definition of such a continuation and how behaves the solution x if (λ, y) tends to the boundary of such a maximal domain of continuation.

Following ideas of HALE and TÁBOAS (see [24, 26, 39], [25, Chapter 17] and [14, Chapter 11]), in this section we will give some answers on the questions stated above in the case

(XI) $\Gamma = \mathbf{S}^1 := \{e^{i\gamma} \in \mathbb{C} : \gamma \in \mathbb{R}\}$.

For general groups and a more geometric and topological approach (in contrast to our analytic approach) see [12].

In this section we suppose assumptions (I) – (X) and (2.4) to be satisfied. We use the notation (similar to (5.7) and (5.8))

$$v := \frac{d}{d\gamma}\Big[S(e^{i\gamma})x_0\Big]_{\gamma=0}, \quad v^* \in X^* : \partial_x F(x_0, \lambda_0)^* v^* = 0, \; \langle v, v^* \rangle = 1. \tag{7.1}$$

Here $\langle \cdot, \cdot \rangle : \tilde{X} \times \tilde{X}^* \to \mathbb{R}$ is the dual pairing, again.

The Lie group S^1 is one dimensional and Abelian. Therefore, if (IX) is valid, each subspace Λ_2 of Λ, which satisfies (VII) and (VIII), is one dimensional (cf. (4.14) and the discussion below this formula). Hence, (VII), (VIII) and (7.1) imply that there exists a $\lambda_* \in \Lambda_2$ such that

$$\langle \partial_\lambda F(x_0, \lambda_0)\lambda_*, v^* \rangle = 1. \tag{7.2}$$

For the sake of simplicity, in this section we will use the following notation (for $z \in Y$)

$$\begin{aligned} \mu_+(z) &:= \max\{\langle \tilde{S}(e^{-i\gamma})z, v^* \rangle : \gamma \in \mathbb{R}\} \\ \mu_-(z) &:= \min\{\langle \tilde{S}(e^{-i\gamma})z, v^* \rangle : \gamma \in \mathbb{R}\}. \end{aligned} \tag{7.3}$$

and the following terminology:

Definition 7.1 A point $z \in Y$ is called nondegenerate of type $\mathcal{T}_l$ if the function $\gamma \in \mathbb{R} \mapsto \langle\, \tilde{S}(e^{-i\gamma})z, v^* \rangle \in \mathbb{R}$ has exactly $2l$ critical points in $[0, 2\pi)$ and if all these critical points are nondegenerate (i.e. if the first derivative of this function vanishes in exactly $2l$ points in $[0, 2\pi)$ and if the second derivative is nonzero in all this points).

Remark 7.2 Obviously, if all critical points of $\varphi(\cdot, z)$ are nondegenerate then there exists a nonnegative integer l such that z is nondegenerate of type $\mathcal{T}_l$ in the sense of Definition 7.1.

The main result of this section is the following

Theorem 7.3 *Suppose* (I) – (XI) *and* (2.4), *and let* $z_0 \in Y$ *be nondegenerate of type* $\mathcal{T}_1$.

Then there exist $\epsilon_0 > 0$, *neighbourhoods* $V \subseteq X$ *of* $\mathcal{O}(x_0)$, $W_1 \subseteq \Lambda_1$ *of* λ_{01}, $W_2 \subseteq \mathbb{R}$ *of zero and* $W \subseteq Y$ *of* z_0 *and* C^{k-1}*-maps* ν_+ *and* ν_- *from* $(-\epsilon_0, \epsilon_0) \times W_1 \times W$ *into* $\mathbb{R}$ *such that*

$$\nu_\pm(\epsilon, \lambda_1, z) = \epsilon\Big[\mu_\pm(z) + O(|\epsilon| + \|\lambda_1 - \lambda_{01}\|)\Big] \text{ for } |\epsilon| + \|\lambda_1 - \lambda_{01}\| \to 0 \tag{7.4}$$

uniformly for $z \in W$, *and that for all* $\epsilon \in (-\epsilon_0, \epsilon_0)$, $\lambda_1 \in W_1$, $\nu \in W_2$ *and* $z \in W$ *the following holds:*

(i) For all $\nu \in (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$ *there exist exactly two solutions* $x \in V$ *of the equation*

$$F(x, \lambda_1 + \widehat{\lambda}_2(\lambda_1) + \nu\lambda_*) = \epsilon z, \tag{7.5}$$

one is linearly stable, the other is linearly unstable. These solutions depend C^k*-smoothly on* ϵ, λ_1, ν *and* z, *and for* $|\nu - \nu_+(\epsilon, \lambda_1, z)| \to 0$ *or* $|\nu - \nu_-(\epsilon, \lambda_1, z)| \to 0$ *they coalesce (saddle node bifurcation).*

(ii) For $\nu \notin (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$ *there do not exist solutions* $x \in V$ *to* (7.5).

Proof For $\gamma \in \mathbb{R}$, $\lambda_1 \in \Lambda_1$ near λ_{01}, $\epsilon, \nu \in \mathbb{R}$ near zero and $z \in Y$ near z_0 we denote

$$\tilde{G}(\gamma, \epsilon, \lambda_1, \nu, z) := \langle G(\lambda_1 + \widehat{\lambda}_2(\lambda_1) + \nu\lambda_*, \epsilon z), v^* \rangle. \tag{7.6}$$

Then Theorem 4.1 implies that $\tilde{G}(\gamma, 0, \lambda_1, 0, z) = 0$ for all γ, λ_1 and z. Hence, we have

$$\tilde{G}(\gamma, \epsilon, \lambda_1, \nu, z) = \epsilon\tilde{G}_\epsilon(\gamma, \epsilon, \lambda_1, \nu, z) + \nu\tilde{G}_\nu(\gamma, \epsilon, \lambda_1, \nu, z) \tag{7.7}$$

with

$$\tilde{G}_\epsilon(\gamma, \epsilon, \lambda_1, \nu, z) := \int_0^1 \partial_\epsilon\tilde{G}(\gamma, t\epsilon, \lambda_1, t\nu, z)dt \text{ and}$$

$$\tilde{G}_\nu(\gamma, \epsilon, \lambda_1, \nu, z) := \int_0^1 \partial_\nu\tilde{G}(\gamma, t\epsilon, \lambda_1, t\nu, z)dt.$$

Let P and $\tilde{P}$ be the projectors which satisfy (2.2) and (2.5). Then we have (cf. (5.8) and (7.1)) $\tilde{P}\tilde{x} = \tilde{x} - \langle \tilde{x}, v^* \rangle$ for $\tilde{x} \in \tilde{X}$, and (3.5), (7.2), (7.3), (7.6) and (7.7) imply

$$\begin{aligned} \tilde{G}_\epsilon(\gamma, 0, \lambda_{01}, 0, z) &= -\langle \tilde{S}(e^{-i\gamma})z, v^* \rangle \\ \tilde{G}_\nu(\gamma, 0, \lambda_{01}, 0, z) &= 1. \end{aligned} \tag{7.8}$$

In order to solve equation (7.5) for $x \approx \mathcal{O}(x_0)$, $\epsilon \approx 0$, $\lambda_1 \approx \lambda_{01}$, $\nu \approx 0$ and $z \approx z_0$, we have to solve the equation

$$\tilde{G}(\gamma, \epsilon, \lambda_1, \nu, z) = 0, \ \gamma \in \mathbb{R}, \ \epsilon \approx 0, \ \lambda_1 \approx \lambda_{01}, \ \nu \approx 0, \ z \approx z_0. \tag{7.9}$$

From (7.7) and (7.8) follows that there exists a constant $c > 0$ such that for all solutions to (7.9) it holds $|\nu| \leq c\epsilon$. Therefore, without loosing solutions we may substitude $\nu = \epsilon\mu$ in (7.9) and, after that, divide by ϵ. So we get the equivalent equation

$$\tilde{G}_\epsilon(\gamma, \epsilon, \lambda_1, \epsilon\mu, z) + \mu\tilde{G}_\mu(\gamma, \epsilon, \lambda_1, \epsilon\mu, z) = 0, \ \gamma \in \mathbb{R}, \ \epsilon \approx 0, \ \lambda_1 \approx \lambda_{01}, \ z \approx z_0. \tag{7.10}$$

First, we determine the singular solutions to (7.10), i.e. the solutions to (7.10) such that

$$\partial_\gamma \tilde{G}_\epsilon(\gamma, \epsilon, \lambda_1, \epsilon\mu, z) + \mu\partial_\gamma \tilde{G}_\nu(\gamma, \epsilon, \lambda_1, \epsilon\mu, z) = 0. \tag{7.11}$$

From (7.8) follows that, for $\epsilon = 0$, $\lambda_1 = \lambda_{01}$ and $z = z_0$, the system (7.10), (7.11) has the following form:

$$\begin{aligned} -\langle \tilde{S}(e^{-i\gamma})z, v^* \rangle + \mu &= 0 \\ -\tfrac{d}{d\gamma}\langle \tilde{S}(e^{-i\gamma})z, v^* \rangle &= 0. \end{aligned}$$

Hence, because z_0 is nondegenerate of type $\mathcal{T}_1$ and because of the Implicit Function Theorem, it holds (7.10), (7.11) iff

$$\gamma = \hat{\gamma}_+(\epsilon, \lambda_1, z)(\mathrm{mod}\ 2\pi), \ \ \mu = \hat{\mu}_+(\epsilon, \lambda_1, z) \tag{7.12}$$

or

$$\gamma = \hat{\gamma}_-(\epsilon, \lambda_1, z)(\mathrm{mod}\ 2\pi), \ \ \mu = \hat{\mu}_-(\epsilon, \lambda_1, z). \tag{7.13}$$

Here $\widehat{\gamma}_+$, $\hat{\gamma}_-$, $\hat{\mu}_+$ and $\hat{\mu}_-$ are C^{k-1}-smooth (in arguments with $\epsilon \neq 0$ C^k-smooth) functions such that

$$\begin{aligned} \hat{\mu}_+(0, \lambda_{01}, z) &= \mu_+(z) = \langle \tilde{S}(e^{-i\hat{\gamma}_+(0,\lambda_{01},z)})z, v^* \rangle \\ \hat{\mu}_-(0, \lambda_{01}, z) &= \mu_-(z) = \langle \tilde{S}(e^{-i\hat{\gamma}_-(0,\lambda_{01},z)})z, v^* \rangle. \end{aligned} \tag{7.14}$$

It is easy to verify (using (7.8) and the assumption of nondegeneracy of type $\mathcal{T}_1$ of z_0) that in the solutions (7.12) and (7.13) the saddle node bifurcation theorem (cf., e.g., [14, Theorem 6.2.1]) works. It claims that, if μ decreases from $\hat{\mu}_+(\epsilon, \lambda_1, z)$ or increases from $\widehat{\mu}_-(\epsilon, \lambda_1, z)$, exactly two solutions to (7.10) grow out of the solutions (7.12) or (7.13).

These solutions may be C^k-smoothly continued (because of the Implicit Function Theorem and of the compactness of S^1) for all $\mu \in (\hat{\mu}_-(\epsilon, \lambda_1, z), \hat{\mu}_+(\epsilon, \lambda_1, z))$ (and $\epsilon \approx 0,\ \lambda_1 \approx \lambda_{01}$ and $z \approx z_0$). Other solutions to (7.10) do not exist (again, because of the nondegeneracy of z_0 and because of the compactness of S^1, cf. Remark 5.4). Hence, the theorem (with the exeption of the stability assertions) is proved with $\nu_\pm(\epsilon, \lambda_1, z) := \epsilon\hat{\mu}_\pm(\epsilon, \lambda_1, z)$.

Now, let us prove the stability assertions.

The reduced bifurcation equation is (cf. (5.2), (5.4), (7.1), (7.2) and (7.6) – (7.8))

$$H_0(\gamma, \mu, z) = [\mu - \langle \tilde{S}(e^{-i\gamma})z, v^*\rangle]v = 0.$$

For $z \approx z_0$ and $\mu \in (\mu_-(z), \mu_+(z))$ it has exactly two solutions, and $\langle \partial_\gamma H_0(\gamma, \mu, z), v^*\rangle = -\frac{d}{d\gamma}\langle \tilde{S}(e^{-i\gamma})z, v^*\rangle$ has different signs in these solutions. Hence, Theorem 6.1 yields the claim. ■

Often in applications one is interested in strategies to head for control parameters $\lambda \in \Lambda$ near λ_0 and $y \in Y$ near zero such that (OE) has exactly one linearly stable solution $x \approx \mathcal{O}(x_0)$. Let us describe such a strategy, using Theorem 7.3:

First, verify assumptions (I) – (XI) and (2.4), determine $v \in X$ and $v^* \in X^*$ with (7.1) and $\lambda_* \in \Lambda_2$ with (7.2).

Second, choose a $z_0 \in Y$ which is nondegenerate of type $\mathcal{T}_1$.

Third, choose $\epsilon > 0$ near zero, $\lambda_1 \in \Lambda_1$ near λ_{01} and $\mu \in \mathbb{R}$ with

$$\min\{\langle \tilde{S}(e^{-i\gamma}z, v^*\rangle : \ \gamma \in \mathbb{R}\} < \mu < \max\{\langle \tilde{S}(e^{-i\gamma}z, v^*\rangle : \ \gamma \in \mathbb{R}\}. \tag{7.15}$$

Then $\lambda = \lambda_1 + \widehat{\lambda}_2(\lambda_1) + \epsilon\lambda_*$ and $y = \epsilon z$ are control parameters of the desired type.

For applications of Theorem 7.3 the following two lemmata (which will be proved in the Appendix of this paper) are useful. They describe sufficient conditions for z to be nondegenerate of type $\mathcal{T}_1$ and for μ and to satisfy (7.15).

For $z \in Y$ we define

$$a_j(z) := \frac{1}{\pi}\int_0^{2\pi} \langle \tilde{S}(e^{-i\gamma}z, v^*\rangle \cos j\gamma \, d\gamma, \ b_j(z) := \frac{1}{\pi}\int_0^{2\pi} \langle \tilde{S}(e^{-i\gamma}z, v^*\rangle \sin j\gamma \, d\gamma. \tag{7.16}$$

Then

$$\langle \tilde{S}(e^{-i\gamma}z, v^*\rangle = \frac{a_0(z)}{2} + \sum_{j=1}^{\infty}[a_j(z)\cos j\gamma + b_j(z)\sin j\gamma]. \tag{7.17}$$

Lemma 7.4 *Suppose $k \geq 3$ and*

$$a_1(z)^2 + b_1(z)^2 > \pi^2 \sum_{j=2}^{\infty} j^4(1 + j^2)[a_j(z)^2 + b_j(z)^2]. \tag{7.18}$$

Then z is nondegenerate of type $\mathcal{T}_1$.

Lemma 7.5 *Suppose*

$$[a_1(z)^2 + b_1(z)^2]^{\frac{1}{2}} > |\mu - \frac{a_0(z)}{2}| + \pi\Big[\sum_{j=2}^{\infty} j^2(a_j(z)^2 + b_j(z)^2)\Big]^{\frac{1}{2}}. \tag{7.19}$$

Then (7.15) *holds.*

Remark 7.6 Analogously one can show that condition

$$|\mu - \frac{a_0(z)}{2}|^2 > \pi^2 \sum_{j=1}^{\infty} j^2[a_j(z)^2 + b_j(z)^2] \tag{7.20}$$

implies that (7.15) is not satisfied and, hence, that for $\lambda = \lambda_1 + \widehat{\lambda}_2(\lambda_1) + \epsilon\lambda_*$ and $y = \epsilon z$ there do not exist solutions $x \approx \mathcal{O}(x_0)$ to (OE) (cf. Theorem 7.3(ii)).

Let the assumptions of Theorem 7.3 be satisfied with $X = \tilde{X} = \mathbb{R}^m$, and consider the ordinary differential equation

$$\dot{x} = F(x, \lambda) - y. \tag{7.21}$$

Then the group orbit $\mathcal{O}(x_0)$ is an attracting normally hyperbolic invariant manifold for (7.21) (cf., e.g., [29]). Hence, for $\lambda \approx \lambda_0$ and $y \approx 0$ there exists an attracting normally hyperbolic invarint manifold $M(\lambda, y)$ for (7.21) near $\mathcal{O}(x_0)$ (cf., e.g., [36, 48]). The manifold $M(\lambda, y)$ is diffeomorphic to $\mathcal{O}(x_0)$ and, hence, to $\mathbf{S}^1$. All solutions $x(t)$ of (7.21), which stay near $\mathcal{O}(x_0)$ for all times, move on $M(\lambda, y)$.

Let us consider the dynamics of (7.21) with

$$\lambda = \lambda_1 + \hat{\lambda}_2(\lambda_1) + \epsilon\lambda_*, \quad y = \epsilon z, \quad \epsilon > 0$$

in more detail.

If $\nu \in (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$, then the two stationary solutions to (7.21), described by Theorem 7.3, lie on $M(\lambda, y)$. Hence, they are connected by two heteroclinic orbits. One of these stationary solutions is asymptotically stable, the other is unstable. For $\nu \downarrow \nu_-(\epsilon, \lambda_1, z)$ or $\nu \uparrow \nu_+(\epsilon, \lambda_1, z)$ they coalesce in a nonhyperbolic stationary solution to (7.21) (a saddle node), one of the heteroclinic orbits disappears, and the other changes into a homoclinic orbit from the saddle node.

If $\nu \notin (\nu_-(\epsilon, \lambda_1, z), \nu_+(\epsilon, \lambda_1, z))$, then there do not exist stationary solutions to (7.21) on $M(\lambda, y)$. Hence, $M(\lambda, y)$ is an attracting periodic orbit. For $\nu \uparrow \nu_-(\lambda_1, \epsilon, z)$ or $\nu \downarrow \nu_+(\lambda_1, \epsilon, z)$ this periodic orbit changes into the homoclinic orbit from the saddle node. Especially, its period tends to infinity.

The codimension one bifurcation which occures for $\nu = \nu_-(\lambda_1, \epsilon, z)$ and $\nu = \nu_+(\lambda_1, \epsilon, z)$ is well dicovered. In [2, Chapter 21] it is called "blue loop" and in [4, Chapter 33] "birth of a cycle from a homoclinic orbit of a saddle node" (see also [14, Chapter 10.4]).

Now, consider the case $\epsilon = 0$, i.e. the $\mathbf{S}^1$-equivariant differential equation

$$\dot{x} = F(x, \lambda). \tag{7.22}$$

Denote by $\mathcal{M} := \{\lambda_1 + \hat{\lambda}_2(\lambda_1) : \lambda_1 \approx \lambda_{01}\}$ the bifurcation hypersurface in Λ corresponding to Theorem 4.1 (cf. (4.9) and (4.10)).

Because of Theorem 4.1, for $\lambda \in \mathcal{M}$ we have $M(\lambda, 0) = \mathcal{O}(\hat{x}_0(\lambda))$, i.e. the invariant manifold $M(\lambda, 0)$ for (7.22) consists of stationary solutions only.

For $\lambda \notin \mathcal{M}$, $M(\lambda, 0)$ is an attracting periodic orbit for (7.22) and simultaneously a group orbit. Hence, it is the orbit of a solution of the type

$$x(t) = S(e^{i\alpha(\lambda)t})x_*(\lambda), \tag{7.23}$$

usually called rotating wave or relative equilibrium. For λ on opposite sides of the hypersurface $\mathcal{M}$, the frequency α of the rotating wave (7.23) has opposite signs, i.e. the solution (7.23) rotates in opposite directions. Moreover, it holds

$$\alpha(\lambda)\langle\frac{d}{dt}[S(e^{it})x_*(\lambda)]_{t=0}, v^*\rangle = \langle F(x_*(\lambda), \lambda), v^*\rangle$$

and, hence,

$$\alpha(\lambda) = O(\mathrm{dist}(\lambda, \mathcal{M})) \text{ for } \mathrm{dist}(\lambda, \mathcal{M}) \longrightarrow 0 \tag{7.24}$$

(cf. (7.1)), i.e. the period of the rotating wave solution (7.23) tends to infinity if the distance of the control parameter λ to $\mathcal{M}$ tends to zero. This is the so-called "freezing phenomenon"(cf. [20]).

Remark 7.7 The generalization of the results of this section to problems of the type (3.9) with (3.10) (cf. Remarks 3.5, 5.10 and 6.3) is straightforward. One has to replace everywhere the term $\langle\tilde{S}(e^{-i\gamma})z, v^*\rangle$ by the term $-\langle\partial_y F(x_0, \lambda_0, 0)T(e^{-i\gamma})z, v^*\rangle$. For example, (7.3) must be replaced by

$$\begin{aligned}\mu_+(z) &:= -\min\{\langle\partial_y F(x_0, \lambda_0, 0)T(e^{-i\gamma})z, v^*\rangle : \gamma \in \mathbb{R}\}\\ \mu_-(z) &:= -\max\{\langle\partial_y F(x_0, \lambda_0, 0)T(e^{-i\gamma})z, v^*\rangle : \gamma \in \mathbb{R}\}.\end{aligned}$$

(cf. (5.17 and (5.18)).

Remark 7.8 A similar to Theorem 7.3, but more complicated result holds if the assumption "z_0 is nondegenerate of type $\mathcal{T}_1$" in Theorem 7.3 is replaced by "z_0 is nondegenerate of type $\mathcal{T}_l$". In that case the system (7.10), (7.11) has not only two, but $2l$

solution families

$$\gamma = \hat{\gamma}_j(\epsilon, \lambda_1, z) \ (\text{mod}\ 2\pi), \quad \mu = \hat{\mu}_j(\epsilon, \lambda_1, z) \quad \text{for} \ \ j = 1, ..., 2l,$$

and it holds $\hat{\mu}_j(\epsilon, \lambda_1, z) < \hat{\mu}_{j'}(\epsilon, \lambda_1, z)$ for $j > j'$ and $\epsilon > 0$ and

$$\begin{aligned} \hat{\mu}_1(0, \lambda_{01}, z) &= \max\{\langle \tilde{S}(e^{-i\gamma})z, v^*\rangle : \gamma \in \mathbb{R}\} = \langle \tilde{S}(e^{-i\hat{\gamma}_1(0,\lambda_{01},z)}z, v^*\rangle \\ \hat{\mu}_{2l}(0, \lambda_{01}, z) &= \min\{\langle \tilde{S}(e^{-i\gamma})z, v^*\rangle : \gamma \in \mathbb{R}\} = \langle \tilde{S}(e^{-i\hat{\gamma}_{2l}(0,\lambda_{01},z)}z, v^*\rangle. \end{aligned}$$

Setting $\nu_j(\epsilon, \lambda_1, z) := \epsilon\mu_j(\epsilon, \lambda_1, z)$ for $j = 1, ..., 2l$, one can describe the solution behavior of (7.5) with $\epsilon \approx 0$, $\lambda_1 \approx \lambda_{01}$, $\nu \approx 0$ and $z \approx z_0$ in the following way:

For $\nu \in (\nu_{2l}(\epsilon, \lambda_1, z), \nu_1(\epsilon, \lambda_1, z))$ there exist at least two (but a finite number of) solutions $x \approx \mathcal{O}(x_0)$ to (7.5). If the control parameter $(\epsilon, \lambda_1, \nu, z)$ intersects one of the hypersurfaces $\nu = \nu_j(\epsilon, \lambda_1, z)$, then the number of solutions $x \approx \mathcal{O}(x_0)$ changes generically by two (saddle node bifurcations). If $(\epsilon, \lambda_1, \nu, z)$ does not belong to one of these hypersurfaces, then the number of solutions $x \approx \mathcal{O}(x_0)$ is even, half of them are linearly stable, the other's are linearly unstable (for related results see [26, Theorem 1.1], [42, Theorem 8.5.6] and [14, Theorem 11.5.1]).

A Appendix

Let us use the notation of Section 7. Let $z \in Y$ be fixed. We denote $\phi(\gamma) := \langle \tilde{S}(\gamma)^{-1}z, v^*\rangle$ and leave out the argument z in the notation (7.16). Then (cf. (7.17))

$$\phi(\gamma) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j \cos j\gamma + b_j \sin j\gamma),$$

and Parceval's identity yields

$$\int_0^{2\pi} \phi(\gamma)^2 d\gamma = \pi\Big[\frac{a_0^2}{2} + \sum_{j=1}^{\infty}(a_j^2 + b_j^2)\Big]. \tag{A.1}$$

Further, the following inequality (corresponding to the continuous embedding of the Sobolev space $W^{1,2}(0, 2\pi)$ into $C([0, 2\pi])$ is easily proved:

$$\max_\gamma |\phi(\gamma)| \leq [\pi \int_0^{2\pi} |\phi'(\gamma)|^2 d\gamma]^{1/2} + \frac{1}{2\pi}|\int_0^{2\pi} \phi(\gamma) d\gamma|.$$

Hence, (A.1) gives

$$\max_\gamma |\phi(\gamma)| \leq \frac{|a_0|}{2} + \pi\Big[\sum_{j=1}^{\infty} j^2(a_j^2 + b_j^2)\Big]^{1/2} \tag{A.2}$$

Let us prove Lemma 7.5.

The desired inequality (7.15) is satisfied if the equation $\phi(\gamma) - \mu = 0$ has at least two solutions. This is the case if the equation $\phi(\gamma) - \mu - a_1 \cos\gamma - b_1 \sin\gamma = -a_1 \cos\gamma - b_1 \sin\gamma$ has at least two solutions, i.e. if

$$\begin{aligned}&\min_\gamma(-a_1 \cos\gamma - b_1 \sin\gamma) < \\ &< \max_\gamma |\phi(\gamma) - \mu - a_1 \cos\gamma - b_1 \sin\gamma| < \max_\gamma(-a_1 \cos\gamma - b_1 \sin\gamma).\end{aligned} \tag{A.3}$$

But (A.3) is equivalent to

$$\max_\gamma |\frac{a_0}{2} - \mu + \sum_{j=2}^{\infty}(a_j \cos j\gamma + b_j \sin j\gamma)| < \sqrt{a_1^2 + b_1^2}. \tag{A.4}$$

Applying (A.2) (with a_0 replaced by $a_0 - 2\mu$ and $a_1 = b_1 = 0$) we get that (A.4) is fulfilled if (7.19) is fulfilled.

Now, let us prove Lemma 7.4.

For $\gamma, \delta \in \mathbb{R}$ denote

$$\varphi(\gamma, \delta) := \frac{1}{\sqrt{a_1^2 + b_1^2}} \Big[a_1 \cos\gamma + b_1 \sin\gamma + \delta[\phi(\gamma) - a_1 \cos\gamma - b_1 \sin\gamma]\Big].$$

Obviously, $\varphi(\cdot, 0)$ is nondegenerate of type $\mathcal{T}_1$. Hence, $\varphi(\cdot, 1) = \frac{1}{\sqrt{a_1^2+b_1^2}}\phi$ is nondegenerate of type $\mathcal{T}_1$ if for all $\gamma \in \mathbb{R}$ and $\delta \in [0, 1]$ the following condition is true:

$$\text{If } \partial_\gamma \varphi(\gamma, \delta) = 0 \text{ then } \partial_\gamma^2 \varphi(\gamma_0, \delta_0) \neq 0. \tag{A.5}$$

Suppose (A.5) is not true, i.e. $\partial_\gamma \varphi(\gamma_0, \delta_0) = \partial_\gamma^2 \varphi(\gamma_0, \delta_0) = 0$ for some $\gamma_0 \in \mathbb{R}$ and $\delta_0 \in [0, 1]$. Then there exists a $\gamma_* \in R$ such that

$$0 = \partial_\gamma \varphi(\gamma_0, \delta_0) = \cos(\gamma_0 + \gamma_*) + \frac{\delta_0}{\sqrt{a_1^2 + b_1^2}} \Big[\phi'(\gamma_0) + a_1 \sin\gamma_0 - b_1 \cos\gamma_0\Big],$$

i. e.

$$1 - \sin^2(\gamma_0 + \gamma_*) = \frac{\delta_0^2}{a_1^2 + b_1^2} \Big[\phi'(\gamma_0) + a_1 \sin\gamma_0 - b_1 \cos\gamma_0\Big]^2, \tag{A.6}$$

and

$$0 = \partial_\gamma^2 \varphi(\gamma_0, \delta_0) = -\sin(\gamma_0 + \gamma_*) + \frac{\delta_0}{\sqrt{a_1^2 + b_1^2}} \Big[\phi''(\gamma_0) + a_1 \cos\gamma_0 + b_1 \sin\gamma_0\Big]. \tag{A.7}$$

But (A.6) and (A.7) yield

$$a_1^2 + b_1^2 \leq \Big[\sum_{j=2}^{\infty} j(-a_j \sin j\gamma + b_j \cos j\gamma)\Big]^2 + \Big[\sum_{j=2}^{\infty} j^2(-a_j \sin j\gamma - b_j \cos j\gamma)\Big]^2.$$

Applying (A.2), we get a contradiction to (7.18).

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