Weierstraß–Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

On an information–type inequality for the Hellinger process

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submitted: 15th February 1996

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Preprint No. 227
Berlin 1996

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ABSTRACT

Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered space with two probability measures P and P' on $(\Omega, \mathcal{F})$. Let X be a d-dimensional locally square-integrable semimartingale relative to P and P' with the canonical decomposition $X = X_0 + M + A$ and $X = X_0 + M' + A'$ respectively. We give a lower bound for the Hellinger process $h(\frac{1}{2}; P, P')$ of order $1/2$ between P and P' in terms of A, A' and the quadratic characteristic of M and M'. This result implies simple sufficient conditions for the entire separation of measures in a linear regression model with martingale errors.

1. Main Results

Let a pair of probability measures P and P' be given on a filtered space $(\Omega, \mathcal{F}, \mathbb{F})$ with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. Let $X = (X^i)_{i\leq d}$ be a càdlàg adapted d-dimensional process on $(\Omega, \mathcal{F}, \mathbb{F})$. We assume that X^i is a locally square-integrable semimartingale with respect to P and P' for each $i \leq d$, i.e. $X^i = X_0^i + M^i + A^i = X_0^i + M'^i + A'^i$, where M^i (respectively, M'^i) is a locally square-integrable martingale relative to P (respectively, P') and A^i and A'^i are predictable processes with finite variation over compact intervals.

The aim of this paper is to give a lower bound for the Hellinger process $h(\frac{1}{2}; P, P')$ of order $1/2$ between P and P' when only a partial information on P and P' is available, namely, when we know only the processes $A = (A^i)_{i\leq d}$, $A' = (A'^i)_{i\leq d}$ and the quadratic characteristic of $M = (M^i)_{i\leq d}$ and $M' = (M'^i)_{i\leq d}$. Such a situation arises naturally in some statistical models of stochastic processes, see examples in Section 4.

We use traditional notation and concepts of the "general theory of processes". If Y is a càdlàg process, the associated jump process is $\Delta Y_t = Y_t - Y_{t-}$ for $t > 0$, and $\Delta Y_0 = 0$. The symbol $\cdot$ denotes the (Lebesgue–Stieltjes or stochastic) integral and the symbol $*$ denotes the integral with respect to a random measure:

$$H \cdot Y_t = \int_0^t H_s \, dY_s \qquad \text{and} \qquad W * \mu_t = \int_0^t \int_{\mathbb{R}^d} W(s, x) \mu(ds, dx).$$

The transpose of a matrix U is U^{T}, and vectors are column matrices. We refer to [8] for the unexplained terminology and notation.

Let us recall (see [8, Definition III.5.8]) that a generalized increasing process is a process Y such that it is $[0, \infty]$-valued and $Y_0 = 0$ and its paths are non-decreasing, and if $T = \inf(t : Y_t = \infty)$ then Y is right-continuous on $[\![0, T[\![$. Given a measure Q, we say that a generalized increasing process Y^1 strictly dominates a generalized increasing process Y^2 (Q-a.s.), and we write $Y^1 \succ Y^2$ (Q-a.s.) or

$Y^2 \prec Y^1$ (Q-a.s.), if paths of the difference $Y^1 - Y^2$ (with $\infty - \infty = \infty$) are non-decreasing Q-a.s.

To simplify the formulation of the main result we assume here that $d = 1$. Let $\langle M \rangle$ (respectively, $\langle M' \rangle$) be the quadratic characteristic of M (respectively, M') with respect to P (respectively, P'). Set $B = A - A'$ and $C = (\langle M \rangle + \langle M' \rangle)/2$. There exists a predictable function γ with values in $[0, \infty]$ such that

$$B = \gamma \cdot C + I(\gamma = \infty) \cdot B \qquad (P + P')\text{-a.s.} \tag{1.1}$$

(the Lebesgue decomposition of dB_t with respect to dC_t).

THEOREM 1.1. *Let γ be a predictable function satisfying* (1.1). *There exists a version h of $h(\frac{1}{2}; P, P')$ such that*

$$\frac{1}{(1 - \Delta h)^2} \cdot h \succ \frac{1}{8} \gamma^2 \cdot C \qquad (P + P')\text{-a.s.} \tag{1.2}$$

REMARK 1.1: Let $z = (z_t)_{t \geq 0}$ (respectively, $z' = (z'_t)_{t \geq 0}$) be the density process of the measure P (respectively, P') with respect to $Q = (P + P')/2$ and $\Gamma = \{(t, \omega) : z_{t-}(\omega) > 0, z'_{t-}(\omega) > 0\} \cup [\![0]\!]$. We call a process h a version of the Hellinger process $h(\frac{1}{2}; P, P')$ of order $1/2$ if h is a predictable generalized increasing process and coincides with the Hellinger process in the strict sense, of order $1/2$, on Γ. Moreover, we shall always assume that $\Delta h \leq 1$ identically on $\{h < \infty\}$ (which is not a real restriction since $\Delta h \leq 1$ on Γ). The left-hand side of (1.2) is assumed to be equal ∞ at the time t for ω such that $h_t(\omega) = \infty$ or $\tau(\omega) \leq t$, where $\tau(\omega) = \inf(t : \Delta h_t(\omega) = 1)$.

The proof of Theorem 1.1 is based on auxiliary results which are of independent interest. First, we construct a predictable increasing process k with values in $[0, \infty]$, closely related to $h(\frac{1}{2}; P, P')$. In particular, if h is the strict version of $h(\frac{1}{2}; P, P')$, then

$$h \prec k \prec 2h - [h, h]. \tag{1.3}$$

Next, we construct a sequence of stopping times T_n such that $\{h < \infty\} = \cup_n [\![0, T_n]\!]$ $(P + P')$-a.s., and a new reference measure Q^n equivalent to $P + P'$ with the following properties. Let z^n and z'^n be the density processes of P and P' with respect to Q^n. Then $z^n = z^n_0 \mathcal{E}(V^n)$ and $z'^n = z'^n_0 \mathcal{E}(-V^n)$ on $[\![0, T^n]\!]$, where V^n is a Q^n-locally square-integrable martingale and $\mathcal{E}(\cdot)$ stands for the Doléans exponential. This fact has an important consequence: if Y is a semimartingale with respect to P and P' (and hence with respect to Q^n) with the triplets $T = (B, C, \nu)$ and $T' = (B', C', \nu')$ of predictable characteristics respectively, then its triplet $T^n = (B^n, C^n, \nu^n)$ of predictable characteristics with respect to Q^n can be expressed Q^n-a.s. on the set $\Gamma \cap [\![0, T^n]\!]$ as follows

$$B^n = \frac{B + B'}{2}, \qquad C^n = \frac{C + C'}{2}, \qquad \nu^n = \frac{\nu + \nu'}{2}.$$

In particular, X is a Q^n-locally square-integrable martingale with the canonical decomposition $X = X_0 + (M + M')/2 + (A + A')/2$ on $\Gamma \cap [\![0, T^n]\!]$.

The connection between the objects just introduced and the initial problem is explained as follows. On the one hand, the Q^n-quadratic characteristic $\langle M + M', M + M'\rangle$ and the Q^n-mutual characteristic $\langle M + M', V^n\rangle$ can be expressed explicitly in terms of B and C on $\Gamma \cap [\![0, T_n]\!]$. On the other hand, the Q^n-quadratic characteristic $\langle V^n, V^n\rangle$ coincides with k on $[\![0, T_n]\!]$. In combination with the Kunita–Watanabe inequality this leads to the inequality

$$\frac{1}{1 - \Delta k} \cdot k \succ \frac{1}{4}\gamma^2 \cdot C \qquad (P + P')\text{-a.s.} \tag{1.4}$$

on $\cup_n [\![0, T_n]\!]$. Now (1.2) follows easily from (1.3) and (1.4).

Inequalities (1.2) and (1.4) can be regarded as a generalization of some information-type inequalities used to obtain some form of the Cramér–Rao inequality, to the case of filtered spaces: compare (1.4) with the inequality in [14, Lemma 1, p. 127] and (1.2) with the inequalities in [11], [19, Section 5.4], [14, Corollary 1, p. 128].

Theorem 1.1 (and its multi-dimensional version) should be compared with another lower bound for the Hellinger process expressed in terms of the triplets of predictable characteristics of X with respect to P and P' (Theorem IV.3.39 in [8], cf. also [15]). Of course, our bound is more rough in general and it can be deduced directly from the bound in terms of the triplets.

The expression standing on the left in (1.2) is sometimes difficult to use. We suggest the following estimate for it, though the inequality is very rough for the continuous part of h. $\mathcal{E}(-h)$ is the Doléans exponential of $-h$:

$$\mathcal{E}(-h)_T = \begin{cases} e^{-h_T} \prod_{s \le T}(1 - \Delta h_s) e^{\Delta h_s}, & \text{if} \quad h_T < \infty, \\ 0, & \text{if} \quad h_T = \infty. \end{cases}$$

LEMMA 1.1. *For any stopping time T,*

$$\int_0^T \frac{1}{(1 - \Delta h_s)^2}\, dh_s \le \frac{1}{\mathcal{E}(-h)_T^2}. \tag{1.5}$$

Section 2 contains some auxiliary results mentioned above. A multi-dimensional version of Theorem 1.1 and its proof will appear in Section 3. In Section 4 we shall consider applications of Theorem 1.1 to semimartingale regression models.

2. Auxiliary Results

2.1. Preliminaries

Let $(\Omega, \mathcal{F})$ be a measurable space with two probability measures P and P'. Let Q be another probability measure on $(\Omega, \mathcal{F})$ such that

$$P \ll Q, \quad P' \ll Q. \tag{2.1}$$

We consider the Radon–Nikodým derivatives

$$z = \frac{dP}{dQ}, \qquad z' = \frac{dP'}{dQ}.$$

Let $\varphi: \mathbb{R}_+^2 \to \mathbb{R}_+$ and $\psi: \mathbb{R}_+^2 \to \mathbb{R}_+$ be functions defined by

$$\varphi(u,v) = \frac{1}{2}(\sqrt{u} - \sqrt{v})^2,$$

$$\psi(u,v) = \frac{1}{2}\frac{(u-v)^2}{u+v}$$

(here and in the sequel 0/0=0). It is easy to check that

$$\varphi \le \psi, \qquad 2\varphi - \psi = \frac{(\sqrt{u}-\sqrt{v})^4}{2(u+v)} \ge 0. \tag{2.2}$$

Let us recall that the variation distance $\|P - P'\|$ and the Hellinger distance $\rho(P, P')$ between P and P' are defined by

$$\|P - P'\| = E_Q|z - z'|, \qquad \rho^2(P,P') = E_Q\varphi(z,z').$$

$H(P,P')$ denotes the Hellinger integral of order 1/2 between P and P':

$$H(P,P') = E_Q\sqrt{zz'} = 1 - \rho^2(P,P').$$

We also introduce another distance $\kappa(P,P')$ defined by

$$\kappa^2(P,P') = E_Q\psi(z,z').$$

This distance arises naturally in some estimation problems, see, e.g., [13] and [14], and was used by Hellinger himself [6]. All the above distances do not depend on the measure Q satisfying (2.1). It follows immediately from (2.2) that

$$\rho^2(P,P') \leq \kappa^2(P,P') \leq 2\rho^2(P,P') - \rho^4(P,P').$$

2.2. Definition and properties of the process k

In the rest of this section, we consider a filtered space $(\Omega, \mathcal{F}, \mathbb{F})$ with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$, and two fixed probability measures P and P' on $(\Omega, \mathcal{F})$. Let Q be another probability measure on $(\Omega, \mathcal{F})$. We assume that P and P' are locally absolutely continuous with respect to Q:

$$P \overset{\text{loc}}{\ll} Q, \qquad P' \overset{\text{loc}}{\ll} Q. \tag{2.3}$$

Let $z = (z_t)_{t\geq 0}$ (respectively, $z' = (z'_t)_{t\geq 0}$) be the density process of the measure P (respectively, P') with respect to Q. The processes z and z' are non-negative Q-martingales. We denote by z^c and z'^c the continuous martingale parts of z and z', relative to Q. We also denote by $\nu^{(z,z')}$ the Q-compensator of the jump measure of the bi-dimensional process (z, z'). It is known that $\nu^{(z,z')}$ only charges the set $\Lambda = \{(\omega, t, x, y) : t > 0,\ x \geq -z_{t-}(\omega),\ x = 0 \text{ if } z_{t-}(\omega) = 0,\ y \geq -z'_{t-}(\omega),\ y = 0 \text{ if } z'_{t-}(\omega) = 0\}$ [8, Theorem IV.1.33]. Set

$$S_n = \inf(t : z_t \wedge z'_t < 1/n), \qquad S = \lim_n \uparrow S_n, \qquad \Gamma = \cup_n [\![0, S_n]\!]. \tag{2.4}$$

Note that, up to a Q-evanescent set, $\Gamma = \{z_- > 0, z'_- > 0\} \cup [\![0]\!]$.

Let us recall that a version of the Hellinger process $h(\frac{1}{2}; P, P')$ of order 1/2 between P and P' is given by

$$\begin{aligned} h' = \frac{1}{8}\left\{\frac{1}{z_-^2} \cdot \langle z^c, z^c\rangle - \frac{2}{z_- z'_-} \cdot \langle z^c, z'^c\rangle + \frac{1}{z'^2_-} \cdot \langle z'^c, z'^c\rangle\right\} \\ + \varphi\left(1 + \frac{x}{z_-}, 1 + \frac{y}{z'_-}\right) * \nu^{(z,z')} \end{aligned} \tag{2.5}$$

(Theorem IV.1.33 in [8]) and

$$h = 1_\Gamma \cdot h' \tag{2.6}$$

is the Hellinger process in the strict sense, of order 1/2, between P and P'. Put

$$\begin{aligned} k' = \frac{1}{4}\left\{\frac{1}{z_-^2} \cdot \langle z^c, z^c\rangle - \frac{2}{z_- z'_-} \cdot \langle z^c, z'^c\rangle + \frac{1}{z'^2_-} \cdot \langle z'^c, z'^c\rangle\right\} \\ + \psi\left(1 + \frac{x}{z_-}, 1 + \frac{y}{z'_-}\right) * \nu^{(z,z')} \end{aligned} \tag{2.7}$$

and

$$k = 1_\Gamma \cdot k'. \tag{2.8}$$

PROPOSITION 2.1. a) *k is a predictable increasing $[0,\infty]$-valued process,*

$$\Gamma \subseteq \{k < \infty\} = \{h < \infty\} \qquad (P+P')\text{-a.s.},$$

$$h \prec k \prec 2h - [h,h] \qquad (P+P')\text{-a.s.}. \tag{2.9}$$

b) *Up to a Q-null set, $\Delta k \leq 1$ and $\{\Delta k = 1\} = \{\Delta h = 1\} \subseteq [\![S]\!] \cap \Gamma$.*

c) *The process k does not depend upon the measure Q satisfying* (2.3), *in the following sense: if $\overline{Q}$ is another measure with $Q \overset{\text{loc}}{\ll} \overline{Q}$, and if k and $\overline{k}$ are the processes computed through Q and $\overline{Q}$, then k and $\overline{k}$ are Q-indistinguishable.*

Thus, the process k is defined uniquely, up to a $(P+P')$-evanescent set, regardless of the dominating measure Q.

PROOF: a) All the statements except of the second inequality in (2.9) follow immediately from (2.2), (2.5)–(2.8) and the corresponding properties of the Hellinger process. Moreover,

$$2h - k = \frac{2\varphi^2(1+x/z_-, 1+y/z'_-)}{(1+x/z_-)+(1+y/z'_-)} 1_\Gamma * \nu^{(z,z')}. \tag{2.10}$$

But, for any predictable stopping time T,

$$\begin{aligned}(\Delta h_T)^2 &= \left\{\int \varphi\left(1+\frac{x}{z_{T-}}, 1+\frac{y}{z'_{T-}}\right)\nu^{(z,z')}(\{T\}\times dx)\right\}^2 \\ &\leq \int \frac{\varphi^2(1+x/z_{T-}, 1+y/z'_{T-})}{(1+x/z_{T-})+(1+y/z'_{T-})}\nu^{(z,z')}(\{T\}\times dx) \\ &\quad \times \int \{(1+x/z_{T-})+(1+y/z'_{T-})\}\nu^{(z,z')}(\{T\}\times dx)\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\int \{(1+x/z_{T-})+(1+y/z'_{T-})\}\nu^{(z,z')}(\{T\}\times dx) \\ &= E_Q\left(\left(1+\frac{\Delta z_T}{z_{T-}}\right)1_{\{\Delta z_T \neq 0\}}\middle|\mathcal{F}_{T-}\right) + E_Q\left(\left(1+\frac{\Delta z'_T}{z'_{T-}}\right)1_{\{\Delta z'_T \neq 0\}}\middle|\mathcal{F}_{T-}\right) \\ &\leq E_Q\left(\frac{z_T}{z_{T-}}\middle|\mathcal{F}_{T-}\right) + E_Q\left(\frac{z'_T}{z'_{T-}}\middle|\mathcal{F}_{T-}\right) = 2\end{aligned} \tag{2.12}$$

on $\{\omega\colon (\omega, T(\omega)) \in \Gamma\}$. Now $2h - k \succ [h,h]$ readily follows from (2.10)–(2.12).

b) The result follows from (2.9) and the corresponding assertions for the Hellinger process, see Lemma IV.1.30 in [8].

c) The proof is essentially the same as the proofs of Theorem IV.1.22 and Proposition IV.1.45 in [8], and may be omitted.

REMARK 2.1: Let

$$K' = \frac{1}{4}\left\{\frac{1}{z_-^2}\cdot\langle z^c, z^c\rangle - \frac{2}{z_- z'_-}\cdot\langle z^c, z'^c\rangle + \frac{1}{z'^2_-}\cdot\langle z'^c, z'^c\rangle\right\} + \sum_{s\le\cdot}\psi\left(1+\frac{\Delta z_s}{z_{s-}}, 1+\frac{\Delta z'_s}{z'_{s-}}\right) \tag{2.13}$$

and

$$K = 1_\Gamma \cdot K'. \tag{2.14}$$

From the definitions of K and k, if T is a stopping time, the stopped process K^T is Q-locally integrable if and only if $k_T < \infty$ Q-a.s., and k^T is the Q-compensator of K^T in this case.

REMARK 2.2: There is a relationship between the process k and the distance κ. In particular, if P (respectively, P') is the law of a sequence of independent random variables with a distribution ρ_n (respectively, ρ'_n), then using the standard embedding of the discrete-time model into the continuous-time one, we obtain

$$k_n = \sum_{1\le p\le n} \kappa^2(\rho_p, \rho'_p).$$

But it should be noted that, though the quantity $\kappa^2(P, P')$ is the f-divergence of P and P' with $f(x) = (x-1)^2/2(1+x)$, our process k do not coincide with the so-called (f, g)-process introduced in [20]. On the other hand, the second summand in the right-hand side of (2.13) coincides with the process $j(f)$ ([8, § IV.1d]) on $\{z'_- > 0\}$ and may have an additional jump when the density z' jumps to 0.

REMARK 2.3: Assume that a d-dimensional process X is given on $(\Omega, \mathcal{F})$, which is a semimartingale relative to P and P'. Similarly to the processes $h(\alpha; P, P')$ and $i(\psi; P, P')$ (see [8, Section IV.3] and also [15]), one can define a process k^0 in terms of the local characteristics of X such that $k \succ k^0$ ($(P+P')$-a.s.) in general and $k = k^0$ if P and P' are the unique solutions of some martingale problems based upon X.

Assume now that $Q = (P + P')/2$, and let ν^z be the Q-compensator of the jump measure of the process z. Since $z+z' = 2$, $z^c + z'^c = 0$, $\langle z^c, z^c\rangle = \langle z'^c, z'^c\rangle = -\langle z^c, z'^c\rangle$, $\Delta z' = -\Delta z$ and neither $\langle z^c, z^c\rangle$ nor ν^z charge the complement of Γ in this case, we obtain from (2.7), (2.8), (2.13) and (2.14) that

$$K = \frac{1}{4}\left(\frac{1}{z_-} + \frac{1}{z'_-}\right)^2\cdot\langle z^c, z^c\rangle + \sum_{s\le\cdot}\psi\left(1+\frac{\Delta z_s}{z_{s-}}, 1-\frac{\Delta z_s}{z'_{s-}}\right). \tag{2.15}$$

and

$$k = \frac{1}{4}\left(\frac{1}{z_-} + \frac{1}{z'_-}\right)^2 \cdot \langle z^c, z^c\rangle + \psi\left(1 + \frac{x}{z_-}, 1 - \frac{x}{z'_-}\right) * \nu^z, \tag{2.16}$$

where $z' = 2 - z$.

2.3. Definition and properties of a new reference measure

In this subsection the setting is the same as in the previous one. It will be convenient to take $Q = (P + P')/2$; then $z + z' = 2$. The processes K and k are given by (2.15) and (2.16), S_n, S and Γ are defined by (2.4). Put $\Sigma = \{k < \infty\}$.

Let us recall that if a predictable function H belongs to the class $L^1_{\text{loc}}(z, Q)$, i.e. if $(H^2 \cdot [z, z])^{1/2}$ is a Q-locally integrable increasing process, then the stochastic integral $H \cdot z$ is well defined and is a Q-local martingale, see, e.g., [7, § II.2.b].

PROPOSITION 2.2. *Let*

$$H = \frac{1}{2}\left(\frac{1}{z_-} - \frac{1}{z'_-}\right) 1_\Gamma.$$

There is an increasing sequence (T_n) of predictable stopping times, such that

$$\Sigma = \cup_n [\![0, T_n]\!] \quad Q\text{-a.s.} \tag{2.17}$$

and the following statements hold for all n.

a)

$$H 1_{[\![0,T_n]\!]} \in L^1_{\text{loc}}(z, Q).$$

b) *Let*

$$N^n = \left(H 1_{[\![0,T_n]\!]}\right) \cdot z \qquad \text{and} \qquad Z^n = \mathcal{E}(N^n).$$

Then Z^n is a Q-uniformly integrable martingale and $Q(\{\inf_t Z^n_t > 0\}) = 1$.

REMARK 2.4: Of course, it follows from this proposition that there is a process N, unique (up to Q-indistinguishability) on the set Σ, such that

$$H 1_{[\![0,T]\!]} \in L^1_{\text{loc}}(z, Q)$$

and

$$N^T = \left(H 1_{[\![0,T]\!]}\right) \cdot z$$

for every stopping time T such that $[\![0, T]\!] \subseteq \Sigma \quad Q$-a.s. In particular, $N^{T_n} = N^n$. Similarly, there is a process Z, unique (up to Q-indistinguishability) on the set Σ, such that $Z = \mathcal{E}(N)$ on Σ and $Z^{T_n} = Z^n$ for all n.

PROOF: Set

$$T_n = \inf(t\colon k_t \geq n). \tag{2.18}$$

The stopping time T_n is predictable and $T_n > 0$ for each n; moreover, (2.17) holds.

First, we shall prove that

$$H^n = H1_{[\![0,T_n[\![} \in L^1_{\mathrm{loc}}(z,Q) \qquad \text{for all } n. \tag{2.19}$$

It is easy to check that

$$\frac{\Delta z_s}{z_{s-}} - \frac{\Delta z_s}{z'_{s-}} = \frac{\Delta z_s}{z_{s-}} - \frac{\Delta z_s}{2 - z_{s-}} \geq -1 \tag{2.20}$$

on Γ, hence $H^n \Delta z \geq -1/2$. Therefore, (2.19) holds if (and only if) the increasing process

$$C^n = (H^n)^2 \cdot \langle z^c, z^c \rangle + \sum_{s \leq \cdot} \left(1 - \sqrt{1 + H^n_s \Delta z_s}\right)^2$$

is Q-locally integrable ([7, Corollary 2.57]). Both $1 + H^n_s \Delta z_s = 1 + \frac{1}{2}\left(\frac{\Delta z_s}{z_{s-}} - \frac{\Delta z_s}{z'_{s-}}\right)$ and 1 always lie between $1 + \frac{\Delta z_s}{z_{s-}}$ and $1 - \frac{\Delta z_s}{z'_{s-}}$, hence

$$\begin{aligned}\left(1 - \sqrt{1 + H^n_s \Delta z_s}\right)^2 &\leq \left(\sqrt{1 + \frac{\Delta z_s}{z_{s-}}} - \sqrt{1 - \frac{\Delta z_s}{z'_{s-}}}\right)^2 \\ &= 2\varphi\left(1 + \frac{\Delta z_s}{z_{s-}}, 1 - \frac{\Delta z_s}{z'_{s-}}\right) \leq 2\psi\left(1 + \frac{\Delta z_s}{z_{s-}}, 1 - \frac{\Delta z_s}{z'_{s-}}\right),\end{aligned}$$

and C^n is strictly dominated by $2K^{T_n-}$ ($K^{T_n-} := 1_{[\![0,T_n[\![} \cdot K$). Since $E_Q K_{T_n-} = E_Q k_{T_n-} \leq n$ due to (2.18), C^n is a Q-integrable increasing process and (2.19) follows.

Moreover, the above arguments show that the Q-compensator of C^n is dominated by $2k^{T_n-}$ and, hence, bounded by $2n$. Let $\overline{N}^n = H^n \cdot z$ and $\overline{Z}^n = \mathcal{E}(\overline{N}^n)$. Since $\Delta \overline{N}^n \geq -1/2$ (see (2.20)), it follows from Theorem 12 in [9] (see also Corollaries 8.17 and 8.30 in [7]) that $\overline{Z}^n$ is a Q-uniformly integrable martingale and

$$Q(\{\inf_t \overline{Z}^n_t > 0\}) = 1. \tag{2.21}$$

To complete the proof of a), it is enough to show that

$$\left| \frac{\Delta z_{T_n}}{z_{T_n-}} - \frac{\Delta z_{T_n}}{z'_{T_n-}} \right| 1_{\{T_n < \infty\}} \in L^1(Q). \tag{2.22}$$

Since $|y-x| \le x+y-xy$ for $0 \le x \le 2$, $0 \le y \le 2$, (2.22) follows from

$$\begin{aligned} &E_Q\left\{\left|\frac{\Delta z_{T_n}}{z_{T_n-}} - \frac{\Delta z_{T_n}}{z'_{T_n-}}\right| 1_{\{T_n<\infty\}}\middle|\mathcal{F}_{T_n-}\right\} \\ &\le \left|\frac{1}{z_{T_n-}} - \frac{1}{z'_{T_n-}}\right| E_Q(z_{T_n-} + z_{T_n} - z_{T_n-}z_{T_n}|\mathcal{F}_{T_n-})1_{\{T_n<\infty\}} \\ &= |z'_{T_n-} - z_{T_n-}|1_{\{T_n<\infty\}} \le 2. \end{aligned} \tag{2.23}$$

Now, $Z^n = \overline{Z}^n + \Delta Z^n 1_{[\![T_n,\infty[\![}$, $\Delta Z^n 1_{\{T_n<\infty\}} = Z^n_{T_n-}\Delta N_{T_n}1_{\{T_n<\infty\}}$, and $Z^n_{T_n-} = \overline{Z}^n_{T_n-} \in L^1(Q)$ from what precedes, so (2.23) implies that $\Delta Z^n 1_{\{T_n<\infty\}} \in L^1(Q)$ and, therefore, Z^n is a Q-uniformly integrable martingale. The final statement follows from (2.21) and (2.20).

REMARK 2.5: Similarly to the proof above, one can check that

$$\frac{1}{z_-}1_{\Gamma\cap[\![0,T_n]\!]} \in L^1_{\text{loc}}(z,Q), \qquad \frac{1}{z'_-}1_{\Gamma\cap[\![0,T_n]\!]} \in L^1_{\text{loc}}(z,Q) = L^1_{\text{loc}}(z',Q),$$

so there are processes n and n', unique (up to Q-indistinguishability) on the set Σ, such that n^T and n'^T are Q-local martingales and

$$n^T = \left(\frac{1}{z_-}1_{\Gamma\cap[\![0,T]\!]}\right)\cdot z, \qquad n'^T = \left(\frac{1}{z'_-}1_{\Gamma\cap[\![0,T]\!]}\right)\cdot z'$$

for every stopping time T such that $[\![0,T]\!] \subseteq \Sigma$ $\;Q$-a.s. Evidently, we have

$$N = \frac{n+n'}{2}$$

on Σ, where N is the process introduced in Remark 2.4. Moreover,

$$z = z_0\mathcal{E}(n) \qquad \text{and} \qquad z' = z'_0\mathcal{E}(n')$$

on Σ. Indeed, it follows from the definition of n that $z = z_0 + z_- \cdot n$ on Γ; on the other hand, $1_{\Gamma^c}\cdot z$ and $(z_-1_{\Sigma\cap\Gamma^c})\cdot n = 0$.

From now on we consider a sequence (T_n) of stopping times, satisfying Proposition 2.2, and fix some n. For simplicity, put $T = T_n$, $\overline{Z} = Z^n$, $\overline{N} = N^n$, $\overline{n} = n^T$, $\overline{n}' = n'^T$. It follows from Proposition 2.2 that $E_Q\overline{Z}_\infty = 1$ and $\overline{Z}_\infty > 0$ $\;Q$-a.s. So we can define a new probability measure $\overline{Q}$ by

$$d\overline{Q} = \overline{Z}_\infty\, dQ,$$

$\overline{Q}$ being equivalent to $Q = (P + P')/2$. Let $\overline{z}$ and $\overline{z}'$ be the density processes of P and P' with respect to $\overline{Q}$; $\overline{z}$ and $\overline{z}'$ are non-negative $\overline{Q}$-uniformly integrable martingales and

$$z = \overline{z}\overline{Z}, \qquad z' = \overline{z}'\overline{Z}. \tag{2.24}$$

Since Q and $\overline{Q}$ are equivalent, a semimartingale with respect to one of these measures is also a semimartingale with respect to another one with the same quadratic variation; moreover, stochastic integrals with respect to Q and $\overline{Q}$ coincide (see, e.g., [7, Chapter 7]). For this reason, we will not mention the reference measure when dealing with quadratic variations, stochastic integrals or Doléans exponentials.

THEOREM 2.1. *Let*

$$V = \frac{\overline{n} - \overline{n}'}{2} - \frac{1}{1 + \Delta\overline{N}} \cdot \left[\frac{\overline{n} - \overline{n}'}{2}, \overline{N}\right]. \tag{2.25}$$

Then V is a $\overline{Q}$-locally square-integrable martingale, its quadratic characteristic $\langle V, V\rangle$ with respect to $\overline{Q}$ coincides with the process k on $[\![0,T]\!]$ and

$$\overline{z} = z_0\mathcal{E}(V) \qquad \textit{and} \qquad \overline{z}' = z_0'\mathcal{E}(-V) \qquad \textit{on} \quad [\![0,T]\!]. \tag{2.26}$$

REMARK 2.6: In the terminology of Chitashvili, Lazrieva and Toronjadze [1], V is the L-transformation of $(\overline{n} - \overline{n}')/2$.

PROOF: Of course, V is a Q-semimartingale. After some elementary calculations from (2.25) we obtain

$$V + \overline{N} + [V, \overline{N}] = \overline{n} \qquad \text{and} \qquad -V + \overline{N} - [V, \overline{N}] = \overline{n}'.$$

Now, by Yor's formula for the product of Doléans exponentials,

$$z_0\mathcal{E}(V)\overline{Z} = z_0\mathcal{E}(V)\mathcal{E}(\overline{N}) = z_0\mathcal{E}(V + \overline{N} + [V, \overline{N}]) = z_0\mathcal{E}(\overline{n}) = z^T \tag{2.27}$$

and

$$z_0'\mathcal{E}(-V)\overline{Z} = z_0'\mathcal{E}(-V)\mathcal{E}(\overline{N}) = z_0'\mathcal{E}(-V + \overline{N} - [V, \overline{N}]) = z_0'\mathcal{E}(\overline{n}') = z'^T, \tag{2.28}$$

and (2.26) follows from (2.24), (2.27) and (2.28). Moreover, since both $\overline{z}$ and $\overline{z}'$ are $\overline{Q}$-martingales, (2.26) implies that V is a $\overline{Q}$-local martingale on the set $[\![0,T]\!] \cap \big(\{\overline{z}_- > 0\} \cup \{\overline{z}'_- > 0\}\big)$. But $V = V^T$ and $\{\overline{z}_- > 0\} \cup \{\overline{z}'_- > 0\} = \{z_- > 0\} \cup \{z'_- > 0\} = \Omega \times \mathbb{R}_+$, hence V is a $\overline{Q}$-local martingale.

To complete the proof it is enough to check that $[V,V]-k^T$ is a $\overline{Q}$-local martingale. Some uncomplicated calculations yield the relation

$$(1+\Delta\overline{N})\cdot[V,V]=K^T,$$

or, equivalently,

$$\overline{Z}\cdot[V,V]=\overline{Z}_-\cdot K^T.$$

Hence, $\overline{Z}\cdot[V,V]-\overline{Z}_-\cdot k^T$ is a Q-local martingale. Since $[V,V]\overline{Z}=[V,V]_-\cdot\overline{Z}+\overline{Z}\cdot[V,V]$, $[V,V]\overline{Z}-\overline{Z}\cdot[V,V]$ is a Q-local martingale. Since k is predictable with finite variation, $k^T\overline{Z}=k^T\cdot\overline{Z}+\overline{Z}_-\cdot k^T$ and $k^T\overline{Z}-\overline{Z}_-\cdot k^T$ is also a Q-local martingale. Combining all these relations, we conclude that $([V,V]-k^T)\overline{Z}$ is a Q-local martingale, and the claim follows.

Relation (2.26) is a characteristic property of the measure $\overline{Q}$. This property has many consequences. We start with the most important one.

THEOREM 2.2. *Let X be a d-dimensional semimartingale on $(\Omega,\mathcal{F},\mathbb{F},P)$ and on $(\Omega,\mathcal{F},\mathbb{F},P')$, with respective characteristics (B,C,ν) and (B',C',ν'), associated with the same truncation function h. Then X is a semimartingale on $(\Omega,\mathcal{F},\mathbb{F},\overline{Q})$ with characteristics $(\overline{B},\overline{C},\overline{\nu})$ associated with h, satisfying*

$$\overline{B}=\frac{B+B'}{2},\qquad \overline{C}=\frac{C+C'}{2},\qquad \overline{\nu}=\frac{\nu+\nu'}{2} \tag{2.29}$$

on $\Gamma\cap[\![0,T]\!]$.

REMARK 2.7: The same proof as below yields an explicit expression for $(\overline{B},\overline{C},\overline{\nu})$ also on $[\![0,T]\!]\setminus\Gamma$. Moreover, it is easy to give an explicit expression for $(\overline{B},\overline{C},\overline{\nu})$ outside $[\![0,T]\!]$ as well (cf. Theorem III.3.40 in [8]).

PROOF: The process X is a Q-semimartingale (see, e.g., Theorem III.3.40 in [8]) and hence a $\overline{Q}$-semimartingale, the characteristics being denoted by $(\overline{B},\overline{C},\overline{\nu})$. Let A be an increasing process such that $\overline{C}^{ij}=\overline{c}^{ij}\cdot A$. Now we shall apply Girsanov's theorem for semimartingales (see, e.g., Theorem III.3.24 in [8]) to the pairs $(\overline{Q},P)$ and $(\overline{Q},P')$. If μ is a random measure on $\mathbb{R}_+\times\mathbb{R}^d$, we denote by M_μ the Doléans measure on $(\Omega\times\mathbb{R}_+\times\mathbb{R}^d,\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R}^d)$ associated with μ and $\overline{Q}$:

$$M_\mu(W)=E_{\overline{Q}}(W*\mu_\infty).$$

First, we have $C=\overline{C}$ P-a.s. and $C'=\overline{C}$ P'-a.s. Therefore, $C=\overline{C}$ on $\{\overline{z}_->0\}$ and $C'=\overline{C}$ on $\{\overline{z}'_->0\}$ $\overline{Q}$-a.s., in particular, $\overline{C}=(C+C')/2$ on Γ $\overline{Q}$-a.s.

Next, put

$$Y=M_\mu x\left(\frac{\overline{z}}{\overline{z}_-}1_{\{\overline{z}_->0\}}\middle|\widetilde{\mathcal{P}}\right)\qquad\text{and}\qquad Y'=M_\mu x\left(\frac{\overline{z}'}{\overline{z}'_-}1_{\{\overline{z}'_->0\}}\middle|\widetilde{\mathcal{P}}\right), \tag{2.30}$$

where μ^X is the jump measure of X. By Girsanov's theorem, $\nu = Y\overline{\nu}$ P-a.s. and $\nu' = Y'\overline{\nu}$ P'-a.s., therefore,

$$1_{\{\overline{z}_->0\}}\nu = Y1_{\{\overline{z}_->0\}}\overline{\nu} \qquad \text{and} \qquad 1_{\{\overline{z}'_->0\}}\nu' = Y'1_{\{\overline{z}'_->0\}}\overline{\nu} \qquad \overline{Q}\text{-a.s.} \quad (2.31)$$

Since

$$\frac{\overline{z}}{\overline{z}_-}1_{\{\overline{z}_->0\}} = 1 + \Delta V \qquad \text{and} \qquad \frac{\overline{z}'}{\overline{z}'_-}1_{\{\overline{z}'_->0\}} = 1 - \Delta V$$

on $[\![0,T]\!]$ by Theorem 2.1, (2.30) implies

$$Y + Y' = 2 \tag{2.32}$$

M_{μ^X}-a.s. and hence $M_{\overline{\nu}}$-a.s. on $[\![0,T]\!]$. Combining (2.31) and (2.32), we obtain $\overline{\nu} = (\nu + \nu')/2$ on $\Gamma \cap [\![0,T]\!]$.

Finally, let predictable processes $\beta = (\beta^i)_{i\le d}$ and $\beta' = (\beta'^i)_{i\le d}$ satisfy the relations

$$\langle \overline{z}^c, X^{i,c}\rangle = \Big(\sum_{j\le d} \overline{c}^{ij}\beta^j\overline{z}_-\Big)\cdot A, \tag{2.33}$$

$$\langle \overline{z}'^c, X^{i,c}\rangle = \Big(\sum_{j\le d} \overline{c}^{ij}\beta'^j\overline{z}'_-\Big)\cdot A, \tag{2.34}$$

where $\overline{z}^c$, $\overline{z}'^c$ and $X^{i,c}$ are continuous martingale parts of $\overline{z}$, $\overline{z}'$ and X^i relative to $\overline{Q}$, and $\langle \overline{z}^c, X^{i,c}\rangle$ and $\langle \overline{z}'^c, X^{i,c}\rangle$ are the corresponding mutual characteristics relative to $\overline{Q}$. By Girsanov's theorem,

$$B^i = \overline{B}^i + \Big(\sum_{j\le d} \overline{c}^{ij}\beta^j\Big)\cdot A + h^i(x)(Y-1) * \overline{\nu} \qquad P\text{-a.s.}, \tag{2.35}$$

$$B'^i = \overline{B}^i + \Big(\sum_{j\le d} \overline{c}^{ij}\beta'^j\Big)\cdot A + h^i(x)(Y'-1) * \overline{\nu} \qquad P'\text{-a.s.} \tag{2.36}$$

Therefore, (2.35) and (2.36) hold $\overline{Q}$-a.s. on $\{\overline{z}_- > 0\}$ and $\{\overline{z}'_- > 0\}$ respectively. Since

$$\overline{z}^c = \overline{z}_- \cdot V^c \qquad \text{and} \qquad \overline{z}'^c = -\overline{z}'_- \cdot V^c$$

on $[\![0,T]\!]$ by Theorem 2.1, where V^c is a continuous martingale part of V relative to $\overline{Q}$, it follows from (2.33) and (2.34) that one may take

$$\beta'^j = -\beta^j \qquad \text{on} \qquad [\![0,T]\!]. \tag{2.37}$$

Now the relation $\overline{B} = (B + B')/2$ on $\Gamma \cap [\![0,T]\!]$ follows from (2.32) and (2.35)–(2.37).

Let us recall that a d-dimensional process X is called a special semimartingale relative to (say) P with the canonical decomposition $X = X_0 + M + A$, $M = (M^i)_{i\le d}$, $A = (A^i)_{i\le d}$, if M^i is a local martingale with respect to P and A^i is a predictable process with finite variation over compact intervals for every $i \le d$. If, moreover, M^i is a locally square-integrable martingale with respect to P, then X is said to be a locally square-integrable semimartingale relative to P.

COROLLARY 2.1. *Let X be a d-dimensional special semimartingale relative to P and P' with the canonical decomposition $X = X_0 + M + A$ and $X = X_0 + M' + A'$ respectively. Then X is a special semimartingale on $\Gamma \cap [\![0, T]\!]$ relative to $\overline{Q}$ with the canonical decomposition*

$$X = X_0 + \frac{M + M'}{2} + \frac{A + A'}{2} \qquad \text{on} \qquad \Gamma \cap [\![0, T]\!].$$

The quadratic co-variation $[M^i + M'^i, V]$ (relative to $\overline{Q}$) is $\overline{Q}$-locally integrable on $\Gamma \cap [\![0, T]\!]$ with the $\overline{Q}$-compensator $A^i - A'^i$ for every $i \leq d$.

PROOF: The first claim immediately follows from Theorem 2.2 and Proposition II.2.29 in [8]. In particular, $M + M'$ is a $\overline{Q}$-local martingale on $\Gamma \cap [\![0, T]\!]$. But $M + M' = 2M + (A - A')$ is also a P-special semimartingale. By Corollary 7.29 in [7] (or Theorem 7.1 in [4]), the process $[M^i + M'^i, \overline{z}]$ is $\overline{Q}$-locally integrable on $\Gamma \cap [\![0, T]\!]$ and $A^i - A'^i$ is the $\overline{Q}$-compensator of $(1/\overline{z}_-) \cdot [M^i + M'^i, \overline{z}]$, and we have $(1/\overline{z}_-) \cdot [M^i + M'^i, \overline{z}] = [M^i + M'^i, V]$ on $\Gamma \cap [\![0, T]\!]$ by Theorem 2.1.

COROLLARY 2.2. *Let X be a d-dimensional locally square-integrable semimartingale relative to P and P' with the canonical decomposition $X = X_0 + M + A$ and $X = X_0 + M' + A'$ respectively. Then X is a locally square-integrable semimartingale on $\Gamma \cap [\![0, T]\!]$ relative to $\overline{Q}$ with the canonical decomposition*

$$X = X_0 + \frac{M + M'}{2} + \frac{A + A'}{2} \qquad \text{on} \qquad \Gamma \cap [\![0, T]\!], \tag{2.38}$$

and the quadratic characteristics $\langle M^i + M'^i, M^j + M'^j \rangle$ and $\langle M^i + M'^i, V \rangle$, $i, j \leq d$, relative to $\overline{Q}$ satisfy the following relations on $\Gamma \cap [\![0, T]\!]$:

$$\langle M^i + M'^i, M^j + M'^j \rangle = 2\langle M^i, M^j \rangle + 2\langle M'^i, M'^j \rangle + [A^i - A'^i, A^j - A'^j], \tag{2.39}$$

$$\langle M^i + M'^i, V \rangle = A^i - A'^i \tag{2.40}$$

where $\langle M^i, M^j \rangle$ (respectively, $\langle M'^i, M'^j \rangle$) is the mutual characteristic of the corresponding processes relative to P (respectively, P').

PROOF: It immediately follows from Theorem 2.2 and Proposition II.2.29 in [8] that X is a locally square-integrable semimartingale on $\Gamma \cap [\![0, T]\!]$, and (2.38) has been proved above. Denote by (B, C, ν), (B', C', ν') and $(\overline{B}, \overline{C}, \overline{\nu})$ the triplets of X associated with a fixed truncation function h relative to P, P' and $\overline{Q}$ respectively. By the same proposition

$$\langle M^i, M^j \rangle = C^{ij} + (x^i x^j) * \nu - \sum_{s \leq \cdot} \Delta A^i_s \Delta A^j_s, \tag{2.41}$$

$$\langle M'^i, M'^j\rangle = C'^{ij} + (x^i x^j) * \nu' - \sum_{s\le\cdot} \Delta A'^i_s \Delta A'^j_s, \tag{2.42}$$

$$\langle M^i + M'^i, M^j + M'^j\rangle = 4\overline{C}^{ij} + 4(x^i x^j) * \overline{\nu} - \sum_{s\le\cdot} (\Delta A^i_s + \Delta A'^i_s)(\Delta A^j_s + \Delta A'^j_s). \tag{2.43}$$

Now (2.39) follows from Theorem 2 and (2.41)–(2.43). Relation (2.40) has been proved in Corollary 2.1.

3. Proofs

In this section we consider a filtered space $(\Omega, \mathcal{F}, \mathbb{F})$ with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$, $\mathcal{F} = \bigvee_{t\ge 0} \mathcal{F}_t$, and two fixed probability measures P and P' on $(\Omega, \mathcal{F})$. The setting is the same as in Subsections 2.2 and 2.3, in particular, $Q = (P + P')/2$ and (2.4) holds.

3.1. A multi-dimensional version of Theorem 1.1

Let $X = (X^i)_{i\le d}$ be a d-dimensional locally square-integrable semimartingale relative to P and P' with the canonical decomposition $X = X_0 + M + A$ and $X = X_0 + M' + A'$ respectively. Set

$$B^i = A^i - A'^i, \qquad C^{ij} = (\langle M^i, M^j\rangle + \langle M'^i, M'^j\rangle)/2, \qquad i,j \le d,$$

where $\langle M^i, M^j\rangle$ (respectively, $\langle M'^i, M'^j\rangle$) is the mutual quadratic characteristic of M^i and M^j (respectively, M'^i and M'^j) relative to P (respectively, P'). Note that $B = (B^i)_{i\le d}$ and $C = (C^{ij})_{i,j\le d}$ are defined uniquely (up to Q-indistinguishability) only on the set Γ; nevertheless, the arguments below are valid for any predictable version of B and C such that B^i and C^{ij} have finite variation over finite intervals.

There are a predictable process $c = (c^{ij})_{i,j\le d}$ with values in the set of all symmetric nonnegative matrices and an increasing predictable process F, with

$$C^{ij} = c^{ij} \cdot F, \qquad Q\text{-a.s.} \tag{3.1}$$

We have a decomposition

$$B = (c\beta) \cdot F + \widetilde{\beta} \cdot F + F', \qquad Q\text{-a.s.}, \tag{3.2}$$

where $\beta = (\beta^i)_{i\le d}$ and $\widetilde{\beta} = (\widetilde{\beta}^i)_{i\le d}$ are predictable, $\widetilde{\beta}_t(\omega)$ is orthogonal in $\mathbb{R}^d$ to the image of $\mathbb{R}^d$ by the linear map associated with the matrix $c_t(\omega)$, the components F'^i have finite variation over compact intervals and the measures

dF_t and dF'^i_t are orthogonal. The processes β and $\widetilde{\beta}$ are not necessarily unique, but the decomposition (3.2) is unique. Finally, define

$$G_t = \begin{cases} (\beta^{\mathrm{T}} c\beta) \cdot F_t, & \text{if} \quad \widetilde{\beta} \cdot F \equiv 0 \quad \text{and} \quad F' \equiv 0 \quad \text{on} \quad [0,t], \\ \infty, & \text{otherwise.} \end{cases} \tag{3.3}$$

It is obvious that G does not depend upon the choice of β and $\widetilde{\beta}$ in (3.2); moreover, it does not depend upon the choice of F, as long as (3.1) holds.

The following theorem is a multi-dimensional version of Theorem 1.1.

THEOREM 3.1. *Assume* (3.1)–(3.3). *There exists a version h of the Hellinger process $h(\frac{1}{2}; P, P')$ such that*

$$\frac{1}{(1-\Delta h)^2} \cdot h \succ \frac{1}{8} G \qquad (P+P')\text{-a.s.}$$

The key result for the proof of Theorem 3.1 is the following lemma.

LEMMA 3.1. *Assume* (3.1)–(3.3). *Then, on the set* $\Gamma \cap \{\Delta k < 1\}$, $\widetilde{\beta} \cdot F \equiv 0$, $F' \equiv 0$ *and the process*

$$\frac{1}{1-\Delta k} \cdot k - \frac{1}{4} G$$

is increasing $(P+P')$*-a.s.*

PROOF: Let (T_n) be a sequence of stopping times, satisfying Proposition 2.2. Since $\cup_n [\![0, T_n]\!] \supseteq \Gamma$ $(P+P')$-a.s., it is sufficient to prove the statement of the lemma on the set $\Gamma \cap [\![0,T]\!] \cap \{\Delta k < 1\}$, where $T = T_n$ for some n.

Let $\overline{Q}$ be a probability measure associated with the density process $\overline{Z} = Z^n$ defined as in Subsection 2.3. The process V is the same as in Theorem 2.1. In the rest of the proof, angle and square brackets and stochastic integrals are taken with respect to $\overline{Q}$, and all the relations hold $\overline{Q}$-a.s.

As it was noted above, the claim does not depend upon the choice of F, as long as (3.1) holds, so we can choose F so that $F' = 0$ and

$$B = (c\beta) \cdot F + \widetilde{\beta} \cdot F. \tag{3.4}$$

Put $w = (c\beta + \widetilde{\beta})(c\beta + \widetilde{\beta})^{\mathrm{T}} \Delta F$. Then

$$\Delta B (\Delta B)^{\mathrm{T}} = w \Delta F \tag{3.5}$$

and the matrix-valued process $[B,B] = ([B^i, B^j])_{i,j \le d}$ has the representation

$$[B,B] = w \cdot F.$$

According to Theorem 2.1 and Corollary 2.2, V and $M^i + M'^i$, $i \leq d$, are $\overline{Q}$-locally square-integrable martingales on $\Gamma \cap [\![0, T]\!]$ and the following relations hold on the same set:

$$\langle M^i + M'^i, M^j + M'^j \rangle = 4C^{ij} + [B^i, B^j] = (4c^{ij} + w^{ij}) \cdot F, \tag{3.6}$$

$$\langle V, M^i + M'^i \rangle = B^i, \tag{3.7}$$

$$\langle V, V \rangle = k. \tag{3.8}$$

According to the multi-dimensional Kunita–Watanabe decomposition, there is a predictable process $H = (H^i)_{i \leq d}$ such that the stochastic integral $H \cdot (M + M')$ with respect to the multi-dimensional locally square-integrable martingale $M + M'$ is well defined and

$$V = H \cdot (M + M') + V^{\perp} \tag{3.9}$$

on $\Gamma \cap [\![0, T]\!]$, where $V^{\perp}$ is a $\overline{Q}$-locally square-integrable martingale with $\langle V^{\perp}, M^i + M'^i \rangle \equiv 0$, $i \leq d$, on $\Gamma \cap [\![0, T]\!]$. Substituting (3.9), (3.6) and (3.7) in the last relation, we get

$$B = (4cH + wH) \cdot F \tag{3.10}$$

on $\Gamma \cap [\![0, T]\!]$. On the other hand, let $\overline{G} = \langle H \cdot (M + M'), H \cdot (M + M') \rangle$. Then (3.6) implies

$$\overline{G} = (4H^{\mathrm{T}} cH + H^{\mathrm{T}} wH) \cdot F \tag{3.11}$$

on $\Gamma \cap [\![0, T]\!]$, and the increasing process $\langle V^{\perp}, V^{\perp} \rangle$ has the form

$$\langle V^{\perp}, V^{\perp} \rangle = \langle V, V \rangle - \langle H \cdot (M + M'), H \cdot (M + M') \rangle = k - \overline{G} \tag{3.12}$$

on $\Gamma \cap [\![0, T]\!]$ (use (3.9) and (3.8)). In particular, $0 \leq \Delta \overline{G} < 1$ on the stochastic interval $\Gamma \cap [\![0, T]\!] \cap \{\Delta k < 1\}$.

By (3.10), (3.5) and (3.11),

$$\Delta B = (4cH + wH)\Delta F = 4cH\Delta F + \Delta B (\Delta B)^{\mathrm{T}} H,$$

$$\Delta \overline{G} = (4H^{\mathrm{T}} cH + H^{\mathrm{T}} wH)\Delta F = H^{\mathrm{T}} \Delta B = (\Delta B)^{\mathrm{T}} H, \tag{3.13}$$

hence,

$$(1 - \Delta \overline{G})\Delta B = 4cH\Delta F \tag{3.14}$$

on $\Gamma \cap [\![0, T]\!]$.

Set $F_t^d = \sum_{0<s\leq t} \Delta F_s$, $F_t^c = F_t - F_t^d$, $B_t^d = \sum_{0<s\leq t} \Delta B_s$, $B_t^c = B_t - B_t^d$, $G_t^d = \sum_{0<s\leq t} \Delta G_s$, $G_t^c = G_t - G_t^d$. We obtain from (3.10) and the definition of w that

$$B^c = (4cH) \cdot F^c.$$

Combining this equality with (3.14), we get

$$(1-\Delta\overline{G})\cdot B=(4cH)\cdot F \tag{3.15}$$

on $\Gamma\cap[\![0,T]\!]$. Comparing (3.4) and (3.15), we obtain

$$((1-\Delta\overline{G})\widetilde{\beta})\cdot F=0 \tag{3.16}$$

and

$$((1-\Delta\overline{G})c\beta)\cdot F=(4cH)\cdot F \tag{3.17}$$

on $\Gamma\cap[\![0,T]\!]$.

Now (3.16) implies $\widetilde{\beta}\cdot F=0$ on $\Gamma\cap[\![0,T]\!]\cap\{\Delta k<1\}$. Applying (3.11) and (3.17) for the continuous part of $\overline{G}$ and (3.13) and (3.4) for the jump part of $\overline{G}$, we get

$$\overline{G}^c=(4H^{\mathrm{T}}cH)\cdot F^c=(H^{\mathrm{T}}c\beta)\cdot F^c,$$

$$\overline{G}^d=\sum_{s\leq\cdot}\Delta G_s=\sum_{s\leq\cdot}H_s^{\mathrm{T}}\Delta B_s=(H^{\mathrm{T}}c\beta)\cdot F^d$$

on $\Gamma\cap[\![0,T]\!]\cap\{\Delta k<1\}$, which yields

$$\overline{G}=(H^{\mathrm{T}}c\beta)\cdot F=(\beta^{\mathrm{T}}cH)\cdot F$$

on the same set. Applying (3.17) again, we conclude that

$$\overline{G}=\frac{1}{4}((1-\Delta\overline{G})\beta^{\mathrm{T}}c\beta)\cdot F=\frac{1}{4}(1-\Delta\overline{G})\cdot G \tag{3.18}$$

on $\Gamma\cap[\![0,T]\!]\cap\{\Delta k<1\}$. Now the claim follows from (3.12) and (3.18).

PROOF OF THEOREM 3.1: Taking into account Remark 1.1, Proposition 2.1b and Lemma 3.1, it is enough to show that

$$\frac{1}{(1-\Delta h)^2}\cdot h-\frac{1}{2}\frac{1}{1-\Delta k}\cdot k$$

is an increasing process on $\Gamma\cap\{\Delta k<1\}$. Let k^c and h^c be the continuous parts of k and h respectively. Proposition 2.1a yields

$$h^c\succ\frac{1}{2}k^c$$

and

$$\frac{1}{2}\frac{\Delta k}{1-\Delta k}\leq\frac{2\Delta h-(\Delta h)^2}{2(1-\Delta h)^2}\leq\frac{\Delta h}{(1-\Delta h)^2} \tag{3.19}$$

on $\Gamma \cap \{\Delta k < 1\}$, and the claim follows.

3.2. Remark

Let (T_n) be a sequence of stopping times, satisfying Proposition 2.2, and $T = T_n$ for some n. Fix $\alpha \in (0,1)$. Similarly to Proposition 2.2 above, one can prove that $\overline{Z}(\alpha) = \mathcal{E}(\alpha n^T + (1-\alpha)n'^T)$ is a Q-uniformly integrable martingale and $Q(\{\inf_t \overline{Z}(\alpha)_t > 0\}) = 1$. Define a probability measure $\overline{Q}(\alpha)$ by

$$d\overline{Q}(\alpha) = \overline{Z}(\alpha)_\infty dQ.$$

If

$$\overline{N}(\alpha) = \alpha n^T + (1-\alpha)n'^T$$

and

$$V(\alpha) = n^T - n'^T - \frac{1}{1+\Delta\overline{N}(\alpha)} \cdot [n^T - n'^T, \overline{N}(\alpha)],$$

then similarly to Theorem 2.1 one can check that the density processes $\overline{z}(\alpha)$ and $\overline{z}'(\alpha)$ of P and P' with respect to $\overline{Q}(\alpha)$ have the representation

$$\overline{z}(\alpha) = z_0 \mathcal{E}((1-\alpha)V(\alpha)) \qquad \text{and} \qquad \overline{z}'(\alpha) = z_0' \mathcal{E}(-\alpha V(\alpha)) \qquad \text{on} \quad [\![0,T]\!].$$

This property implies analogues of Theorem 2.2 and its corollaries; in particular, (2.29) is replaced by

$$\overline{B}(\alpha) = \alpha B + (1-\alpha)B', \qquad \overline{C}(\alpha) = \alpha C + (1-\alpha)C', \qquad \overline{\nu}(\alpha) = \alpha\nu + (1-\alpha)\nu'$$

for the characteristics $(\overline{B}(\alpha), \overline{C}(\alpha), \overline{\nu}(\alpha))$ of X with respect to $\overline{Q}(\alpha)$.

Again, $V(\alpha)$ is a $\overline{Q}(\alpha)$-locally square-integrable martingale. Let $\langle V(\alpha), V(\alpha)\rangle$ be the quadratic characteristic of $V(\alpha)$ with respect to $\overline{Q}(\alpha)$. Then the process $\alpha(1-\alpha)\langle V(\alpha), V(\alpha)\rangle$, coincides with some increasing predictable process $k(\alpha)$ on $[\![0,T]\!]$. This process $k(\alpha)$ can be defined similarly to (2.7) and (2.8), replacing $1/4$ by $\alpha(1-\alpha)$ and ψ by ψ_α in (2.7) with

$$\psi_\alpha(u,v) = \frac{\alpha(1-\alpha)(u-v)^2}{\alpha u + (1-\alpha)v}.$$

Moreover, the proof similar to that of Lemma 3.1 shows that the statement of Lemma 3.1 holds true if one replaces $\frac{1}{4}G$ by $\alpha(1-\alpha)G(\alpha)$, where $G(\alpha)$ is defined similarly to G replacing C by $C(\alpha) = (C^{ij}(\alpha))_{i,j\le d}$ with $C^{ij}(\alpha) = \alpha\langle M^i, M^j\rangle + (1-\alpha)\langle M'^i, M'^j\rangle$. But if $\alpha \neq 1/2$, there is no appropriate relationship (similar to (3.19)) between the jumps of $k(\alpha)$ and the jumps of the Hellinger process of any order.

3.3. Proof of Lemma 1.1

We assume that $h_T < \infty$ and $0 \le \Delta h < 1$ on $[\![0,T]\!]$; otherwise the both sides of (1.5) are infinite.

Set $h_t^d = \sum_{0<s\le t} \Delta h_s$, $h_t^c = h_t - h_t^d$. We have

$$\begin{aligned}\int_0^T \frac{1}{(1-\Delta h_s)^2}\,dh_s^d &= \sum_{s\le T} \frac{\Delta h_s}{(1-\Delta h_s)^2} = \sum_{s\le T}\left\{\frac{1}{(1-\Delta h_s)^2} - \frac{1}{1-\Delta h_s}\right\}\\ &= \sum_{s\le T}\left\{\exp\left(2\log\frac{1}{1-\Delta h_s}\right) - \exp\left(\log\frac{1}{1-\Delta h_s}\right)\right\}\\ &\le \exp\left(2\sum_{s\le T}\log\frac{1}{1-\Delta h_s}\right) - \exp\left(\sum_{s\le T}\log\frac{1}{1-\Delta h_s}\right)\\ &= \prod_{s\le T}\frac{1}{(1-\Delta h_s)^2} - \prod_{s\le T}\frac{1}{1-\Delta h_s}\\ &\le \prod_{s\le T}\frac{1}{(1-\Delta h_s)^2},\end{aligned} \tag{3.20}$$

where we used the fact that the function $f(x) = e^{2x} - e^x$, $x \ge 0$, is convex and $f(0) = 0$; hence $\sum_{i=1}^n f(x_i) \le f(\sum_{i=1}^n x_i)$ for arbitrary n and $x_i \ge 0$, $i = 1, \dots, n$.

Now (3.20) implies

$$\begin{aligned}\int_0^T \frac{1}{(1-\Delta h_s)^2}\,dh_s &= h_T^c + \int_0^T \frac{1}{(1-\Delta h_s)^2}\,dh_s^d \le h_T^c + \prod_{s\le T}\frac{1}{(1-\Delta h_s)^2}\\ &\le \exp(2h_T^c)\prod_{s\le T}\frac{1}{(1-\Delta h_s)^2} = \mathcal{E}(-h)_T^{-2}.\end{aligned}$$

4. Applications to linear regression models

In this section we give applications of what precedes to linear regression models with martingale errors.

4.1. A general linear regression model with martingale errors

Let us consider the following multiple linear regression model:

$$X_t = X_0 + \int_0^t \vartheta^{\mathrm{T}} f_s \, d\langle M\rangle_s + \sigma M_t, \quad t \geq 0,$$

where X is a real-valued observable process, $\vartheta \in \mathbb{R}^k$ is an unknown parameter, f is an observable predictable $\mathbb{R}^k$-valued function, M is a locally square-integrable martingale, $M_0 = 0$, with the observable quadratic characteristic $\langle M\rangle$, σ is a positive number (known or unknown). This model was studied in many papers, see, e.g., [18], [16], [2], [12], [3] and [17].

More formally, we assume that there is a filtered space $(\Omega, \mathcal{F}, \mathbb{F})$ with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$, an adapted càdlàg real-valued process $X = (X_t)_{t\geq 0}$, a predictable increasing process $a = (a_t)_{t\geq 0}$, a predictable $\mathbb{R}^k$-valued function $f = (f_t)_{t\geq 0}$ and a family $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ satisfying the following properties: for every $P \in \mathcal{P}$, there is a vector $\vartheta = \vartheta(P) \in \mathbb{R}^k$ and a number $\sigma^2 = \sigma^2(P) > 0$ such that

(i) $|f|^2 \cdot a_t < \infty$ for all $t \geq 0$ (P-a.s.) and the functions $f^i \cdot a_t$, $t \geq 0$, are linearly independent P-a.s.

(ii) $M = M(P) = X - X_0 - (\vartheta^{\mathrm{T}} f) \cdot a$ is a P-locally square-integrable martingale.

(iii) The quadratic characteristic of $M(P)$ with respect to P is equal to $\sigma^2 a$.

It follows from (i) that $\vartheta(P)$ and $\sigma^2(P)$ are defined uniquely.

Define $F = (ff^{\mathrm{T}}) \cdot a$ and let $\lambda_{\min}(\cdot)$ denote the minimal eigenvalue of a matrix.

Let $P, P' \in \mathcal{P}$. By Theorem 1.1, there is a version h of the Hellinger process $h(\frac{1}{2}; P, P')$ of order 1/2 between P and P' such that

$$\begin{aligned}\frac{1}{(1-\Delta h)^2} \cdot h &\geq \frac{(\vartheta(P) - \vartheta(P'))^{\mathrm{T}} ff^{\mathrm{T}} (\vartheta(P) - \vartheta(P'))}{4(\sigma^2(P) + \sigma^2(P'))} \cdot a \\ &\geq \frac{|\vartheta(P) - \vartheta(P')|^2}{4(\sigma^2(P) + \sigma^2(P'))} \lambda_{\min}(F). \end{aligned} \tag{4.1}$$

The last inequality allows us to find simple sufficient conditions for the entire separation of measures in the considered model.

THEOREM 4.1. *Let $P^n \in \mathcal{P}$ and $P'^n \in \mathcal{P}$, $n \geq 1$, and T_n be a sequence of stopping times such that*

$$\limsup_n \|P^n_{T_n} - P'^n_{T_n}\| < 2(1-\varepsilon), \qquad \varepsilon > 0, \tag{4.2}$$

(the subscript T_n denotes the restriction of a measure to the σ-field F_{T_n}). Assume

$$\Sigma^2 = \limsup_n \left(\sigma^2(P^n) + \sigma^2(P'^n)\right) < \infty \tag{4.3}$$

and

$$\varphi_n^{-2}\lambda_{\min}^{-1}(F_{T_n}) = O_{P^n}(1) \tag{4.4}$$

for some sequence (φ_n) *of positive numbers. Then there is a constant* $C < \infty$ *such that*

$$\limsup_n \varphi_n^{-1}|\vartheta(P^n) - \vartheta(P)| \leq C,$$

and C *depends on* (P'^n) *satisfying* (4.2) *and* (4.3) *only through* ε *and* Σ.

In particular, if (4.3) and (4.4) hold, then $\limsup_n \varphi_n^{-1}|\vartheta(P^n) - \vartheta(P)| = \infty$ implies the entire separation of $(P^n_{T_n})$ and $(P'^n_{T_n})$.

PROOF: Due to (4.1), there is a version h^n of the Hellinger process $h(\frac{1}{2}; P^n, P'^n)$ of order $1/2$ between P^n and P'^n such that

$$\frac{1}{(1-\Delta h^n)^2} \cdot h^n_{T_n} \geq \frac{|\vartheta(P'^n) - \vartheta(P^n)|^2}{4(\sigma^2(P^n) + \sigma^2(P'^n))}\lambda_{\min}(F_{T_n}). \tag{4.5}$$

By Lemma 1.1,

$$\frac{1}{(1-\Delta h^n)^2} \cdot h^n_{T_n} \leq \frac{1}{\mathcal{E}(-h^n)^2_{T_n}}. \tag{4.6}$$

According to the well-known lower estimate for the variation distance, see [10, Theorem 2.1],

$$\begin{aligned} \|P^n_{T_n} - P'^n_{T_n}\| &\geq 2\Big(1 - \sqrt{H(P^n_0, P'^n_0)E\mathcal{E}^n(-h^n)_{T_n}}\Big) \\ &\geq 2\Big(1 - \sqrt{E\mathcal{E}^n(-h^n)_{T_n}}\Big), \end{aligned} \tag{4.7}$$

where E^n is the expectation with respect to P^n and $H(P^n_0, P'^n_0)$ is the Hellinger integral of order $1/2$ between the restrictions of P^n and P'^n to the σ-field $\mathcal{F}_0$. It follows from (4.2) and (4.7) that

$$\liminf_n E^n\mathcal{E}(-h^n)_{T_n} > \varepsilon^2. \tag{4.8}$$

Since $\mathcal{E}(-h^n)_{T_n} \leq 1$, (4.8) implies

$$P^n(\mathcal{E}(-h^n)_{T_n} \geq \varepsilon^2/2) \geq \varepsilon^2/2 \tag{4.9}$$

for n large enough.

On the other hand, (4.4) yields that, for some $\delta > 0$,

$$P^n(\varphi_n^2\lambda_{\min}(F_{T_n}) > \delta) \geq 1 - \varepsilon^2/4 \tag{4.10}$$

for n large enough. Therefore (4.9) and (4.10) give

$$n \geq n_0 \Rightarrow P^n(\varphi_n^2 \lambda_{\min}(F_{T_n})\mathcal{E}(-h^n)_{T_n}^2 \geq \delta\varepsilon^4/4) \geq \varepsilon^2/4. \tag{4.11}$$

Combining (4.5), (4.6) and (4.11), we get

$$n \geq n_0 \Rightarrow |\vartheta(P'^n) - \vartheta(P^n)|^2 \leq 16\delta^{-1}\varepsilon^{-4}(\sigma^2(P^n) + \sigma^2(P'^n))\varphi_n^2,$$

and (4.3) yields the claim with $C = 4\delta^{-1/2}\varepsilon^{-2}\Sigma$.

4.2. Example: autoregressive model of the first order

Assume that the observable process (x_n) is such that

$$x_n = \vartheta x_{n-1} + \varepsilon_n, \qquad n \geq 1, \tag{4.12}$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are independent and identically distributed with a distribution function F, x_0 is an observable square-integrable random variable independent of ε_n, $n \geq 1$, ϑ is a real parameter. It is assumed that

$$0 < \sigma^2(F) = \int x^2 F(dx) < \infty, \qquad \int x F(dx) = 0.$$

This classical scheme is a special case of the model considered in the previous subsection. Indeed, let $\Omega = \{(x_0, x_1, \ldots, x_n, \ldots),\ x_n \in \mathbb{R}\}$, $\mathcal{F}$ is the product σ-field on Ω, $\mathcal{F}_t = \sigma\{x_0, \ldots, x_{[t]}\}$ ($[t]$ is the integer part of t) and let $\mathcal{P}$ consist of the distributions of the sequence (4.12), corresponding to different ϑ and F.

Put

$$X_t = \sum_{j=1}^{[t]} x_j = \vartheta \sum_{j=1}^{[t]} x_{j-1} + \sum_{j=1}^{[t]} (x_j - \vartheta x_{j-1}), \qquad a_t = [t], \qquad f_t = x_{[t]-1}.$$

Then the quadratic characteristic of the martingale $\sum_{j=1}^{[t]}(x_j - \vartheta x_{j-1})$ is $\sigma^2(F)a_t$.

Let

$$\varphi_n(\vartheta) = \begin{cases} n^{-1/2}, & \text{if } |\vartheta| < 1, \\ n^{-1}, & \text{if } |\vartheta| = 1, \\ |\vartheta|^{-n}, & \text{if } |\vartheta| > 1. \end{cases} \tag{4.13}$$

Since the sequence

$$\varphi_n^2(\vartheta) \sum_{j=1}^{n} x_{j-1}^2$$

converges in distribution to a limit with zero mass at 0, Theorem 4.1 yields the following result.

PROPOSITION 4.1. *Let P_n (respectively, P'_n) be the distribution of the sample $(x_0, \ldots, x_n)$ satisfying (4.12) with a parameter ϑ and an error distribution F (respectively, ϑ_n and F_n). If*

$$\limsup_n \|P_n - P'_n\| < 2(1-\varepsilon), \qquad \varepsilon > 0,$$

and

$$\Sigma^2 = \limsup_n \sigma^2(F^n) < \infty,$$

then

$$\limsup_n \varphi_n^{-1}(\vartheta)(\vartheta_n - \vartheta) \le C,$$

where $\varphi_n(\vartheta)$ is defined in (4.13), and C depends only on ϑ, F, ε and Σ.

4.3. Example: Galton–Watson branching process with nonrandom immigration

Assume that we observe the branching process with nonrandom immigration

$$x_n = \sum_{i=1}^{x_{n-1}} y_{n,i} + 1, \qquad n \ge 1, \qquad x_0 = 1, \tag{4.14}$$

where $y_{n,i}$ are independent and identically distributed with a distribution function F on $\{0, 1, 2, \ldots\}$. It is assumed that

$$0 < \sigma^2(F) = \int x^2 F(dx) < \infty.$$

Put $\vartheta(F) = \int x F(dx)$. The distributions of the sequence $(x_1, \ldots, x_n, \ldots)$, corresponding to different F with the above properties, form the class $\mathcal{P}$. Again, this model is a special case of the linear regression model of Subsection 4.1: put

$$X_t = \sum_{j=1}^{[t]} (x_j - 1) = \vartheta(F) \sum_{j=1}^{[t]} x_{j-1} + \sum_{j=1}^{[t]} \sum_{i=1}^{x_{j-1}} (y_{j,i} - \vartheta(F)), \quad a_t = \sum_{j=1}^{[t]} x_{j-1}, \quad f_t = 1,$$

then the quadratic characteristic of the martingale $\sum_{j=1}^{[t]} \sum_{i=1}^{x_{j-1}} (y_{j,i} - \vartheta(F))$ is $\sigma^2(F) a_t$.

Let

$$\varphi_n(\vartheta) = \begin{cases} n^{-1/2}, & \text{if } 0 < \vartheta < 1, \\ n^{-1}, & \text{if } \vartheta = 1, \\ \vartheta^{-n/2}, & \text{if } \vartheta > 1. \end{cases} \tag{4.15}$$

It is known that the sequence

$$\varphi_n^2(\vartheta(F)) \sum_{j=1}^{n} x_{j-1}$$

converges in distribution to a limit with zero mass at 0, so the following result follows from Theorem 4.1 (cf. Lemma 6 in [5]).

PROPOSITION 4.2. *Let P_n (respectively, P'_n) be the distribution of the sample $(x_1, \ldots, x_n)$ satisfying* (4.14) *with an offspring distribution F (respectively, F_n). If*

$$\limsup_n \|P_n - P'_n\| < 2(1-\varepsilon), \qquad \varepsilon > 0,$$

and

$$\Sigma^2 = \limsup_n \sigma^2(F^n) < \infty,$$

then

$$\limsup_n \varphi_n^{-1}(\vartheta(F_n) - \vartheta(F)) \le C,$$

where φ_n is defined according to (4.15) *with $\vartheta = \vartheta(F)$, and C depends only on F, ε and Σ.*

5. Acknowledgments

This research was partially supported by the Russian Foundation for Basic Research under Grant 93-011-1440, and by the International Science Foundation under Grant MMK300. The work was completed when the author was visiting the Weierstraß—Institut für Angewandte Analysis und Stochastik, Berlin. The author wish to thank this institution for its hospitality, and the Deutsche Forschungsgemeinschaft for a support.

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