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A dimension formula for endomorphisms – The Belykh family

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Abstract

In [11] we considered a class of hyperbolic endomorphisms and asked the question whether there exists a physical motivated invariant measure (SRB-measure) and if so we gave a criterion when the map is invertible on a set of full measure. In this work we want to consider a particular example of this class - in fact a 3-parameter family of those - and proof that a.s. the criterion is fulfilled. From this it follows that the Young formulae for the Hausdorff dimension of the SRB-measure holds.

Contents

1 Introduction

In this work we want to apply the general theory from [11] to a specific example. From the first point of view this example seems rather special. But it contains almost all difficulties of the general case and it is the most natural one for further investigations. The results we derive in this section may be generalized to other cases by following the proofs but we need some sort of transversality conditions for which we have no simple criterion in general and moreover we don't know if they are generic. Therefore, we restrict to our example where the transversality condition is proven.

We consider the three-parameter family of the Belykh map and restrict to the set of parameters for which the assumptions of [11] hold. This allows us to investigate the properties of the SBR-measure. In particular, we are interested in the question how the dependence of the Hausdorff dimension of the measure - what is the same as the information dimension - on the parameters is. By our criterion in [11] this is connected to the almost sure invertibility of the map. As we mentioned in [11] it is not to expect that the Young-Pesin formula holds for all parameters of the family. But it seems likely - and we will prove it - that the criterion in [11] holds for almost all parameters in the sense of Lebesgue. Actually, it is sometimes conjectured that the exceptional set is also of first Baire category or even countable. For us the answer to this conjecture even in particular examples is for out of reach at the moment.

We will have two results leading together with the general observations of [11] to a complete picture of the bifurcations of the invertibility picture of the map. We will discuss this picture at the end of this work. The interesting thing is that in theorem 3.1 we prove the invertibility directly and then apply the criterion to get the dimension formula while in theorem 3.3 we prove the dimension formula and then derive the almost sure invertibility with the help of the criterion. So we use the criterion in [11] from both sides what indicates that both statements of the criterion have its own interest and it depends on the application which way we want to read it.

2 The Belykh family

We want briefly remind the Belykh system (see section 0.1.3).

Let us consider the square $Q = [-1,1] \times [-1,1] \in \mathbb{R}^2$ and the map $f : Q \to Q$ defined by

$$f(x,y) = \begin{cases} (\lambda x_1 + 1 - \lambda, \gamma x_2 + 1 - \gamma) & x_2 > kx_1 \\ (\lambda x_1 + (\lambda - 1), \gamma x_2 + (\gamma - 1)) & x_2 < kx_1 \end{cases} \tag{1}$$

with $-1 < k < 1, 1 < \gamma \leq \frac{2}{|k|+1}, 0 < \lambda \leq 1$.

We denote the upper half U and the lower half L by

$$U = \{x = (x_1, x_2) \in Q | x_2 \geq kx_1\}$$

and

$$L = \{x = (x_1, x_2) \in Q | x_2 \leq kx_1\}, \text{ respectively.} \tag{2}$$

Let $\partial U, \partial L$ denote their boundaries, $N^+ = \partial U \cap \partial L$.

For $\lambda \geq \frac{1}{2}$ this map is not injective. See figure

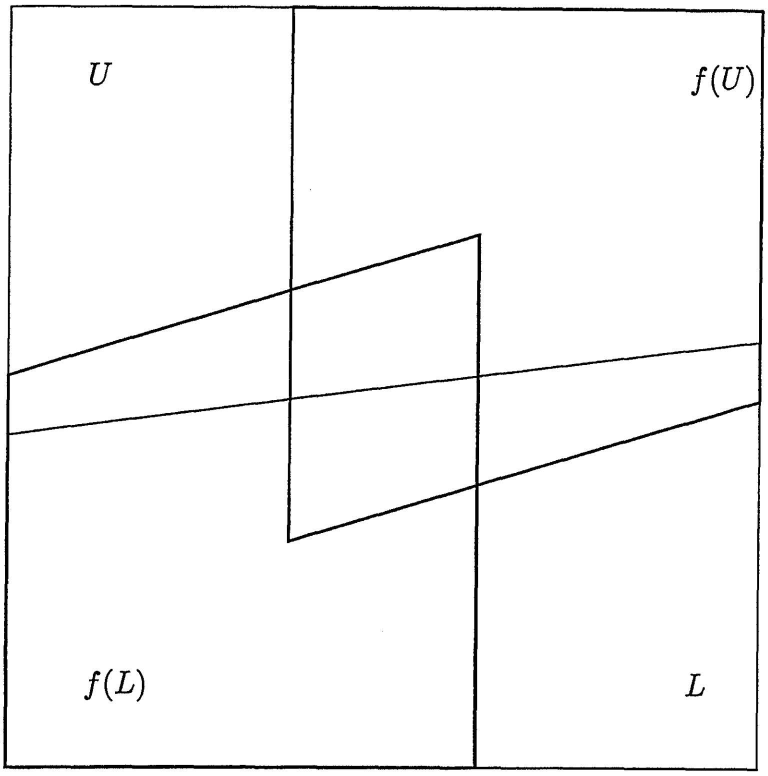


Figure 1

The fat Belykh attractor $\Lambda = \Lambda(\lambda, \gamma, k)$, $(\lambda > \frac{1}{2})$, is defined by

$$\begin{aligned} Q^+ &= \{x \in Q | f^n(x) \notin \partial U \cup \partial L, \quad n = 0, 1, 2, \dots\} \\ D &= \bigcap_{n \geq 0} f^n(Q^+) \quad \text{and finally} \\ \Lambda &= Cl(D) \end{aligned} \tag{3}$$

where $Cl(\cdot)$ denotes the closure. We write for $\Delta > 0, l \geq 1$

$$\begin{aligned} D^n_{\Delta,l} &= \{x \in Q^+ | d(f^k x, \partial U \cup \partial L) \geq l^{-1} e^{-\Delta k}, k = 0, 1, 2, \dots, n\} \\ D^+_{\Delta,l} &= \bigcap_{n \geq 0} D^n_{\Delta,l} \end{aligned} \tag{4}$$

In [11] it is proved that if the maps $f = f_{\lambda,\gamma,k}$ satisfies the conditions (H4) - (H7) then μ_{SBR} exists and

$$\mu_{\text{SBR}} \left(\bigcup_{l \geq 1} D_{\Delta,l} \right) = 1 \tag{5}$$

for sufficiently small Δ.

Remark. It is easy to see that the Belykh map satisfies always the conditions (H4) - (H7) from [11]. The preimages of the singularity line never meet and the constant L in (H7) equals 1 for all iterates f^τ. Since $\gamma > 1$ we find a τ such that $\gamma^\tau > 2$ and (H7) is satisfied for f^τ.

This means there is an open subset of parameters (λ, γ, k) such that f is not injective, exhibits an SBR-measure with $\mu_{\text{SBR}}(\cup D_{\Delta,l}) = 1$ and the sum of the Lyapunov exponents $\chi_s = \log \lambda$ and $\chi_u = \log \gamma$ is less than 0 (dissipativity).

Let us now remind the main result of [] applied to the Belykh map

Theorem 2.1 [11]: *If $f = f_{\lambda,\gamma,k}$ satisfies (H4) - (H7) from [11] then the following is equivalent*

(1) $\dim_H \mu_{SBR} = 1 - \frac{\log\gamma}{\log\lambda}$

(2) f is invertible on a set of full μ_{SBR}-measure.

We also will use the lifted system:

$$\hat{f}_\lambda = \hat{f}_{\lambda,\gamma,k} = \left(f_{\lambda,\gamma,k}(x_1,x_2), \tau\omega + \frac{i}{2} \right)$$
$$\text{with } i = \begin{cases} 0 & (x_1,x_2) \in U \\ 1 & (x_1,x_2) \in L \end{cases}. \tag{6}$$

We write $\hat{Q}, \hat{L}, \hat{Q}^+, \hat{D}, \hat{\Lambda}, \hat{D}^n_{\Delta,l}, \hat{D}^+_{\Delta,l}, \hat{D}^-_{\Delta,l}$ and so on for the similar defined sets for $\hat{f}$. $\pi : \hat{Q} \to Q$ is the canonical projection. The above defined sets for the lifted system are projected via π to the corresponding sets for f. Since these sets depend on the parameters (λ, γ, k) in the case it is important we indicate this dependence by an index or a bracket. In the lifted system we are able to define backward orbits and also to filtrate them

$$\hat{D}^-_{\Delta,l} = \left\{ \hat{x} \in \hat{D} \middle| d(\hat{f}^{-n}(\hat{x}), \hat{N}^+ \subset N) \geq \frac{1}{l} e^{-\Delta n} \right\}. \tag{7}$$

In [8] and [9] it is proved that for

$$\hat{D}^0 = \bigcup_{l\geq 1} \hat{D}^+_{\Delta,l} \cap \bigcup_{l \geq 1} \hat{D}^-_{\Delta,l}$$
$$\hat{\mu}_{\mathrm{SBR}}(\hat{D}^0) = 1 \tag{8}$$

3 The main theorems

In this this section we prove the following

Theorem 3.1 *Let $B_\rho \subset \mathbb{R}^3$ be a ball of radius ρ such that for $(\lambda, \gamma, k) \in B_\rho$ the map $f_{\lambda,\gamma,k}$ fulfills (H4) - (H7) and* $\boxed{\lambda \cdot \gamma^2 < 1}$ *and $\lambda < 0.64$. Then the map is fully invertible on the attractor almost surely.*

Corollary 3.2 *Under the assumption of theorem 3.1 for Lebesgue a.e.* $(\lambda, \gamma, k) \in B_\rho$

$$\dim_H \mu_{\mathrm{SBR}} = 1 - \frac{\log\gamma}{\log\lambda}$$

Proof. Theorem 3.1 says that Lebesgue almost surely the condition ii) of the criterion in [11] is fulfilled. Hence, condition i) holds almost surely. □

Theorem 3.3 *Let $B_\rho \subset \mathbb{R}^3$ be a ball of radius ρ such that for $(\lambda, \gamma, k) \in B_\rho$ the map $f_{\lambda,\gamma,k}$ fulfills (H4) - (H7) and* $\boxed{\lambda \cdot \gamma < 1}$ *and $\lambda < 0.64$. Then for Lebesgue a.e. $(\lambda, \gamma, k) \in B_\rho$*

$$\dim_H \mu_{SBR} = 1 - \frac{\log \gamma}{\log \lambda}. \tag{9}$$

Corollary 3.4 *Under the assumptions of theorem 3.3 for Lebesgue a.e. $(\lambda, \gamma, k) \in B_\rho$ the map $f_{\lambda,\gamma,k}$ is almost surely invertible.*

Proof. Theorem 3.3 says that Lebesgue almost surely the condition i) of the criterion in [11] is fulfilled. Hence, condition ii) holds almost surely. □

Theorem 3.5 *Let $B_\rho \subset \mathbb{R}^3$ be a ball of radius ρ such that for $(\lambda, \gamma, k) \in B_\rho$ the map $f_{\lambda,\gamma,k}$ fulfills (H4) - (H7) and* $\boxed{\lambda \cdot \gamma > 1}$. *Then for Lebesgue a.e. $(\lambda, \gamma, k) \in B_\rho$*

$$\dim_H \mu_{\mathrm{SBR}} = 2.$$

Remarks:

(i) Equation (9) is known as the Young or sometimes the Young-Pesin formula. It was proved in the invertible case in [14]. Actually L.S. Young proved that for an ergodic measure m for a diffeomorphism on a surface

$$\dim_H m = h_m \left(\frac{1}{\chi^u} - \frac{1}{\chi^s} \right).$$

Since $\log \gamma = \chi^n, \log \lambda = \chi^s$ and by the Pesin formula $h_{\mu_{\mathrm{SBR}}} = \log \gamma$ we get the above formula.

(ii) The Lyapunov dimension of a two-dimensional system is defined by

$$\dim_L \Lambda = \min \left(1 - \frac{\log \gamma}{\log \lambda}, 2 \right).$$

Theorem 3.1 - 3.5 tell that the Lyapunov dimension equals the information dimension - i.e. the Hausdorff dimension of the SBR-measure - for almost all Belykh maps. This is the well-known Kaplan-Yorke conjecture for the special case of the Belykh attractor.

(iii) If $(\lambda_0, \gamma_0, k_0) \in H = H_0 \cap \{\lambda \cdot \gamma < 1\} \cap \{\lambda < 0.64\}$ then there is always a ball with center $(\lambda_0, \gamma_0, k_0)$ fulfilling the assumptions of the theorem. Therefore for Lebesgue a.e. parameter in H the Young formula holds.

(iv) It is standard to show that the Hausdorff dimension of Λ is less or equal to $1 - \frac{\log \gamma}{\log \lambda}$. Consequently, for Lebesgue a.e. $(\lambda, \gamma, k) \in B_\rho$

$$\dim_H \Lambda = 1 - \frac{\log \gamma}{\log \lambda} \tag{10}$$

The fact that the SBR-measure has maximal dimension is prior to the constant Jacobian of f along unstable leafs.

(v) In fact, we will prove slightly stronger statements than those in theorem 1 and 2. We will prove that for fixed γ, k for almost all $\lambda \in B_\rho$ the assertions hold.

4 S-coding

In this section we want to define a way how to trace points while the parameters vary. For this we want to introduce a coding space. Since we want to fix γ and k and let only λ vary and on the other hand the "unraveling" of the images proceeds in the stable direction we concentrate our coding to the stable direction. The main problem is that there is no unique coding space as λ varies - except in the case when $k = 0$. Therefore we introduce a coding space into which all the symbolic spaces for various λ are embedded. The basic point is that the orbit of the points are subject to two different maps according to their position in Q. We will code a point by the sequence of the map applied to it.

We consider the linear maps $S_i = S_i^\lambda = S_i^{\lambda,\gamma} = [-1,1] \times (-\infty,\infty) \circlearrowleft, \; i = 1,2$ given by

$$\begin{aligned} S_1^\lambda(x_1,x_2) &= (\lambda(x_1-1)+1, \gamma(x_2-1)+1) \\ S_2^\lambda(x_1,x_2) &= (\lambda(x_1+1)-1, \gamma(x_2+1)-1) \\ \tilde{S}_1^\lambda(x_1) &= \lambda(x_1-1)+1 \\ \tilde{S}_2^\lambda(x_1) &= \lambda(x_1+1)-1. \end{aligned} \tag{11}$$

Then

$$f_{\lambda,\gamma,k}(x_1,x_2) = \begin{cases} S_1^{\lambda,\gamma}(x_1,x_2) & x_2 > kx_1 \\ S_2^{\lambda,\gamma}(x_1,x_2) & x_2 < kx_1 \end{cases}$$

Let us denote by $I_{\underline{i}}^{(n)}, \underline{i} = i_1 i_2 \ldots i_n \ldots$ - a finite or infinite sequence of 1's and 2's, the strip

$$\begin{aligned} I_{i_1\ldots i_n}^{(n)}(\lambda) &= S_{i_1}^\lambda \circ S_{i_2}^\lambda \circ \ldots \circ S_{i_n}^\lambda([-1,1] \times (-\infty,\infty)) \\ &= \tilde{S}_{i_1}^n \circ \ldots \circ \tilde{S}_{i_n}^\lambda([-1,1]) \times (-\infty,\infty). \end{aligned} \tag{12}$$

$I_{\underline{i}}^{(\infty)}(\lambda)$ denotes for infinite $\underline{i}$ the corresponding vertical line which is the intersection $\bigcap I_{\underline{i}}^{(n)}$.

For given $\underline{i} = i_1 \ldots i_n \ldots$ finite or infinite we consider the set

$$R_{\underline{i}}^{(n)}(\lambda) = \left\{ x = (x_1,x_2) \in Q^+ \middle| f^k(x) \in \begin{cases} U & \text{if } i_{k+1} = 1 \\ L & \text{if } i_{k+1} = 2 \end{cases} \quad k = 0,\ldots,n-1 \right\}. \tag{13}$$

Then the $R_{\underline{i}}^{(n)}$ are connected polygons in $I_{\underline{i}}^{(n)}$ or empty. The sets $\hat{R}_{\underline{i}}^{(n)}(\lambda)$ are the analogously defined sets for the lifted system.

The sets $R_{\underline{i}}^{(n)}$ are the images of f^n of the maximal components of continuity of the map f^n. Therefore

$$D = \bigcap_{n=0}^{\infty} \bigcup_{\underline{i} \in \Sigma_2^+} R_{\underline{i}}^{(n)}. \tag{14}$$

Here we denoted by Σ_2^+ the set of all one-sided sequences of symbols 1 and 2. For given λ, γ, k the symbolic coding space for $f_{\lambda,\gamma,k}$ is the set $\Sigma_\lambda = \Sigma_{\lambda,\gamma,k}$ of all infinite sequences $\underline{i} \in \Sigma_2^+$ defined by

$$\Sigma_\lambda = \left\{ \underline{i} \in \Sigma_2 \middle| R_{\underline{i}}^{(n)}(\lambda) \neq \emptyset, \quad n = 1,2,\ldots \right\}.$$

Those, it consists of the forward coding sequences of all points in D.

Notice, that the coding spaces $\Sigma_2^+, \Sigma_\lambda$ are the same for f_λ and $\hat{f}_\lambda$ and $\pi : \hat{Q} \to Q$ projects the sets $\hat{R}_{\underline{i}}^{(n)}(\lambda)$ to $R_{\underline{i}}^{(n)}(\lambda)$.

The idea of the proof of theorem 3.1 is to show that the probability that two different $R_{\underline{i}}^{(n)}$ have non-empty intersection tends to 0 as n tends to infinity. The next characteristic function turns out to be useful.

For a given pair $\underline{i}, \underline{j}$ of length n we introduce the function

$$\chi_{\underline{i},\underline{j}}(\lambda) = \begin{cases} 1 & \text{if } R_{\underline{i}}^{(n)}(\lambda) \neq \emptyset \text{ and } R_{\underline{j}}^{(n)}(\lambda) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

A crucial step is to get estimates on the number of non-empty $R_{\underline{i}}^{(n)}$'s for different λ.

This will be done in the next two lemmata.

Lemma 4.1 *The partition $\{\hat{U}, \hat{L}\}$ of the cube $\hat{Q}$ is a generator for $\hat{D}$.*

Proof. We have to prove that for $\hat{x}, \hat{y} \in \hat{D}, \hat{x} \neq \hat{y}$ there is an $n \in \mathbb{Z}$ such that $\hat{f}^n(x), \hat{f}^n(y)$ are in different elements of the partition $\{\hat{U}, \hat{L}\}$. Since $\hat{x} = (x_1, x_2, w_1) \neq \hat{y} = (y_1, y_2, w_2)$ we have $x_1 \neq y_1$ or $x_2 \neq y_2$ or $w_1 \neq w_2$. We can assume that $\hat{x}$ and $\hat{y}$ are in the same element of $\{\hat{U}, \hat{L}\}$. Otherwise we are done. Let us assume first that $x_1 \neq y_1$. As long as $\hat{f}^{-k}\hat{x}$ and $\hat{f}^{-k}\hat{y}$ stay in the same elements of $\{\hat{U}, \hat{L}\}$ the same maps S_i^λ are applied to $(x_1, x_2)$and (y_1, y_2) and the distance $|x_1 - y_1|$ increases by hyperbolicity by the factor λ^{-k}. But, this cannot happen forever since the diameter of the square is bounded. Hence, there is an $n \in -\mathbb{N} \subset \mathbb{Z}$ with $\hat{f}^n\hat{x}, \hat{f}^n\hat{y}$ in different elements of $\{\hat{U}, \hat{L}\}$.

If $x_2 \neq y_2$ or $w_1 \neq w_2$ then we can argue in the same manner by iterating forwards. □

Since $\{\hat{U}, \hat{L}\}$ is a generator for the expansive map $\hat{f}$ the topological entropy of $\hat{f}$ is given by the following procedure (see [2] and appendix):

Let $\hat{R}^{(n)}(\hat{x}) = \hat{R}_\lambda^{(n)}(\hat{x}) = \{\hat{y} \in \hat{Q} | \hat{f}^k\hat{y}$ is in the same element of $\{\hat{U}, \hat{L}\}$ as $\hat{f}^k\hat{x}$ for $k = 0, 1, \ldots, n\}$. Then

$$\begin{aligned} h_{\text{top}}(\hat{f}) &= \lim_{n\to\infty} \frac{\log(\text{number of distinct } \hat{R}^{(n)}(\hat{x}))}{n} = \\ &= \lim_{n\to\infty} \frac{\log \#\hat{R}^{(n)}(\hat{x})}{n}. \end{aligned}$$

It is easy to see that $\#\hat{R}^{(n)}(\hat{x}) =$ number of distinct $R_{\underline{i}}^{(n)} = \#R_{\underline{i}}^{(n)}$. Hence,

$$h_{\text{top}}(\hat{f}) = \lim_{n\to\infty} \frac{\log \#R_{\underline{i}}^n}{n}. \tag{16}$$

But, the topological entropy is less than the logarithm of the longest stretching rate; consequently

$$h_{\text{top}}(f) \leq \log \gamma. \tag{17}$$

Now we can state the estimate we need.

Lemma 4.2 *For fixed γ, k and given $\varepsilon_1, \varepsilon_2 > 0$ there is a set $Y \subset (0,1)$ with $\mathcal{L}(Y) > 1 - \varepsilon_1$ and a number $n_0 = n_0(\varepsilon_1, \varepsilon_2)$ such that for $\lambda \in Y, n > n_0$*

$$\sum_{\underline{i},\underline{j} \in W^n} \chi_{\underline{i},\underline{j}}(\lambda) \leq (\gamma + \varepsilon_2)^{2n}$$

where $\mathcal{L}$-denotes the Lebesgue measure and $W^n = \{1,2\}^{\mathbb{N}}$ are all words of length n of symbols 1 or 2.

Proof. Let ε, γ, k be given. From the variational principle ([7]) it follows that the entropy of the SBR-measure $h_{\text{top}}(\hat{f}) \geq h_{\hat{\mu}_{\text{SBR}}} = \log \gamma$. On the other hand the topological entropy is always less then $\log \gamma$, hence

$$\log \gamma = h_{\text{top}}(\hat{f}) = \lim_{n \to \infty} \frac{\log \# R_{\underline{i}}^{(n)}}{\log n}. \tag{18}$$

By applying Lusin's theorem we find a set $Y \subset (\frac{1}{2}, 1)$ with $\mathcal{L}(Y) > 1 - \varepsilon$, a number n_0 and a constant $\tilde{C} > 0$ such that

$$\# R_{\underline{i}}^{(n)}(\lambda) \leq (\gamma + \varepsilon_2)^n \quad \text{for all } \lambda \in K, n > n_0.$$

Inserting this into the definition of $\chi_{\underline{i},\underline{j}}$ and into () we get the desired result. □

Proof of theorem 3.1

The method of proof of theorem 3.1 was developed in [10], proof of lemma 2. The crucial step is to trace points as they vary with the parameters and derive that they stay away each other in average. For this we will estimate integrals of the kind

$$\int_A \frac{d\lambda \, d\gamma \, dk}{|p(\lambda, \gamma, k) - q(\lambda, \gamma, k)|^s}.$$

This is known as the potential-theoretic approach.

Let us fix B_ρ according to the assumption of the theorem. Clearly, we can modify the set Y of lemma 4.2 in the way that if we restrict the Lebesgue measure $\mathcal{L}$ to the interval that is the projection of B_ρ to the $\lambda-$ axis then $\mathcal{L}(Y) > 1 - \varepsilon_1$ and then $\lambda \cdot \gamma^2 < 1$ for all $\lambda \in Y$.

Let $\Sigma_\lambda \subset \Sigma_2 = \{0,2\}^{\mathbb{N}}$ be the set of all infinite sequences $\underline{i} = i_1 \dots i_n \dots$ such that $R_{\underline{i}}^{(n)}(\lambda) \neq \emptyset$ for all $n \in \mathbb{N}$. Fix γ, k according to the assumptions. We want to show that the map $\rho_\lambda : \Sigma_\lambda \to [-1,1]$

$$\rho_\lambda(\underline{i}) = \lim_{n \to \infty} P_1 \circ (S_{i_1} \circ \dots \circ S_{i_n}(0)) \tag{19}$$

where P_1 is the projection onto the first coordinate, is injective a.s. This implies the statement. Moreover, it is enough to show that $\rho_\lambda(\underline{i}) \neq \rho_\lambda(\underline{j})$ a.s. if $i_1 \neq j_1$. Finally we will prove that if $R_{\underline{i}}^{(n)}$ and $R_{\underline{j}}^{(n)}$ are both non-empty then $I_{\underline{i}}^{(n)}$ and $I_{\underline{j}}^{(n)}$ will eventually be disjoint almost surely.

Fix $\varepsilon_1, \varepsilon_2 > 0$ and let Y and n_0 be as in lemma 4.2. We fix another $\varepsilon_3 > 0$ and let $\overline{\lambda} = \max(\frac{1}{\gamma^2}, 0, 64)$. We denote by 0_ε^n the ε-neighbourhood of the iterated singularity line:

$$0_\varepsilon^n = \left\{x = (x_1, x_2) \in Q \middle| d(f^k x, \{y_2 = k y_1\}) < \varepsilon \text{ for some } k \in [0, 1, \dots, n]\right\}$$

Let us consider the event

$$E_n = E_n(\varepsilon) = \tag{20}$$

$$= \left\{ \lambda \in \left(\frac{1}{2}, \overline{\lambda}\right) \cap Y \middle| \left(\bigcup_{\underline{i} \in W^{n-1}} (R_{1\underline{i}}^{(n)} \backslash 0_\varepsilon^n) \right) \cap \left(\bigcup_{\underline{j} \in W^{n-1}} (R_{2\underline{j}}^{(n)} \backslash 0_\varepsilon^n) \right) \neq \emptyset \right\}$$

This event means that the map f^n is not one-to-one outside an ε-neighbourhood of the singularity line if we look at images which arise when starting in U and L. Our aim is to prove that the probability of this event tends to 0 as n tends to infinity.

We can proceed:

$$\begin{aligned} \mathcal{L}(E_n) &\leq \sum_{\underline{i},\underline{j} \in W^{n-1}} \mathcal{L}\left(\left(R_{1\underline{i}}^{(n)} \backslash 0_\varepsilon^n\right) \cap \left(R_{2\underline{j}}^{(n)} \backslash 0_\varepsilon^n\right) \neq \emptyset \right) \\ &\leq \sum_{\underline{i},\underline{j} \in W^{n-1}} \int_{(\frac{1}{2}, 0{,}64) \cap Y} \chi_{1\underline{i},2\underline{j}}^{\varepsilon}(\lambda) \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda \end{aligned} \tag{21}$$

where

$$\hat{\chi}_{\underline{i},\underline{j}}(\lambda) = \begin{cases} 1 & \text{if } d\left(I_{1\underline{i}0^\infty}^{(\infty)}, I_{2\underline{j}0^\infty}^{(\infty)}\right) \leq \overline{\lambda}^n \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi_{\underline{i},\underline{j}}^{\varepsilon}(\lambda) = \begin{cases} 1 & \text{if } R_{\underline{i}}^{(n)}(\lambda) \backslash 0_\varepsilon^n \neq \emptyset \text{ and } R_{\underline{j}}^{(n)}(\lambda) \backslash 0_\varepsilon^n \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

Obviously,

$$\chi_{\underline{i},\underline{j}}^{\varepsilon}(\lambda) \leq \chi_{\underline{i},\underline{j}}(\lambda) \quad \text{for } \varepsilon > 0 \tag{23}$$

We want to decouple the functions under the integral sign in (21). This is because we are able to estimate them separately. This is the point where the difficulties arise from the fact that Σ_λ - the symbolic space - changes with λ. Our way to overcome these difficulties is to estimate how fast the symbolics change as λ varies.

Auxiliary lemma. *Let $\lambda > \frac{1}{2}, \varepsilon > 0$ be given. Let x be a point in $Q^+(\lambda)$ and let $\underline{i} = i_0 i_1 \ldots i_n$ be the sequence $i_k = \begin{cases} 1 & \text{if } f_\lambda^k x \in U \\ 2 & \text{if } f_\lambda^k x \in L \end{cases}$. We assume that the distance of $f^k x$ to the boundary of the corresponding $R_{\underline{i}}^{(k)}(\lambda)$ is larger than ε for $k = 0, 1, \ldots, n$. Then for $\delta < \min(\frac{1}{n}, \frac{\varepsilon}{1000})$ and λ' with $\lambda(1-\delta) < \lambda' < \lambda(1+\delta)$ the sets $R_{\underline{i}}^{(k)}(\lambda')$ are non-empty and moreover, $f_{\lambda'}^k(x) \in R_{\underline{i}}^{(k)}(\lambda'), k = 0, 1, \ldots, n$, and $d(f_{\lambda'}^k(x)) < 1000\delta$, $k = 0, 1, \ldots, n$.*

Proof. Let us fix $\varepsilon > 0, n \in \mathbb{N}, \delta < \min(\frac{1}{n}, \frac{\varepsilon}{1000})\, x, \lambda$ according to the assumptions of the lemma. We only have to prove that $d(f_{\lambda'}^k(x), f_\lambda^k(x)) < 1000\delta$ for $k = 0, \ldots, n$. Because, then $f_{\lambda'}^k(x)$ stays always on the same side of the singularity as f_λ^k. we also notice that as long as the same sequence of S_i is applied to x for λ and λ' the coordinates in the γ-(unstable) directions of the images don't change (We have γ fixed!). So we concentrate on the first (λ-) coordinates. We will prove the lemma by induction.

For $k = 0$ the assertion is trivially fulfilled.

Let us assume that the assertion holds for $(k-1)$ and conclude it for k.

The first coordinate of $f_\lambda^l(x)$ is given by

$$\left(f_\lambda^l(x)\right)_1 = \lambda x_1 + (1-\lambda)[j_1, +\lambda j_2 + \ldots + \lambda^{l-1} j_k]$$

where $j_l = \begin{cases} -1 & \text{if } i_l = 1 \\ 1 & \text{if } i_l = 2 \end{cases}$.

Since we assumed the assertion to hold for $l \le k-1$ the maps S_{i_l} applied for λ' are the same as for λ as long as $l \le k-1$. But we also have $d(f_{\lambda'}^{k-1}(x), f_\lambda^{k-1}(x)) < 1000\delta < \varepsilon$ and $d(f_\lambda^{k-1}(x), \{x_2 = kx_1\}) > \varepsilon$. This yields that we apply in the k-th step also the same S_{i_k}. Hence,

$$\begin{aligned} &\left|\left(f_\lambda^k(x)\right)_1 - \left(f_{\lambda'}^k(x)\right)_1\right| = \\ &= \Big|\lambda^k x_1 + (1-\lambda)\left(j_1 + \ldots + \lambda^{k-1} j_k\right) - \\ &\quad - (\lambda')^k x_1 + (1-\lambda')\left(j_1 + \ldots + (\lambda')^{k-1} j_k\right)\Big| \le \\ &\le \Big|\lambda^k (1+\delta)^k x + (1-\lambda(1+\delta))\left(j_1 + \lambda(1+\delta) j_1 + \ldots\right. \\ &\quad \left.\ldots + \lambda^{k-1}(1+\delta)^{k-1} j_k\right) - \left(\lambda^k x + (1-\lambda)(j_1 + \ldots + \lambda^{k-1} j_k)\right)\Big| \end{aligned} \tag{24}$$

Now we have $\delta < \frac{1}{n}$. This gives for $k \le n$

$$\begin{aligned} (1+\delta)^l &= \sum_{m=0}^{l} \binom{l}{m} \delta^m = 1 + l\cdot\delta + \sum_{m=2}^{l} \binom{l}{m} \delta^m \le \\ &\le 1 + l\cdot\delta + \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \cdot l \cdot \delta = 1 + 2l\cdot\delta. \end{aligned}$$

Inserting this into (24) we can continue

$$\begin{aligned} \left|\left(f_{\lambda'}^k(x)\right)_1 - \left(f_\lambda^k(x)\right)_1\right| &\le \Big|\lambda^k(1+2k\delta)x + (1-\lambda(1+\delta))(j_1 + \ldots \\ &\quad \ldots + \lambda^{k-1}(1+2(k-1)\delta) j_k) - \\ &\quad - \lambda^k x + (1-\lambda)(j_1 + \ldots + \lambda^{k-1} j_k)\Big| \le \\ &\le \lambda^k \cdot 2k\delta + (1-\lambda)\sum_{m=1}^{k-1} 2(m-1)\lambda^m \delta + \\ &\quad + \delta \sum_{m=1}^{k-1} \lambda^m (1+2(m-1)\delta)^m \\ &\le 1000\,\delta < \varepsilon \end{aligned} \tag{25}$$

for $\frac{1}{2} < \lambda, \lambda' < 0,64$.

We have proved the assertion for k and hence by complete induction we conclude

$$d\left(f_{\lambda'}^k(x), f_\lambda^k(x)\right) < 1000\delta < \varepsilon, \quad k = 0, \ldots, n$$

This completes the proof of the lemma. □

Let us now fix ε_4 and consider the event $E_n = E_n(\varepsilon_4), n > n_0$. We also fix $\delta < \min(\frac{1}{n}, \frac{\varepsilon_4}{1000})$. In $(\frac{1}{2}, 0.64) \cap Y$ we choose a sequence $\{\lambda_l\}_1^r = \{\lambda_l^{(n)}\}_1^r$ such that for any $\lambda \in (\frac{1}{2}, 0.64) \cap Y$ there is a number n such that

$$\lambda(1-\delta) < \lambda_n < \lambda(1+\delta).$$

Since $1 > \lambda > \frac{1}{2}$ we can find such a sequence $\{\lambda_l\}_1^r$ with cardinality $r < \frac{4}{\delta}$ and $\lambda_{l+1} > \lambda_l$.

Let us consider an interval $I_{l+1} = (\lambda_l, \lambda_{l+1})$. Here we use the convention that $\lambda_0 = \frac{1}{2}, \lambda_{r+1} = 0.64$.

As an immediate consequence of the auxilary lemma we see that for $\lambda(1-\delta) < \lambda_l < \lambda(1+\delta)$

$$\left\{ \underline{i} \middle| R_{\underline{i}}^{(n)}(\lambda_l) \backslash 0^n_{\varepsilon_4 - 1000\delta} \neq \emptyset \right\} \supset \left\{ \underline{i} \middle| R_{\underline{i}}^{(n)}(\lambda) \backslash 0^n_{\varepsilon_4} \right\}.$$

It follows, that

$$\chi_{\underline{i},\underline{j}}^{\varepsilon_4 - 1000\delta}(\lambda_l) \geq \chi_{\underline{i},\underline{j}}^{\varepsilon_4}(\lambda) \quad \text{for } (1-\delta)\lambda < \lambda_l < \lambda(1-\delta).$$

Hence, we have

$$\chi_{\underline{i},\underline{j}}^{\varepsilon_4 - 1000\delta}(\lambda_l) \geq \chi_{\underline{i},\underline{j}}^{\varepsilon_4}(\lambda) \quad \text{on } I_l.$$

Therefore, we can continue in formula (21)

$$\begin{aligned}
\mathcal{L}(E_n(\varepsilon_4)) &\leq \sum_{\underline{i},\underline{j} \in W^{n-1}} \int\limits_{(\frac{1}{2},0.64)\cap Y} \chi_{\underline{i},\underline{j}}^{\varepsilon_4}(\lambda) \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda \\
&\leq \sum_{\underline{i},\underline{j} \in W^{n-1}} \sum_{l=1}^{r+1} \int\limits_{I_l \cap Y} \chi_{\underline{i},\underline{j}}^{\varepsilon_n}(\lambda) \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda \\
&\leq \sum_{\underline{i},\underline{j} \in W^{n-1}} \sum_{l=1}^{r+1} \int\limits_{I_l \cap Y} \chi_{\underline{i},\underline{j}}^{\varepsilon_4 - 1000\delta}(\lambda_l) \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda \qquad (26) \\
&\leq \sum_{l=1}^{r+1} \sum_{\underline{i},\underline{j} \in W^{n-1}} \chi_{\underline{i},\underline{j}}^{\varepsilon_4 - 1000\delta}(\lambda_l) \int\limits_{I_l \cap Y} \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda \\
&\leq \sum_{l=1}^{r+1} \sum_{\underline{i},\underline{j} \in W^{n-1}} \chi_{\underline{i},\underline{j}}(\lambda_l) \int\limits_{I_l \cap Y} \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda
\end{aligned}$$

and by lemma 4.2 ($\lambda_l \in Y$) with $n \geq n_0$

$$\mathcal{L}(E_n(\varepsilon_4)) \leq \sum_{l=1}^{r+1} (\gamma + \varepsilon_2)^{2n} \int\limits_{I_l \cap Y} \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda$$

The integral on the right-hand side of (26) can be transformed in the following way:

$$\begin{aligned}
&\int\limits_{(\frac{1}{2},0.64)} \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda = \mathcal{L}\left(d(I^{(\infty)}_{1\underline{i}0^\infty}, I^{(\infty)}_{1\underline{j}0^\infty} \leq \overline{\lambda}^n \right) = \\
&= \mathcal{L}\left(\left| (1-\lambda) \left[2 + \lambda(\xi_{i_1} + \xi_{j_1}) + \ldots + \lambda^n(\xi_{j_{n-1}} + \xi_{i_{n-1}}) \right] \right| \leq \overline{\lambda}^n \right) \leq \qquad (27) \\
&\leq \mathcal{L}\left(\left| (1-\lambda) \left[2 + \lambda(\xi_{i_1} + \xi_{j_1}) + \ldots + \lambda^n(\xi_{i_{n-1}} + \xi_{j_{n-1}}) \right] \right|^{-s} \geq \overline{\lambda}^{-sn} \right)
\end{aligned}$$

where $0 < s < 1$ and $\xi_l = \begin{cases} -1 & \text{if } l = 1 \\ 1 & \text{if } l = 2 \end{cases}$.

In [13] it is proved that

$$\int\limits_{(\frac{1}{2},0.64)} \frac{d\,\lambda}{|(1-\lambda)[2 + \lambda(\xi_{i_1} + \xi_{j_1}) + \ldots + \lambda^n(\xi_{i_{n-1}} + \xi_{j_{n-1}})]|^s} \leq c_s < \infty$$

where c_s is independent of $n, \underline{i}$ and $\underline{j}$.

Therefore we can continue in (27) by using Chebyshev's inequality:

$$\begin{aligned} &\int\limits_{(\frac{1}{2},0.64)} \hat{\chi}_{\underline{i},\underline{j}}(\lambda) d\lambda \\ &\leq \int\limits_{(\frac{1}{2},0.64)} \frac{d\,\lambda}{|(1-\lambda)[2+\lambda(\xi_{i_1}+\xi_{j_1})+\ldots+\lambda^n(\xi_{i_{n-1}}+\xi_{j_{n-1}})]|^s} \times \overline{\lambda}^{ns} \\ &\leq C_s \cdot \overline{\lambda}^{ns}. \end{aligned}$$

Inserting this into (26) we have

$$\mathcal{L}(E_n(\varepsilon_4)) \leq \sum_{l=1}^{r+1} (\gamma+\varepsilon_2)^{2n} \cdot \lambda_l^{ns} \leq (\tfrac{4}{\delta}+1)(\gamma+\varepsilon_2)^{2n} \cdot (\max \lambda_l)^{ns} \tag{28}$$

Now we want to look at the condition that the orbit stays at least ε_4 off the singularity line. The condition (H3) implies that the two-dimensional Lebesgue measure ν of the points that stay away from the singularity line is large:

$$\nu\left(\bigcup_{l\geq 1} D_{\Delta,l}\right) = 1 \qquad \text{for all small enough } \Delta \tag{29}$$

(see [11]).

Let $\varepsilon_5 > 0$ and Δ be chosen that $e^{\Delta} \cdot \gamma^2(\max \lambda_l) < 1$ and l such $\nu(D_{\Delta,l}) \geq 1-\varepsilon_5$. This is possible since $\gamma^2 \cdot \lambda < 1$ by the assumption of the theorem. Let now n grow and ε_4 depend on n:

$$\varepsilon_4(n) = \frac{1}{l} e^{-\Delta n}.$$

Letting n tend to infinity and observing that for large enough n we can choose $\delta_n = \frac{\varepsilon_4(n)}{1000} \leq \min(\frac{1}{n}, \frac{\varepsilon_4}{1000})$ and $E_{n+1}(\varepsilon_4) \subset E_n(\varepsilon_4)$ we obtain for $0 < s < 1$:

$$\begin{aligned} \mathcal{L}\left(\bigcap_{n=0}^{\infty} E_n(\varepsilon_4)\right) &= \lim_{n\to\infty} \left[\left(\frac{4}{\delta_n}+1\right)(\gamma+\varepsilon_2)^{2n}(\max \lambda_l)^{ns}\right] = \\ &= \lim_{n\to\infty} \left[(4000 l \cdot e^{\Delta n}+1)(\gamma+\varepsilon_2)^{2n}(\max \lambda_l)^{ns}\right]. \end{aligned} \tag{30}$$

Since $(\lambda_l, \gamma, k) \in B_\rho, \lambda \cdot \gamma^2 < 1$ for all $(\lambda, \gamma, k) \in B_\rho$ we can choose ε_2 and Δ small enough, n_0 according to lemma 4.2 large enough such that

$$e^{\Delta}(\gamma+\varepsilon_2)^2(\max \lambda_l) < 1.$$

Also we can find an $s < 1$ such that

$$e^{\Delta}(\gamma+\varepsilon_2)^2(\max \lambda_l)^s < 1 \tag{31}$$

Therefore the limit on the right-hand-side of (30) tends to 0 and hence,

$$\mathcal{L}\left(\bigcap_{n=0}^{\infty} E_n(\varepsilon_4(n))\right) = 0 \tag{32}$$

i.e. for almost every $\lambda \in (\frac{1}{2}, 0.64) \cap Y$ there is an n with $\lambda \neq E_n(\varepsilon_4)$. But this means that if x and y are points in U or L, respectively, and $d(f_\lambda^k(x), \{z_2 = kz_1\}) > \varepsilon_4(n)$

and $d(f_\lambda^k(y), \{z_2 = kz_1\}) > \varepsilon_4(n), k = 0, 1, \ldots, n$ - i.e., for instance $x, y \in D_{\Delta,l}$ - and $f_\lambda^k(x) \in R_{\underline{i}}^k(\lambda), f_\lambda^k(y) \in R_{\underline{j}}^k(\lambda)$ for some sequences $\underline{i}, \underline{j} \in \sum_\lambda$and $k = 0, 1, \ldots .n$; then $R_{\underline{i}}^k(\lambda) \cap R_{\underline{j}}^k(\lambda') = \emptyset$.

Letting now ε_5 tend to 0 we can conclude that for almost every $\lambda \in Y$ and almost every x and y in Q there is an n such that if $f_{\underline{i}}^k(x) \in R_{\underline{i}}^{(k)}(\lambda), f_{\underline{j}}^k(y) \in R_{\underline{j}}^{(k)}(\lambda)$ for appropriate chosen $\underline{i}$ and $\underline{j}, k = 0, 1, \ldots, n$ then $R_{\underline{i}}^{(n)}(\lambda) \cap R_{\underline{j}}^{(n)}(\lambda) = \emptyset$. Let us now assume for this - almost surely chosen - λ there is a pair of points x and $y \in Q^+$ which are mapped under some iteration of f_λ - say f_λ^n - onto the same point.

This means that there are sets

$$R_{\underline{i}}^{(n)} = \{z \in Q^+ | f_\lambda^k z \in U \Leftrightarrow f_\lambda^k x \in U \qquad k = 0, 1, \ldots, n\}$$
$$R_{\underline{j}}^{(n)} = \{z \in Q^+ | f_\lambda^k z \in U \Leftrightarrow f_\lambda^k y \in U \qquad k = 0, 1, \ldots, n\}$$

(Note that $f^k z, f^k x, f^k y$ are always contained in U or L since $x, y \in Q^+$) with non-empty intersection

$$R_{\underline{i}}^{(n)} \cap R_{\underline{j}}^{(n)} \neq \emptyset \quad \text{for all large enough } n.$$

Since both $R_{\underline{i}}^{(n)}$ and $R_{\underline{j}}^{(n)}$ are open there exists an non-empty open set $G \subset R_{\underline{i}}^{(n)} \cap R_{\underline{j}}^{(n)}$. But this yields that there are at least two non-empty open sets G_1 and G_2 which are mapped under f_λ^n onto the same set G. They have both positive Lebesgue measure what contradicts (32).

Finally, let ε_2 and then ε_1 tend to 0 we derive the assertion of the theorem. □

5 Proof of theorem 3.3 and 3.5

We will proof both theorems simultaneously.

The following proof relies on the potential-theoretic definition of the Hausdorff dimension of a measure. This approach is widely used to get dimension results for parameters dependend system ([10], [3]). As far as we know in all the considered cases there was a canonical symbolic space with a canonical measure which where projected to the invariant set with the invariant measure. The only dependence on the parameter was that of the projection map. In our case the situation is more delicate. We have no canonical symbolic space. It varies as the parameters change as we have explained in section 2.4. Although we could embed all these spaces into Σ_2^+ - the space of all 1, 2-sequences - we still have the problem that the measure varies with the parameter. Moreover, usually these measures have disjoint Borel supports - this is definitely true for ergodic measures. Therefore, we have to discuss their dependence on the parameter, especially, we have to prove that the depend continuously in the weak topology and estimate their rate of convergence.

Let us fix a ball $B_\rho \subset \mathbb{R}^3$ according to the assumptions of the theorem and let $\mathcal{L}^1$ denote the Lebesgue measure restricted to B_ρ^1 and normalized. Let $\gamma, k \in B_\rho$ be fixed and $B_\rho^1 \subset (\frac{1}{2}, 0.64)$ the projection of the straight $(\lambda, \gamma, k) \cap B_\rho$ onto $\mathbb{R}$. The proof of theorem 3.3 and 3.5 will consist of several steps:

Step 0: Ergodicity

In a - as far as we know not yet published - manuscript Sataev claims that the Belykh attractor is ergodic with respect to the SBR-measure. Therefore, we restrict our proof to the case that μ_{SBR} is ergodic. This makes the notations and calculations much more convenient and transparent. It is only a technical problem to generalize the proof for the non-ergodic case and does not involve new ideas. One way is to restrict the SBR-measure to an ergodic component or to use the ergodic decomposition in all calculations.

STEP 1: The restriction to a set of "controllable" points

From the assumptions of the theorem and [11] we know that for all $\lambda \in B^1_\rho$

$$\mu^{(\lambda)}_{\mathrm{SBR}} \left(\bigcup_{l \geq 1} \hat{D}^-_{\Delta,l}(\lambda) \right) = 1. \tag{33}$$

We want to note that the sets $\hat{D}^-_{\Delta,l} = \hat{D}^-_{\Delta,l}(\lambda)$ depend on λ!. Let us fix Δ small and $\varepsilon_1 > 0$. Then there is for all $\lambda \in B^1_\rho$ an $l_0 = l_0(\lambda)$, depending measurably on λ, such that

$$\mu^{(\lambda)}_{\mathrm{SBR}} \left(\hat{D}^-_{\Delta,l}(\lambda) \right) > 1 - \varepsilon_1 \quad \text{for all } l \geq l_0.$$

Since the SBR-measure is concentrated on $\hat{D}$ we have

$$\mu^{(\lambda)}_{\mathrm{SBR}} \left(\hat{D}^-_{\Delta,l}(\lambda) \cap \hat{D}(\lambda) \right) > 1 - \varepsilon_1. \tag{34}$$

By Lusin's theorem for given $\varepsilon_2 > 0$ we can find a number $l_1 = l_1(\varepsilon_1, \varepsilon_2)$ - independent of λ - and a set $Z_1 \subset B^1_\rho$ such that

$$\begin{aligned} &i) \quad l_0(\lambda) < l_1 \quad \text{for } \lambda \in Z_1 \\ &ii) \quad \mathcal{L}^1(Z_1) > 1 - \varepsilon_2 \end{aligned} \tag{35}$$

where $\mathcal{L}^1$ denotes the normalized one-dimensional Lebesgue measure restricted to B^1_ρ.

STEP 2: The measure of the set $\hat{R}^{(n)}(\hat{x})$

In section 2.4 we have defined the sets $\hat{R}^{(n)}(\hat{x}) = \hat{R}^{(n)}(\hat{x}, \lambda) = \hat{R}^{(n)}_\lambda(\hat{x})$ as the elements of the partition $\bigvee_{i=0}^{n} \hat{f}^{-n}\{\hat{U}, \hat{L}\} =: \widehat{\{\hat{U}, \hat{L}\}}^n_0 =: \hat{\mathcal{R}}^{(n)} = \hat{R}^{(n)}_\lambda$. We also showed that the entropy of $\{\hat{U}, \hat{L}\}$ is the maximal possible - the topological entropy:

$$H\left(\{\hat{U}, \hat{L}\}, \hat{f}_\lambda \right) = h_{\mathrm{top}}(\hat{f}_\lambda) = \log \gamma. \tag{36}$$

Since $\{\hat{U}, \hat{L}\}$ is generating (see lemma. 4.1) we have by the theorem of Kolmogorov and Sinai (see appendix) that it also carries the metric entropy of the SBR-measure:

$$h_{\hat{\mu}_{\mathrm{SBR}}}(\hat{f}_\lambda) = h_{\hat{\mu}_{\mathrm{SBR}}} \left(\{\hat{U}, \hat{L}\}, \hat{f}_\lambda \right) = \log \gamma. \tag{37}$$

To this situation we apply the Shannon-McMillan-Breiman theorem:

Lemma 5.1 (Shannon-McMillan-Breiman). *For λ fixed, $\varepsilon_3 > 0$ there is an $n_0 = n_0(\lambda) = n_0(\lambda, \varepsilon_3)$ such that for $n > m > n_0$*

$$\hat{\mu}^{(\lambda)}_{\mathrm{SBR}} \left[\bigcap_{q=m}^{n} \left\{ \bigcup \hat{R}^{(q)}_\lambda(\hat{x}) \middle| \exp\{-q \log(\gamma - \varepsilon_3)\} < \hat{\mu}_{\mathrm{SBR}}(\hat{R}^{(q)}_\lambda(\hat{x})) < \exp\{-q \log(\gamma + \varepsilon_3)\} \right\} \right]$$
$$> 1 - \varepsilon_3$$

The next lemma follows easily from the continuous dependence of f_λ on λ (say in the C^1-topology which respects the singularities).

Lemma 5.2 *The function $n_0 : B^1_\rho \to \mathbb{N}$ is measurable.*

This means we can apply Lusin's theorem:

Lemma 5.3 *For given $\varepsilon_4 > 0$ there is a set $Z_2 \subset Z_1$ and an $n_1 = n_1(\varepsilon_4, \varepsilon_3) \in \mathbb{N}$ - independent of λ - such that*

$$\begin{aligned} &i) \quad n_1 > n_0(\lambda, \varepsilon_3) \quad \textit{for all } \lambda \in Z_2 \\ &ii) \quad \mathcal{L}^1(Z_2) > 1 - \varepsilon_2 - \varepsilon_4 \end{aligned}$$

Step 3: Continuous dependence of the measure

In this section the crucial point is that the auxiliary lemma helps to control the measure of points staying apart from the singularities when λ is changed. So we have to ensure that an essential part of the points is covered by cylinder sets containing enough "good" points and then check the variation of the measure of those rectangles. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4, \lambda \in Z_2, n \geq m > n_1(\varepsilon_3, \varepsilon_4), \Delta$ small, $l > l_1(\varepsilon_1, \varepsilon_2)$ be fixed.

By (34) and (35) we have that most of the atoms of $\hat{R}^{(q)}_\lambda, l_1 < m \leq q \leq n$, contain points from $\hat{D}^-_{\Delta,l}(\lambda)$.

$$\hat{\mu}^{(\lambda)}_{\text{SBR}} \left[\bigcap_{q=m}^{n} \left\{ \bigcup \text{ atoms } \hat{R}^{(q)}_\lambda(\hat{x}) \text{ of } \hat{\mathcal{R}}^{(q)}_\lambda \middle| \hat{R}^{(q)}_\lambda(\hat{x}) \cap \hat{D}^-_{\Delta,l}(\lambda) \neq \emptyset \right\} \right] > 1 - \varepsilon_1. \tag{38}$$

In other words this means that most atoms of the partition $\hat{\mathcal{R}}^{(n)}$ contain a significant part of points from $\hat{D}^-_{\Delta,l}$.

Further lemma 5.1 tells us that most atoms have measure approximately γ^{-n}. The previous observations conglomerate to the following statement:

Let $\hat{\mathcal{G}}^n_m = \hat{\mathcal{G}}^n_m(\lambda)$ (and $\hat{\mathcal{G}}^n = \hat{\mathcal{G}}^n(\lambda) := \hat{\mathcal{G}}^n_n(\lambda)$) be the set of atoms $\hat{R}^{(q)}_\lambda(\hat{x})$ of $\hat{\mathcal{R}}^{(q)}_\lambda$ with $m \leq q \leq n$ the properties

$$\begin{aligned} &i) \quad \nu^{(\lambda)}_k(\hat{R}^{(q)}_\lambda(\hat{x})) < 2\nu^{(\lambda)}_k(\hat{R}^{(q)}_\lambda(\hat{x}) \cap \hat{D}^-_{\Delta,l,q}(\lambda)) \quad m \leq k \leq q \\ &ii) \quad (\gamma - \varepsilon_3)^{-q} < \hat{\mu}_{\text{SBR}}(\hat{R}^{(q)}_\lambda(\hat{x})) < (\gamma + \varepsilon_3)^{-q} \end{aligned} \tag{39}$$

where $\hat{D}^-_{\Delta,l,q}(\lambda) = \{\hat{x} \in \hat{K} | \exists \hat{f}^{-k}(\hat{x}) \text{ and } d(\hat{f}^{-k}\hat{x}, N^+) > \frac{1}{2}e^{-\Delta k}\}$ for $k = 0, 1, \ldots, q$.

Lemma 5.4 *For given $\varepsilon_5 > 0$ there is a set $Z_2 \subset Z_1$ with $\mathcal{L}^1(Z_2) > 1 - \varepsilon_2 - \varepsilon_4 - \varepsilon_5$ and an $n_2 = n_2(\varepsilon_1, \ldots, \varepsilon_4) \in \mathbb{N}$ such that for $\lambda \in Z_2$ and $n_2 < m \leq n$*

$$\hat{\mu}^{(\lambda)}_{\text{SBR}} \left(\bigcap_{q=m}^{n} \bigcup_{\hat{\mathcal{G}}^q(\lambda)} \hat{R}^{(q)}(\hat{x}) \right) > 1 - 3\varepsilon_1 - \varepsilon_3. \tag{40}$$

Proof of lemma 5.4. We first fix λ. Let $A = A(\lambda)$ be the set

$$A = \left\{ \hat{x} \in \hat{D}(\lambda) \middle| \exists n(\hat{x}) \text{ s.t. } \forall n(\hat{x}) < k \leq q : \nu_k^{(\lambda)}(\hat{R}_\lambda^{(q)}) < 2\nu_k^{(\lambda)} \left(\hat{R}_\lambda^{(q)} \cap \hat{D}^-_{\Delta,l,q}(\lambda) \right) \right\}.$$

Then

$$B = \left\{ \hat{x} \in \hat{D}(\lambda) \middle| \exists k_i(\hat{x}) \to \infty,\ q_i \geq k_i \text{ with } \nu_{k_i}^{(\lambda)}(\hat{R}_\lambda^{(q_i)}) < 2\nu_{k_i}^{(\lambda)} \left(\hat{R}_\lambda^{(q_i)} \cap \hat{D}^-_{\Delta,l,q_i}(\lambda) \right) \right\}$$

is its complement.

We are going to prove that there is a number $\tilde{n}_2 = \tilde{n}_2(\lambda)$ such that

$$\hat{\mu}_{\text{SBR}}^{(\lambda)} \left\{ \hat{x} \in A \middle| n(\hat{x}) < \tilde{n}_2(\lambda) \right\} \geq 1 - 3\varepsilon_1.$$

we fix $\delta > 0$ small and choose $\tilde{n}_2(\lambda) = \tilde{n}_2(\lambda, \delta)$ such that for $A' = \{\hat{x} \in A | n(\hat{x}) < \tilde{n}_2(\lambda)\}$

$$\hat{\mu}_{\text{SBR}}(A') > \hat{\mu}_{\text{SBR}}(A) - \delta. \tag{41}$$

We observe that the partition elements $\hat{R}_\lambda^{(n)}$ have the net property - i.e. $\hat{R}_\lambda^{(n)}(\hat{x}) \cap \hat{R}_\lambda^{(m)}(\hat{y}) = \emptyset$ or $\hat{R}_\lambda^{(n)}(\hat{x}) \subset \hat{R}_\lambda^{(m)}(\hat{y})$ or $\hat{R}_\lambda^{(n)}(\hat{x}) \supset \hat{R}_\lambda^{(m)}(\hat{y})$. This yields that we can find a finite partition $\mathcal{P}(A)$ of A' by cylinders $\hat{R}_\lambda^{(q_i)}(\hat{x})$

$$\hat{\mu}_{\text{SBR}} \left(\bigcup_{\mathcal{P}(A)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) > \hat{\mu}_{\text{SBR}}(A') - 2\delta. \tag{42}$$

Since $\hat{\mu}_m$ converges weakly to $\hat{\mu}_{\text{SBR}}$ there is a number $m > \tilde{n}_2$ such that

$$\hat{\mu}_m \left(\bigcup_{\mathcal{P}(A)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) > \hat{\mu}_{\text{SBR}} \left(\bigcup_{\mathcal{P}(A)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) - \delta. \tag{43}$$

Also we can find a partition $\mathcal{P}(B)$ of B by cylinders with the properties:

i) $\hat{\mu}_{\text{SBR}} \left[\left(\bigcup_{\mathcal{P}(A)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) \cap \left(\bigcup_{\mathcal{P}(B)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) \right] < \delta$

ii) $\left(\bigcup_{\mathcal{P}(A)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) \cup \left(\bigcup_{\mathcal{P}(B)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) = \hat{Q}$

iii) $\hat{\mu}_{\text{SBR}} \left(\bigcup_{\mathcal{P}(B)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) < \hat{\mu}_{\text{SBR}}(B) + \delta = 1 - \hat{\mu}_{\text{SBR}}(A) + \delta < 1 - \hat{\mu}_m \left(\bigcup_{\mathcal{P}(A)} \hat{R}_\lambda^{(q_i)}(\hat{x}) \right) + 4\delta$

iv) if $\hat{R}_\lambda^{(q_i)}(\hat{x}) \in \mathcal{P}(B)$ then $q_i \geq m$ and there is a $k_i, m \leq k_i \leq q_i$ such that

$$\hat{\nu}_{k_i} \left(\hat{R}_\lambda^{(q_i)}(\hat{x}) \right) > 2\hat{\nu}_{k_i} \left(\hat{R}_\lambda^{(q_i)}(\hat{x}) \cap \hat{D}^-_{\Delta,l,q_i} \right).$$

We note that

$$\hat{\nu}_k \left(\hat{R}_\lambda^{(q_i)}(\hat{x}) \right) > 2\hat{\nu}_k \left(\hat{R}_\lambda^{(q_i)}(\hat{x}) \cap \hat{D}^-_{\Delta,l,q_i} \right).$$

if $\hat{R}_\lambda^{(q_i)}(\hat{x}) \in \mathcal{P}(B)$ and $k = 0, 1, \ldots, q_i$ holds, since f has constant Jacobian and all the maps $\hat{f}^{-k}, k = 0, \ldots, q_i$ are diffeomorphisms when restricted to $\hat{R}_\lambda^{(q_i)}(\hat{x})$.

By condition (H3) we have

$$\begin{aligned}
&\hat{\nu}_k\left(\bigcup_i\left(\hat{R}_\lambda^{(q_i)}(\hat{x})\cap\hat{D}^-_{\Delta,l,q_i}\right)\right)\\
&=\nu\left(\bigcup_i\left\{\hat{x}\in\hat{Q}\middle|\hat{f}^{q_i}(\hat{x})\in\hat{R}_\lambda^{(q_i)}\text{ and }d\left(\hat{f}^n(\hat{x}),N^+\right)\geq\frac{1}{l}e^{-\Delta q_i}\;\; n=0,1,\dots,q\right\}\right)\geq\\
&\geq 1-\nu\left\{\bigcup_{n=0}^{\infty}\hat{f}^{-n}\left(U\left(\frac{1}{l}e^{-\Delta n},N^+\right)\right)\right\}\geq\\
&\geq\hat{\mu}_{\text{SBR}}\left(\hat{D}^-_{\Delta,l}\right)>1-\varepsilon_1
\end{aligned}\tag{44}$$

On the other hand for $0\leq k\leq n$

$$1\leq\hat{\nu}_k\left(\bigcup_{\mathcal{P}(A)}\hat{R}_\lambda^{(q_i)}(\hat{x})\right)+\hat{\nu}_k\left(\bigcup_{\mathcal{P}(B)}\hat{R}_\lambda^{(q_i)}(\hat{x})\right)\tag{45}$$

and

$$\hat{\nu}_k\left(\hat{R}_\lambda^{(q_i)}(\hat{x})\right)\geq\hat{\nu}_k\left(\hat{R}_\lambda^{(q_i)}(\hat{x})\cap\hat{D}^-_{\delta,l,q_i}\right),\qquad\hat{R}_\lambda^{(q_i)}(\hat{x})\in\mathcal{P}(A)\cup\mathcal{P}(B).\tag{46}$$

Combining (44) to (46) we see that for all $0\leq k\leq m$

$$\hat{\nu}_k\left(\bigcup_{\mathcal{P}(B)}\hat{R}_\lambda^{(q_i)}(\hat{x})\right)<2\varepsilon_1+\delta.\tag{47}$$

Hence,

$$\hat{\mu}_m\left(\bigcup_{\mathcal{P}(B)}\hat{R}_\lambda^{(q_i)}(\hat{x})\right)<2\varepsilon_1+\delta.\tag{48}$$

This implies

$$\hat{\mu}_m\left(\bigcup_{\mathcal{P}(A)}\hat{R}_\lambda^{(q_i)}(\hat{x})\right)<1-2\varepsilon_1-\delta\tag{49}$$

and, by (43)

$$\hat{\mu}_{\text{SBR}}(A)>1-2\varepsilon_1-2\delta.\tag{50}$$

Let $\bar{n}_2(\lambda)=\tilde{n}_2(\lambda,\frac{\varepsilon_1}{2})$ then

$$\hat{\mu}_{\text{SBR}}^{(\lambda)}\left\{\hat{x}\in\hat{Q}\middle|n(\hat{x})<\bar{n}_2(\hat{x})\right\}>1-\frac{5}{2}\varepsilon_1.$$

Applying Lusin's theorem to the measurable function $\bar{n}_2(\lambda)$ for $\varepsilon_5>0$ we can find a number n_2 and a set $Z_2\subset Z_1$ with

$$\mathcal{L}^1(Z_2)>1-\varepsilon_2-\varepsilon_4-\varepsilon_5$$

and for $n_2 \leq m \leq n$

$$\hat{\mu}^{(\lambda)}_{\text{SBR}} \left(\bigcap_{q=m}^{n} \bigcup_{\hat{\mathcal{G}}^q(\lambda)} \hat{R}^{(q)}_{\lambda}(\hat{x}) \right) >$$

$$> \hat{\mu}^{(\lambda)}_{\text{SBR}} \Big(A \cap \bigcap_{q=m}^{n} \Big\{ \bigcup \hat{R}^{(q)}_{\lambda}(\hat{x}) \Big| \exp\{-q\log(\gamma-\varepsilon_3)\} <$$

$$< \hat{\mu}_{\text{SBR}}(\hat{R}^{(q)}_{\lambda}(\hat{x})) < \exp\{-q\log(\gamma+\varepsilon_3)\} \Big\} \Big)$$

$$> 1 - 3\varepsilon_1 - \varepsilon_3$$

□

Let us now investigate the dependence of the measures $\nu^{(\lambda)}_k$ on the parameter λ. Let $\hat{R}^{(n)}_{\lambda}(\hat{x}) \in \hat{\mathcal{G}}^n(\lambda)$. Let $\hat{y} \in \hat{R}^{(n)}_{\lambda}(\hat{x}) \cap \hat{D}^{-}_{\Delta,l}(\lambda)$. Then all its preimages exist, in particular $\hat{z}_k = \hat{f}^{-k}_{\lambda}(\hat{x}) \in \hat{Q} \backslash \partial\hat{U} \cap \partial\hat{L}, k = 0,1,\ldots,n$ and for $\hat{f}^{k}_{\lambda}(\hat{z}_n) = \hat{z}_{n-k}$ we have

$$d\left(\hat{f}^k_{\lambda}(\hat{z}_n), \partial\hat{U} \cap \partial\hat{L}\right) \geq \frac{1}{l} e^{-\Delta(n-k)} \quad k = 0,1,\ldots,n. \tag{51}$$

This means we are in the situation where the auxiliary lemma can be applied to the points $\hat{z}_{n-k}$.

Note that $\hat{R}^{(n)}_{\lambda}(\hat{x})$ are the images of the maximal components of continuity of $\hat{f}^n_{\lambda}$. This yields that $\hat{f}^{-k}_{\lambda}$ is continuous on $\hat{R}^{(n)}_{\lambda}(\hat{x})$ for $k = 0,1,\ldots,n$.

According to the auxiliary lemma we fix $\delta < \frac{\delta}{1-\delta} < \min\{\frac{1}{n}, \frac{1}{1000l} e^{-\Delta n}\}$. We consider a λ' with $(1-\delta)\lambda < \lambda' < (1+\delta)\lambda$. Then the auxilary lemma gives that $\hat{f}^k_{\lambda'}(\hat{z}_{n-k}) \in \hat{R}^{(n)}_{\underline{i}}(\lambda')$, for all $\hat{z}_{n-k} = \hat{f}^{-k}_{\lambda}\hat{y}$, where $\hat{R}^{(n)}_{\underline{i}}(\lambda) = \hat{R}^{(n)}_{\lambda}(\hat{x})$ is the corresponding continouity component for λ'. Also we have

$$\begin{aligned} \hat{\nu}^{(\lambda')}_k \left(\hat{R}^{(n)}_{\underline{i}}(\lambda')\right) &= \hat{\nu}\left(\hat{f}^{-k}_{\lambda'}(\hat{R}^{(n)}_{\underline{i}}(\lambda'))\right) \geq \\ &\geq \hat{\nu}\left(\hat{f}^{-k}_{\lambda'}(\hat{R}^{(n)}_{\underline{i}}(\lambda')) \cap \hat{f}^{-k}_{\lambda}(\hat{D}^{-}_{\Delta,l,n}(\lambda))\right) = \\ &= \hat{\nu}\left(\hat{f}^{-k}_{\lambda}(\hat{R}^{(n)}_{\underline{i}}(\lambda)) \cap \hat{f}^{-k}_{\lambda}(\hat{D}^{-}_{\Delta,l,n}(\lambda))\right) = \\ &= \hat{\nu}\left(\hat{f}^{-k}_{\lambda}(\hat{R}^{(n)}_{\lambda}(\hat{x})) \cap \hat{f}^{-k}_{\lambda}(\hat{D}^{-}_{\Delta,l,n}(\lambda))\right) = \\ &= \hat{\nu}\left(\hat{f}^{-k}_{\lambda}(\hat{R}^{(n)}_{\lambda}(\hat{x}) \cap \hat{D}^{-}_{\Delta,l,n}(\lambda))\right) = \\ &= \hat{\nu}^{(\lambda)}_k \left(\hat{R}^{(n)}_{\lambda}(\hat{x}) \cap \hat{D}^{-}_{\Delta,l,n}(\lambda)\right) = \\ &\geq \frac{1}{2}\nu^{(\lambda)}_k \left(\hat{R}^{(n)}_{\lambda}(\hat{x})\right). \end{aligned} \tag{52}$$

For this chain of inequalities we used only the definition of the measures ν_k, the above observation that the map $\hat{f}^k_{\lambda'}\hat{f}^{-k}_{\lambda}$ is continous at least on $\hat{R}^{(n)}_{\lambda}(\hat{x}) \cap \hat{D}^{-}_{\Delta,l,n}$ and that $\hat{R}^{(n)}_{\lambda}(\hat{x}) \in \hat{\mathcal{G}}^n$ what in particular means that it has large intersection with the set $\hat{D}^{-}_{\Delta,l,n}$.

Observing that $\hat{R}^{(n)}_{\underline{i}}(\lambda') = \hat{R}^{(n)}_{\lambda'}(\hat{w})$ for $\hat{w} = \hat{f}^k_{\lambda'}(\hat{z}_{n-k}), k = 0,1,\ldots,n$, and $\frac{\delta}{1-\delta} < \min\{\frac{1}{n}, \frac{1}{1000l} e^{\Delta n}\}$ we have $(1-\frac{\delta}{1-\delta})\lambda' < \lambda < (1+\frac{\delta}{1-\delta})\lambda'$ fulfills the conditions of the auxilary lemma, too.

Hence, if $\lambda' \in Z_2$ we can apply it to the reverse situation - i.e. we start with the map $\hat{f}_{\lambda'}$ - and can derive

$$\hat{\nu}^{(\lambda)}_k \left(\hat{R}^{(n)}_{\underline{i}}(\lambda)\right) = \nu^{(\lambda)}_k \left(\hat{R}^{(n)}_{\lambda}(\hat{x})\right) \geq \frac{1}{2}\nu^{(\lambda')}_k \left(\hat{R}^{(n)}_{\lambda'}(\hat{x})\right) \quad k = 0,1,\ldots,n. \tag{53}$$

Proceeding as in the definition of the SBR-measure by setting

$$\hat{\mu}_m^{(\lambda)} = \frac{1}{m}\sum_{k=0}^{m-1}\hat{\nu}_k^{(\lambda)} \tag{54}$$

and

$$\hat{\mu}_{m,n_2}^{(\lambda)} = \frac{1}{m-n_2}\sum_{n_2}^{m-1}\hat{\nu}_k^{(\lambda)} \tag{55}$$

we obtain

$$\frac{1}{2}\hat{\mu}_{m,n_2}^{(\lambda')}\left(\hat{R}_{\lambda'}^{(n)}(\hat{w})\right) \leq \hat{\mu}_{m,n_2}^{(\lambda)}\left(\hat{R}_{\lambda}^{(n)}(\hat{x})\right) \leq 2\hat{\mu}_{m,n_2}^{(\lambda')}\left(\hat{R}_{\lambda'}^{(n)}(\hat{w})\right) \tag{56}$$

as long as $\lambda, \lambda' \in Z_2$, $(1-\delta)\lambda < \lambda' < (1+\delta)\lambda$, $\delta < \min\{\frac{1}{n}, \frac{1}{1000l}e^{-\Delta n}\}$ and $n_2 \leq m \leq n$.

Since $\hat{R}_{\underline{i}}^{(l)} = \bigcup\left\{\hat{R}_{\underline{j}}^{(n)} | i_1 = j_1 \ldots i_l = j_l\right\}, n \geq l$ we finally have the

Lemma 5.5 *Let* $n \geq l, m > n_2(\varepsilon_1, \ldots, \varepsilon_5), \frac{\delta}{1-\delta} < \min\{\frac{1}{n}, \frac{1}{1000l}e^{-\Delta n}\}, \lambda, \lambda' \in Z_2, (1-\delta)\lambda < \lambda' < (1+\delta)\lambda, \hat{R}_{\underline{i}}^{(l)}(\lambda) \in \hat{\mathcal{G}}^{(l)}(\lambda)$. *Then*

$$\frac{1}{2}\hat{\mu}_{m,n_2}^{(\lambda')}\left(\hat{R}_{\underline{i}}^{(l)}(\lambda')\right) \leq \hat{\mu}_{m,n_2}^{(\lambda)}\left(\hat{R}_{\underline{i}}^{(l)}(\lambda)\right) \leq 2\hat{\mu}_{m,n_2}^{(\lambda')}\left(\hat{R}_{\underline{i}}^{(l)}(\lambda')\right).$$

Combining the previous lemma with (40) we get

Corollary 5.6 *Under the assumptions of lemma 5.4 and* $n > p > n_2(\varepsilon,\ldots,\varepsilon_5)$

$$\hat{\mu}_{m,n_2}^{(\lambda')}\left[\bigcap_{q=p}^{n}\left\{\bigcup \hat{R}_{\underline{i}}^{(q)}(\lambda') | \hat{R}_{\underline{i}}^{(q)}(\lambda) \in \hat{\mathcal{G}}^{q}(\lambda)\right\}\right] > \frac{1}{2}(1 - 3\varepsilon_1 - \varepsilon_3).$$

For fixed λ the measures $\hat{\mu}_{m,n_2}^{(\lambda)}$ converge weakly to the SBR-measure $\hat{\mu}_{\text{SBR}}^{(\lambda)}$:

$$\lim_{m\to\infty}\hat{\mu}_m^{(\lambda)} = \lim_{m\to\infty}\hat{\mu}_{m,n_2}^{(\lambda)} = \hat{\mu}_{\text{SBR}}^{(\lambda)}.$$

Since the sets $\hat{R}_{\underline{i}}^{(k)}(\lambda)$ are open the weak convergence yields that for given k_0 there is an $m_0(\lambda, k_0) > n_2$ such that for all $m > m_0, k \leq k_0$

$$\frac{2}{3}\hat{\mu}_{m,n_2}^{(\lambda)}\left(\hat{R}_{\underline{i}}^{(k)}(\lambda)\right) \leq \hat{\mu}_{\text{SBR}}^{(\lambda)}\left(\hat{R}_{\underline{i}}^{(k)}(\lambda)\right) \leq \frac{3}{2}\hat{\mu}_{m,n_2}^{(\lambda)}\left(\hat{R}_{\underline{i}}^{(k)}(\lambda)\right). \tag{57}$$

It is an easy exercise to see that the number $m_0(\lambda, k_0)$ depends measurable on λ. Therefore, Lusin's theorem improves lemma 5.4 and corollary 5.6 to:

Lemma 5.7 *For given* ε_6 *there is a number* $m_1 = m_1(\varepsilon_6) > n_2(\varepsilon_1, \ldots, \varepsilon_5)$ *and a set* $Z_3 \subset Z_2$ *such that*

i) $\mathcal{L}(Z_3) > \mathcal{L}(Z_2) - \varepsilon_6$

ii) *for* $n \geq m_1$, $\lambda, \lambda' \in Z_3$, $(1-\delta)\lambda < \lambda' < (1+\delta)\lambda$, $\frac{\delta}{1-\delta} < \min\{\frac{1}{n}, \frac{1}{1000l}e^{-\Delta n}\}$, $m_1 \leq k \leq n$, $\hat{R}_{\underline{i}}^{(k)}(\lambda) \in \hat{\mathcal{G}}^{(k)}(\lambda)$

$$\frac{1}{3}\hat{\mu}_{\text{SBR}}^{(\lambda')}\left(\hat{R}_{\underline{i}}^{(k)}(\lambda')\right) \leq \hat{\mu}_{\text{SBR}}^{(\lambda)}\left(\hat{R}_{\underline{i}}^{(k)}(\lambda)\right) \leq 3\hat{\mu}_{\text{SBR}}^{(\lambda')}\left(\hat{R}_{\underline{i}}^{(k)}(\lambda')\right)$$

iii) $\hat{\mu}_{\text{SBR}}^{(\lambda')}\left[\bigcap_{k=m_1}^{n}\left\{\bigcup \hat{R}_{\underline{i}}^{(k)}(\lambda') | \hat{R}_{\underline{i}}^{(k)}(\lambda) \in \hat{\mathcal{G}}_{\lambda}^{l}\right\}\right] > \frac{1}{3}(1 - 3\varepsilon_1 - \varepsilon_3).$

Step 4: Dimension estimates

We like to estimate the stable dimension $\delta^s(\lambda)$ of the SBR-measure $\mu_{\text{SBR}}^{(\lambda)}$. In [11] we discussed the existence of $\delta^s(\lambda)$ and proved the formula

$$\dim_H \mu_{\text{SBR}}^{(\lambda)} = 1 + \delta^s(\lambda). \tag{58}$$

Here we want to prove that

$$\delta^s(\lambda) = \min\left(-\frac{\log\gamma}{\log\lambda}, 1\right) \quad \text{almost surely.} \tag{59}$$

Let $\hat{x} \in \hat{D}^-_{\Delta,l}$, $\hat{x} = (x_1, x_2, \omega)$, let $\hat{\mu}_{\hat{x}}^{s,(\lambda)}$ denote the conditional measure of $\mu_{\text{SBR}}^{(\lambda)}$ on the square $\{x_1\} \times [-1,1] \times [0,1]$ and $\mu_x^{s,(\lambda)} = \hat{\mu}_{\hat{x}}^{s,(\lambda)} \circ \pi^{-1}$ be the projection onto the straight $\{x_1\} \times [-1,1] \times \{0\}$. Then if we set $\hat{y} = (y_1, y_2, \kappa), \hat{z} = (z_1, z_2, \zeta)$

$$\begin{aligned} &\int_{\Omega_\lambda}\int_{\Omega_\lambda} \frac{d\mu_x^{s,(\lambda)}(y_1,y_2)\, d\mu_x^{s,(\lambda)}(z_1,z_2)}{|(y_1,y_2)-(z_1,z_2)|^s} = \\ &= \int_{\hat{\Omega}_\lambda}\int_{\hat{\Omega}_\lambda} \frac{d\hat{\mu}_x^{s,(\lambda)}(\hat{y})\, d\hat{\mu}_x^{s,(\lambda)}(\hat{z})}{|\pi(\hat{y})-\pi(\hat{z})|^s} < \infty \\ &\Longleftrightarrow \\ &s < \delta^s(\lambda) \end{aligned} \tag{60}$$

provided $\hat{\mu}_{\text{SBR}}^{(\lambda)}(\hat{\Omega}_\lambda) > 0$, (see appendix).

We want to prove a slightly stronger statement (which in fact is equivalent in the case of the Belykh map). Namely, we claim that for $s(\lambda) = \min\left(-\frac{\log\gamma}{\log\lambda}, 1\right)$, $\varepsilon_7 > 0$

$$\int_{\hat{\Omega}_\lambda}\int_{\hat{\Omega}_\lambda} \frac{d\hat{\mu}_{\text{SBR}}^{(\lambda)}(\hat{y})\, d\hat{\mu}_{\text{SBR}}^{(\lambda)}(\hat{z})}{|y_1 - z_1|^{s(\lambda)-\varepsilon_7}} < \infty \tag{61}$$

for almost all λ and some sets $\hat{\Omega}_\lambda$ of positive measure. Because (61) implies that

$$\int_{\hat{\Omega}_\lambda}\int_{\hat{\Omega}_\lambda} \frac{d\hat{\mu}_x^{s,(\lambda)}(\hat{y})\, d\hat{\mu}_x^{s,(\lambda)}(\hat{z})}{|\pi(\hat{y})-\pi(\hat{z})|^{s(\lambda)-\varepsilon_7}} < \infty \tag{62}$$

for almost every $\hat{x}$ this yields the theorem.

Step 5: Proof of claim (61)

We fix $\varepsilon_i > 0, i = 1, \ldots, 7$. We are going to prove that

$$\int_{Z_3}\int_{\hat{\Omega}_\lambda}\int_{\hat{\Omega}_\lambda} \frac{d\hat{\mu}_{\text{SBR}}^{s,(\lambda)}(\hat{y})\, d\hat{\mu}_{\text{SBR}}^{s,(\lambda)}(\hat{z})\, d\lambda}{|y_1 - z_1|^{s(\lambda)-\varepsilon_7}} < \infty \tag{63}$$

for $s(\lambda) = \min\left(-\frac{\log\gamma}{\log\lambda}, 1\right)$ and some sets $\hat{\Omega}_\lambda$ of positive measure.

This gives the assertion of the claim for a.e. $\lambda \in Z_3$. The claim then follows by letting ε_2 and ε_4 tend to 0. As in the proof of theorem 3.1 the crucial point is that we can estimate (see [13])

$$\int_{(\frac{1}{2},0,64)} \frac{d\lambda}{|\tilde{S}_{\underline{i}}^{(\lambda)} - \tilde{S}_{\underline{j}}^{(\lambda)}|^s} = c_s < \infty, \quad \text{for all } \underline{i}, \underline{j} \in \Sigma_2^+, i_1 \neq j_1, s < 1. \tag{64}$$

We first approximate the function $s(\lambda)-\varepsilon_7$ by a step function. We choose a partition $\mathcal{J}$ of the interval $(\frac{1}{2}, 0, 64)$ into finitely many intervals $J_p = (\alpha_{p-1}, \alpha_p), p = 1, \ldots, P$, such that for $s_p = \max_{\lambda \in J_p} s(\lambda) - \frac{\varepsilon_7}{2}$ we have $s_p > s(\lambda) - \varepsilon_7$ on J_p, $p = 1, \ldots, P$.

Hence, it is enough to prove that

$$\int\limits_{Z_3 \cap J_p} \int\limits_{\hat{\Omega}_\lambda} \int\limits_{\hat{\Omega}_\lambda} \frac{d\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}(\hat{y}) d\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}(\hat{z}) d\lambda}{|y_1 - z_1|^{s_p}} < \infty \tag{65}$$

for some sets $\hat{\Omega}_\lambda$ with $\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}(\hat{\Omega}_\lambda) > 0$ for $\lambda \in Z_3$.

We have seen that $\tilde{S}^{(\lambda)}_{\underline{i}}$ is the first coordinate of a point on the attractor iff $\hat{R}^{(n)}_{\underline{i}}(\lambda) \neq \emptyset$ for all n.

Let $\hat{y} \in \hat{R}^{(k)}_{\underline{i}}(\lambda) \cap \hat{D}^-_{\Delta,l}$ for $\underline{i} = i_1 \ldots i_k$. Then there is a point $q(y, \lambda) \in [-1, 1]$ such that $\tilde{S}^{(\lambda)}_{\underline{i}}(q(y, \lambda)) = y_1$ and $\tilde{S}^{(\lambda')}_{\underline{i}}(q(y, \lambda)) = y_1$ is the first coordinate of a point in $\hat{R}^{(k)}_{\underline{i}}(\lambda')$ provided λ' is near λ. This way we have defined a map $\tilde{\Xi}_{\lambda'}$: $(\hat{R}^{(k)}_{\underline{i}}(\lambda) \cap \hat{D}^-_{\Delta,l})_1 \to (\hat{R}^{(k)}_{\underline{i}}(\lambda'))_1$ on the first coordinates which has all properties derived above. This map is the **tracing map**.

We start with the estimation made in [13] (see also the proof of theorem 3.1):

Let $\underline{i} = i_0 \ldots i_k$, $\hat{y} \in \hat{R}^{(k+1)}_{\underline{i}1}(\lambda)$, $\hat{z} \in \hat{R}^{(k+1)}_{\underline{i}2}(\lambda)$. Then

$$\begin{aligned}
&\int\limits_{J_p \cap Z_3} \frac{d\lambda'}{|\tilde{\Xi}_{\lambda'}(y_1) - \tilde{\Xi}_{\lambda'}(z_1)|^{s_p}} \\
&\leq \int\limits_{J_p} \frac{d\,\lambda}{|(1-\lambda)[(\xi_{i_0} + \xi_{j_0}) + \lambda(\xi_{i_1} + \xi_{j_1}) + \ldots + \lambda^n(\xi_{i_{n-1}} + \xi_{j_{n-1}}) + \cdots]|^{s_p}} \leq \\
&\leq \left(\max_{J_p} \lambda\right)^{-ks_p} \int\limits_{J_p} \frac{d\,\lambda}{|(1-\lambda)[2 + \lambda(\xi_{i_{k+2}} + \xi_{j_{k+2}}) + \ldots + \lambda^n(\xi_{i_{k+n+1}} + \xi_{j_{k+n+1}}) + \cdots]|^{s_p}} \\
&\leq c_{s_p}(\max_{J_p}(\lambda))^{-ks_p}
\end{aligned} \tag{66}$$

we now fix $0 \leq p \leq P, n \geq k \geq m_1$, define $\frac{\delta_k}{1-\delta_k} = \min\{\frac{1}{k}, \frac{1}{1000l} e^{-sk}\}$ and choose a sequence $\{\lambda_t\}_1^r = \{\lambda^{(k)}_t\}_1^{(k)}$ such that

$$\begin{aligned}
&i) \quad \lambda^{(k)}_t \in Z_3 \cap J_p \quad t = 1, \ldots, r(k) \\
&ii) \quad \text{For any } \lambda \in Z_3 \text{ there is number } t \text{ with} \\
&\qquad (1-\delta_k)\lambda^{(k)}_t < \lambda < (1+\delta_k)\lambda^{(k)}_t.
\end{aligned} \tag{67}$$

We always can choose the sequence in the way that its cardinality $r(k)$ is less than $\frac{4}{\delta_k}$ and $\lambda^{(k)}_{t+1} > \lambda^{(k)}_t$. We make the convention $\lambda^{(k)}_0 = \alpha_{p-1}, \lambda^{(k)}_{r+1} = \alpha_p$, with $J_p = (\alpha_{p-1}, \alpha_p)$, and $I^{(k)}_t = (\lambda^{(k)}_{t-1}, \lambda^{(k)}_t), t = 1, \ldots, r(k) + 1$.

For $\lambda^{(k)}_t \in \{\lambda^{(k)}_t\}_1^{r(k)}, m_1 \leq k \leq n$ we define

$$\hat{\Gamma}^k_{\lambda^{(k)}_t} = \bigcup_{\hat{g}^{(k)}(\lambda^{(k)}_t)} \hat{R}^{(k)}_{\underline{i}}(\lambda^{(k)}_t) \cap \hat{D}^-_{\Delta,l}(\lambda^{(k)}_t).$$

Then

$$\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}\left(\hat{\Gamma}^k_{\lambda^{(k)}_t}\right) > (1 - 3\varepsilon_1 - \varepsilon_3).$$

Let now λ be arbitrary in J_p. Then by ii) of (67) we find a $t = t(\lambda)$ (if there are more than one we choose the smallest one) with

$$(1-\delta_k)\lambda_t^{(k)} < \lambda < (1+\delta_k)\lambda_t^{(k)}$$

and the map $\Xi_\lambda : \hat{\Gamma}^k_{\lambda_t^{(k)}} \to \Xi_\lambda\left(\hat{\Gamma}^k_{\lambda_t^{(k)}}\right) =: \hat{\Gamma}^k_\lambda$

$$\Xi_\lambda(y_1, y_2, \omega) = \left(\tilde{\Xi}_\lambda(y_1), y_2, \omega\right) \tag{68}$$

is well defined and by lemma 5.7 has the property

$$\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}\left(\hat{\Gamma}^k_\lambda\right) > \frac{1}{3}\hat{\mu}^{(\lambda_t)}_{\mathrm{SBR}}\left(\hat{\Gamma}^k_{\lambda_t^{(k)}}\right) > \frac{1}{3}(1-3\varepsilon_1-\varepsilon_3). \tag{69}$$

Using lemma 5.7 we obtain for $0 \le k \le n, \underline{i} = i_0 \dots i_{k-1}$

$$\begin{aligned}
&T^{k,t,p}_{\underline{i}} = \\
&= \int\limits_{Z_3\cap I_t^{(k)}} \int\limits_{\hat{R}^{(k)}_{\underline{i}1}(\lambda)\cap\hat{\Gamma}^k_\lambda} \int\limits_{\hat{R}^{(k)}_{\underline{i}2}(\lambda)\cap\hat{\Gamma}^k_\lambda} \frac{d\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}(\hat{y})d\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}(\hat{z})d\lambda}{|y_1-z_1|^{s_p}} \le \\
&\le 3 \int\limits_{Z_3\cap I_t^{(k)}} \int\limits_{\hat{R}^{(k)}_{\underline{i}1}(\lambda_t^{(k)})\cap\hat{\Gamma}^k_{\lambda_t^{(k)}}} \int\limits_{\hat{R}^{(k)}_{\underline{i}2}(\lambda_t^{(k)})\cap\hat{\Gamma}^k_{\lambda_t^{(k)}}} \frac{d\hat{\mu}^{(\lambda_t^{(k)})}_{\mathrm{SBR}}(\hat{y})d\hat{\mu}^{(\lambda_t^{(k)})}_{\mathrm{SBR}}(\hat{z})d\lambda}{|\tilde{\Xi}_\lambda y_1 - \tilde{\Xi}_\lambda z_1|^{s_p}} \le \\
&\le 3 \int\limits_{\hat{R}^{(k)}_{\underline{i}1}(\lambda_t^{(k)})\cap\hat{\Gamma}^k_{\lambda_t^{(k)}}} \int\limits_{\hat{R}^{(k)}_{\underline{i}2}(\lambda_t^{(k)})\cap\hat{\Gamma}^k_{\lambda_t^{(k)}}} \left(\int\limits_{Z_3\cap I_t^{(k)}} \frac{d\lambda}{|\tilde{\Xi}_\lambda y_1 - \tilde{\Xi}_\lambda z_1|^{s_p}} \right) d\hat{\mu}^{(\lambda_t^{(k)})}_{\mathrm{SBR}}(\hat{y})d\hat{\mu}^{(\lambda_t^{(k)})}_{\mathrm{SBR}}(\hat{z})d\lambda \\
&\le 3c_{s_p}\left(\max_{J_p}\lambda\right)^{-ks_p} \hat{\mu}^{(\lambda_t^{(k)})}_{\mathrm{SBR}}\left(\hat{R}^{(k)}_{\underline{i}1}(\lambda_t^{(k)})\cap\hat{\Gamma}^k_{\lambda_t^{(k)}}\right) \hat{\mu}^{(\lambda_t^{(k)})}_{\mathrm{SBR}}\left(\hat{R}^{(k)}_{\underline{i}2}(\lambda_t^{(k)})\cap\hat{\Gamma}^k_{\lambda_t^{(k)}}\right).
\end{aligned} \tag{70}$$

We remember the definition of $\hat{\mathcal{G}}^k(\lambda)$ and we use the fact that $\lambda \in Z_3 \subset Z_2$ we see that

$$\hat{\mu}^{(\lambda_t^{(k)})}_{\mathrm{SBR}}\left(\hat{R}^{(k)}_{\underline{i}j}(\lambda_t^{(k)})\cap\hat{\Gamma}^k_{\lambda_t^{(k)}}\right) < (\gamma-\varepsilon_3)^{-k} \quad j=1,2. \tag{71}$$

This yields

$$T^{k,t,p}_{\underline{i}} \le 3c_{s_p}(\max_{J_p}\lambda)^{-ks_p}(\gamma-\varepsilon_3)^{-k}\hat{\mu}^{(\lambda_t^{(k)})}_{\mathrm{SBR}}\left(\hat{R}^{(k)}_{\underline{i}}(\lambda_t^{(k)})\cap\hat{\Gamma}^k_{\lambda_t^{(k)}}\right). \tag{72}$$

We are now going to estimate the original integral (63). Let

$$U^{(p)}_N = U^{(p)}_N\left(\hat{\Gamma}^k_\lambda\right) = \int\limits_{Z_3\cap J_p} \sum_{k=m_1}^{N} \sum_{\underline{i}=i_0\dots i_{k-1}} \int\limits_{\hat{R}^{(k)}_{\underline{i}1}(\lambda)\cap\hat{\Gamma}^k_\lambda} \int\limits_{\hat{R}^{(k)}_{\underline{i}2}(\lambda)\cap\hat{\Gamma}^k_\lambda} \frac{d\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}(\hat{y})d\hat{\mu}^{(\lambda)}_{\mathrm{SBR}}(\hat{z})d\lambda}{|y_1-z_1|^{s_p}}. \tag{73}$$

By choosing ε_7 and ε_3 small enough and P large - i.e. the partition of $(\frac{1}{2}, 0.64)$ consists of very small intervals J_p - we can achieve that

$$\left(\max_{J_p}\lambda\right)^{-s_p}(\gamma-\varepsilon_3)^{-1} = \tau < 1. \tag{74}$$

Let now $n > \max\{N, m_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6)\}$. Then we can proceed

$$\begin{aligned} U_N^{(p)} &\le \sum_{k=m_1}^{N} \sum_{\underline{i}=i_0\ldots i_{k-1}} \int_{Z_3\cap J_p} \int_{\hat{R}_{\underline{i}1}^{(k)}(\lambda)\cap\hat{\Gamma}_\lambda^k} \int_{\hat{R}_{\underline{i}2}^{(k)}(\lambda)\cap\hat{\Gamma}_\lambda^k} \frac{d\hat{\mu}_{\mathrm{SBR}}^{(\lambda)}(\hat{y})d\hat{\mu}_{\mathrm{SBR}}^{(\lambda)}(\hat{z})d\lambda}{|y_1 - z_1|^{s_p}} \le \\ &\le \sum_{k=m_1}^{N} \sum_{\underline{i}=i_0\ldots i_{k-1}} \sum_{t=1}^{m} T_{\underline{i}}^{k,t,p} \le \\ &\le \sum_{k=m_1}^{N} \sum_{\underline{i}=i_0\ldots i_{k-1}} \sum_{t=1}^{m} 3c_{s_p}\tau^k \hat{\mu}_{\mathrm{SBR}}^{(\lambda_t^{(k)})} \left(\hat{R}_{\underline{i}}^{(k)}(\lambda_t^{(k)}) \cap \hat{\Gamma}^k_{\lambda_t^{(k)}}\right). \end{aligned} \tag{75}$$

For Δ small by the choice of the sequence $\{\lambda_t^{(k)}\}_1^{r(k)}$, $r(k) < \frac{4}{\delta_k} = 4000l \cdot e^{\Delta k}$ we can continue

$$\begin{aligned} U_N^{(p)} &\le 3c_{s_p} \sum_{t=1}^{m} \sum_{k=m_1}^{N} \tau^k \sum_{\underline{i}=i_0\ldots i_{k-1}} \hat{\mu}_{\mathrm{SBR}}^{(\lambda_t^{(k)})} \left(\hat{R}_{\underline{i}}^{(k)}(\lambda_t^{(k)}) \cap \hat{\Gamma}^k_{\lambda_t^{(k)}}\right) \le \\ &\le 3c_{s_p} \sum_{k=m_1}^{N} \left(\frac{4}{\delta_k} + 1\right)\tau^k \le \\ &\le C_p \sum_{k=m_1}^{\infty} 4000 l e^{\Delta k}\tau^k. \end{aligned} \tag{76}$$

If we take Δ so small that $e^{\Delta}\tau < 1$ we have a uniform bound for U_N:

$$U_N^{(p)} \le U^{(p)} = C_p \sum_{k=m_1}^{\infty} 4000l \left(e^{\Delta}\tau\right)^k < \infty. \tag{77}$$

Hence, for $\lambda \in J_p \cap Z_3, N > m_1$, $\varepsilon_1, \varepsilon_3$ small enough and

$$\hat{\Gamma}_\lambda(N) = \bigcap_{q=m_1}^{N} \bigcup_{\underline{i}=i_0\ldots i_q} \hat{R}_{\underline{i}}^q(\lambda) \cap \hat{\Gamma}_\lambda^q$$

holds

$$\begin{aligned} &i) \quad \hat{\mu}_{\mathrm{SBR}}^{(\lambda)}\left(\hat{\Gamma}_\lambda(N)\right) > \frac{1}{3}(1 - 3\varepsilon_1 - \varepsilon_3) > \frac{1}{4} \\ &ii) \quad U_N^{(p)}\left(\hat{\Gamma}_\lambda(N)\right) = U_N^{(p)}\left(\hat{\Gamma}_\lambda^{(k)}\right). \end{aligned} \tag{78}$$

Using the σ-additivity of the SBR-measure we derive that the set

$$\hat{\Gamma}_\lambda(\infty) = \bigcap_{A=m_1}^{\infty} \bigcup_{N=A}^{\infty} \hat{\Gamma}_\lambda(N) = \bigcap_{A=m_1}^{\infty} \hat{\Gamma}_\lambda(N) \tag{79}$$

has positive measure:

$$\hat{\mu}_{\mathrm{SBR}}^{(\lambda)}\left(\hat{\Gamma}_\lambda(\infty)\right) > \frac{1}{4}. \tag{80}$$

Finally, we choose cylinder sets $\hat{R}_{\underline{i}}^{m_1}(\lambda_t) \in \hat{\mathcal{G}}_{\lambda_t}^{m_1}, 1 \le t \le r(m_1), \delta_{m_1} = \frac{1}{4000l}e^{-\Delta m_1}$, with

$$\hat{\mu}_{\mathrm{SBR}}^{(\lambda_t^{(k)})}\left(\hat{R}_{\underline{i}}^{m_1}(\lambda_t^{(k)}) \cap \hat{\Gamma}_{\lambda_t^{(k)}}(\infty)\right) \ge \beta_{m_1} > 0. \tag{81}$$

This can be done if $\varepsilon_1, \varepsilon_3$ are small by lemma 5.7.

Then the sets

$$\hat{\Omega}_\lambda = \hat{R}_{\underline{i}}^{m_1}(\lambda) \cap \hat{\Gamma}_\lambda(\infty) \quad \text{for } (1-\delta_{m_1})\lambda_t^{(k)} < \lambda < (1+\delta_{m_1})\lambda_t^{(k)}$$

have still positive measure:

$$\hat{\mu}_{\text{SBR}}^{(\lambda)}\left(\hat{\Omega}_\lambda\right) > \frac{1}{4}\hat{\mu}_{\text{SBR}}^{(\lambda_t^{(k)})}\left(\hat{\Omega}_{\lambda_t^{(k)}}\right) \geq \frac{1}{4}\beta_{m_1} > 0. \tag{82}$$

Putting all this together we are able to estimate

$$\begin{aligned}
&\int\limits_{Z_3}\int\limits_{\hat{\Omega}_\lambda}\int\limits_{\hat{\Omega}_\lambda} \frac{d\hat{\mu}_{\text{SBR}}^{s,(\lambda)}(\hat{y})d\hat{\mu}_{\text{SBR}}^{s,(\lambda)}(\hat{z})d\lambda}{|y_1 - z_1|^{s(\lambda)-\varepsilon_7}} \leq \\
&\leq \sum_{p=1}^{P}\int\limits_{Z_3\cap J_p}\int\limits_{\hat{\Omega}_\lambda}\int\limits_{\hat{\Omega}_\lambda} \frac{d\hat{\mu}_{\text{SBR}}^{s,(\lambda)}(\hat{y})d\hat{\mu}_{\text{SBR}}^{s,(\lambda)}(\hat{z})d\lambda}{|y_1 - z_1|^{s_p}} \leq \\
&\leq \sum_{p=1}^{P}\sup_N U_N^{(p)}\left(\hat{\Gamma}_\lambda^{(k)}\right) \leq P\max U^{(p)} < \infty.
\end{aligned} \tag{83}$$

The potential-theoretic characterization of the Hausdorff dimension (see appendix) tells us that for Lebesgue a.e. $\lambda \in Z_3$

$$\dim_H \mu_{\text{SBR}}^{(\lambda)} \geq \min\left(1 - \frac{\log\gamma}{\log\lambda}, 2\right) - \varepsilon_7. \tag{84}$$

Letting first ε_7 then $\varepsilon_6, \varepsilon_4$ and ε_1 tend to 0 we finish the proof of theorem 3.3 and 3.5. □

6 Concluding Remarks

6.1 The Bifurcation Picture of Invertibility

We want to consider the question to what extend Belykh maps are invertible for given parameters λ, γ, k.

- If $\lambda < \frac{1}{2}$ it is obvious that $f_{\lambda,\gamma,k}$ is invertible on $f(Q\backslash N^+)$ no matter what values take γ and k (as long as the map is defined). Let this set of parameters be denoted by

$$A = \{(\lambda,\gamma,k) | \lambda < \frac{1}{2},\ |k| < 1,\ 1 < \gamma < \frac{1}{|k|+1}\}.$$

- If $\lambda\gamma^2 < 1$ then theorem 3.1 says that almost every f with respect to the Lebesgue measure on the parameter space is fully invertible when restricted to the non-closed attractor D. We denote this parameter set by

$$B = \{(\lambda,\gamma,k) | \lambda > \frac{1}{2},\ |k| < 1,\ 1 < \gamma < \frac{1}{|k|+1}\lambda\gamma^2 < 1\}.$$

For this set we loose the invertibility on $f(Q\backslash N^+)$ but still have invertibility on the limit set.

- If λ increases further to $\lambda\gamma < 1 < \lambda\gamma^2$ then we loose invertibility on D but still almost every f with respect to the Lebesgue measure on the parameter space is invertible on a set of full SBR-measure. We write

$$C = \{(\lambda, \gamma, k) | \lambda > \frac{1}{2}, \ |k| < 1, \ 1 < \gamma < \frac{1}{|k|+1}, \lambda\gamma < 1 < \lambda\gamma^2\}.$$

- The criterion in [11] tells us that for parameters from the complement of the set $A \cup B \cup C$ the map f is never invertible even when restricted to a set of full measure.

This gives the following picture (for $k = 0$):

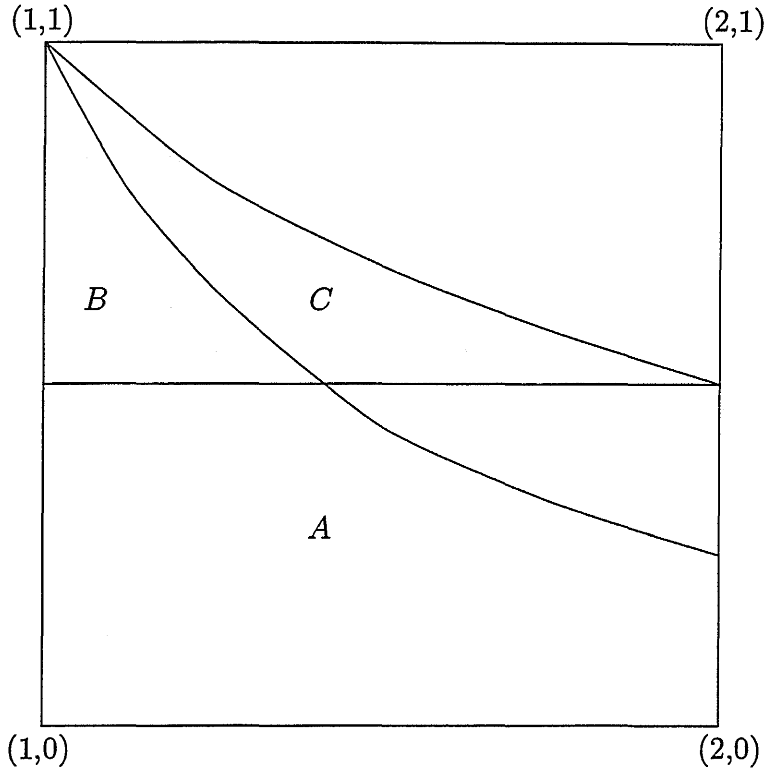


Figure 2

In the appendix we have included Fig. 3-5 which show the Belykh map for parameters in A, C and the complement of $A \cup B \cup C$, respectively.

As we mentioned in the remarks of section 2.3 we proved for the Belykh family the validity of the Kaplan-Yorke conjecture.

6.2 Generalizations

Our proof of theorem 3.3 and 3.5 of this work is based on some special properties of the Belykh family. We restricted ourselves to this map because these special properties make the formulae and technical details more transparent. Also the proofs contain all ideas which are needed to prove a more general result. Below we will discuss how this could be done.

The first property is having constant Jacobian. To avoid this condition is only technical problem. We have to follow a similar way as in the proof that conditions (H4) – (H7) imply condition (H3) in [11] where we first proved the constant Jacobian case and then explained how to use uniform bounds on the ratio of the growth rate of the Jacobian along the orbits of points from the same local stable (unstable) manifold. This technique is now standard.

The next properties are more serious. We do not know whether similar results hold for generic families fulfilling conditions (H1) – (H3) only.

The following condition (H8) we need to get continuous dependence of the SBR-measure on the parameters and to estimate the convergence rate.

(H8) i) The family f_t is defined w.r.t. the same sets $M \supset K \supset N$, $K \backslash N = K_1 \cup \cdots \cup K_r$ (compare with (H1)) and fulfills (H1), (H2) and (H3) uniformly in t.

ii) The partition $\{K_1, \cdots, K_r\}$ is generating.

iii) f_t depends continuously in the topology defined in the appendix on t.

The condition (H8) is similar to those Sataev [9] used to prove continuous dependence of the SBR-measure. Unfortunately, we cannot use his results because there are no estimates we need for the rate of convergence.

The last condition is more tricky. The Belykh family possesses for $\lambda \in (\frac{1}{2}, 0.64)$ a certain transversality condition which was used in [13] to estimate the integrals (66) and (27). We have to assume this condition for the considered family to hold.

Definition 6.1 *We say a parameter family $g_t(x,y) : X_t \times X_t \to \mathbb{R}$, $t \in T \subset \mathbb{R}^d$, T is the open parameter space, (X_t, μ_t) are probability spaces, is a.s. transversal w.r.t. the family $\{\mu_t\}$ if for all $t \in T$ and $\mu_t \times \mu_t$ a.e. $(x,y) \in X \times X$ there is a neighborhood $U_{x,y}(t) \subset T$ such that $g(t) = g_t(x,y) : T \to \mathbb{R}$ is C^2 and $\det Dg \neq 0$ whenever $g(t) = 0$.*

Then (H9) can be formulated as follows:

(H9) The parameter family $g_t(\hat{x}, \hat{y}) = \Xi_t(\hat{x}) - \Xi_t(\hat{y})$, where Ξ_t is the tracing map defined in section 2.5 step 3, is transversal w.r.t $\hat{\mu}^{(t)}_{\mathrm{SBR}}$.

We have no idea if property (H9) is a generic property in any sense. Moreover, we don't have a simple general criterion for a family of maps to satisfy (H9). But there are other examples where a stronger condition than (H9) is proved and used to get dimension results (see [12], [1]).

The above conditions (H8) and (H9) are together with (H1)-(H3) all conditions we need to derive general results analoguous to theorem 3.3 and 3.5.

Concerning theorem 3.1 the situation is different. The theorem is definitely not true for projections of the solenoid. The reason is that for the Belykh family each lifted unstable manifold of x_0 $\hat{W}^{(u)}(x_0, \omega)$ projects to one and the same vertical line. Therefore we either have to make a corresponding condition on our system or the result would be more restrictive, f.i. a.e. map restricted to almost all stable manifolds is invertible. Also we have to change condition $\lambda \cdot \gamma^2 < 1$ to $\lambda \exp\{2h_{\mathrm{top}}\} < 1$.

This is because we have to calculate the number of all cylinders $\hat{R}^{(k)}_{\lambda}$ which is given assymptoticly in terms of the topological entropy rather than the positive Lyapunov exponent ($\log \gamma = h_{\mathrm{top}}$ is a consequence of the constant Jacobian on unstable manifolds).

The generalization to more than two-dimensional systems should involve more knowledge on the dimension theory of higher-dimensional diffeomorphisms. Especially, to get nice dimension formulae one would like to have the Kaplan-Yorke conjecture to hold for generic higher-dimensional systems. Also it seems to be vital to have the Ruelle-Eckmann conjecture to hold. The first conjecture is to have a higher-dimensional Pesin-Young formula - the Lyapunov dimension formula - for a generic system with an SBR-measure and the second deals with the possibility to add stable and unstable dimension to the dimension of the system.

Another interesting direction for further investigations would be the description of the exceptional set of parameters. Even in the case of the fat Belykh attractor - i.e. the case where $\Sigma_\lambda \equiv \Sigma_2^+$ - very little is known about the exceptional set. The only examples of known exceptional values form a countable set - the set of reciprocies of Pisot - Vijayaraghavan numbers.

7 Appendix

A Some Terminology in Dynamical Systems

We consider piecewise smooth maps $f : K \to f(K) \subset K$, where K is an open finite-dimensional submanifold with compact closure of a manifold M. N is a finite union of smooth submanifolds and $K \backslash N = K_1 \cup \cdots \cup K$, with K_i - open, $i = 1, \ldots, r$, and $f|_{K_i}$ (the restriction of f to K_i) is a C^2-diffeomorphism. Let us denote this class by $S^2(K, N)$. We will use the topology defined by the basis of neighbourhoods

$$U(f, \varepsilon, M_1, \ldots, M_r) = \left\{ g \in S^2(K, N) | \sum_{i=1}^{r} ||f|_{M_i} - g|_{M_i}||_{C^2} < \varepsilon \right\}$$

where $\varepsilon > 0, M_i$ are regular compact subsets of K_i. $g^0 = id, g^{n+1} = g^n \circ g$ and if g is a homeomorphism we write $g^{-n} = (g^{-1})^n$. Sometimes we use g^{-n} for the full preimage i.e. $g^{-n}(Y) = \{z | g^n z \in Y\}$.

Let $K^+ = \{x \in K | \exists f^n(x)$ for all $n \geq 0\}$ be the set of points whose trajectory never hits the singularities. Then the non-closed attractor D of f is the set

$$D = \bigcap_{n \geq 0} f^n K^+$$

and the attractor Λ its closure.

A set A is called invariant if $f(A) = A$, clearly. D is invariant.

B Measures - Invariance and Ergodicity

A Borel probability measure μ is called invariant if $\mu(f^{-1}(A)) = \mu(A)$ for all Borel-measurable sets A. An invariant measure μ is called ergodic if for all measurable invariant sets A either $\mu(A) = 0$ or $\mu(A) = 1$.

C Entropy

Let $f : X \to X$ be a Borel measurable map. Let U be an open cover of the compact space X. We write

$$H(\mathfrak{U}) = \log N(\mathfrak{U})$$

where $N(\mathfrak{U})$ denotes the smallest cardinality of a subcover of $\mathfrak{U}$. We consider the expression

$$H(\mathfrak{U}, f) = \lim_{N \to \infty} H\left(\bigvee_{i=0}^{N-1} f^{-i}(\mathfrak{U}) \right)$$

where for two open covers $\mathfrak{U} = \{U_\alpha\}$ and $\mathfrak{U}' = \{U'_\beta\}$ of X $\mathfrak{U} \vee \mathfrak{U}'$ is the cover consisting of elements $U_\alpha \cap U'_\beta$.

Definition C.1 *The quantity*

$$h_{top}(f) = \sup\left\{H(\mathfrak{U}, f) \middle| \mathfrak{U} \text{ an open cover of } \right\}$$

is called the topological entropy *of f.*

Definition C.2 *For a partition $\mathcal{P}$ the quantity*

$$H_\mu(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P)$$

is called the entropy of the partition $\mathcal{P}$.

Definition C.3 *For a partition $\mathcal{P}$ of finite entropy and an invariant measure μ the value*

$$h_\mu(\mathcal{P}, f) = \lim_{N \to \infty} \frac{1}{N} H_\mu \left(\bigvee_{i=0}^{N-1} f^{-i}(\mathcal{P}) \right)$$

is called the entropy of $\mathcal{P}$ w.r.t. f.

Definition C.4 *If μ is invariant the expression*

$$h_\mu(f) = \sup\left\{h_\mu(\mathcal{P}, f) \middle| \mathcal{P} \text{ is a partition with } H(\mathcal{P}) < \infty\right\}$$

is called the entropy of μ.

Proposition C.5 $h_\mu(f) \leq h_{top}(f)$.

Definition C.6 *Let $Y \subset X$ and f be invertible on Y. An at most countable partition $\mathcal{P}$ of X is called a generator for Y if*

i) each $P \in \mathcal{P}$ is regular ($\mathrm{Cl}\,\mathrm{int}\, P = P$)

ii) for $x, y \in Y$ the assumption $f^i(x)$ and $f^i(y)$ stay in the same atom $P(f^i(x)) = P(f^i(y))$ for all $i \in \mathbb{Z}$ implies that $x = y$

iii) $\mu\left(\bigcup_{P_i \in \mathcal{P}} \partial P_i\right) = 0$

Remark The last definition is stronger than the usually given ones and contains topological generators as well as measurable generators.

The next theorem shows the importance of generators. The second part is called the Kolmogorov-Sinai theorem.

Theorem C.7 *Let $\mathcal{P}$ be a generating partition. Then* *i) if $\mathcal{P}$ is finite then*

$$h_{top}(f) = H(\mathcal{P}, f) := \lim_{N\to\infty} \frac{1}{N} \tilde{N} \left(\bigvee_{i=1}^{N-1} f^{-i}\mathcal{P} \right)$$

where $\tilde{N}$ is the number of elements of a partition.

ii) if $H_\mu(\mathcal{P}) < \infty$ then

$$h_\mu(f) = h_\mu(\mathcal{P}, f).$$

Next we will give a non-standard version of the Shannon-McMillan-Breiman theorem for ergodic measures.

Theorem C.8 *Let μ be ergodic and $\mathcal{P}$ be a finite partition. Then for $\varepsilon > 0$ there is a number $n_0 \in \mathbb{N}$ such that for all $n \geq m > n_0$*

$$\mu\Big(\bigcap_{q=m}^{n} \Big\{ \bigcup \Big[\text{ atoms } P^{(q)} \text{ of } \bigvee_{i=0}^{q-1} \mathcal{P} \big| \exp\{-g(h_\mu(\mathcal{P}, f) + \varepsilon)\} >$$
$$< \mu(P^{(q)}) < \exp\{-n(h_\mu(\mathcal{P}, f) - \varepsilon)\} \Big] \Big\} \Big) > 1 - \varepsilon.$$

From general ergodic theory follows the next lemma which says that the "present" is contained in the "future" if the entropy of the system is 0. Thus a zero entropy system is completely deterministic.

Lemma C.9 *$h_\mu(f) = 0$ if and only if for any finite entropy partition $\mathcal{P}$*

$$\mathfrak{A}(\mathcal{P}) \subset \bigvee_{i=1}^{\infty} f^{-i}\left(\mathfrak{A}(\mathcal{P})\right)$$

where $\mathfrak{A}(\mathcal{P})$ is the sub-σ-algebra generated by $\mathcal{P}$.

For more details of entropy theory we refer to the standard literature [see f.i. [2], [6]]

D The Uniformity Theorems of Lusin and Egorov

The references to the following two theorems are somewhat confusing in the literature. One of them belongs to Lusin the other to Egorov. These theorems are consequences of the σ-additivity of the measure μ.

Theorem D.10 *Let ϕ be a real-valued function of the space (X, μ) then ϕ is measurable if and only if for all $\varepsilon > 0$ there is a closed set E whose complement has measure less than ε such that ϕ is continuous on E.*

Theorem D.11 *Let $\{\phi_n\}$ be a sequence of measurable functions converging pointwise on the space (X, μ) then for all $\varepsilon > 0$ there is a measurable set E whose complement has measure less than ε on which $\{\phi_n\}$ converges uniform.*

Detailed information on these theorems can be found in [5].

We will use the following particular version of theorem D.11.

Theorem D.12 *Let $n : (0, 1) \to \mathbb{N}$ be a measurable function with $n(x) < \infty$ for all $x \in \mathbb{R}$. Then for $\varepsilon > 0$ there is a set $Y \subset (0, 1)$ and a number n_0 such that*

i) $\mathcal{L}^1(Y) > 1 - \varepsilon$

ii) $n(x) \leq n$ for all $x \in Y$.

E A Density Lemma for Borel Measures

The following lemma is the analogue of the Lebesgue density lemma for Borel measures. It is proved in [4] chapter 4.

Lemma E.13 *Let (X,μ) be a Borel space and $g \in L^1(\mu)$. We define*

$$g_\delta(x) = \frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} g \, d\mu$$

then

$$g_\delta(x) \longrightarrow g(x) \qquad \mu - \text{ a.e.}$$

F Hausdorff dimension

For a subset Y of a metric space X the s-dimensional ($s \in [0,\infty]$) Hausdorff measure is defined as

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \inf, \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^s \middle| F \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \max_i(\text{diam } U_i) < \delta \right\}$$

It is easy to see that there is a unique $s_0 = s_0(F)$ such that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{for } s < s_0 \\ 0 & \text{for } s > s_0 \end{cases}$$

This number s_0 is called the Hausdorff dimension of F and is denoted by $\dim_H F$.

Let μ be a Borel probability measure on X. Then the Hausdorff dimension of the measure μ is defined by

$$\dim_H \mu = \inf \left\{ \dim_H Y \middle| \mu(Y) = 1 \right\}.$$

Clearly, if $\mu(A) > 0$ then

$$\dim_H \mu\lfloor A \le \dim_H \mu$$

and

$$\dim_H \mu = \sup \left\{ \dim_H \mu\lfloor A \middle| \mu(A) > 0 \right\}. \tag{85}$$

Let $\underline{\delta}_\mu(x)$ denote the lower pointwise dimension of μ:

$$\underline{\delta}_\mu(x) = \liminf_{\varepsilon\to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}.$$

The next lemma is usually known as Frostman's lemma:

Lemma F.14 *If $\underline{\delta}_\mu(x) \ge \delta$ for a set of points x of positive measure then*

$$\dim_H \mu \ge \delta.$$

One can even prove a stronger statement

Proposition F.15 $\dim_H \mu = \operatorname{ess\,sup} \underline{\delta}_\mu(x) = \sup \left\{ \delta \middle| \mu \left\{ x \middle| \underline{\delta}_\mu(x) \ge \delta \right\} > 0 \right\}.$

Sometimes it is convenient to use the potential theoretic approach to calculate the dimension. This approach is based on the following facts.

Theorem F.16 *If*

$$\int_x \frac{d\mu(y)}{|x-y|^s} < \infty$$

then $\underline{\delta}_\mu(x) \geq s$.

Corollary F.17 *If*

$$\int_X \int_X \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty$$

then

$$\dim_H \mu \geq s.$$

The next theorem is the combination of theorem F.16 and (85).

Theorem F.18 *Let* $A \subset X$ *be a set of positive measure. Let moreover*

$$\int_A \int_A \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty$$

then

$$\dim_H \mu \geq s.$$

A survey of the methods and results in dimension theory can be found in [3]. There are contained actually stronger results than those stated above. But for our purposes the here stated versions are satisfactory.

Date: Wed May 12 12:53:25 1993
x Range = [-1, 1]:   y Range = [-1, 1]
Initial Conditions: (x, y, iter)=(-0.678161, -0.212598, 0)
Parameters: (k, lambda, gamma)=(0.2, 0.4, 1.5)
Num Pts = 60003

Belykh

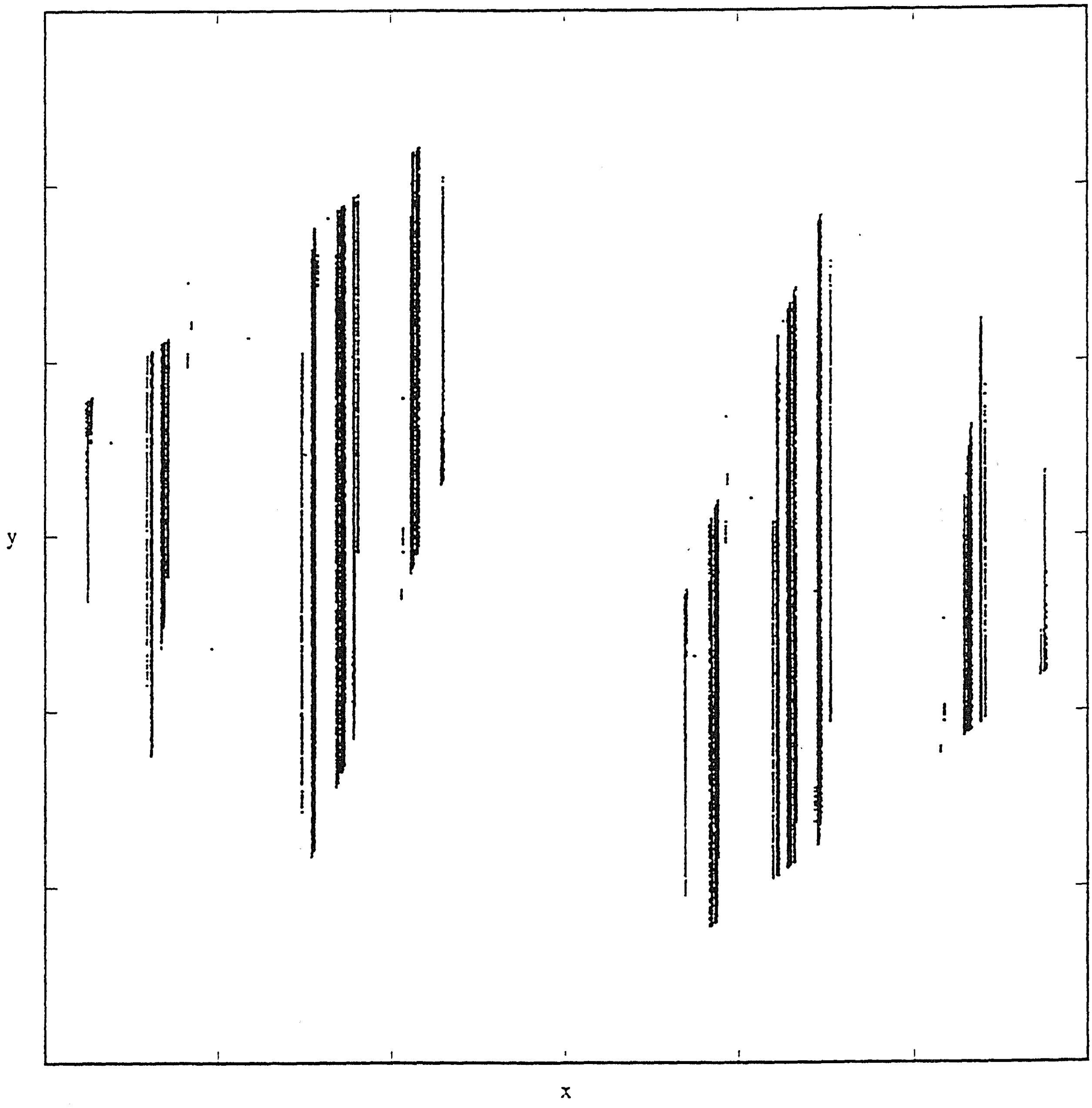
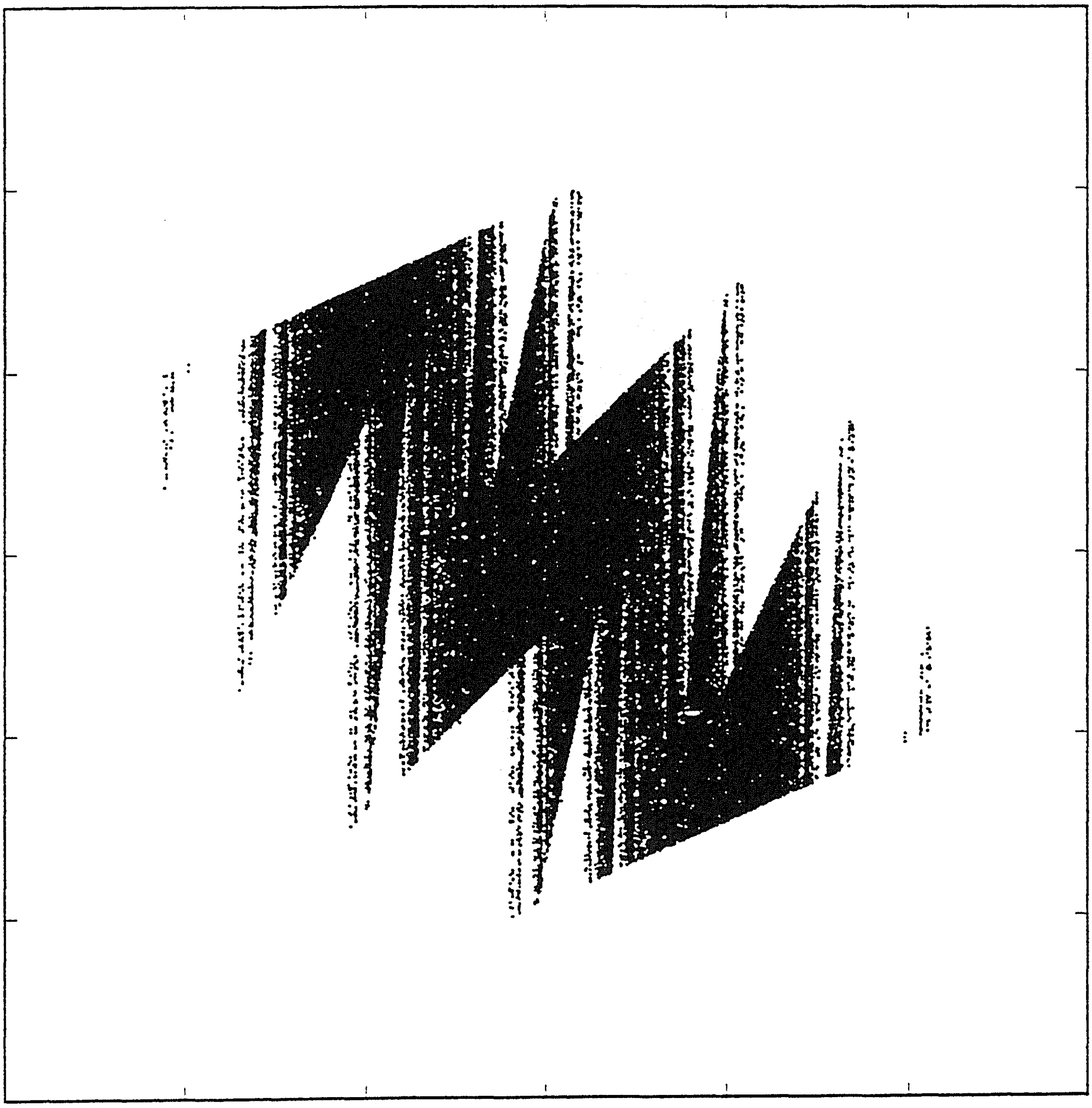


Figure 3

Date: Wed May 12 13:56:02 1993
x Range = [-1, 1]; y Range = [-1, 1]
Initial Conditions: (x, y, iter)=(-0.0541069, 0.0817271, 0)
Parameters: (k, lambda, gamma)=(0.2, 0.68, 1.5)
Num Pts = 100001

Belykh

y



x

Figure 4

Date: Wed May 12 13:07:26 1993
x Range = [-1, 1];   y Range = [-1, 1]
Initial Conditions: (x, y, iter)=(-0.298851, 0.553806, 0)
Parameters: (k, lambda, gamma)=(0.2, 0.9, 1.5)
Num Pts = 60003

Belykh

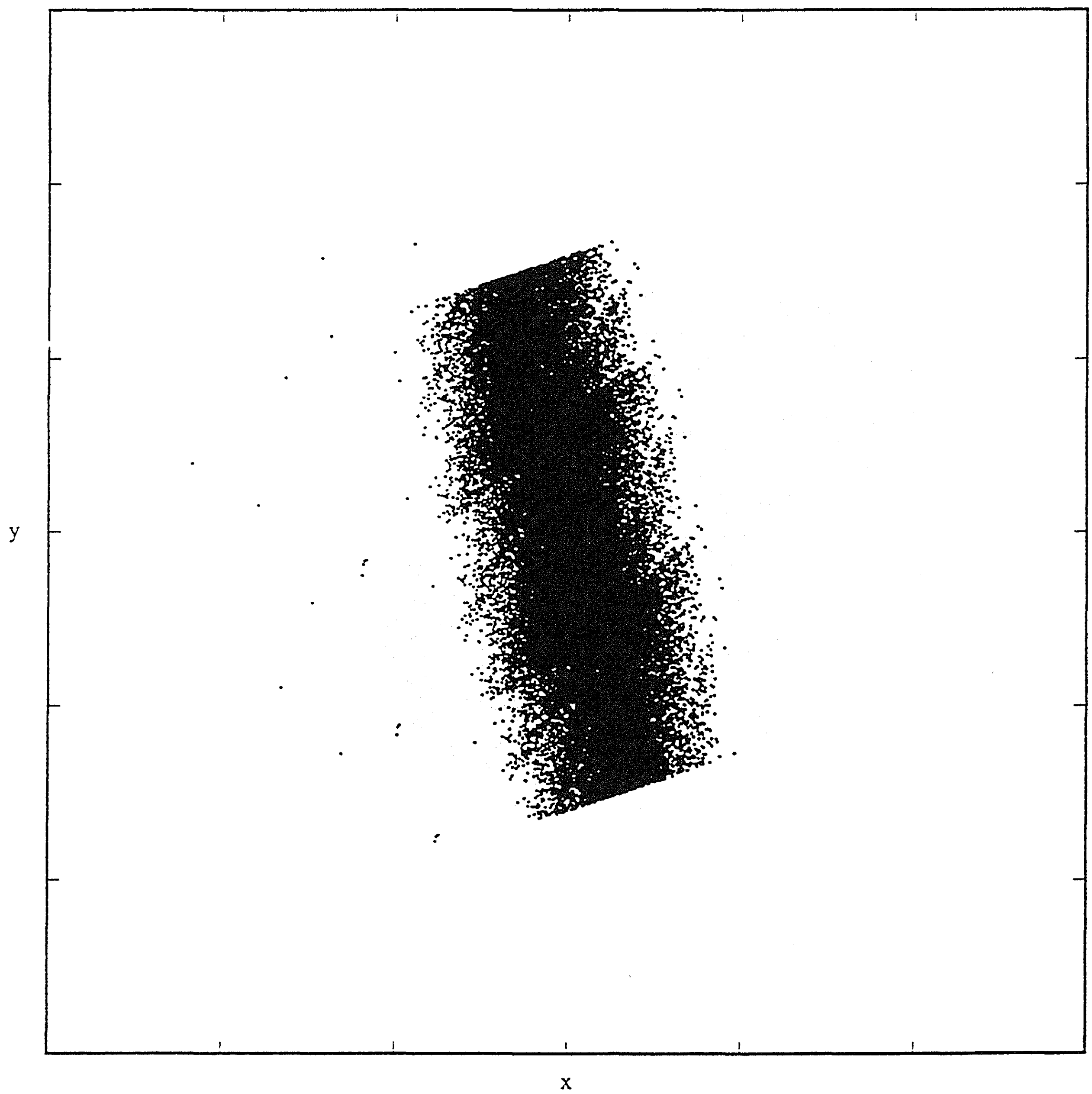


Figure 5

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