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Stability of Pulses on Optical Fibers with Phase-Sensitive Amplifiers

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Abstract

Pulse stability is crucial to the effective propagation of information in a soliton-based optical communication system. It is shown in this paper that pulses in optical fibers, for which attenuation is compensated by phase-sensitive amplifiers, are stable over a large range of parameter values. A fourth-order nonlinear diffusion model due to Kath and co-workers is used. The stability proof invokes a number of mathematical techniques, including the Evans function and Grillakis' functional analytic approach.

Keywords. solitons, nonlinear optical pulse propagation, optical fibers, stability
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1 Introduction

Attenuation of pulses in optical fibers is a major obstacle to the efficacy of a nonlinear optical communication system. It has been suggested recently by Kath and co-workers, see [13], that periodically-spaced phase-sensitive amplifiers offer a realistic alternative to Erbium-doped amplifiers, see Hasegawa and Kodama [10] as well as Mollenauer and co-workers [15]. These amplifiers affect different parts of the signal according to phase, so that the part of the signal in phase with the amplifier is most accentuated and that out of phase is actually attenuated. The detailed mechanism is discussed by Kath in [13]. The amplifiers are placed approximately every 20-50 km and the typical dispersion length of an optical fiber is 200-500 km. It is thus reasonable to treat the amplifiers as closely spaced, see Figure 1. In the mathematical formulation, the amplification can be treated as a rapidly-varying function of space. It is shown in [13], [14] that the equations can then be written

$$\frac{\partial q}{\partial z} + \frac{i}{2}\frac{\partial^2 q}{\partial T^2} + i\left|q\right|^2 q - \gamma q + \frac{1}{\epsilon} h\left(\frac{z}{\epsilon}\right) q + \frac{1}{\epsilon} f\left(\frac{z}{\epsilon}\right) e^{i\phi(z)} q^* = 0, \tag{1}$$

where q is the envelope of the electric field and γ is the linear loss rate in the fiber. The convention of optical scientists is used here so that the evolution variable is z which measures distance along the fiber; and the variable $T \in \mathbf{R}$ represents time in a translating frame. The functions h and f encode the amplification, which is expressed using delta functions at points where the amplifiers are located. It is based upon the formula

$$q_{out} = (\cosh\alpha) q_{in} + e^{i\phi}(\sinh\alpha) q_{in}^*,$$

where $\phi = \phi(z)$ is the reference phase associated with the amplifier. The spacing of the amplifiers is of the order εl, where l is $O(1)$. On account of the rapidly-varying terms in the equations, it is natural to perform averaging and Kath et al. [13] use a multi-scale averaging procedure to derive a fourth order equation that governs the amplitude of the scaled in-phase component of the wave:

$$U = \left(\frac{1 - e^{-2\Gamma l}}{2\Gamma l}\right) Re\left(q e^{-\frac{i\phi}{2}}\right), \tag{2}$$

where $\Gamma = \varepsilon\gamma$. In the limit $\Gamma l \to 0$ the equation has a useful structure — it can be factored into second order operators. Since we systematically exploit this structure, we write the

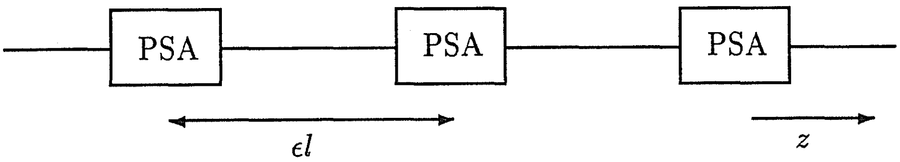

Figure 1: Phase sensitive amplifiers denoted by PSA periodically spaced along an optical fiber. The distance of the amplifiers is of the order ϵl compared to the dispersion length scale z.

full equation as a perturbation of this case:

$$\frac{\partial U}{\partial \zeta} + \Big(\frac{\partial^2}{\partial T^2} + 2U^2 - (2a^2 - \eta^2)\Big)\Big(\frac{\partial^2}{\partial T^2} + 2U^2 - \eta^2\Big)U \qquad (3)$$
$$+ \; 4\sigma\Big(3U\Big(\frac{\partial U}{\partial T}\Big)^2 + U^2\frac{\partial^2 U}{\partial T^2}\Big) = 0,$$

where the variable ζ is a long length scale $\zeta = \varepsilon k z$, k is $O(1)$, and

$$\sigma = 1 - \frac{\tanh \Gamma l}{\Gamma l}.$$

We consider here exclusively the case where $\sigma = 0$, which is tantamount to assuming that the amplifiers are spaced together closely in the asymptotic limit, or that the damping is weak. Indeed, we see that as $\Gamma l \to 0$ then $\sigma \to 0$. In order to explain the parameters a and η in (3), we introduce the quantity $\Delta\alpha$ which measures the $O(\varepsilon^2)$ discrepancy between the decay and amplification rates:

$$\alpha - \Gamma l = \varepsilon^2 \Delta\alpha\Big(\frac{l^2}{2\tanh \Gamma l}\Big).$$

With the characteristic frequency of the amplifier depending linearly on the frequency, we set $a^2 = d\phi/dz$ and determine η from the formula

$$(a^2 - \eta^2)^2 = 4\Delta\alpha. \qquad (4)$$

Since the scaling of close spacing of amplifiers is used in the averaging procedure, the further assumption that $\sigma \to 0$ appears to be of questionable value. However, this limit is so useful that it turns out to be advantageous to view the full equations as a perturbation of this limiting case. In a companion paper to this we consider the case of non-zero σ and, in particular, the possibility of multiple pulses existing on such a fiber with both σ small, by analytic techniques, and σ large, by numerical techniques.

Steady state solutions of (3) can easily be found and these represent the propagation of information along the fiber. The steady state equation can be solved by a function R that satisfies

$$\frac{\partial^2 R}{\partial T^2} + 2R^3 - \eta^2 R = 0 \qquad (5)$$

and such is easily found having the form

$$R(T) = \eta \operatorname{sech} \eta T,$$

where

$$\eta = \pm\sqrt{a^2 \pm \sqrt{4\Delta\alpha}},$$

It is clear that $R(T)$ has the form of a pulse, but it should be noted that, even though these solutions are described as steady states, they are actually independent of space and not time.

Of interest here is the stability of these pulses. We use the linearization

$$\frac{\partial Q}{\partial \zeta} + LQ = 0$$

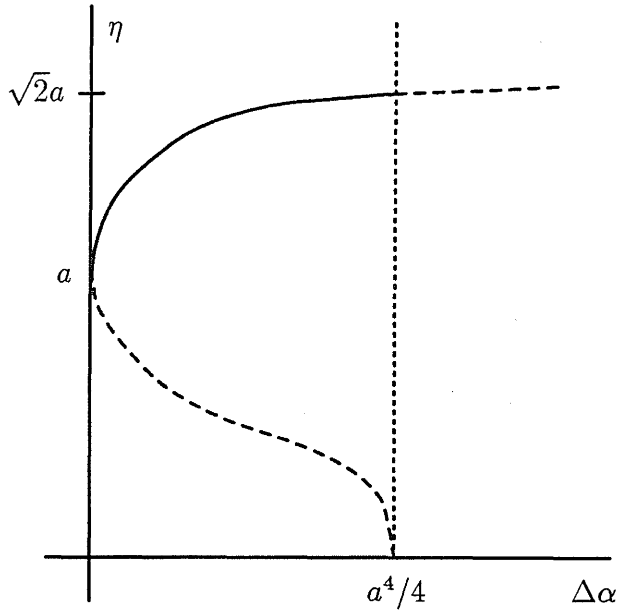

Figure 2: The bifurcation diagram. Dotted lines correspond to unstable $R(T)$, while solid lines correspond to stable $R(T)$.

of equation (3) at the pulse $R(T)$ as the avenue for assessing stability. The criterion for stability is that the linearization L has no spectrum in the closed left half plane other than a simple eigenvalue at zero due to invariance under time translation. This criterion guarantees that perturbations of the wave, in the full nonlinear problem, decay to a translate of the wave, see Henry [11], Bates and Jones [3] — which is nonlinear stability for the family of waves that consists of the pulse and all its translates. Note that the physicists' convention in the sign of the linear operator, entailing that stable spectrum has positive real part, is used here.

These steady state solutions can be organized in a bifurcation diagram, see Figure 2, which is suggestive of stability. It is shown by Kath and Kutz [14] that the lower branch is unstable in the neighborhood of the point where the bifurcation curve touches the axis, namely $(a^4/4, 0)$. More importantly, they show that, in the neighborhood of the turning point $(0, a)$, the lower branch is unstable while the upper branch is stable. They also observe that the upper branch suffers a radiation instability, i.e. the essential spectrum moves into the right half plane, as $\Delta\alpha$ passes through $a^4/4$. This is a curious phenomenon as it occurs at the same value of damping as that at which the lower branch bifurcates subcritically from the zero state. Whether this radiation instability is accompanied by a bifurcation is not presently known; it is a question concerning bifurcation from the essential spectrum.

The main issue addressed in the present paper is the stability of the remainder of the upper branch between the region near the turning point, where stability is guaranteed by the result of Kath and Kutz [14], and the point at which the radiation instability sets in. The difficulty in the analysis is that the linearization of (3) is not self-adjoint and thus many of the standard techniques of stability theory are unavailable. Indeed, there is even the potential for oscillatory instabilities, which are notoriously difficult to prevent.

For the sake of completeness, we give a precise definition of stability for this pulse. The term asymptotic stability would be more appropriate as decay to some translate of the pulse is sought. The natural underlying space is $L^2(\mathbf{R})$ with the usual norm denoted $\|u\| = \left\{\int_{\mathbf{R}} |u|^2 \; dT\right\}^{\frac{1}{2}}$. Indeed, as q is the field envelope, the intensity of the electric field is $\|q\|$. The L^2 norm of q is thus the energy of the pulse, compare [14, p. 3]. In particular, $q \in L^2(\mathbf{R})$. By (2), then $U \in L^2(\mathbf{R})$, too.

Definition 1 *The pulse $R(T)$ is said to be stable if there is a $\delta > 0$ so that if $\|u(0,T) - R(T)\| < \delta$ then there is a $b > 0$ so that $\|u(z,T) - R(T+b)\| \to 0$ as $z \to +\infty$.*

The physical significance of stability is apparent. Only stable pulses can be realistically expected to be relevant to communications models as other pulses necessarily respond to perturbations, so readily supplied by their environment, by losing their form. By the same token, any potential instabilities are important to understand. In this paper we show, however, that the entire upper branch up to the point of the radiation instability consists of stable pulses. Thus the stability noted by Kath and Kutz [14] is not just a local phenomenon and a sizeable range of parameter values is available at which a stable pulse exists. In particular, the over-amplification $\Delta\alpha$ can be chosen arbitrarily within the interval $(0, a^4/4)$. The main result of this paper can be encapsulated in the following theorem.

Theorem 1 *If $a < \eta < \sqrt{2}\alpha$, then the pulse on the upper branch i.e.,*

$$R = \eta \operatorname{sech} \eta T,$$

is stable relative to equation (3).

A central issue for optical communications is whether the fiber under consideration is capable of supporting multiple pulses. It is obviously of interest for a fiber to be able to carry multiple pulses as these represent the propagation of many pieces of information along the fiber. In the companion paper, it is shown that multiple pulses bifurcate from the base pulses on the upper branch as σ is increased from zero. The stability of the base pulse allows us to ascertain which of these multiple pulses are stable.

In section 2, the stability is set up and the key relationship of the linearized version of (3) at the pulse to the linearized nonlinear Schrödinger equation is exposed. As a by-product of this relationship, the instability of the lower branch is easily concluded from results of Jones [12] and Grillakis [7]. The strategy of the stability proof is a continuation argument. It is shown that as η increases from a no instability can occur. In other words, every possibility of an eigenvalue crossing the imaginary axis, which would entail an instability, is ruled out. There are two separate cases to consider. In section 3, it is shown that no eigenvalue passes through 0 and in section 4 it is shown that no eigenvalue can pass through the imaginary axis at a point other than zero, thus ruling out the possibility of any oscillatory instability. Two different approaches are used. In section 3, the key technique is the Evans function and in section 4 functional analytic techniques due to Grillakis are applied.

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2 Linearization

The stability of the pulse to perturbations in the initial conditions is ascertained from collecting sufficient information about the linearization of the evolution equations, in this case equation (3). In order to compute this equation, write (3) with $\sigma = 0$, as

$$\frac{\partial U}{\partial \zeta} + \Phi(U) = 0, \tag{6}$$

where

$$\Phi(U) = (D^2 + 2U^2 - (2a^2 - \eta^2))(D^2 + 2U^2 - \eta^2)U.$$

Rewriting Φ as

$$(D^2 - (2a^2 - \eta^2))(D^2U + 2U^3 - \eta^2 U) + 2U^2(D^2U + 2U^3 - \eta^2 U),$$

we calculate the derivative of Φ as

$$\begin{aligned} D\Phi(R)Q &= (D^2 - (2a^2 - \eta^2))(D^2Q + 6R^2Q - \eta^2 Q) \\ &\quad + 4RQ(D^2R + 2R^3 - \eta^2 R) + 2R^2(D^2Q + 6R^2Q - \eta^2 Q), \end{aligned}$$

and, since $D^2R + 2R^3 - \eta^2 R = 0$, see(5), we see that $D\Phi(R)$ can also be factored as

$$LQ = D\Phi(R)Q = (D^2 + 2R^2 - (2a^2 - \eta^2))(D^2 + 6R^2 - \eta^2)Q. \tag{7}$$

The linearized evolution equation can thus be written

$$\frac{\partial Q}{\partial \zeta} + LQ = 0.$$

For stability it must be shown that the spectrum of the operator L lies in the open right half plane $\{Re\lambda > 0\}$ except for a simple eigenvalue at zero, which is inevitable due to translation.

The factored form of L proves critical in our analysis of its spectrum. A factoring of exactly the same form appears in the linearization of the nonlinear Schrödinger equation, see Weinstein [17, 18]. We borrow the notation for the factors from that context:

$$L_+ = -(D^2 + 6R^2 - \eta^2) \tag{8}$$

$$L_- = -(D^2 + 2R^2 - (2a^2 - \eta^2)), \tag{9}$$

so that $L = L_- L_+$. The spectra of both L_+ and L_- are readily determined as they are second order operators of a simple form. Appropriate qualitative information, such as number of negative eigenvalues, can be determined using Sturm-Liouville theory, or eigenvalues can be exactly computed using the explicit form of R. The spectrum of L_+ consists of essential spectrum which is $\{\lambda : \lambda \geq \eta^2\}$ together with an eigenvalue at $\lambda = 0$ (with eigenfunction $R'(T)$). Since $R'(T)$ has one zero, Sturm-Liouville theory implies that L_+ has exactly one negative eigenvalue. (Indeed L_+ has negative eigenvalue $-3\eta^2$ with eigenfunction

$$R_2 = R^2(T) = \eta^2 \operatorname{sech}^2 \eta T.) \tag{10}$$

The spectrum of L_- includes essential spectrum $\{\lambda : \lambda \geq 2a^2 - \eta^2\}$ and one eigenvalue at $\lambda = 2(a^2 - \eta^2)$, with eigenfunction $R(T)$. There are no other eigenvalues λ of L_- with $\lambda \leq 0$. Indeed, for $a \neq \eta$, two linearly independent solutions of $L_- r = 0$ are given in (20) using the transformation (12). Because both of them are unbounded, no eigenvalue except for $\lambda = 2(a^2 - \eta^2)$ can cross the imaginary axis. Note that on the upper branch L_- has exactly one eigenvalue with $\lambda < 0$ and on the lower branch its spectrum is entirely positive.

Unfortunately, it is far from clear how to build the spectrum of L from those of its component parts L_+ and L_-. Indeed, this issue has led to considerable research, see Weinstein [17, 18], Jones [12], Strauss, Shatah and Grillakis [8, 9] and Grillakis [6, 7]. The operator L is not even self-adjoint and spectrum that is not real is a definite possibility, see Grillakis [7]. The criterion of Jones [12] can be restated as follows. Define the following quantities:

1. P = number of eigenvalues of L_+ in $\lambda \leq 0$,
2. Q = number of eigenvalues of L_- in $\lambda \leq 0$.

It was shown in [12] that if $P - Q > 1$ then L has an eigenvalue $\lambda < 0$, and thus the pulse is unstable. In fact, the result of [12] has strict inequalities in the definitions of P and Q, but it also holds with P and Q as defined here. From the above considerations, it follows then that the lower branch consists of unstable pulses and we have proved the following theorem.

Theorem 2 *If $0 < \eta < a$ then the pulse $R(T)$ is unstable.*

It should be noted that this result is far simpler than the stability of the upper branch and almost every technique we apply, each providing a different piece of the stability puzzle for the upper branch, entails this instability result.

In order to prove stability, as stated in Theorem 1, it suffices to verify the following:

1. If $\lambda \in \sigma(L)$ and $\lambda \neq 0$ then $Re\lambda > 0$,
2. $\lambda = 0$ is a simple eigenvalue.

The sufficiency of these conditions follows from Henry [11] or Bates and Jones [3] as the linearized operator $-L$ satisfies the general conditions needed, such as being the generator of a C^0-semigroup.

The first step in verifying the above conditions is to calculate the essential spectrum of L, denoted $\sigma_e(L)$. This is a standard calculation that follows the lines described in Henry [11, appendix to section 5, Thm. A.2]. It is easily calculated that

$$\sigma_e(L) = \left\{\lambda : \lambda \geq \eta^2(2a^2 - \eta^2)\right\}.$$

The fact that $\sigma_e(L)$ is real is deceptive in that one might then expect L to be self-adjoint. However, it is not and Grillakis [7] shows that in a problem with the same formal structure non-real eigenvalues are a possibility.

The strategy for the proof of Theorem 1 is to exploit the known result of Kath and Kutz [14] that $R(T)$ is stable if $\eta > a$, but η is sufficiently close to a. There are then only two possibilities for an instability to occur:

(A) As η increases, an eigenvalue crosses through 0 into $Re\lambda < 0$,

(B) as η increases, a pair of eigenvalues crosses $Re\lambda = 0$ at $\lambda \neq 0$.

Note that no eigenvalue of L with non-positive real part can come in from infinity via continuation. Indeed, let $J = -D^4$ and $K = -(J + L)$. Then, J is sectorial and $J^{-1/2}K$ is bounded uniformly in a and η. Hence, by [11, Thm. 1.4.4 and Cor. 1.4.5], $-L = J + K$ is sectorial, too, with its spectrum contained in a sector which is independent of η and a. In particular, any eigenvalue of L in the closed left half plane has to be the continuation of an eigenvalue that crossed the imaginary axis.

The next section is devoted to proving that (A) cannot happen. In the final section, it is shown that, even though non-real eigenvalues are a possibility, they must all have positive real part so (B) also cannot occur.

3 Real Spectrum

We consider the Evans function $E(\lambda)$ for the operator L evaluated at the pulse solution $R(T)$. From the general theory developed in Evans [4, 5], for the case of nerve impulse equations, and in Alexander, Gardner and Jones [1], for the more general case, eigenvalues of L are zeroes of the Evans function. Here we develop the Evans function along the lines of Pego and Weinstein [16] and Alexander and Sachs [2]. The details differ from the approach of Alexander, Gardner and Jones [1], but the resulting Evans function is the same (see Pego and Weinstein [16, Proposition 1.15]), and the stability theory holds. The Evans function is defined in terms of solutions of the eigenvalue equation, written as a first order system, and solutions of its adjoint. The eigenvalue equation for L is given by

$$(D^2 + 2R^2 - (2a^2 - \eta^2))(D^2 + 6R^2 - \eta^2)r = \lambda r. \tag{11}$$

For ease of computation, it behooves us to introduce new parameters and variables as follows

$$\mu = \frac{\eta}{\sqrt{2a^2 - \eta^2}}, \quad t = T\sqrt{2a^2 - \eta^2}, \quad \Lambda = \frac{\lambda}{\sqrt{2a^2 - \eta^2}}, \tag{12}$$

by which (11) becomes

$$(D^2 + 2R^2 - 1)(D^2 + 6R^2 - \mu^2)r = \Lambda r. \tag{13}$$

Here, with an abuse of notation, we have also used D to denote $\partial/\partial t$ and $R = \mu \operatorname{sech} \mu t$. Note that the value $\eta = a$ now corresponds to $\mu = 1$. The adjoint equation of (13) is then

$$(D^2 + 6R^2 - \mu^2)(D^2 + 2R^2 - 1)\tilde{u} = \Lambda \tilde{u}. \tag{14}$$

We need to view both (13) and (14) as first order system of ODEs. As such we write them, respectively, in the form

$$X' = A_\Lambda(t)X, \tag{15}$$

and

$$Y' = -YA_\Lambda(t), \tag{16}$$

where

$$A_\Lambda(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mu^2 - 6R^2 & 1 & 0 & 0 \\ \Lambda & 1-2R^2 & 0 & 0 \end{pmatrix}.$$

Indeed, denoting $X = (r, q, s, u)$ and $Y = (\tilde{r}, \tilde{q}, \tilde{s}, \tilde{u})$, we see that

$$\begin{array}{rclcrcl} r' & = & s & \quad & (D^2 + 6R^2 - \mu^2)\, r & = & q \\ q' & = & u & \quad & (D^2 + 2R^2 - 1)\, q & = & \Lambda r, \end{array} \tag{17}$$

and

$$\begin{array}{rclcrcl} \tilde{s}' & = & -\tilde{r} & \quad & (D^2 + 6R^2 - \mu^2)\, \tilde{s} & = & \Lambda \tilde{u} \\ \tilde{u}' & = & -\tilde{q} & \quad & (D^2 + 2R^2 - 1)\, \tilde{u} & = & \tilde{s}, \end{array} \tag{18}$$

once X and Y solve (15) and (16), respectively. Below, we construct $X_1(\Lambda, t)$, $X_3(\Lambda, t)$, linearly independent solutions of (15), which decay to 0 as $t \to +\infty$, and linearly independent solutions $Y_1(\Lambda, t)$, $Y_3(\Lambda, t)$ of (16) that decay to 0 as $t \to -\infty$ (the rationale for the unorthodox indexing will become apparent below). One can easily check from the eigenvalues of the limiting matrix $\lim_{t\to\pm\infty} A_\lambda(t)$ that two independent solutions are expected in each case. In such a case, the Evans function is defined in terms of exterior products. See [1] for the theory and [2] for an explicit computation. The exterior product $X_1 \wedge X_3$ is a six-dimensional vector

$$X_1 \wedge X_3 = (a_{11}, a_{12}, a_{13}, a_{14}, a_{23}, a_{24}),$$

where

$$a_{ij} = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{3i} & x_{3j} \end{vmatrix},$$

with x_{mi}, $i = 1, \ldots, 4$ the components r, q, s, u of X_m in the obvious way. Similarly, the six-dimensional exterior product $Y_1 \wedge Y_3$ is defined. The Evans function is then

$$E(\Lambda) = (Y_1 \wedge Y_3) \cdot (X_1 \wedge X_3), \tag{19}$$

which ostensibly depends on t, but which can be easily shown, by differentiating with respect to t and using (15) and (16), to depend actually only on Λ. The theory of the Evans function, see [1], guarantees that eigenvalues of L correspond to zeroes of $E(\Lambda)$, and moreover that the order of the zero equals the multiplicity of the eigenvalue.

Lemma 1 *If $\mu \neq 1$ then $E'(0) \neq 0$.*

It follows directly from the Lemma that no eigenvalue can pass through zero on the upper branch, so that possibility (A) above is eliminated. Indeed, if an eigenvalue were to pass through zero the order of the zero of the Evans function at $\Lambda = 0$ would have to be greater than or equal to 2, and hence $E'(0)$ would be zero. By the same token, the instability of the lower branch also follows from the Lemma as, from the work of Kath and Kutz [14], we know that there is exactly one negative eigenvalue near the two bifurcation points ($\eta = 0$ and $\eta = a$) and no other unstable eigenvalues. For any of the pulses on the lower branch to be stable, an eigenvalue would have to pass through zero, which again is impossible.

The remainder of this section is devoted to proving the Lemma. This is done by constructing explicit solutions of (15) and (16).

Proof. We first construct solutions of (13) and (14). The function $r_1(t) = R'(t)$ is a solution of

$$(D^2 + 6R^2 - \mu^2)r = 0,$$

as can be easily verified and hence it satisfies (13). A second solution of (13) can be found by reduction of order

$$r_2(t) = \frac{\operatorname{sech}^2 \mu t\,(9\cosh \mu t - \cosh 3\mu t - 12\mu t \sinh \mu t)}{8\mu^3}.$$

If $\mu \neq 1$ then two linearly independent solutions of

$$(D^2 + 2R^2 - 1)\tilde{u} = 0$$

are

$$\begin{aligned} \tilde{u}_3(t) &= e^t(1 - \mu \tanh \mu t) \\ \tilde{u}_4(t) &= e^{-t}(1 + \mu \tanh \mu t). \end{aligned} \tag{20}$$

For each equation a further two solutions can be found by variation of parameters. Noting that the Wronskian of $r_1(t)$ and $r_2(t)$ is 1, as is always the case for reduction of order, and that the Wronskian of $\tilde{u}_3$ and $\tilde{u}_4$ is $W = 2(\mu^2 - 1)$. Accordingly the computations are valid for $\mu \neq 1$.

In particular, using standard formulae for variation of parameters, we set

$$\begin{aligned} r_3(t) &= -r_1(t)\int_0^t r_2(\tau)\tilde{u}_4(\tau)\,d\tau - r_2(t)\int_t^\infty r_1(\tau)\tilde{u}_4(\tau)\,d\tau \\ r_4(t) &= -r_1(t)\int_0^t r_2(\tau)\tilde{u}_3(\tau)\,d\tau + r_2(t)\int_{-\infty}^t r_1(\tau)\tilde{u}_3(\tau)\,d\tau \\ \tilde{u}_1(t) &= (1/W)\left(-\tilde{u}_3(t)\int_\infty^t r_1(\tau)\tilde{u}_4(\tau)\,d\tau + \tilde{u}_4(t)\int_{-\infty}^t r_1(\tau)\,\tilde{u}_3(\tau)\,d\tau\right) \\ \tilde{u}_2(t) &= (1/W)\left(-\tilde{u}_3(t)\int_0^t r_2(\tau)\tilde{u}_4(\tau)\,d\tau + \tilde{u}_4(t)\int_0^t r_2(\tau)\,\tilde{u}_3(\tau)\,d\tau\right), \end{aligned}$$

where $X_i = (r_i, q_i, s_i, u_i)$ and $Y_i = (\tilde{r}_i, \tilde{q}_i, \tilde{s}_i, \tilde{u}_i)$, see (17) and (18). An inspection of the above described solutions will convince the reader that $X_1(t)$ and $X_3(t)$ decay to 0 as $t \to +\infty$. Moreover, as solutions of the adjoint $Y_1(t)$ and $Y_3(t)$ decay to 0 as $t \to -\infty$. As in Pego and Weinstein [16], we apply the Melnikov method to conclude the formula

$$E'(0) = -\int_{-\infty}^{+\infty} (Y_1(t) \wedge Y_3(t)) \frac{\partial A_\Lambda^{(2)}(t)}{\partial \Lambda} (X_1(t) \wedge X_3(t))dt, \tag{21}$$

where $A_\Lambda^{(2)}(t)$ is the 6×6 matrix induced by $A_\Lambda(t)$ on the exterior power space $\Lambda^2(\mathbf{R}^4)$, see again [2] for details. The matrix $\partial A_\Lambda^{(2)}(t)/\partial\Lambda$ is rather sparse and has only two non-zero components. One checks then that the integrand in (21) can be written as

$$\begin{aligned} &\det\begin{pmatrix} \tilde{q}_3(t) & \tilde{u}_3(t) \\ \tilde{q}_1(t) & \tilde{u}_1(t) \end{pmatrix} \det\begin{pmatrix} r_1(t) & r_3(t) \\ q_1(t) & q_3(t) \end{pmatrix} \\ + \; &\det\begin{pmatrix} \tilde{s}_3(t) & \tilde{u}_3(t) \\ \tilde{s}_1(t) & \tilde{u}_1(t) \end{pmatrix} \det\begin{pmatrix} r_1(t) & r_3(t) \\ s_1(t) & s_3(t) \end{pmatrix}. \end{aligned}$$

This simplifies greatly. The relevant facts are $q_1 = 0$, $\tilde{s}_3 = 0$, $\tilde{q}_j = -\tilde{u}_j'$, $s_i = r_i'$, $q_3 = \tilde{u}_4$, $\tilde{s}_1 = r_1$ and the expressions for r_3 and $\tilde{u}_3$ above. The integrand (21) then reduces to

$$r_1(t)\tilde{u}_4(t)\int_{-\infty}^{t} r_1(\tau)\tilde{u}_3(\tau)\,d\tau + r_1(t)\tilde{u}_3(t)\int_{t}^{\infty} r_1(\tau)\tilde{u}_4(\tau)\,d\tau. \tag{22}$$

Note that the t derivative of

$$\int_{-\infty}^{t} r_1(\tau)\tilde{u}_3(\tau)\,d\tau \int_{t}^{\infty} r_1(\tau)\tilde{u}_4(\tau)\,d\tau,$$

is the difference of the terms in (22) so that both terms of (22) integrate to the same value. Therefore, integrating the second term of (22), written out in full, using the above expressions for r_1, $\tilde{u}_1$, $\tilde{u}_2$, and changing the independent variable yields

$$\begin{aligned} -\frac{1}{2}E'(0) &= \frac{1}{\kappa^2}\int_{-\infty}^{\infty}\Big[e^{\kappa t}(\kappa - \tanh t)\tanh t\,\operatorname{sech} t\times \\ &\quad\times\int_{t}^{\infty} e^{-\kappa\tau}(\kappa + \tanh\tau)\tanh\tau\,\operatorname{sech}\tau\,d\tau\Big]\,dt, \end{aligned} \tag{23}$$

where $\kappa = 1/\mu$. Note that the derivative of $e^{-\kappa t}\tanh t\,\operatorname{sech} t$ is

$$-\kappa e^{-\kappa t}\tanh t\,\operatorname{sech} t + e^{-\kappa t}\operatorname{sech}^3 t - e^{-\kappa t}\tanh^2 t\,\operatorname{sech} t,$$

so that, integrating by parts, the inner integral in (23) is

$$\int_{t}^{\infty} e^{-\kappa\tau}\operatorname{sech}^3\tau\,d\tau + e^{-\kappa t}\tanh t\,\operatorname{sech} t. \tag{24}$$

Note that

$$y_\kappa(t) = \int_{t}^{\infty} e^{-\kappa(\tau - t)}\operatorname{sech}^3\tau\,d\tau \tag{25}$$

is the solution of the differential equation

$$y_\kappa'(t) - \kappa y_\kappa(t) = \operatorname{sech}^3 t, \qquad y_\kappa(\infty) = 0. \tag{26}$$

From (25), we note that if $\kappa' > \kappa \geq 0$, then, pointwise,

$$0 < y_{\kappa'}(t) < y_\kappa(t). \tag{27}$$

Moreover

$$y_0 = \frac{\pi}{4} - \arctan(\tanh t/2) - \frac{\operatorname{sech} t\tanh t}{2},$$

so that all $y_\kappa(t)$ are bounded. Hence, the integral (23) can be written

$$\begin{aligned} &\int_{-\infty}^{\infty}(\kappa - \tanh t)\tanh t\,\operatorname{sech} t\; y_\kappa(t)\,dt \\ &\quad + \kappa\int_{-\infty}^{\infty}\tanh^2 t\,\operatorname{sech}^2 t\,dt - \int_{-\infty}^{\infty}\tanh^3 t\,\operatorname{sech}^2 t\,dt. \end{aligned} \tag{28}$$

The second integral of (28) has value $2\kappa/3$ and the third one is zero, since the integrand is an odd function. Consider the first integral of (28). Note that the derivative of $y_k(t)\,\tanh t\,\operatorname{sech} t$ is

$$y_\kappa(t)\,\operatorname{sech}^3 t - y_\kappa(t)\,\tanh^2 t\,\operatorname{sech} t + y_\kappa'(t)\,\tanh t\,\operatorname{sech} t$$

so, integrating by parts and using (26), the first integral of (28) can be written

$$y_\kappa(t)\ \tanh t \operatorname{sech} t\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} y_\kappa(t)\ \operatorname{sech}^3 t\ dt - \int_{-\infty}^{\infty} \tanh t \operatorname{sech}^4 t\ dt. \tag{29}$$

Here the first and third terms are zero. Thus, in sum, (23) equals

$$\frac{2\kappa}{3} - \int_{-\infty}^{\infty} y_\kappa(t)\ \operatorname{sech}^3 t\ dt. \tag{30}$$

From (27), we see that (30) is an increasing function of κ. Moreover direct calculation of (23) with $\kappa = 1$ establishes that it is zero. Hence (23) is positive for $\kappa > 1$ ($\mu < 1$) and negative for $\kappa < 1$ ($\mu > 1$). Thus the quantity $E'(0)$ is non-zero if $\mu \neq 1$, and the lemma is proved. □

4 Complex Spectrum

In order to determine if eigenvalues of L pass through the imaginary axis as η is increased from a, we apply a completely different technique. The following is an application of the technique developed by Grillakis [7]. We do not need the full power of the theory and can prove the desired result by an adaptation of one of the key parts of the technique. Recalling that L can be expressed as $L = L_- L_+$, where the factors are given by (9) and (8), respectively, the eigenvalue problem is written as $L_- L_+ \phi = \lambda\phi$. If $a < \eta < \sqrt{2}a$ then zero is not an eigenvalue of L_- and so L_- is invertible. Thus the eigenvalue problem can be rewritten as

$$L_+\phi = \lambda L_-^{-1}\phi, \tag{31}$$

which we express using the notation $B = L_+$, $C = L_-^{-1}$ as

$$(B - \lambda C)\phi = 0. \tag{32}$$

Note that both B and C are self-adjoint. Throughout, we assume that $Im\lambda \neq 0$.

We decompose (32) into equations on the spaces $\mathbf{Q} = \operatorname{span}\{R\}$ and $\mathbf{P} = \mathbf{Q}^\perp$, where $R(T)$ is the pulse solution (1). Let $\pi: L^2(\mathbf{R}) \to \mathbf{P}$ be the natural projection and set $p = \pi(\phi)$, so that

$$\phi = p + \alpha R,$$

where α is a complex scalar. With the notation $B_1 = \pi B$ and $C_1 = \pi C$, the eigenvalue equation (32) projected onto $\mathbf{P}$ becomes

$$(B_1 - \lambda C_1)p + \alpha B_1 R = 0. \tag{33}$$

In deriving (33) we have used the fact that R is an eigenfunction of L_- with eigenvalue $2(a^2 - \eta^2)$, which implies that

$$CR = \frac{1}{2(a^2 - \eta^2)} R \tag{34}$$

and hence $C_1 R = \pi C R = 0$. Projecting onto $\mathbf{Q}$ we obtain

$$\langle (B - \lambda C)(p + \alpha R), R\rangle = 0. \tag{35}$$

Since C is self-adjoint and (34), $\langle Cp, R\rangle = 0$; thus (35) can be simplified to

$$\langle Bp, R\rangle + \alpha\left[\langle BR, R\rangle - \lambda\langle CR, R\rangle\right] = 0. \tag{36}$$

From (34), we calculate

$$\langle CR, R\rangle = \frac{1}{2(a^2-\eta^2)}\langle R, R\rangle = -\beta^2,$$

where β is some non-zero real number. Since B and π are self-adjoint and $\pi p = p$, the first term of (36) equals $\langle p, B_1 R\rangle$. Equation (36) thus reads

$$\langle p, B_1 R\rangle + \alpha\left[\langle BR, R\rangle + \lambda\beta^2\right] = 0. \tag{37}$$

The sign of $\langle BR, R\rangle$ can be established easily. Since R satisfies (5), $L_+R = -4R^3$. Hence,

$$\langle BR, R\rangle = -4\int_{-\infty}^{+\infty} R^4\,dt < 0. \tag{38}$$

The next step is to determine an expression for $\langle p, B_1R\rangle$, given that p satisfies (33). The approach is to use the Spectral Theorem to obtain an expression for p; this is the heart of the technique. Note that since C leaves $\mathbf{P}$ invariant the operator C_1 is self-adjoint. It is also positive since its negative eigenspace $\mathbf{Q}$ has been factored out. Thus C_1 has a square root $C_1^{\frac{1}{2}}$; similarly C_1^{-1} has a square root, denoted $C_1^{-\frac{1}{2}}$. Equation (33) can thus be rewritten as

$$C_1^{\frac{1}{2}}\left[H_1 - \lambda\right]C_1^{\frac{1}{2}}p + \alpha B_1 R = 0, \tag{39}$$

where

$$H_1 = C_1^{-\frac{1}{2}} B_1 C_1^{-\frac{1}{2}}. \tag{40}$$

Concerning the operator H_1 we have the following lemma.

Lemma 2 *H_1 is self-adjoint and $H_1 \geq 0$.*

Proof. It is easily checked that H_1 is self-adjoint. To check that it is non-negative, we calculate

$$\langle H_1\phi, \phi\rangle = \langle C_1^{-\frac{1}{2}} B_1 C_1^{-\frac{1}{2}}\phi, \phi\rangle = \langle B_1 C_1^{-\frac{1}{2}}\phi, C_1^{-\frac{1}{2}}\phi\rangle.$$

Thus, if we can show that $\langle Bp, p\rangle \geq 0$ for every $p \in \mathbf{P}$, then the desired result follows.

It is convenient for the remainder of the proof to factor out by the kernel $\operatorname{span}\{R'\}$ of B. It is a general fact that if $B : \mathcal{H} \to \mathcal{H}$ is a self-adjoint non-singular operator on a Hilbert space (here $\mathcal{H} = L^2/\operatorname{span}\{R'\}$), with $BR_2 = -\theta R_2$, $\theta > 0$ (see (10)) and $B|_{R_2^\perp} > 0$ for $R_2 \neq 0$, then for $R \in \mathcal{H}$ with

$$\langle B^{-1}R, R\rangle < 0 \tag{41}$$

we have $\langle B\phi, \phi\rangle > 0$ for all non-zero $\phi \in \mathbf{P} = R^\perp$. For write

$$R = \begin{pmatrix} R_- \\ R_+ \end{pmatrix}, \qquad \phi = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix},$$

where R_-, $\phi_- \in \text{span}\{R_2\}$, R_+, $\phi_+ \in \mathbf{P}$. Write

$$\begin{aligned} B^{-1}R &= B^{-1}\begin{pmatrix} R_- \\ R_+ \end{pmatrix} = \begin{pmatrix} -R_-/\theta \\ B_+^{-1}R_+ \end{pmatrix}, \\ B\phi &= B\begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} = \begin{pmatrix} -\theta\phi_- \\ B_+\phi_+ \end{pmatrix}. \end{aligned}$$

We are given

$$\langle \phi_-, R_- \rangle + \langle \phi_+, R_+ \rangle = 0,$$
$$\langle -\frac{1}{\theta}R_-, R_- \rangle + \langle B_+^{-1}R_+, R_+ \rangle < 0.$$

Suppose also

$$\langle B\phi, \phi \rangle = \langle -\theta\phi_-, \phi_- \rangle + \langle B_+\phi_+, \phi_+ \rangle \leq 0$$

for some nonzero $\phi \in \mathcal{H}$. Then

$$\begin{array}{rcccll} \langle \phi_-, R_- \rangle^2 & = & \langle \phi_+, R_+ \rangle^2 & = & \langle B_+^{\frac{1}{2}}\phi_+, B_+^{-\frac{1}{2}}R_+ \rangle^2, & (i) \\ \frac{1}{\theta}\langle R_-, R_- \rangle & > & \langle B_+^{-1}R_+, R_+ \rangle & = & \langle B_+^{-\frac{1}{2}}R_+, B_+^{-\frac{1}{2}}R_+ \rangle, & (ii) \\ \theta\langle \phi_-, \phi_- \rangle & \geq & \langle B_+\phi_+, \phi_+ \rangle & = & \langle B_+^{\frac{1}{2}}\phi_+, B_+^{\frac{1}{2}}\phi_+ \rangle. & (iii) \end{array} \tag{42}$$

Since the "minus" subspace is 1-dimensional, the LHS(ii)×LHS(iii)=LHS(i). By the Cauchy inequality, RHS(ii)×RHS(iii)≥RHS(i). Chasing around, this is a contradiction. (Note: the converse of this general result is also valid.)

Hence it is sufficient to prove (41). Differentiating (5) with respect to $\hat{\eta} = \eta^2$, we find $B^{-1}R = -dR/d\hat{\eta}$, so that

$$\langle B^{-1}R, R \rangle = -\frac{1}{2}\frac{d}{d\hat{\eta}}\langle R, R \rangle = -\frac{1}{2}\frac{d}{d\hat{\eta}}\int_{-\infty}^{\infty} \eta^2 \operatorname{sech}^2 \eta t \, dt = -\frac{1}{2\eta}.$$

Thus (41) is verified. Hence $B_1 \geq 0$ on $\mathbf{P}$ and the lemma is proved. □

Now (39) can be written

$$[H_1 - \lambda]\, C_1^{\frac{1}{2}} p = -\alpha C_1^{-\frac{1}{2}} B_1 R. \tag{43}$$

We see immediately from (43) and the above lemma that, if λ is an eigenvalue of L with non-zero imaginary part, then $\alpha \neq 0$. But also, with λ so set, the operator $H_1 - \lambda$ is invertible, and hence (43) can be rewritten as

$$p = -\alpha C_1^{-\frac{1}{2}} [H_1 - \lambda]^{-1} (C_1^{-\frac{1}{2}} B_1 R). \tag{44}$$

The Spectral Theorem can be applied to obtain, from (44), an integral expression for p. If $\{E_\rho\}_{\rho \in \mathbf{R}}$ is a spectral resolution for H_1, then we have

$$p = -\alpha \int_{-\infty}^{+\infty} \frac{C_1^{-\frac{1}{2}} dE_\rho (C_1^{-\frac{1}{2}} B_1 R)}{\rho - \lambda}. \tag{45}$$

This can be substituted into $\langle p, B_1 R\rangle$ to obtain

$$\langle p, B_1 R\rangle = \alpha \int_{-\infty}^{+\infty} \frac{dv(\rho)}{\lambda - \rho}, \tag{46}$$

where $dv(\rho) = \langle dE_\rho(C_1^{-\frac{1}{2}} B_1 R), C_1^{-\frac{1}{2}} B_1 R\rangle$. Note that $\text{supp}(dv) \subset \{\rho \geq 0\}$ since $H_1 \geq 0$. Using (46), (37) becomes

$$\int_{-\infty}^{+\infty} \frac{dv(\rho)}{\lambda - \rho} + \langle BR, R\rangle + \lambda\beta^2 = 0, \tag{47}$$

for $\lambda \notin \{\rho \geq 0\}$, where $\alpha \neq 0$ has been cancelled from each term.

Now we consider whether expression (47) is consistent with having an eigenvalue of non-positive real part. To this end, suppose that $Re\lambda \leq 0$ and $Im\lambda \neq 0$. Since $\text{supp}(dv) \subset \{\rho \geq 0\}$, the first term of (47) is negative, as is the second term by (38). The real part of the third term would then have to be positive in order for (47) to hold. However, it is not and we have a contradiction. Thus, any eigenvalue of L with non-zero imaginary part must have positive real part. This completes the proof of Theorem 1. □

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