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Fast and slow waves in the FitzHugh-Nagumo equation

Martin Krupa*‡ Björn Sandstede† Peter Szmolyan *‡

Abstract

It is known that the FitzHugh-Nagumo equation possesses fast and slow travelling waves. Fast waves are perturbations of singular orbits consisting of two pieces of slow manifolds and connections between them, whereas slow waves are perturbations of homoclinic orbits of the unperturbed system. We unfold a degenerate point where the two types of singular orbits coalesce forming a heteroclinic orbit of the unpertubed system. Let c denote the wave speed and ϵ the singular perturbation parameter. We show that there exists a C^2 smooth curve of homoclinic orbits of the form $(c, \epsilon(c))$ connecting the fast wave branch to the slow wave branch. Additionally we show that this curve has a unique non-degenerate maximum. Our analysis is based on a Shilnikov coordinates result, extending the Exchange Lemma of Jones and Kopell. We also prove the existence of inclination-flip points for the travelling wave equation thus providing the evidence of the existence of n-homoclinic orbits (n-pulses for the FitzHugh-Nagumo equation) for arbitrary n.

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1 Introduction

Travelling wave solutions are basic patterns of reaction diffusion equations. These waves correspond to heteroclinic or homoclinic orbits of related ODE problems. Frequently such orbits can be found by means of singular perturbation theory. One of the prototypical examples is the FitzHugh-Nagumo equation:

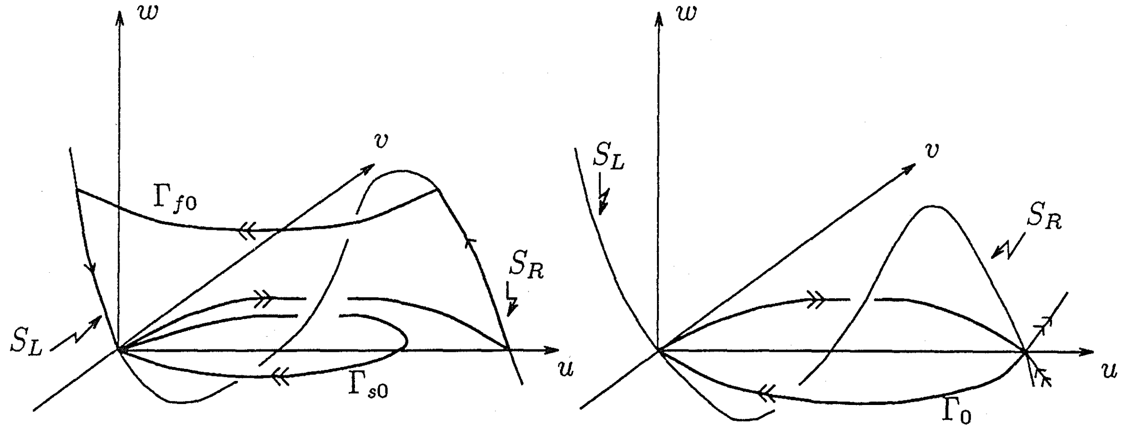
$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \epsilon(u - \gamma w). \end{aligned} \tag{1.1}$$

The equation (1.1) is a simplified version of the Hodgkin-Huxley equation used as a model of nerve axon dynamics. *Travelling waves* for (1.1) are solutions of the form $(u,w)(x,t) = (u,w)(\xi)$, $\xi = x + ct$. Here c is the *wave speed* of the travelling wave and is assumed to be positive. Looking for travelling waves is equivalent to searching for bounded solutions of the following ODE:

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= \frac{\epsilon}{c}(u - \gamma w), \end{aligned} \tag{1.2}$$

It is assumed that γ and ϵ are non-negative and that $f(u)$ is a cubic nonlinearity with $f(0) = 0$. In this work we consider a specific choice of f, namely $f(u) = u(1-u)(u-a)$, where $a > 0$ is a real parameter. Our results can be easily extended to a more general setting.

Note that $p_0 = (0,0,0)$ is an equilibrium of (1.2) regardless of the values of c and a and becomes a hyperbolic saddle for $\epsilon > 0$. The equilibrium point p_0 corresponds to the stationary solution $(u,w) = (0,0)$ of (1.1), which is the rest state of the system. An interesting question is whether there are homoclinic orbits of (1.2) doubly asymptotic to p_0. Such homoclinic orbits correspond to travelling waves of (1.1) having the form of a pulse and approaching the rest state for $\xi \to \pm\infty$. For $\epsilon \ll 1$ one can look for homoclinic orbits using singular perturbation theory. More specifically, when $\epsilon = 0$ one can construct singular orbits of (1.2) containing the equilibrium p_0. Here we consider two types of singular orbits shown in Fig. 1a. The orbit Γ_{f0} exists for $c = c^* \neq 0$ and the orbit Γ_{s0} can be found in the limit $\epsilon = \frac{\epsilon}{c} = c = 0$. Homoclinic



(a) Γ_{f0} and Γ_{s0} (b) Γ_0

Figure 1: The singular orbits of (1.2).

orbits of (1.2) can be sought as perturbations of the orbits Γ_{f0} and Γ_{s0}. Homoclinic orbits obtained by perturbing Γ_{f0} are referred to as *fast waves* and the ones found by perturbing Γ_{s0} are refered to as *slow waves.* The existence of fast waves was proved by Hastings [10] using classical singular perturbation theory, by Carpenter [2] using Conley index and by Langer [17] using a combination of analytic and geometric methods. Later an elegant geometric proof of the result was given by Jones, Kopell and Langer [13]. Their analysis was based on a technical result called the Exchange Lemma, which describes the behavior of certain invariant manifolds transverse to the orbit Γ_{f0} as they travel along a slow manifold of (1.2). The most general version of the Exchange Lemma can be found in the work of Tin, Kopell and Jones [26], see also [15] and [25]. Recently a more elementary proof was given by Brunovsky [1].

The known results on the existence of fast and slow waves are illustrated by the solid lines in the bifurcation diagram in Fig. 2 [28]. Yanagida [28] conjectured that bifurcation curves corresponding to the slow wave and to the fast wave are connected (see the dashed line in Fig. 2). The conjecture of Yanagida cannot be proved in general using local analytic methods since it requires global information on the vector field. However, when $a = \frac{1}{2}$ the singular orbits Γ_{f0} and Γ_{s0} coalesce

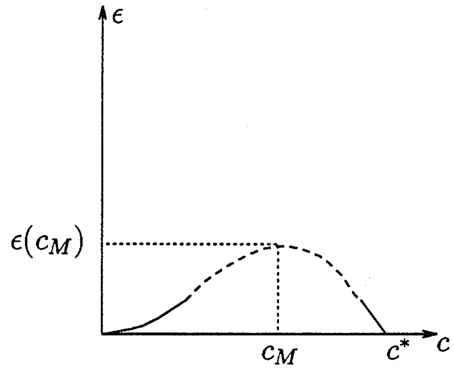


Figure 2: The bifurcation curves: known (solid line) and conjectured (dashed line).

forming a heteroclinic cycle Γ_0, connecting p_0 and $p_1 = (1, 0, 0)$, see Fig 1b. In this article we prove the existence of a connection between fast waves and slow waves for $(a, \frac{\epsilon}{c}, c) \approx (\frac{1}{2}, 0, 0)$. More precisely we show that for every fixed $a < \frac{1}{2}$ sufficiently close to $a = \frac{1}{2}$ there exists a curve in the (ϵ, c) space of the form $(c, \epsilon(c))$, $0 \leq c \leq c^*$, such that for each $(c, \epsilon(c))$ the equation (1.2) has a homoclinic orbit near Γ_0. Equivalently there exists a surface of homoclinic orbits in the (c, ϵ, a) space bounded by the curves $c = 0$ and $c = c^*(a)$, see Fig. 4. For $c \approx c^*(a)$ the homoclinic orbit corresponds to a fast wave and for $c \approx 0$ to a slow wave. We also show that the function $\epsilon(c)$ has a unique nondegenerate maximum.

The main technical part of our work is proving the existence of Shilnikov coordinates [6] in the vicinity of a slow manifold. Our result is in many aspects similar to the Exchange Lemma, it has however the advantage of describing the behavior of trajectories for which the passage time near the slow manifold is uniformly bounded as $\epsilon \to 0$. This allows us to handle the codimension two problem mentioned above. Our result gives also a more analytic and formula oriented way of proving the existence of fast waves. In fact the Exchange Lemma in its full generality can be proved by using standard estimates on Shilnikov coordinates, see Szmolyan [24].

Using the information we obtain about the existence of homoclinic orbits we show the existence of inclination-flip homoclinic orbits. An inclination-flip point is, roughly speaking, a codimension two point corresponding to a degeneracy of the center bundle around the homoclinic orbit. The unfolding of an inclination-flip point may contain complex dynamics [11], [22], including *n-homoclinic orbits*, i.e. orbits which pass n-

times near the singularity before closing up. This complex dynamics is likely to occur for the equation (1.2). To actually prove its existence it would be necessary to verify a non-degeneracy condition – a difficult task, since the location of the inclination-flip points is not known. The n-homoclinic orbits of (1.2) would correspond to n-pulse travelling waves of (1.1).

Jones [12] and Yanagida [28] proved independently that fast waves are asymptotically stable as solutions of (1.1). Since slow waves are unstable [8] it follows that along the curve $(c, \epsilon(c))$ a stability change must occur. In a companion article Sandstede [21] proves that the stability change occurs precisely for the point $(c_M, \epsilon(c_M))$ corresponding to the maximum of the curve $\epsilon(c)$. The result of [21] has an interesting consequence for our problem. Numerical experiments suggest that inclination-flip points occur for values of $c < c_M$. Consequently the n-pulse solutions created as a result of the inclination-flip bifurcation must all be unstable as solutions of (1.1), since the basic solution from which they bifurcate is unstable.

This article is organized as follows. Section 2 contains background information on fast waves and slow waves. In Section 3 we formulate the main results of this article. In Section 4 we prove the main bifurcation result using the result on Shilnikov coordinates. In this section we also present a proof of the existence of fast waves using Shilnikov coordinates. In Section 5 we prove the existence of Shilnikov coordinates and derive their asymptotic expansion. In Section 6 we prove the existence of inclination-flip points. Section 7 summarizes the results of the article and presents some open problems.

2 Background

In this section we present background information on the existence of fast and slow waves. We begin with a few general remarks concerning the travelling wave equation (1.2).

When $\epsilon = 0$ (1.2) has a curve of equilibria S defined by the conditions $v = 0$, $w = f(u)$ and containing the points $p_0 = (0,0,0)$ and $p_1 = (1,0,0)$. When $\epsilon \neq 0$ (1.2) still has the equilibrium at p_0 and may have one or two additional equilibria, depending on the value of γ. In this section we assume that γ is small, so that p_0

is the only equilibrium of (1.2). The results discussed below extend to the other case with some (trivial) restrictions. Let S_L and S_R denote neighborhoods of p_0 and p_1, respectively, in S. If S_L and S_R are not too big they are normally hyperbolic and have two dimensional stable and unstable manifolds. Fenichel theory [9] implies that for $\epsilon \neq 0$ S_L and S_R persist as locally invariant manifolds with stable and unstable manifolds depending smoothly on ϵ and other parameters. We will now briefly describe the construction of fast waves and slow waves, denoted by Γ_f and Γ_s, respectively.

2.1 Fast Waves

The orbits Γ_f are close to singular orbits of (1.2) which are obtained in the following way. Set $\epsilon = 0$. For every $0 < a < \frac{1}{2}$ there exists $c^*(a)$ such that there is a heteroclinic connection from p_0 to p_1. Typically for this $c^*(a)$ no connection from p_1 to p_0 exists, but one can choose $w^* > 0$ so that there is a connection in the $w = w^*$ plane from the equilibrium in S_R to the equilibrium in S_L. Hence there exists a singular closed orbit (a collection of trajectories) of (1.2) consisting of the connection $p_0 \to p_1$, the piece of S_R from p_1 to $S_R \cap \{w = w^*\}$, the connection in $\{w = w^*\}$ from S_R to S_L and the piece of S_L from $S_L \cap \{w = w^*\}$ to p_0 (see Fig. 1a). Let Γ_{f0} be this singular orbit. The following celebrated theorem has been the subject of many mathematical investigations.

Theorem 1 *For a fixed value of $0 < a < \frac{1}{2}$ there exists a unique curve in the (ϵ, c) plane of 1-homoclinic orbits Γ_f emanating from the point $(0, c^*(a))$. The orbits Γ_f are close to Γ_{f0} and converge to Γ_{f0} as $c \to c^*(a)$ and $\epsilon \to 0$.*

A modern proof of Theorem 1 was given in Jones, Kopell and Langer [13]. The difficult part of the proof consists of analyzing the behavior of trajectories near the manifold S_R. This difficulty was solved in [13] by means of the so-called Exchange Lemma which we state below. Improved and more general versions of the Exchange Lemma were proved by Jones and Kopell [14], Jones, Kaper and Kopell [15], Tin, Jones and Kopell [26]. Alternative proofs of the result were given by Brunovsky [1] and Szmolyan [24]. In the proof of the Exchange Lemma one uses the *Fenichel coordinates* [9], [15], which are defined in a small neighborhood of S_R. In these coordinates (1.2) has the form:

$$\begin{aligned} \dot{x} &= -A^s(x,y,z)x \\ \dot{y} &= A^u(x,y,z)y \\ \dot{z} &= \epsilon(1 + B(x,y,z)xy), \end{aligned} \tag{2.3}$$

for $(x, y, z) \in W \times [-\Omega, \Omega]$, where W is a small neighborhood of $(0,0)$ and $\Omega > 0$. The functions A^s and A^u are positive and bounded away from 0. The Fenichel coordinates are C^k smooth for arbitrary k. Let $\Sigma_1 = \{(x, y, z) : x = \Delta\}$, $\Sigma_2 = \{(x, y, z) : y = \Delta\}$ for some $\Delta > 0$ sufficiently small. Consider the following extension of the equation (2.3):

$$\begin{aligned} \dot{x} &= -A^s(x,y,\zeta)x \\ \dot{y} &= A^u(x,y,\zeta)y \\ \dot{\zeta} &= \epsilon(U + B(x,y,\zeta)xy), \end{aligned} \tag{2.4}$$

where x and y are as above, $\zeta \in \mathbb{R}^k$ and $U = (1, 0, \ldots, 0) \in \mathbb{R}^k$. The sections Σ_1 and Σ_2 are defined analogously as for (2.3). In the standard proof of Theorem 1 one needs $k = 2$ and $\zeta = (z, c)$. Note that the sets $\{x = 0\}$ and $\{y = 0\}$ correspond to $W^u(S_R)$ and $W^s(S_R)$, respectively. Here we state a version of the Exchange Lemma applicable to (2.4). For the general result see [15].

Theorem 2 *Let $M_\epsilon \subset \mathbb{R}^{2+k}$ be a two dimensional manifold invariant for the flow of (2.4). Assume that $N_\epsilon = M_\epsilon \cap \Sigma_1$ intersects $\{y = 0\}$ transversely. Let $p = (x_0, y_0, \zeta_0) \in N_\epsilon$ be a point whose trajectory intersects Σ_2 at some point $p(T)$ after a time $T = \mathcal{O}(\frac{1}{\epsilon})$. Then, for some $C > 0$ and for $\epsilon > 0$ sufficiently small, the manifold M_ϵ is $\mathcal{O}(e^{-\frac{C}{\epsilon}})$ C^1 close to $\{x = 0, \zeta^i = \zeta_0^i, i > 1\}$ at $p(T)$.*

The statement of Theorem 2 is illustrated in Fig. 3.

The existence of fast waves is now proved as follows. For fixed ϵ let M_ϵ be the union of $W^s(p_0)$ taken over different values of $c \approx c^*$ and let $N_\epsilon = M_\epsilon \cap \Sigma_1$. A Melnikov computation shows that N_0 is transverse to $W^s(S_R)$. This transversality persists for $\epsilon > 0$. Using Theorem 2 we conclude that N_ϵ is carried by the flow to a manifold which is exponentially C^1 close to $W^u(S_R)$. Finally, computing another Melnikov integral, we establish transversality of the intersection of $W^u(S_R)$ and $W^s(S_L)$. It

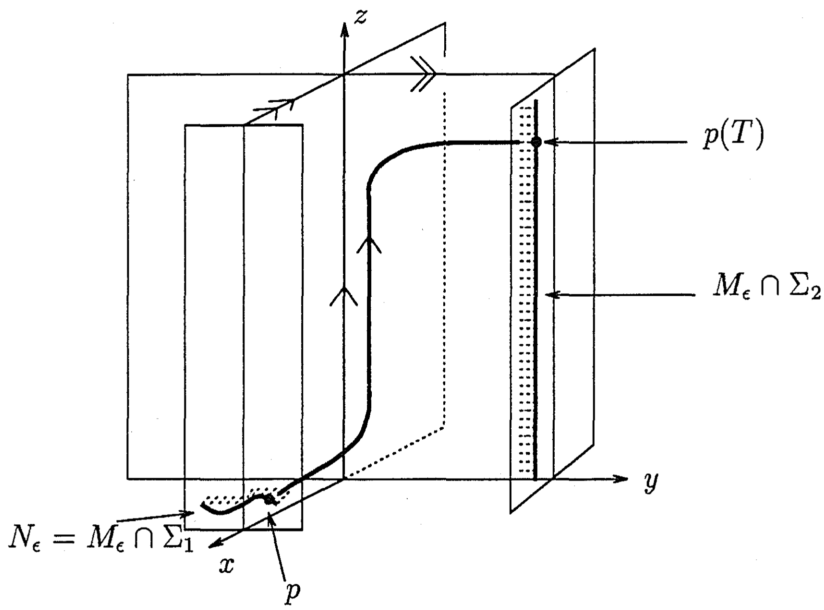


Figure 3: The Exchange Lemma.

follows that M_ϵ and $W^s(S_L)$ intersect transversally for small enough ϵ and these intersections correspond to fast waves Γ_f.

2.2 Slow waves.

Set $\delta = \frac{\epsilon}{c}$ and consider the limit $\delta = c = 0$ in (1.2). In this limit the first two equations of (1.2) form a hamiltonian system and, for $0 < a < \frac{1}{2}$, have a nondegenerate planar homoclinic orbit to the equilibrium at $(0,0)$, see Fig. 1a. We denote this homoclinic orbit by Γ_{s0}. It can be shown that the Melnikov coefficient with respect to the parameter c does not vanish, see Section 4. This implies that, when $\delta = 0$, the surfaces formed by stable and unstable manifolds of $(0,0)$ taken for all values of $c \approx 0$ intersect transversally. Hence these manifolds must intersect transversally for $\delta \neq 0$, [9], [23]. Clearly these intersections give homoclinic orbits, which are the slow waves Γ_s.

3 Statement of the results

3.1 The main result

As mentioned in the introduction the objective of this work is to show that for $a \approx \frac{1}{2}$ there exists a surface of homoclinic orbits of (1.2) bounded by a wedge in the $\epsilon = 0$

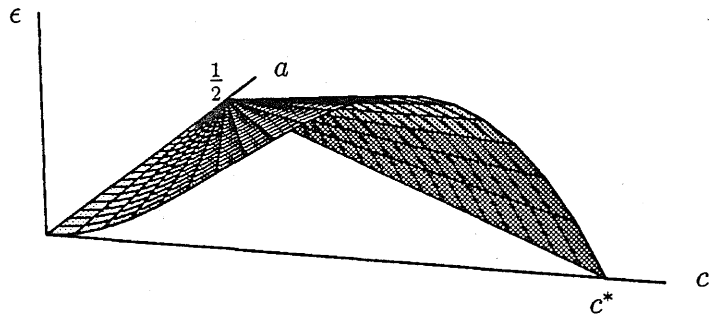


Figure 4: The surface $\mathcal{H}$ of homoclinic orbits.

plane given by the lines $c = 0$ and $c = c^*(a)$, see Fig 4. The homoclinic orbits near the $c = 0$ boundary correspond to slow waves and the ones near the $c = c^*(a)$ boundary correspond to fast waves. Consequently, we consider the case $0 < \epsilon \ll c$ and let $\delta = \frac{\epsilon}{c}$. Note that when c is bounded away from 0 the limits $\delta = 0$ and $\epsilon = 0$ are equivalent. We will often use δ as an independent parameter and define ϵ by $\epsilon = \delta c$.

For $\delta = c = 0$ (1.2) has a heteroclinic cycle Γ_0 connecting the equilibria p_0 and p_1 contained in the plane $w = 0$, see Fig. 1b. We will look for homoclinic orbits to p_0 which are close to Γ_0. Let $N(\Gamma_0)$ be a sufficiently small tubular neighborhood of Γ_0. We say that a homoclinic orbit is *n-homoclinic* if its winding number with respect to $N(\Gamma_0)$ is n. The following theorem is the main result of this article:

Theorem 3 *There exists a surface $\mathcal{H}$ in the (ϵ, c, a) space defined near the point $(\epsilon, c, a) = (0, 0, \frac{1}{2})$ in the region $\epsilon > 0$, $c > 0$, $a < \frac{1}{2}$ such that for every $(\epsilon, c, a) \in \mathcal{H}$ there exists a 1-homoclinic orbit $\Gamma(\epsilon, c, a)$ near Γ_0. The surface $\mathcal{H}$ and the family $\Gamma(\epsilon, c, a)$ have the following properties:*

(i) *The map $(\epsilon, c, a) \to \Gamma(\epsilon, c, a)$ is continuous and* $\lim_{c,\delta \to 0,\, a \to \frac{1}{2}} \Gamma(\epsilon, c, a) = \Gamma_0$.

(ii) $\Gamma(\epsilon, c, a)$ *is unique, that is if Γ is a 1-homoclinic orbit of (1.2) with (ϵ, c, a) sufficiently near to $(0, 0, \frac{1}{2})$ and Γ is sufficiently near Γ_0 then $(\epsilon, c, a) \in \mathcal{H}$ and* $\Gamma = \Gamma(\epsilon, c, a)$.

(iii) *For $a \approx \frac{1}{2}$ fixed the curve $\mathcal{H}_a = \{(\epsilon, c) : (\epsilon, c, a) \in \mathcal{H}\}$ has the form $\mathcal{H}_a = \{(\epsilon_a(c), c) : c \in [0, c^*(a)]\}$, where ϵ_a is a C^2-smooth function of c.*

(iv) *For every* $a \approx \frac{1}{2}$ *the function* ϵ_a *has a unique nondegenerate maximum in* $(0, c^*(a))$. *Moreover* $\epsilon_a'(0) = 0$ *(see Fig. 2)*

(v) *As* $c \to 0$ *the orbit* $\Gamma(\epsilon, c, a)$ *approaches the planar homoclinic orbit* Γ_{s0} *and as* $c \to c^*(a)$ *the orbit* $\Gamma(\epsilon, c, a)$ *approaches the singular orbit* Γ_{f0}.

Remark 1 Let $a \approx \frac{1}{2}$ be fixed. For c in $(0, c^*(a))$ close to 0 the homoclinic orbit $\Gamma(\epsilon, c, a)$ corresponds to a slow wave Γ_s. For c near $c^*(a)$ the orbit $\Gamma(\epsilon, c, a)$ corresponds to a fast wave Γ_f.

3.2 Shilnikov coordinates.

In order to prove Theorem 3 we need to understand the flow near the point p_1 for $\delta > 0$. Recall that the existence of the fast wave Γ_f away from the point $a = \frac{1}{2}$ can be proved using the Exchange Lemma (Theorem 2). The assertion of the Exchange Lemma holds for passage times from Σ_1 to Σ_2 of the order $\mathcal{O}(\frac{1}{\delta})$. However, to prove the existence of the surface $\mathcal{H}$ we also need to understand the behavior of trajectories for which the passage time is bounded away from ∞ uniformly in δ. To handle this situation we use an analytic approach based on the method of Lin [16], [20]. The following theorem establishes the existence of Shilnikov coordinates [6] for the flow of (2.4) from Σ_1 to Σ_2 (see Fig 3) and is the main technical result needed in the proof of Theorem 3.

Our starting point is an arbitrary singularly perturbed system in Fenichel normal form, analogous to (2.4).

$$\begin{aligned} \dot{x} &= -A^s(x,y,z)x \\ \dot{y} &= A^u(x,y,z)y \\ \dot{z} &= \delta(U + B(x,y,z)xy), \end{aligned} \tag{3.5}$$

where $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m$, $U = (1, 0, \dots, 0) \in \mathbb{R}^m$, and m is a positive integer. We assume that the functions A^s and A^u are uniformly bounded and bounded away from 0. Let $\hat{F} = (\hat{F}^1, \hat{F}^2, \hat{F}^3)$ denote the RHS of (3.5). The vector field $\hat{F}$ depends on a multidimensional parameter $\lambda \approx \lambda_0$ and is C^{k+1} in $(x, y, z, \delta, \lambda)$, $k \geq 0$. For the FitzHugh-Nagumo problem $\lambda = (c, a)$. Let $\Sigma_1 = \{(x, y, z) \,|\, x = \Delta\}$, $\Sigma_2 = \{(x, y, z) \,|\, y = \Delta\}$.

Theorem 4 *Fix $\Delta > 0$ small. For every sufficiently large T and sufficiently small $\delta > 0$ and z there exists a unique solution $p(t)$ of (3.5) with flight time T from Σ_1 to Σ_2 such that*

$$
\begin{aligned}
p(0) &= (\Delta, a_1 e^{-\alpha_u^* T}, z) + \mathcal{O}(e^{-\alpha^* T}(0, e^{-\alpha_u^* T}, 0)) \\
p(T) &= (a_2 e^{-\alpha_s^* T}, \Delta, z + \delta T U) + \mathcal{O}(e^{-\alpha^* T}(e^{-\alpha_s^* T}, 0, \delta)).
\end{aligned}
$$

where a_1, a_2, α_u^* and α_s^* are positive, C^k smooth functions of (T, δ, λ, z) and α^* is a positive constant. Moreover

$$
\begin{aligned}
D_{\nu_1 \ldots \nu_l} p(0) &= D_{\nu_1 \ldots \nu_l}(\Delta, a_1 e^{-\alpha_u^* T}, z) + \mathcal{O}(e^{-\alpha^* T}(0, e^{-\alpha_u^* T}, 0)) \\
D_{\nu_1 \ldots \nu_l} p(T) &= D_{\nu_1 \ldots \nu_l}(a_2 e^{-\alpha_s^* T}, \Delta, z + \delta T U) + \mathcal{O}(e^{-\alpha^* T}(e^{-\alpha_s^* T}, 0, \delta)),
\end{aligned}
$$

where $\nu_j = T, \lambda, z$ or δ, $j = 1, \ldots, l$ and $l \in \{1, \ldots, k\}$.

Remark 2 Consider the vector field

$$
\begin{aligned}
\dot{x} &= -\alpha_s^* x \\
\dot{y} &= \alpha_u^* y \\
\dot{z} &= \delta U
\end{aligned}
\tag{3.6}
$$

The first hit map from Σ_1 to Σ_2 given by the flow of (3.6) is the first order approximation of the map given by the Shilnikov coordinates. Consequently, Theorem 4 can be seen as a justification of using the simple model of the flow in studying the dynamics of (1.2) near S_R.

The proof of Theorem 4 is given in Section 5.

3.3 Inclination-flip points.

In Section 6 we prove the existence of inclination-flip points for equation (1.2). We now review the basic definitions and results on inclination flip bifurcations and subsequently state the result proved in this article. Consider a differential equation

$$
\dot{x} = F(x), \quad x \in \mathbb{R}^3
$$

Let $\Gamma = \{\gamma(t) \ : \ t \in \mathbb{R}\}$ be a homoclinic orbit asymptotic to a saddle point p. Suppose that $DF(p)$ has three real eigenvalues

$$\lambda_{ss} < \lambda_s < 0 < \lambda_u,$$

with eigenvectors e_{ss}, e_s, e_u. Generically the orbit Γ is tangent at p to the principal directions e_u and e_s. Choose e_u oriented according to the direction of the flow and e_s opposite to the direction of the flow. Consider the bundle of stable directions $Z(t) = T_{\gamma(t)}W^s(p)$ and a continuous vector field consisting of vectors $n(t)$ normal to $Z(t)$ such that for any t sufficiently large $n(t) \cdot e_u > 0$. The bundle $Z(t)$ is *orientable* if for any t sufficiently large $n(-t) \cdot e_s > 0$ and is *nonorientable* if for any t sufficiently large $n(-t) \cdot e_s < 0$. The orbit Γ is twisted if Z is orientable and nontwisted if Z is nonorientable [5].

A point of transition between a twisted homoclinic orbit and a non-twisted one, occuring when $Z(t)$ tends to the plane spanned by e_u and e_s as $t \to -\infty$, is called an *inclination-flip* point.

In Section 6 we prove the following theorem.

Theorem 5 *There exist values of (γ, δ, c, a) for which the equation (1.2) has an inclination-flip homoclinic orbit.*

Remark 3 For the eigenvalue configuration of (1.2), that is $-\lambda_{ss} \gg -\lambda_s$, $\lambda_u \gg -\lambda_s$ a generic unfolding of an inclination-flip point contains very complicated dynamics, including n-homoclinic orbits for arbitrary n (orbits passing near P $n-1$ times before closing up), Smale horseshoes and Hénon-like attractors [11], [18]. We are unfortunately not able to check the nondegeneracy condition necessary to guarantee the existence of this dynamics, see however [3] for a possible approach. Based on the results of Nii [19] we can conclude the existence of 2-homoclinic orbits near the inclination-flip point. In particular the n-homoclinic orbits correspond to travelling waves with n humps for the FitzHugh-Nagumo equation.

4 Bifurcation analysis

4.1 Melnikov computations.

In this section we carry out the Melnikov computations necessary to prove Theorems 3 and 1. As we are searching for homoclinic orbits to the equilibrium p_0 we must locate the intersections of $W^u(p_0)$ with $W^s(p_0)$. Recall the definitions of the sections Σ_1 and Σ_2. Let Π denote the first hit map from Σ_1 to Σ_2. Observe that $W^u(p_0) \cap W^s(p_0) \neq \emptyset$ if and only if $\Pi(W^u(p_0) \cap \Sigma_1) \in W^s(S_L) \cap \Sigma_2$. The strategy of our proof is to determine when this inclusion takes place. Thus,0, we need to compute the positions of the manifolds $W^u(p_0) \cap \Sigma_1$ and $W^s(S_L) \cap \Sigma_2$ and understand the action of the map Π. The necessary information on Π is given by Theorem 4. In this subsection we determine the positions of $W^u(p_0) \cap \Sigma_1$ and $W^s(S_L) \cap \Sigma_2$ using Melnikov analysis.

Let (x, y, z) denote globally defined coordinates which are obtained from (u, v, w) by a transformation yielding Fenichel coordinates near p_1 and equal to the identity outside a small neighborhood of p_1. Note that when $\delta = 0$ the plane $w = 0$ is invariant and the transformation to (x, y, z) coordinates for $\delta = 0$ does not alter the w coordinate. Define α by $\alpha = a - \frac{1}{2}$. It follows that the manifolds $W^u(p_0) \cap \Sigma_1$ and $W^s(S_L) \cap \Sigma_2$ have the following representation in the (x, y, z) coordinates.

$$W^u(0) \cap \Sigma_1 = (\Delta, \nu_0\alpha + \nu_1 c + \nu_2\delta, \mathcal{O}(\delta)) + R_1$$
$$W^s(S_L) \cap \Sigma_2 = (\nu_3\alpha + \nu_4 c + \nu_5 z, \Delta, z + \mathcal{O}(\delta)) + R_2,$$

where R_1 is quadratic in (c, α, δ) and R_2 is quadratic in (c, α, z, δ). We prove the following result.

Proposition 1 *The coefficients $\nu_0, \ldots, \nu_5$ do not vanish. Moreover ν_0, ν_1, ν_2, ν_3, $\nu_5 < 0$ and $\nu_4 > 0$.*

Proof. We perform the required Melnikov computations for the original (u, v, w) coordinates and then argue that the coefficients in the (x, y, z) coordinates have the same sign. The coefficients ν_0, ν_1, ν_3, ν_4 and ν_5 can be computed for $\delta = 0$. Hence we set $\delta = 0$ and consider the planar problem

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - f(u) + w. \end{aligned} \tag{4.1}$$

Let $X^r(t)$ and $X^l(t)$ denote the heteroclinic connections from p_0 to p_1 and from p_1 to p_0 respectively. These connections exist for $\delta = c = w = \alpha = 0$. We compute the distances from $W^u(p_0)$ to $W^s(p_1)$ in Σ_1 and from $W^u(p_1)$ to $W^s(p_0)$ in Σ_2 up to first order in c, w and α. Write $X^r(t) = (u^r(t), v^r(t), w^r(t))$. Consider the adjoint equation of (4.1) with respect to the connection X^r:

$$\dot{\psi} = \begin{pmatrix} 0 & \frac{df}{du}(u^r(t)) \\ -1 & 0 \end{pmatrix} \psi \tag{4.2}$$

and let Ψ^r be a bounded solution of (4.2) pointing to the outside of Γ_0. Let $\lambda \in \{\alpha, c, w\}$ and let F_0 denote the RHS of (4.1). Then

$$M_\lambda^r = \int_{-\infty}^{\infty} D_\lambda F_0(X^r(t)) \cdot \Psi^r(t) dt$$

measures the distance from $W^u(p_0)$ to $W^s(p_1)$ at first order in λ [27], [16]. Up to multiplication by a constant $\Psi^r = (-\dot{v}^r(t), \dot{u}^r(t)) = (-\dot{v}^r(t), v^r(t))$. We compute

$$\begin{aligned} M_c^r &= \int_{-\infty}^{\infty} v^r(t)^2 dt > 0 \\ M_w^r &= \int_{-\infty}^{\infty} v^r(t) dt = \int_0^1 du^r = 1 \\ M_\alpha^r &= -\int_{-\infty}^{\infty} u^r(t)(u^r(t) - 1) v^r(t) dt = -\int_0^1 u^r(u^r - 1) du^r = \frac{1}{6}. \end{aligned}$$

To compute the Melnikov coefficients relative to the connection X^l we consider the adjoint equation of (4.1) with respect to the connection X^l. The function $\Psi^l = (-\dot{v}^l(t), \dot{u}^l(t))$ is a bounded solution of this equation. In a similar manner as in the case of X^r we obtain

$$\begin{aligned} M_c^l &= \int_{-\infty}^{\infty} v^l(t)^2 dt > 0 \\ M_w^l &= \int_{-\infty}^{\infty} v^l(t) dt = \int_1^0 du^l = -1 \\ M_\alpha^l &= -\int_{-\infty}^{\infty} u^l(t)(u^l(t) - 1) v^l(t) dt = -\int_0^1 u^l(u^l - 1) du^l = -\frac{1}{6}, \end{aligned}$$

where M_c^l, M_w^l and M_α^l measure the distance from $W^u(p_1)$ to $W^s(p_0)$.

We now compute ν_2 as the derivative of the distance from $W^u(p_0)$ to $W^s(S_R)$ with respect to δ at $\delta = 0$. The adjoint equation of (1.2) along $X_r(t)$ is

$$\begin{pmatrix} \dot{\psi}^1 \\ \dot{\psi}^2 \\ \dot{\psi}^3 \end{pmatrix} = \begin{pmatrix} 0 & f'(u^r(t)) & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix}. \tag{4.3}$$

The space of bounded solutions of (4.3) is two dimensional, spanned by $(-\dot{v}(t),\, \dot{u}(t),\, -u(t))$ and $(0,0,1)$. Let $\Psi = (-\dot{v}(t), \dot{u}(t), 1 - u(t))$. The function Ψ is the unique, up to multiplication by a constant, bounded solution of (4.3) satisfying the condition $\lim_{t\to\infty} \Psi(t) = 0$. It follows that $\Psi(t)$ is normal to $T_{X^r(t)}W^s(S_R)$ for $t \in \mathbb{R}$. Moreover Ψ points to the outside of $W^s(S_R)$, that is its inner product with the eigenvector of $DF(p_1)$ pointing opposite to the flow on X^r is positive. It now follows from Melnikov theory that the dependence on δ of the distance from $W^u(p_0)$ to $W^s(S_R)$ is given by the following integral:

$$M_\delta = \int_{-\infty}^{\infty} D_\delta F \cdot \Psi dt$$

Since X^r is contained in the plane $w = 0$ it follows that $M_\delta = \int_{-\infty}^{\infty} (1-u(t))u(t)dt > 0$.

The corresponding Melnikov coefficients in the $(x,\, y,\, z)$ coordinates are of the same sign. To see this note that the transformation to the $(x,\, y,\, z)$ coordinates is a composition of the translation $u \to \hat{u} + 1$, a linear transformation having no effect on the signs of the Melnikov coefficients and a transformation H_1, which is bounded in the C^1 norm, independently of the size of the neighborhood on which the Fenichel coordinates are defined. Hence the relevant Melnikov coefficients have the form:

$$\tilde{M}_\lambda = \int_{-\infty}^{T} \frac{\partial f}{\partial \lambda} \cdot \Psi dt + \int_{T}^{\infty} * \cdot \Psi dt,$$

where T can be made arbitrarily large and $*$ is some unknown expression bounded independently of T. Recall that $\Psi(t)$ approaches 0 exponentially fast as $t \to \infty$. It follows that when T is made sufficiently large the signs of the relevant Melnikov coefficients remain unaltered.

We now consider the coefficients $\nu_0, \ldots, \nu_5$. The quantities ν_0, ν_1 and ν_2 measure the distance from $W^u(p_0)$ to $W^s(S_R)$ along the coordinate y, which is oriented opposite to Ψ^r (see Fig. 3). Hence the signs of ν_0, ν_1 and ν_2 are opposite to the signs of M^r_α, M^r_c and M^r_δ respectively. Consequently ν_0, ν_1, $\nu_2 < 0$. The quantities ν_3, ν_4 and ν_5 measure the distance from $W^s(S_R)$ to $W^u(S_L)$ along the coordinate x, which is oriented opposite to Ψ^l. The Melnikov coefficients relative to X^l measure the distance from $W^u(S_L)$ to $W^s(S_R)$ along Ψ^l. Hence the signs of ν_3, ν_4, ν_5 are the same as the signs of M^l_α, M^l_c, and M^l_w. □

Remark 4 For $c = 0$ the time reversal symmetry of (4.1) implies that

$$\mathrm{dist}(W^u(p_0), W^s(p_1)) = \mathrm{dist}(W^u(p_1), W^s(p_0)).$$

This implies that when $\delta = c = w = 0$ the dependence of $W^u(0) \cap \Sigma_1$ and $W^s(S_L) \cap \Sigma_2$ on α is the same. In particular $\nu_0 = \nu_3$. This equality will not be altered by the transformation to the Fenichel coordinates.

To prove Theorem 1 we need similar information on the behavior of the invariant manifolds of p_0 and S_R when $a \not\approx \frac{1}{2}$. Let c^* and w^* be as introduced in Section 2 and let $(0, 0, z^*)$ be the representation of the point $(f^{-1}(w^*), 0, w^*)$ in the Fenichel coordinates near S_R. Fix a. For c near c^* and z near z^* we consider the following expansions:

$$\begin{aligned} W^u(0) \cap \Sigma_1 &= (\Delta, \eta_1(c - c^*) + \eta_2\delta, \mathcal{O}(\delta) + R_1) \\ W^s(S_L) \cap \Sigma_2 &= (\Delta, \eta_3(c - c^*) + \eta_4(w - w^*), R_2), \end{aligned}$$

where R_1 is quadratic in $(c - c^*, \delta)$ and R_2 is quadratic in $(c - c^*, w - w^*)$ and linear in δ. We have the following result:

Proposition 2 *The coefficients $\eta_1, \ldots, \eta_4$ do not vanish. Moreover η_1, η_2, $\eta_4 < 0$ and $\eta_3 > 0$.*

Proof. Similar to the proof of Proposition 1. □

4.2 Proof of Theorem 3

In this section we prove Theorem 3 using Proposition 1 and Theorem 4. The assertion of Proposition 1 implies that the manifolds $W^u(p_0) \cap \Sigma_1$ and $W^s(S_L) \cap \Sigma_2$ can be expressed in the following way.

$$\begin{aligned} W^u(p_0) \cap \Sigma_1 &= (\Delta, -\kappa_0\alpha - \kappa_1 c - \kappa_2\delta, \mathcal{O}(\delta)) \\ W^s(S_L) \cap \Sigma_2 &= (-\kappa_0\alpha + \kappa_3 c - \kappa_4 z + \mathcal{O}(\delta), \Delta, z + \mathcal{O}(\delta)), \end{aligned}$$

where $\kappa_0 = -\nu_0 + \mathcal{O}(\alpha)$ and depends only on α, $\kappa_1 = -\nu_1 + \mathcal{O}(c, \alpha, z, \delta)$, $\kappa_2 = -\nu_2 + \mathcal{O}(c, \alpha, z, \delta)$, $\kappa_3 = \nu_4 + \mathcal{O}(c, \alpha, z, \delta)$ and $\kappa_4 = -\nu_5 + \mathcal{O}(c, \alpha, z, \delta)$.

We now use the information about the local map $\Pi : \Sigma_1 \to \Sigma_2$ provided by Theorem 4. Recall that for any large enough T Theorem 4 guarantees the existence of a solution $X(t)$ of (3.5) such that

$$
\begin{aligned}
X(0) &= (\Delta, b_1 e^{-\alpha_u^* T}, z) \\
X(T) &= (b_2 e^{-\alpha_s^* T}, \Delta, z + \delta T + \mathcal{O}(\delta e^{-\alpha^* T})),
\end{aligned}
$$

where $b_i = a_i + \mathcal{O}(e^{-\alpha^* T})$, $i = 1, 2$, a_i do not vanish and are independent of T. Similar estimates hold for the first and second derivatives of $X(0)$ and $X(T)$ with respect to T, c, α, δ and z. The time reversibility of (4.1) for $\delta = c = 0$ implies that $b_1 = b_2$ and $\alpha_u^* = \alpha_s^*$ when $\delta = c = 0$. A 1-homoclinic solution of (1.2) is determined by the equations:

$$
\begin{aligned}
(b_1 e^{-\alpha_u^* T}, z) &= (-\kappa_0 \alpha - \kappa_1 c - \kappa_2 \delta, \mathcal{O}(\delta)) \\
(b_2 e^{-\alpha_s^* T}, \tilde{z}) &= (-\kappa_0 \alpha + \kappa_3 c - \kappa_4 \tilde{z}, \tilde{z}) \\
\tilde{z} &= z + \delta T + \mathcal{O}(\delta).
\end{aligned}
$$

Substituting the expression for $\tilde{z}$ we obtain the equations:

$$
b_1 e^{-\alpha_u^* T} = -\kappa_0 \alpha - \kappa_1 c - \kappa_2 \delta \tag{4.4a}
$$

$$
b_2 e^{-\alpha_s^* T} = -\kappa_0 \alpha + \kappa_3 c - \kappa_4 \delta T + \mathcal{O}(\delta) \tag{4.4b}
$$

Fix $\alpha < 0$ and recall that $\epsilon = \delta c$. Let $c^*(\alpha)$ be the solution of the equation obtained by setting $\delta = 0$ and $T = \infty$ in (4.4a), that is $0 = -\kappa_0 \alpha - \kappa_1 c$. Theorem 3 is a consequence of the following proposition:

Proposition 3 *The system of equations (4.4) defines ϵ as a C^2 smooth function of c mapping $[0,\ c^*(\alpha)]$ into $\mathbb{R}^+$ with the following properties:*

(i) $\epsilon(0) = \epsilon(c^*(\alpha)) = 0$,

(ii) $\dfrac{d\epsilon}{dc}(0) = 0$, $\dfrac{d\epsilon}{dc}(c^*(\alpha)) < 0$

(iii) *ϵ has a unique maximum in $[0, c^*(\alpha)]$.*

Proof. We subtract (4.4b) from (4.4a) obtaining the expression

$$\kappa_4 \delta T + \mathcal{O}(\delta) = (\kappa_1 + \kappa_3)c + b_1 e^{-\alpha_u^* T} - b_2 e^{-\alpha_s^* T}. \tag{4.5}$$

Observe that (4.5) can be solved for δT as a function of $\frac{1}{T}$ and c. Let $\vartheta(c) = \kappa_1(c,\alpha,0,0) + \kappa_3(c,\alpha,0,0)$. It follows that

$$\delta(c,T) = \frac{1}{\kappa_4(0)T}\left(\vartheta(c)c + \hat{b}_1 e^{-\alpha_1 T} - \hat{b}_2 e^{-\alpha_2 T}\right) \cdot \Psi(c, \frac{1}{T}), \tag{4.6}$$

where $\Psi(c,\frac{1}{T})$ is a smooth function of c and $\frac{1}{T}$, $\alpha_1(c,T) = \alpha_u^*(c,T,0)$, $\alpha_2(c,T) = \alpha_s^*(c,T,0)$, $\hat{b}_1(c,T) = b_1(c,T,0)$ and $\hat{b}_2(c,T) = b_2(c,T,0)$. Due to time reversibility for $c = \delta = 0$ we have $\alpha_1(0,T) = \alpha_2(0,T)$ and $\hat{b}_1(0,T) = \hat{b}_2(0,T)$.

We now proceed as follows. We substitute (4.6) in (4.4a) and show that the resulting expression defines T as an increasing function of c mapping $[0, c^*(\alpha)]$ into $\mathbb{R}^+$ and such that $T \to \infty$ as $c \to c^*(\alpha)$. By substituting $T(c)$ back into (4.6) we obtain δ as a function of c with $\delta(0) = \delta(c^*(\alpha)) = 0$ and $\frac{d\delta}{dc}(0)$ bounded. Next we show that $\frac{d\epsilon}{dc} < 0$ for $c \geq T^2 e^{-\alpha_u^* T}$ and $\frac{d\epsilon}{dc} > 0$ for $0 < c \leq e^{-2\alpha_u^* T}$. Finally we show that, for $c \neq 0$, $\frac{d\epsilon}{dc} = 0$ implies $\frac{d^2\epsilon}{dc^2} < 0$. These properties imply the assertions (i), (ii) and (iii).

We now substitute $\delta(c,T)$ into (4.4a) thus obtaining an expression of the form

$$\Theta(c,T) = 0, \tag{4.7}$$

where

$$\Theta(c,T) = b_1 e^{-\alpha_u^* T} + \kappa_0 \alpha + \kappa_1 c + \kappa_2 \delta. \tag{4.8}$$

We show that (4.7) defines T as a function of c by showing that $\frac{\partial}{\partial T}\Theta(c,T) \neq 0$. From the definition of α_u^* and the C^2 estimates on the local coordinates we conclude that

$$\frac{\partial}{\partial T}\Theta(c,T) = \left(-\hat{b}_1 \alpha_u^* e^{-\alpha_u^* T} + \kappa_2(0)\frac{\partial \delta}{\partial T}(c,T)\right)(1 + \mathcal{O}(\delta T + \frac{1}{T} + c)). \tag{4.9}$$

Also

$$\frac{\partial \delta}{\partial T}(c,T) = -\frac{1}{\kappa_4(0)T^2}\left(\vartheta(c)c + \hat{b}_1 e^{-\alpha_1 T} - \hat{b}_2 e^{-\alpha_2 T}\right)\cdot\left(\Psi(c,\frac{1}{T}) + \mathcal{O}(\frac{1}{T})\right) \quad (4.10)$$
$$+ \quad \frac{1}{\kappa_4(0)T}\left(-\alpha_1\hat{b}_1 e^{-\alpha_1 T} + \alpha_1\hat{b}_2 e^{-\alpha_2 T}\right)\cdot\left(\Psi(c,\frac{1}{T})\right)\cdot\left(1+\mathcal{O}(\delta T + \frac{1}{T} + c)\right).$$

Suppose $c \leq \frac{1}{T^2}$. Consider the expression $\hat{b}_1 e^{-\alpha_1 T} - \hat{b}_2 e^{-\alpha_2 T}$. Note that $\hat{b}_1 - \hat{b}_2$ and $\alpha_1 - \alpha_2$ are of the order $\mathcal{O}(c)$. Since $cT = \mathcal{O}(\frac{1}{T})$ we have the following estimate:

$$e^{(\alpha_1-\alpha_2)T} = 1 + \mathcal{O}(cT). \quad (4.11)$$

It follows that

$$\hat{b}_1 e^{-\alpha_1 T} - \hat{b}_2 e^{-\alpha_2 T} = ce^{-\alpha_1 T}\mathcal{O}(\frac{1}{T}). \quad (4.12)$$

Hence $-\hat{b}_1 e^{-\alpha_1 T}$ and $-\frac{\kappa_2(0)}{\kappa_4(0)}\vartheta(0)\frac{c}{T^2}$ are the dominating terms in $\frac{\partial}{\partial T}\Theta(c,T)$. It follows that $\frac{\partial}{\partial T}\Theta(c,T) < 0$. When $c > \frac{1}{T^2}$ then $-\frac{\kappa_2(0)}{\kappa_4(0)}\vartheta(0)\frac{c}{T^2}$ is the dominant term in $\frac{\partial}{\partial T}\Theta(c,T)$. It follows that $\frac{\partial}{\partial T}\Theta(c,T) < 0$ for any $c \in (0, c^*(\alpha))$. Consequently (4.7) defines T as a smooth function of c. Moreover

$$\frac{\partial}{\partial T}\Theta(c,T)\frac{dT}{dc} = -\kappa_1(0) + \mathcal{O}(c + \delta T + \frac{1}{T}).$$

Hence $\frac{dT}{dc} > 0$, that is T is increasing as a function of c. Recall that $\epsilon = \delta c$. We have

$$\frac{d\epsilon}{dc} = c\frac{d\delta}{dc} + \delta = c(\frac{\partial \delta}{\partial c} + \frac{\partial \delta}{\partial T}\frac{dT}{dc}) + \delta, \quad (4.13)$$

and

$$\frac{\partial \delta}{\partial c} = \frac{1}{\kappa_4(0)T}\vartheta(0) + \mathcal{O}(e^{-\alpha_1 T} + e^{-\alpha_2 T} + c)$$
$$+\frac{1}{\kappa_4(0)T}\left(\vartheta(c)c + \hat{b}_1 e^{-\alpha_1 T} - \hat{b}_2 e^{-\alpha_2 T}\right)\cdot\frac{\partial \Psi(c,\frac{1}{T})}{\partial c}. \quad (4.14)$$

We claim that $c \geq T^2 e^{-\alpha_1 T}$ implies $\dfrac{d\epsilon}{dc} < 0$. Note that if $c \geq T^2 e^{-\alpha_1 T}$ then

$$\frac{dT}{dc} = \frac{\kappa_1(0)\kappa_4(0)}{\kappa_2(0)\vartheta(0)}\frac{T^2}{c}(1 + \hat{b}_1\alpha_1 + \mathcal{O}(c + \frac{1}{T} + \delta T)).$$

It follows that

$$\frac{\partial\delta}{\partial T}\frac{dT}{dc} = -\frac{\kappa_1(0)}{\kappa_2(0)}(1 + \hat{b}_1\alpha_1 + \mathcal{O}(c + \frac{1}{T} + \delta T)),$$

whereas the terms δ and $c\dfrac{\partial\delta}{\partial c}$ are $\mathcal{O}(\dfrac{c}{T})$. The claim follows. We now restrict our attention to (c, T) satisfying

$$c < T^2 e^{-\alpha_1 T}. \tag{4.15}$$

We first show that $\epsilon(c)$ has a maximum. Suppose that $c \leq e^{-2\alpha_1 T}$. Then

$$\frac{dT}{dc} = \frac{\kappa_1(0)}{\hat{b}_1\alpha_1}e^{\alpha_1 T}(1 + \mathcal{O}(\delta T + \frac{1}{T} + c)), \tag{4.16}$$

so that $c^2\dfrac{dT}{dc} = \mathcal{O}(e^{-3\alpha_1 T})$. It follows that the terms δ and $c\dfrac{\partial\delta}{\partial c}$ dominate in the expression for $\dfrac{d\epsilon}{dc}$. Hence $\dfrac{d\epsilon}{dc} > 0$. It follows that $\epsilon(c)$ has a maximum for $0 < c < T^2 e^{-\alpha_1 T}$.

We will now show that $\dfrac{d\epsilon}{dc} = 0$ implies $\dfrac{d^2\epsilon}{dc^2} < 0$. This implies that the function $\epsilon(c)$ has a unique nondegenerate maximum. To this end we write:

$$\frac{d\epsilon}{dc} = c\frac{dT}{dc}g(c),$$

where

$$g(c) = \frac{1}{c}\frac{\partial\delta}{\partial T} + (\frac{dT}{dc})^{-1}(\frac{\delta}{c} + \frac{\partial\delta}{\partial c}).$$

Clearly $g(c) = 0$ is equivalent to $\dfrac{d\epsilon}{dc} = 0$. Suppose $g(c) = 0$. Then

$$\frac{d^2\epsilon}{dc^2} = c\frac{dT}{dc}\frac{dg}{dc}(c).$$

We show that $\dfrac{dg}{dc}(c) < 0$. It follows from (4.12) that $|\dfrac{d}{dc}\dfrac{1}{c}\dfrac{\partial\delta}{\partial T}| = \mathcal{O}(\dfrac{1}{T^2})$. We compute

$$\frac{d}{dc}((\frac{dT}{dc})^{-1}(\frac{\delta}{c} + \frac{\partial\delta}{\partial c})) = -(\frac{dT}{dc})^{-2}\frac{d^2T}{dc^2}(\frac{\delta}{c} + \frac{\partial\delta}{\partial c}) + (\frac{dT}{dc})^{-1}\frac{d}{dc}(\frac{\delta}{c} + \frac{\partial\delta}{\partial c}).$$

The equations (4.12) and (4.16) imply that $|T^2(\frac{dT}{dc})^{-1}\frac{d}{dc}(\frac{\delta}{c}+\frac{\partial\delta}{\partial c})| = \mathcal{O}(\frac{1}{T^2})$. We now estimate $(\frac{dT}{dc})^{-2}\frac{d^2T}{dc^2}(\frac{\delta}{c}+\frac{\partial\delta}{\partial c})$. By differentiating the equation $\Theta = 0$ with respect to c we obtain

$$-\frac{\partial\Theta}{\partial c} = \frac{\partial\Theta}{\partial T}\frac{\partial T}{\partial c}.$$

Differentiating further we obtain

$$-\frac{\partial^2\Theta}{\partial c^2} - 2\frac{\partial^2\Theta}{\partial c\partial T} = \frac{\partial^2\Theta}{\partial T^2}\left(\frac{dT}{dc}\right)^2 + \frac{d^2T}{dc^2}\frac{\partial\Theta}{\partial T}.$$

From (4.9), (4.10) and (4.11) we obtain the following formula.

$$\frac{\partial\Theta}{\partial T} = (-\hat{b}_1\alpha_1 e^{-\alpha_1 T} - \frac{1}{\kappa_4(0)}\kappa_2(0)\vartheta(0)\frac{c}{T^2})(1+\mathcal{O}(\delta T + \frac{1}{T} + c)) \tag{4.17}$$

It follows that

$$\begin{aligned}\frac{\partial^2\Theta}{\partial T^2} &= (\hat{b}_1\alpha_1^2 e^{-\alpha_1 T} + 2\frac{1}{\kappa_4(0)}\kappa_2(0)\vartheta(0)\frac{c}{T^3})(1+\mathcal{O}(\delta T + \frac{1}{T} + c)) \\ \frac{\partial^2\Theta}{\partial T\partial c} &= (-\frac{1}{\kappa_4(0)}\kappa_2(0)\vartheta(0)\frac{1}{T^2} + \mathcal{O}(e^{-\alpha_1 T}))(1+\mathcal{O}(\delta T + \frac{1}{T} + c)).\end{aligned} \tag{4.18}$$

Note also that $|\frac{\partial^2\Theta}{\partial T^2}|$ is bounded. It follows from (4.15), (4.17) and (4.18) that

$$(\frac{dT}{dc})^{-2}\frac{d^2T}{dc^2} = -\frac{\partial^2\Theta}{\partial T^2}\cdot(\frac{\partial\Theta}{\partial T})^{-1} + \mathcal{O}(\frac{1}{T^2}) = C_0(1+\mathcal{O}(\delta T + \frac{1}{T} + c)),$$

where $C_0 \geq \frac{\hat{b}_1\alpha_1^2}{\hat{b}_1\alpha_1 + \frac{1}{\kappa_4(0)}\kappa_2(0)\vartheta(0)}$. It follows that $(\frac{dT}{dc})^{-2}\frac{d^2T}{dc^2}(\frac{\delta}{c}+\frac{\partial\delta}{\partial c}) > \frac{C_0}{2T}$. Hence $\frac{dg}{dc} < 0$ for T sufficiently large. □

Remark 5 The maximum of $\epsilon(c)$ must occur when the three expressions in (4.13), namely $c\frac{\partial\delta}{\partial c}$, $\frac{\partial\delta}{\partial T}\frac{dT}{dc}$ and δ are of the same order. This happens when $c = \mathcal{O}(Te^{-\alpha_1 T})$.

Remark 6 Note that for fixed α the variable T varies monotonically in the interval $[T^*(\alpha), \infty]$. It follows from (4.8) that $T^*(\alpha) \to \infty$ as $\alpha \to 0$. Consequently (4.6) implies that $\delta(c) \to 0$ uniformly in $c \in [0, c^*(\alpha)]$, $\alpha \to 0$.

4.3 Proof of Theorem 1

We look for homoclinic orbits Γ_f as perturbations of the singular orbit Γ_{f0} shown in Figure 1a. Using Proposition 2 we express $W^u(p_0) \cap \Sigma_1$ and $W^s(S_L) \cap \Sigma_2$ in the following way:

$$W^u(p_0) \cap \Sigma_1 = (\Delta, -\mu_1(c-c^*) - \mu_2\delta, \mathcal{O}(\delta))$$
$$W^s(S_L) \cap \Sigma_2 = (\mu_3(c-c^*) - \mu_4(z-z^*) + \mathcal{O}(\delta), \Delta, z + \mathcal{O}(\delta)),$$

where $\mu_1 = -\eta_1 + \mathcal{O}(c-c^*, z-z^*, \delta)$, $\mu_2 = -\eta_2 + \mathcal{O}(c-c^*, z-z^*, \delta)$, $\mu_3 = \eta_3 + \mathcal{O}(c-c^*, z-z^*, \delta)$ and $\mu_4 = -\eta_4 + \mathcal{O}(c-c^*, z-z^*, \delta)$. The condition $\Pi(W^u(p_0) \cap \Sigma_1) \subset W^s(S_L) \cap \Sigma_2$ for the existence of a homoclinic orbit yields the bifurcation equation:

$$b_1 e^{-\alpha_u^* T} = -\mu_1(c-c^*) - \mu_2\delta \tag{4.19a}$$
$$b_2 e^{-\alpha_s^* T} = \mu_3(c-c^*) - \mu_4(\delta T - z^*) + \mathcal{O}(\delta). \tag{4.19b}$$

We solve (4.19b) for δT as a function of $\frac{1}{T}$ and c by the implicit function theorem, obtaining the following expression:

$$\delta = \frac{1}{T} z^* + \mathcal{O}\left(\frac{1}{T^2} + (c-c^*)\frac{1}{T}\right). \tag{4.20}$$

Next we substitute (4.20) in (4.19a) and solve by the implicit function theorem for $\frac{1}{T}$ as a function of $c-c^*$, obtaining the following expression:

$$\frac{1}{T} = -\frac{\mu_1(0)}{\mu_2(0) z^*}(c-c^*) + \mathcal{O}((c-c^*)^2). \tag{4.21}$$

Combining (4.20) and (4.21) we obtain the expression for δ:

$$\delta = -\frac{\mu_1(0)}{\mu_2(0)}(c-c^*) + \mathcal{O}((c-c^*)^2). \tag{4.22}$$

□

Remark 7 Since $\dfrac{\mu_1(0)}{\mu_2(0)} > 0$ it follows that the curve in the (c, ϵ) plane corresponding to fast waves has negative slope. This information is crucial for the proof of stability of the fast wave.

5 Shilnikov coordinates.

In this section we prove Theorem 4.

We will make use of the following properties of (3.5).

1. The slow manifold $S_R = \{(0,0,z) : z \in \mathbb{R}^m\}$ is invariant. The stable and unstable manifolds $W^s(S_R)$ and $W^u(S_R)$ are the planes $\{y=0\}$ and $\{x=0\}$ respectively.

2. Restricted to $W^s(S_R)$ the equation (3.5) has the form

$$\begin{aligned} \dot{x} &= -A^s(x,0,z)x \\ y &= 0 \\ \dot{z} &= \delta U \end{aligned} \tag{5.1}$$

   Let $q_+(t, z_0, \delta)$ denote the solution to (5.1) with initial condition $(\Delta, 0, z_0)$.

3. Restricted to $W^u(S_R)$ the equation (3.5) has the form

$$\begin{aligned} x &= 0 \\ \dot{y} &= A^u(0,y,z)y \\ \dot{z} &= \delta U \end{aligned} \tag{5.2}$$

   Let $q_-(t, z_0, \delta)$ denote the solution to (5.2) with initial condition $(0, \Delta, z_0)$.

The proof of Theorem 4 consists of the following steps:

(i) proving the existence of a unique solution of the Shilnikov problem for specified (T, z, λ, δ).

(ii) obtaining the estimates on the solution and its derivatives.

For convenience of notation we replace T by $2T$, that is look for a solution p of (3.5) satisfying $p(0) \in \Sigma_0$ and $p(2T) \in \Sigma_1$. To prove (i) we consider two solutions of (3.5) $p_+(t)$ and $p_-(t)$, defined on $[0,T]$ and on $[T,2T]$, respectively, with $p_+(0) \in \Sigma_1$ and $p_-(2T) \in \Sigma_2$, see Fig. 5. The solution of the Shilnikov problem is obtained by finding p_+ and p_- satisfying the matching condition $p_+(T) = p_-(T)$. This apparently

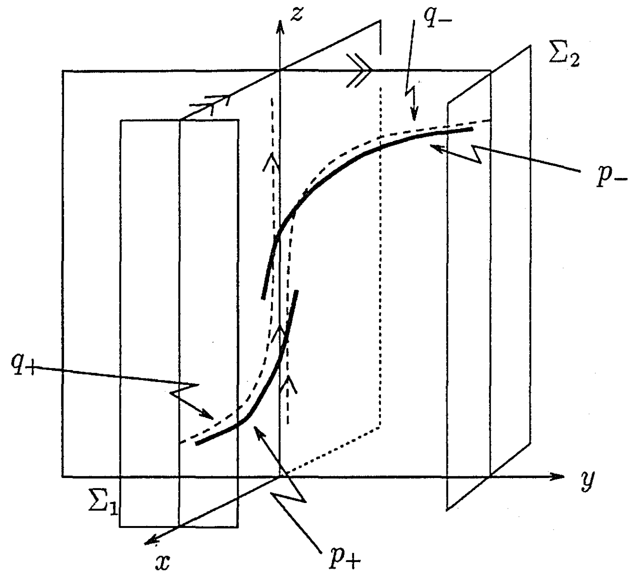


Figure 5: The solutions p_+ and p_-.

roundabout way of proving (i) is helpful in dealing with (ii). We will show that $p_\pm$ remain very close to $q_\pm$. Using this information we can, in a fairly straightforward way, derive estimates separately for p_+ and p_-. Consider the equations

$$\dot{y} = A^u(q_+)y \tag{5.3}$$

and

$$\dot{x} = A^s(q_-)x \tag{5.4}$$

Let $\tilde{\Phi}^u_+(t,s)$ and $\tilde{\Phi}^s_-(t,s)$ be the transition matrices of (5.3) and (5.4) respectively. The proof of (ii) proceeds in two steps. Write $p = (p^1, p^2, p^3)$ where p^1 and p^2 correspond to the one-dimensional coordinates x and y and p^3 corresponds to the m dimensional vector coordinate z. We show that $p^2_+(0)$ and $p^1_-(2T)$ are approximately given by:

$$p^2_+(0) \approx \tilde{\Phi}^u_+(0,T)q^2_-(-T,0,z_0+2\delta T) \tag{5.5}$$

$$p^1_-(2T) \approx \tilde{\Phi}^s_-(0,-T)q^1_+(T,0,z_0). \tag{5.6}$$

In addition $p^3_-(2T) = z_0 + 2\delta TU + \mathcal{O}(e^{-\mathrm{const}\,T})$. Similar expressions hold for the derivatives of $p^2_+(0)$, $p^1_-(2T)$ and $p^3_-(2T)$. The second step in the proof of (ii) is to obtain the exact information on the asymptotic behavior of q_+, q_-, $\tilde{\Phi}^u_+$ and $\tilde{\Phi}^s_-$. We discuss the estimates for $q_\pm$ in detail, leaving the similar analysis of $\tilde{\Phi}^u_+$ and $\tilde{\Phi}^s_-$ to the reader.

This section is organized as follows. We begin by proving a result on the asymptotic behavior of $q_{\pm}$. Next we consider the linearizations of (3.5) around q_+ and q_- and prove a result on exponential trichotomies for these equations. The proof of the existence of a solution to the Shilnikov problem follows. We first solve the linear nonhomogeneous problem and then the nonlinear problem in suitable function spaces. Finally we derive the estimates on the Shilnikov variables.

Fix $\gamma \geq 0$ small. We introduce the following functions.

$$\begin{aligned}
&\alpha_s^\gamma(z) = A^s(0,0,z) - \gamma \\
&\beta_s^\gamma(t,\tau,z_0) = -\int_\tau^t \alpha_s^\gamma(\delta\sigma + z_0)d\sigma \\
&\alpha_u^\gamma(z) = A^u(0,0,z) - \gamma \\
&\beta_u^\gamma(\tau,t,z_0) = \int_t^\tau \alpha_u^\gamma(\delta\sigma + z_0)d\sigma \\
&\alpha^\gamma(z) = \min\{\alpha_u^\gamma(z), \alpha_s^\gamma(z)\} \\
&\beta^\gamma(t,\tau) = -\int_\tau^t \alpha^\gamma(\delta\sigma)d\sigma
\end{aligned} \tag{5.7}$$

Let $\beta_s^\gamma(t,z_0) = \beta_s^\gamma(t,0,z_0)$, $\beta_u^\gamma(t,z_0) = \beta_u^\gamma(t,0,z_0)$, $\beta^\gamma(t,z_0) = \beta^\gamma(t,0,z_0)$.

Fix α^* independent of γ and satisfying $0 < \alpha^* < \min_{z\in\mathbb{R}^m}\{\alpha_s^\gamma(z), \alpha_u^\gamma(z)\}$. We have the following result.

Lemma 1 *There exist functions $b_j(\lambda, z)$, $j = 1,2$, such that*

$$\begin{aligned}
q_+^1(t) &= b_1 e^{\beta_s^0(t)} + \mathcal{O}(e^{\beta_s^0(t) - \alpha^* t}), \quad t \geq 0 \\
q_-^2(t) &= b_2 e^{\beta_u^0(t)} + \mathcal{O}(e^{\beta_u^0(t) + \alpha^* t}), \quad t \leq 0
\end{aligned}$$

The functions b_j are differentiable with respect to (λ, z) and $b_j \neq 0$ for λ near λ_0 and $z \in R^m$. Moreover

$$\begin{aligned}
D^l_{\nu_1\ldots\nu_l} q_+^1(t) &= D^j_{\nu_1\ldots\nu_l}(b_1 e^{\beta_s^0(t)}) + \mathcal{O}(e^{\beta_s^0(t) - \alpha^* t}), \quad t \geq 0 \\
D^l_{\nu_1\ldots\nu_l} q_-^2(t) &= D^j_{\nu_1\ldots\nu_l}(b_2 e^{\beta_u^0(t)}) + \mathcal{O}(e^{\beta_u^0(t) + \alpha^* t}), \quad t \leq 0
\end{aligned}$$

where $\nu_j \in \{\delta, \lambda, z\}$, $j = 1, \ldots, l$ and $1 \leq l \leq k+1$.

Proof The proof is similar to the proof of Lemma 1.5 in [20]. Here it will be carried out for q_+^1. The argument for q_-^2 is similar. Let $h(x,\delta t) = A^s(0,0,\delta t+z) - A^s(x,0,\delta t+z)$. Clearly $|h(x,\delta)| \leq Cx^2$ for some constant C. The equation (5.1) is equivalent to:

$$\dot{x} = -A^s(0,0,\delta t + z)x + h(x,\delta t). \tag{5.8}$$

Consider $t \in [0, T_\delta]$. Let $\Theta^s(t,\tau)$ be the transition matrix of the linearization of (5.8), that is $\Theta^s(t,\tau)x = e^{\beta_s^0(t,\tau)}x$. For $T \in \mathbb{R}$ we have

$$q_+^1(t) = \Theta^s(t,T)q_+^1(T) + \int_T^t \Theta^s(t,\tau)h(q_+^1(\tau),\delta\tau)d\tau.$$

We define

$$\lim_{T\to\infty} b_1(\lambda, z) = \Theta^s(0,T)q_+^1(T).$$

It follows that

$$q_+^1(t) = \Theta^s(t,0)b_1 + \int_\infty^t \Theta^s(t,\tau)h(q_+^1(\tau),\delta\tau)d\tau. \tag{5.9}$$

From $\Delta = q_+^1(0)$ we conclude

$$b_1 = \Delta - \int_\infty^0 \Theta^s(0,\tau)h(q_+^1(\tau),\delta\tau)d\tau. \tag{5.10}$$

Note that for arbitrarily small positive γ there exists a constant K independent of Δ such that

$$\begin{aligned} |q_+^1(t,z,\delta)| &< Ke^{\beta_s^\gamma(t,z)}\Delta \\ |q_-^2(t,z,\delta)| &< Ke^{\beta_u^\gamma(t,z)}\Delta, \end{aligned} \tag{5.11}$$

Similar estimates hold for the derivatives of q_+^1 and q_-^2 with respect to δ and the other parameters. Using (5.11) we obtain

$$\begin{aligned} &\left|\int_\infty^t \Theta^s(t,\tau)h(q_+^1(\tau),\delta\tau)d\tau\right| \leq \\ &KC\Delta^2 e^{\beta_s^\gamma(t)} \int_\infty^t e^{\beta_s^\gamma(\tau)+2\gamma\tau}d\tau = \Delta^2\mathcal{O}(e^{\beta_s^\gamma(t)-\alpha^* t}). \end{aligned} \tag{5.12}$$

Hence, for Δ small enough, $b_1 \neq 0$. Additionally we conclude from (5.10) that b_1 is C^k smooth as a function of (δ, λ, z).

To prove the statements concerning the derivatives of q_+^1 observe that for every $\gamma > 0$ there exists a constant $\tilde{K}$ such that the estimate

$$|D_{\nu_1 \ldots \nu_l} \Theta^s(t,\tau)x| \leq \tilde{K} e^{\beta_s^\gamma(t,\tau)} |x|$$

holds for all partial derivatives up to order $k+1$. The estimates on the derivatives of q_+^1 are now obtained by differentiating (5.9) with respect to the desired parameter and using estimates analogous to (5.11). Finally, to show that the derivatives of b_1 exist and are continuous we differentiate (5.10). □

We now linearize the equations (3.5) around the solutions $q_\pm(t, z_0, \delta)$. The linearization of (3.5) around $q_+(t, z_0, \delta) = (q_+^1(t, z_0, \delta), 0, z_0 + \delta t)$ is given by

$$\begin{aligned}
\dot{x} &= -A^s(q_+)x - D_x A^s(q_+) q_+^1 x + D_y A^s(q_+) q_+^1 y + D_z A^s(q_+) q_+^1 z \\
\dot{y} &= A^u(q_+) y \\
\dot{z} &= \delta B(q_+) q_+^1 y
\end{aligned} \tag{5.13}$$

Similarily for $q_-(t, w, \delta) = (0, q_-^2(t, z_0, \delta), z_0 + \delta t)$ we obtain

$$\begin{aligned}
\dot{x} &= -A^s(q_-)x \\
\dot{y} &= A^u(q_-)y + D_x A^u(q_-) q_-^2 x + D_y A^u(q_-) q_-^2 y + D_z A^u(q_-) q_-^2 z \\
\dot{z} &= \delta B(q_-) q_-^2 x.
\end{aligned} \tag{5.14}$$

Let $\underline{0} \in \mathbb{R}^m$ be the zero vector. We have

Lemma 2 *The linear equations (5.13) and (5.14) have exponential trichotomies determined by the projections $P_\pm^s(t), P_\pm^u(t), P_\pm^c(t)$ and the exponential rates $\beta_s^\gamma(t)$, $\beta_u^\gamma(t)$. There exists $K > 0$ such that the following properties hold:*

1.

$$\begin{aligned}
P_\pm^s(0)(x,y,z) &= (x, 0, \underline{0}), \\
P_\pm^u(0)(x,y,z) &= (0, y, \underline{0}), \\
P_\pm^c(0)(x,y,z) &= (0, 0, z),
\end{aligned}$$

2.

$$
\begin{aligned}
&P_+^s(t)(x,y,z) = (x,0,\underline{0}) + \mathcal{O}(e^{\beta_s^\gamma(t)}(|y|+|z|),0,\underline{0})\\
&P_+^u(t)(x,y,z) = (0,y,\underline{0}) + \mathcal{O}(e^{\beta_s^\gamma(t)}|y|,0,e^{\beta^\gamma(t)}\delta|y|)\\
&P_+^c(t)(x,y,z) = (0,0,z) + \mathcal{O}(e^{\beta_s^\gamma(t)}(|z|+e^{\beta^\gamma(t)}\delta|y|),0,e^{\beta^\gamma(t)}\delta|y|)\\
&|\Phi_+(t,\tau)P_+^s(\tau)p| \leq Ke^{\beta_s^\gamma(t,\tau)}|p|, \quad 0\leq\tau\leq t\\
&|\Phi_+(\tau,t)P_+^u(t)p| \leq Ke^{\beta_u^\gamma(\tau,t)}|p|, \quad 0\leq\tau\leq t
\end{aligned}
$$

3.

$$
\begin{aligned}
&P_-^s(t)(x,y,z) = (x,0,\underline{0}) + \mathcal{O}(0,e^{\beta_u^\gamma(t)}|x|,e^{\beta^\gamma(t)}\delta|x|)\\
&P_-^u(t)(x,y,z) = (0,y,\underline{0}) + \mathcal{O}(0,(e^{\beta_u^\gamma(t)}(|x|+|z|),\underline{0})\\
&P_-^c(t)(x,y,z) = (0,0,z) + \mathcal{O}(e^{\beta_u^\gamma(t)}(|z|+e^{\beta^\gamma(t)}\delta|x|),0,e^{\beta^\gamma(t)}\delta|x|)\\
&|\Phi_-(\tau,t)P_-^u(t)p| \leq Ke^{\beta_u^\gamma(\tau,t)}|p|, \quad \tau\leq t\leq 0\\
&|\Phi_-(t,\tau)P_-^s(t)p| \leq Ke^{\beta_s^\gamma(t,\tau)}|p|, \quad \tau\leq t\leq 0.
\end{aligned}
$$

Proof. We carry out the proof of *2.* The other case is similar. Let $\Phi_+(t,s)$ denote the transition matrices of (5.13) and $\tilde{\Phi}_+(t,s)$ the transition matrices for

$$
\begin{aligned}
\dot{x} &= -(A^s(q_+) + (D_xA^s(q_+)q_+^1))x\\
\dot{y} &= A^u(q_+)y
\end{aligned}
\tag{5.15}
$$

Since (5.15) is completely decoupled its transition matrix has the form

$$
\begin{pmatrix} \tilde{\Phi}_+^s(t,s) & 0\\ 0 & \tilde{\Phi}_+^u(t,s) \end{pmatrix},
$$

where $\tilde{\Phi}_+^s(t,s)$ and $\tilde{\Phi}_+^u(t,s))$ are the transition matrices of the first and the second equation in (5.15) respectively. Using the variation of constants formula we obtain

$$
\begin{aligned}
y(t,s) &= \tilde{\Phi}_+^u(t,s)y_0\\
z(t,s) &= z_0 + \delta\int_s^t B(q_+(\sigma))q_+^1(\sigma)y(\sigma,s)d\sigma\\
x(t,s) &= \tilde{\Phi}_+^s(t,s)x_0 - \int_s^t \tilde{\Phi}_+^s(t,\sigma)\Big(D_yA^s(q_+(\sigma))q_+^1(\sigma)y(t,s)\\
&\qquad +D_zA^s(q_+(\sigma))q_+^1(\sigma)z(t,s)\Big)d\sigma,
\end{aligned}
\tag{5.16}
$$

where $(x(t,s), y(t,s), z(t,s)) = \Phi(t,s)(x_0, y_0, z_0)$. Let $P_+^s(0)$, $P_+^u(0)$ and $P_+^c(0)$ be defined by *1* in the lemma. We define $P_+^s(t)$ by:

$$P_+^s(t) = \Phi(t,0)P_+^s(0)\Phi(0,t) \quad t \geq 0.$$

The other projections are defined analogously. We use this formula and (5.16) to derive the projections. Note that the y and z coordinates of $\Phi(t,s)(x_0, y_0, z_0)$ are given by the first and the second equation in (5.16) respectively. We have

$$\begin{aligned}
P_+^s(t)(x_0, y_0, z_0) &= (x^s(t), 0, 0) \\
P_+^u(t)(x_0, y_0, z_0) &= (x^u(t), y_0, z^u(t)) \\
P_+^c(t)(x_0, y_0, z_0) &= (x^c(t), 0, z^c(t))
\end{aligned}$$

where,

$$\begin{aligned}
&x^s(t) = x_0 - \tilde{\Phi}_+^s(t,0)\int_t^0 \tilde{\Phi}_+^s(0,\sigma)\Big(D_yA^s(q_+(\sigma))q_+^1(\sigma)\tilde{\Phi}_+^u(\sigma,t)y_0 \\
&+D_zA^s(q_+(\sigma))q_+^1(\sigma)[z_0 + \delta\int_t^\sigma B(q_+(\tau))q_+^1(\tau)\tilde{\Phi}_+^u(\tau,t)y_0 d\tau]\Big)d\sigma \\
&x^u(t) = \int_0^t \tilde{\Phi}_+^s(t,\sigma)\Big(D_yA^s(q_+(\sigma))q_+^1(\sigma)\tilde{\Phi}_+^u(\sigma,t)y_0 \\
&+D_zA^s(q_+(\sigma))q_+^1(\sigma)[\delta\int_0^\sigma B(q_+(\tau))q_+^1(\tau)\tilde{\Phi}_+^u(\tau,t)y_0 d\tau]\Big)d\sigma \\
&z^u(t) = \delta\int_0^t B(q_+(\sigma))q_u^+(\sigma)\Phi_+^u(\sigma,t)y_0 d\sigma \\
&x^c(t) = -\int_0^t \tilde{\Phi}_+^s(t,\sigma)D_zA^s(q_+(\sigma))q_+^1(\sigma)[z_0 + \delta\int_\sigma^0 B(q_+(\tau))q_u^+(\tau)\tilde{\Phi}_+^u(\tau,t)y_0 d\tau]d\sigma \\
&z^c(t) = z_0 + \delta\int_t^0 B(q_+(\tau))q_u^+(\tau)\tilde{\Phi}_+^u(\tau,t)y_0 d\tau
\end{aligned}$$

It follows that the projection operators have the required properties. In order to get the estimates on the contraction and expansion note that they hold for $\tilde{\Phi}_+(t,s)$. The equations (5.16) and the form of P_+^u and P_+^s imply that they also hold for $\Phi_+(t,s)$. The lemma follows. $\square$

Remark 8 The existence of the trichotomies of Lemma 2 ia a consequence of the robustness of the hyperbolicity properties of $\tilde{\Phi}^u$ and $\tilde{\Phi}^s$. Next to the existence of the trichotomies we also need the estimates of the asymptotic behavior of the projections $P^s_\pm$, $P^u_\pm$ and $P^c_\pm$, which can only be obtained via a direct proof. These estimates will be used in Lemma 6.

As outlined at the beginning of this section we consider solutions $p_\pm(t)$ of (3.5) with $p_+(0) = (\Delta, y_0, z_0)$ and $p_-(2T) \in \Sigma_1$. Let $X = (X^1, X^2, X^3)$. We write $p_\pm$ in the form

$$p_\pm = X_\pm + q_\pm. \tag{5.17}$$

More precisely

$$\begin{aligned} p_+(t) &= X_+(t) + q_+(t, z_0) \;\; T \geq t \geq 0 \\ p_-(t) &= X_-(t - 2T) + q_-(t - 2T, z_0 + 2\delta T) \;\; 2T \geq t \geq T. \end{aligned} \tag{5.18}$$

Let $\mathcal{A}_\pm(t) = D\hat{F}(q_\pm(t))$. The matrices $\mathcal{A}_+(t)$ and $\mathcal{A}_-(t)$ are defined by the right hand sides of the equations (5.13) and (5.14) respectively. The equation (3.5) is equivalent to the pair of equations

$$\begin{aligned} \dot{X}_+ &= \mathcal{A}_+ X_+ + F_+(t, X_+), \quad t \geq 0 \\ \dot{X}_- &= \mathcal{A}_- X_- + F_-(t, X_-), \quad t \leq 0, \end{aligned} \tag{5.19}$$

where $F_\pm(\cdot, X_\pm) = \hat{F}(X_\pm + q_\pm) - \hat{F}(q_\pm) - \mathcal{A}_\pm X_\pm$. Our goal is to find, for each choice of T, z_0 and δ, solutions $X_\pm$ to (5.19) such that:

(a) $X^3_+(0) = z_0$,

(b) $X_+(T) - X_-(-T) = (-q^1_+(T), q^2_-(-T), 0)$.

The conditions (a) and (b) imply that the function $p(t)$ given by concatenating the functions p_+ and p_- given in (5.18) is a solution of (3.5). We will show that $p(t)$ satisfies the conditions of Theorem 4.

We begin by solving the linearized problem. Consider the equations

$$\dot{X} = \mathcal{A}_\pm(t) X + h_\pm(t) \tag{5.20}$$

We introduce the following function spaces:

$$
\begin{aligned}
V_+^s &= \{g : [0,T] \to \mathbb{R} \mid \sup_{T\geq t\geq 0} e^{\beta_s^\gamma(t)}|g(t)| = \|g\|_+^s < \infty\} \\
V_+^u &= \{g : [0,T] \to \mathbb{R} \mid \sup_{T\geq t\geq 0} e^{\beta_u^\gamma(t,T)}|g(t)| = \|g\|_+^u < \infty\} \\
V_-^u &= \{g : [-T,0] \to \mathbb{R} \mid \sup_{-T\leq t\leq 0} e^{\beta_u^\gamma(t)}|g(t)| = \|g\|_-^u < \infty\} \\
V_-^s &= \{g : [-T,0] \to \mathbb{R} \mid \sup_{-T\leq t\leq 0} e^{\beta_s^\gamma(t,-T)}|g(t)| = \|g\|_-^s < \infty\}
\end{aligned}
$$

For $h : [0,T] \to \mathbb{R}^m$ let

$$\|h\|_+^c = T \sup_{t\in[0,T]} |h(t)|$$

and for $h : [-T,0] \to \mathbb{R}^m$ let

$$\|h\|_-^c = T \sup_{t\in[-T,0]} |h(t)|.$$

Let $\Phi_\pm^s(t,\tau) = \Phi(t,\tau)P_\pm^s(\tau)$, $\Phi_\pm^u(t,\tau) = \Phi(t,\tau)P_\pm^u(\tau)$ and $\Phi_\pm^c(t,\tau) = \Phi(t,\tau)P_\pm^c(\tau)$. Let

$$
\begin{aligned}
V_+ &= \{g : [0,T] \to \mathbb{R}^{m+2} \mid g_+^1 \in V_+^s \text{ and } g_+^2 \in V_+^u\} \\
V_- &= \{g : [-T,0] \to \mathbb{R}^{m+2} \mid g_-^1 \in V_-^s \text{ and } g_-^2 \in V_-^u\}
\end{aligned}
$$

We write $g : [0,T] \to \mathbb{R}^{m+2}$ in components as $g = (g^1, g^2, g^3)$. For $g \in V_\pm$ let $\|g\|_\pm = \|g^1\|_\pm^s + \|g^2\|_\pm^u + \|g^3\|_\pm^c$. Let $V = V_+ \times V_-$. For $g = (g_+, g_-) \in V$ let $\|g\| = \|g_+\|_+ + \|g_-\|_-$. We assume that $(h_+, h_-) \in V$.

Let $A = A^1e_1 + A^2e_2 + C$ be some constant vector in $\mathbb{R}^{m+2}$, where $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$ and C is orthogonal to e_1 and e_2. Consider the following solutions of (5.20_+) and (5.20_-).

$$
\begin{aligned}
Y_+(t) &= \Phi_+^u(t,T)A^2e_2 + \int_T^t \Phi_+^u(t,s)h_+(s)ds + \int_0^t \Phi_+^c(t,s)h_+(s)ds \\
&\quad + \int_0^t \Phi_+^s(t,s)h_+(s)ds \qquad (5.21) \\
Y_-(t) &= \Phi_-^u(t,-T)A^1e_1 + \int_{-T}^t \Phi_-^u(t,s)h_-(s)ds + \int_0^t \Phi_-^c(t,s)h_-(s)ds + \\
&\int_0^t \Phi_-^u(t,s)h_-(s)ds + \Phi_-^c(t,0)C.
\end{aligned}
$$

Lemma 3 *Let $d = (d^1, d^2, d^3) \in \mathbb{R}^{m+2}$. For T sufficiently large the linear equation*

$$Y_+(T) - Y_-(-T) = d \tag{5.22}$$

has a unique solution.

Proof Using Lemma 2 we can write

$$\begin{aligned} Y_+(T) &= A^2 e_2 + \mathcal{O}(e^{\beta_s^\gamma(T)}|A^2|, 0, e^{\beta^\gamma(T)}|A^2|\delta) \\ &\quad + \int_0^T \Phi_+^c(T,s)h_+(s)ds + \int_0^T \Phi_+^s(T,s)h_+(s)ds \\ Y_-(-T) &= A^1 e_1 + C + \mathcal{O}(0, e^{\beta_u^\gamma(-T)}(|A^1| + |C|), e^{\beta^\gamma(T)}\delta|A^1|) - \\ &\quad \int_{-T}^0 \Phi_-^c(-T,s)h_-(s)ds - \int_{-T}^0 \Phi_-^u(-T,s)h_-(s)ds. \end{aligned}$$

It now follows that (5.22) is equivalent to

$$d = L \cdot A + v, \tag{5.23}$$

where

$$\begin{aligned} v = &\int_0^T \Phi_+^s(T,s)h_+(s)ds + \int_0^T \Phi_+^c(T,s)h_+(s)ds \\ &+ \int_{-T}^0 \Phi_-^u(T,s)h_-(s)ds + \int_{-T}^0 \Phi_-^c(T,s)h_-(s)ds, \end{aligned}$$

and

$$L = \begin{pmatrix} -1 & 0 & \underline{0} \\ 0 & 1 & \underline{0} \\ \underline{0}^{\mathrm{T}} & \underline{0}^{\mathrm{T}} & -\mathrm{Id}_{m\times m} \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{O}(e^{\beta_s^\gamma(T)}) & \underline{0} \\ \mathcal{O}(e^{\beta_u^\gamma(-T)}) & 0 & \mathcal{O}(e^{\beta_u^\gamma(-T)}) \\ \mathcal{O}(e^{\beta^\gamma(T)}\delta) & \mathcal{O}(e^{\beta^\gamma(T)})\delta & 0_{m\times m} \end{pmatrix},$$

where $\underline{0} = (0, \ldots, 0) \in \mathbb{R}^m$, $\mathrm{Id}_{m\times m}$ is the $m \times m$ identity matrix and $0_{m\times m}$ is the $m \times m$ matrix with all entries equal to 0. For sufficiently large T L^{-1} exists, and $A = L^{-1}(d - v)$. □

Let $\mathcal{L} : \mathbb{R}^3 \times V \to V$ denote the linear operator assigning to (d, h_+, h_-) the solutions of (5.20) (Y_+, Y_-) satisfying the condition (5.22). The operator $\mathcal{L}$ depends C^{k+1} smoothly on (δ, λ, z) and its derivatives also define linear operators mapping $\mathbb{R}^3 \times V$ to V. Let $\|\mathcal{L}\|$ denote the usual operator norm with respect to the norm $|\cdot| + \|\cdot\|$ on $\mathbb{R}^3 \times V$.

Lemma 4 *The norms* $\|\mathcal{L}\|$ *and* $\|D_{\nu_1\ldots\nu_l}\mathcal{L}\|$, $\nu_j \in \{\delta, \lambda, z\}$, $j = 1, \ldots, l$, $l \in \{1, \ldots, k+1\}$ *are uniformly bounded in* T.

Proof We derive a more precise expression for the solution of (5.23). We have the following estimates

$$
\begin{aligned}
\int_0^t \Phi_+^s(t,s)h_+(s)ds &= \mathcal{O}(e^{\beta_s^\gamma(t)}(\|h_+^1\|_+^s + |h_+|), 0, 0) \\
\int_0^t \Phi_+^c(t,s)h_+(s)ds &= \mathcal{O}(t|h_+^3| + \delta e^{\beta^\gamma(t)}|h_+^2|)(e^{\beta_s^\gamma(t)}, 0, 1) \\
\int_T^t \Phi_+^u(t,s)h_+(s)ds &= \mathcal{O}(e^{\beta_u^\gamma(t,T)}\|h_+^2\|_+^u)(e^{\beta_s^\gamma(t)}, 1, \delta e^{\beta^\gamma(t)}) \\
\int_{-t}^0 \Phi_-^u(-t,s)h_-(s)ds &= \mathcal{O}(0, e^{\beta_u^\gamma(-t)}(\|h_-^2\|_-^u + |h_-|), 0), \\
\int_{-t}^0 \Phi_-^c(-t,s)h_-(s)ds &= \mathcal{O}(t|h_-^3| + \delta e^{\beta^\gamma(t)}|h_-^1|)(0, e^{\beta_u^\gamma(-t)}, 1) \\
\int_{-T}^t \Phi_-^u(t,s)h_-(s)ds &= \mathcal{O}(e^{\beta_s^\gamma(t,-T)}\|h_+^1\|_-^s)(1, e^{\beta_u^\gamma(t)}, \delta e^{\beta^\gamma(t)}).
\end{aligned}
\tag{5.24}
$$

A computation shows that

$$
L^{-1} = \begin{pmatrix} -1 & 0 & \underline{0} \\ 0 & 1 & \underline{0} \\ \underline{0}^{\mathrm{T}} & \underline{0}^{\mathrm{T}} & -\mathrm{Id}_{m\times m} \end{pmatrix} + \begin{pmatrix} \mathcal{O}(e^{\beta^\gamma(T)+\beta_u^\gamma(-T)}\delta) & \mathcal{O}(e^{\beta_s^\gamma(T)}) & \mathcal{O}(e^{\beta_s^\gamma(T)+\beta_u^\gamma(-T)}) \\ \mathcal{O}(e^{\beta_u^\gamma(-T)}) & 0 & \mathcal{O}(e^{\beta_u^\gamma(-T)}) \\ \mathcal{O}(e^{\beta^\gamma(T)}\delta) & \mathcal{O}(e^{\beta^\gamma(T)}\delta) & \mathcal{O}(e^{\beta^\gamma(T)+\beta_u^\gamma(-T)\delta}) \end{pmatrix}.
$$

Using (5.24) with $t = T$ we obtain the following estimate

$$
\begin{aligned}
A = (-d^1, d^2, -d^3) &+ \mathcal{O}(e^{\beta^\gamma(T)}|d| + (\|h_+^1\|_+^s + |h_+|)e^{\beta_s^\gamma(T)}(1, e^{\beta_u^\gamma(-T)}, e^{\beta^\gamma(T)}\delta) \\
&+ (\|h_-^u\|_-^u + |h_-|)e^{\beta_u^\gamma(-T)}(e^{\beta_s^\gamma(T)}, 1, e^{\beta^\gamma(T)}\delta) \\
&+ (\|h_+^c\|_+^c + \|h_-^c\|_-^c + \delta e^{\beta^\gamma(T)}(|h_+^2| + |h_-^1|))(e^{\beta_s^\gamma(T)}, e^{\beta_u^\gamma(-T)}, \underline{1})).
\end{aligned}
\tag{5.25}
$$

It follows from (5.24) and (5.25) that $\mathcal{L}$ is uniformly bounded in T. To obtain the result for the derivatives of $\mathcal{L}$ we need to obtain results analogous to Lemma 2 for the corresponding derivatives of Φ_+ and Φ_-. Such results are proved by combining the arguments used in the proof of Lemma 2 and the already available information on Φ_+ and Φ_-. See also [20, Lemma 1.1]. The remaining part of the proof is similar to the proof of boundedness of $\mathcal{L}$. □

We now define the Nemitskii operators $G_\pm$ by the formula:

$$G_\pm(\delta, z, X_\pm)(t) = F_\pm(t, \delta, z, X_\pm(t)).$$

Let $\mathcal{G}$ be the functional defined by the formula:

$$\mathcal{G}(X_+, X_-) = (X_+, X_-) - \mathcal{L}(d, G_+(X_+), G_-(X_-), \delta).$$

For fixed T we consider the following nonlinear equation:

$$\mathcal{G}(X_+, X_-) = 0. \tag{5.26}$$

Note that (5.26) is trivially satisfied for $d = 0$ and $(X_+, X_-) = (0, 0)$. We will show that (5.26) can be solved for (X_+, X_-) by the implicit function theorem and that the solution exists for (d, T) belonging to a neighborhood of $(0, \infty)$. The solution (X_+, X_-) of (5.26) satisfying

$$d = (-q_+^1(T), q_-^2(-T), 0). \tag{5.27}$$

will provide the solution sought in Theorem 4. We begin by proving the following proposition.

Proposition 4 *The equation (5.26) has a unique solution $X = (X_+, X_-)$ depending C^k on $(T, \delta, \lambda, z, d)$. Moreover $\|X\|$ and $\|D_{\nu_1 \ldots \nu_l} X\|$, $\nu_j \in \{T, \lambda, z, d\}$, $j = 1, \ldots, l$, $l \in \{0, \ldots, k\}$ are uniformly bounded in T. The partial derivatives involving differentiation with respect to δ grow with at most polynomial rate as $T \to \infty$. This solution is defined on the set of the form $W \times (T_0, \infty)$, where W is a neighborhood of $(\delta, \lambda, z, d) = (0, \lambda_0, 0, 0)$ and $T_0 > 0$ is sufficiently large.*

The proof of Proposition 4 is based on the following lemma.

Lemma 5 *Fix $T > 0$. The operators G_+ and G_- are C^k smooth mappings of $\mathbb{R}^4 \times V_+ \to V_+$ and $\mathbb{R}^4 \times V_- \to V_-$, respectively. The norms of the derivatives of G_+ and G_- with respect to (X_+, X_-) and (λ, z) are uniformly bounded in T. The partial derivatives involving differentiation with respect to δ grow with at most polynomial rate as $T \to \infty$.*

Proof The proof is similar to the proof of Lemma 3.13 in [20]. Here we sketch the proof referring to [20] for more details. We carry out the proof for the case of G_+. The other case is similar.

Using the form of F_+ one obtains the following estimates

$$\begin{aligned} F^1_+(t,\delta,z,X_+) &= \mathcal{O}((|X^1_+| + |q^1_+||X_+|)|X_+|) \\ F^2_+(t,\delta,z,X_+) &= \mathcal{O}(|X^2_+||X_+|) \\ F^3_+(t,\delta,z,X_+) &= \mathcal{O}(\delta(|X^1_+| + |q^1_+|)|X^+_2|). \end{aligned} \tag{5.28}$$

It follows that

$$\begin{aligned} \|G^2_+(\delta,z,X_+)\|^u_+ &= \|F^2_+(\cdot,\delta,z,X_+)\|^u_+ = \mathcal{O}(\|X^2_+\|^u_+|X_+|) \\ \|G^1_+(\delta,z,X_+)\|^u_+ &= \|F^1_+(\cdot,\delta,z,X_+)\|^u_+ = \mathcal{O}((\|X^1_+\|^s_+ + \|q^1_+\|^s_+)|X_+| \\ \|G^3_+(\delta,z,X_+)\|^c_+ &= T|F^3_+(t,\delta,z,X_+)| = \mathcal{O}(e^{\beta^\gamma(T)}(\|X^1_+\|^s_+ + \|q^1_+\|^s_+)\|X^2_+\|^u_+) \end{aligned} \tag{5.29}$$

The estimate (5.29) implies that $G_+(\delta,z,X_+) \in V_+$. We now show differentiability of G_+. Continuity follows from differentiability. We claim that

$$DG_+(\delta,z,X_+)h = DF_+(\cdot,\delta,z,X_+)h. \tag{5.30}$$

Consider G^2_+. We claim that partial derivatives of G^2_+ with respect to X^1_+ and X^2_+ have the form

$$\begin{aligned} D_{X^1}G^2_+(\delta,z,X_+) &= D_{X^1}F^2_+(\cdot,\delta,z,X_+) \\ D_{X^2}G^2_+(\delta,z,X_+) &= D_{X^2}F^2_+(\cdot,\delta,z,X_+) \end{aligned} \tag{5.31}$$

Recall that if DG_+ exists then it must be a bounded operator mapping $\mathbb{R}^4 \times V_+$ to V_+. For the partial derivatives this requirement means

$$\begin{aligned} \|D_{X^1}G^2_+(\delta,z,X_+)h\|^u_+ &\le C\|h\|^s_+ \\ \|D_{X^2}G^2_+(\delta,z,X_+)h\|^u_+ &\le C\|h\|^u_+ \end{aligned} \tag{5.32}$$

for some constant C. We need to know that such inequalities hold for the right hand sides of the equations (5.31). It follows from the form of F_+ that $D_{X^1}F^2_+(\cdot,\delta,z,X_+) = \mathcal{O}(\|X^2_+\|^u_+)$. Hence

$$\begin{aligned} &\|D_{X^1}F^2_+(\cdot,\delta,z,X_+)h\|^u_+ = \sup_{T\ge t\ge 0} e^{\beta^\gamma_u(T,t)}|D_{X^1}F^2_+(\cdot,\delta,z,X_+)||h| \\ &\le \mathcal{O}(\|X^u\|^u_+)|h| \le C\|h\|^s_+ \end{aligned}$$

and

$$\|D_{X^2}F_+^2(\cdot,\delta,z,X_+)h\|_+^u = \sup_{T\geq t\geq 0} e^{\beta_u^\gamma(T,t)}|D_{X^2}F_+^2(\cdot,\delta,z,X_+)h| \leq C\|h\|_+^u.$$

Note that C is independent of T. We have shown that the right hand sides of (5.31) define bounded operators on the right spaces. We now need to show that these indeed are the required derivatives. Let $h \in V_+^s$.

$$\|G_+^2(X_+^1+h,X_+^2,X_+^3) - G_+^2(X_+) - D_{X^1}F_+^2(\cdot,X_+)h\|_+^u \frac{1}{\|h\|_+^u} \leq$$

$$\sup_{T\geq t\geq 0} |F_+^2(t,X_+^1+h,X_+^2,X_+^3) - F_+^2(t,X_+) - D_{X^1}F_+^2(t,X_+)h|e^{\beta_u^\gamma(T,t)}\frac{1}{\|h\|_+^u} \leq$$

$$\sup_{T\geq t\geq 0} e^{\beta_u^\gamma(T,t)}\int_0^1 |D_{X^1}F_+^2(t,X_+^1+\tau_1 h,X_+^2,X_+^3) - D_{X^1}F_+^2(t,X_+)|d\tau_1 \leq$$

$$\sup_{T\geq t\geq 0}\ \sup_{\tau_1\in[0,1]}$$

$$\int_0^1 |D_{X^2}D_{X^1}(F_+^2(t,X_+^1+\tau_1 h,\tau_2 X_+^2,X_+^3) - F_+^2(t,X_+^1,\tau_2 X_+^2,X_+^3))|d\tau_2\|X_+^2\|_+^u \to 0$$

$$\text{as } \|h\|_+^s \to 0.$$

In the last inequality we used the fact that $F_+^2(X_+^1,0,X_+^3)=0$, which follows from (5.28). The first equation in (5.31) follows. The proof of the second inequality in (5.31) is similar but does not require the use of (5.28). Using the same methods one proves that partial derivatives of G^j, $j=1,2,3$, with respect to X^i, $i=1,2,3$ and with respect to the parameters are obtained by taking the corresponding partial derivatives of F^j and composing them with X_+. In the estimates other than the ones corresponding to $D_{X_j}G^j$, $j=1,2$, (5.28) must be used. For second order partial derivatives the argument is similar, however estimates analogous to (5.28) must be obtained for partial derivatives other than $D_{X_j}G^j$, $j=1,2$. This is done using the form of F_+ and Lemma 1. The situation is similar for partial derivatives of any order $0<l\leq k+1$. We conclude that G_+ is C^k since partial derivatives up to order $k+1$ exist. The norms of the partial derivatives are uniformly bounded in T except for the derivatives with respect to δ. For this case we have

$$\begin{aligned} D_\delta F_+^1(t,\delta,z,X_+) &= \mathcal{O}(t(|X_+^1|+|q_+^1|+|D_\delta q_+^1|)|X_+|) \\ D_\delta F_+^2(t,\delta,z,X_+) &= \mathcal{O}(t|X_+^2||X_+|) \\ D_\delta F_+^3(t,\delta,z,X_+) &= \mathcal{O}(\delta(te^{\beta^\gamma(T,0)})). \end{aligned} \tag{5.33}$$

It follows that $D_\delta F_+^3$ is bounded. However for the other two derivatives we obtain

$$\begin{aligned}\|D_\delta F_+^1\|_+^s = \mathcal{O}(T(\|X_+^1\|_+^s + \|q_+^1\|_+^s + \|D_\delta q_+^1\|_+^s)|X_+|)\\ \|D_\delta F_+^2(\cdot,\delta,z,X_+)\|_+^u = \mathcal{O}(T\|X_+^2\|_+^u|X_+|).\end{aligned} \tag{5.34}$$

The lemma follows. □

Proof of Proposition 4 Let $T > 0$ be fixed. Since $\mathcal{G} = \mathcal{L} \circ (G_+, G_-)$ it follows that $\mathcal{G}$ is a C^k smooth function of $(\delta, \lambda, z, d, X_+, X_-)$. It follows from the form of F_+ and Lemma 5 that $DG_{\pm(X_\pm)}(\delta, \lambda, z, 0, 0, 0) = 0$. Hence $D\mathcal{G}_{(X_+,X_-)}(\delta, \lambda, z, 0, 0, 0)$ is invertible. By the implicit function theorem [4] (5.26) has a unique solution defined on a neighborhood of $(0, \lambda_0, 0, 0, 0, 0)$ in V. It follows from Proposition 4 and Lemma 4 that the estimates on the derivatives of $\mathcal{G}$ with respect to d, X_+ and X_- are uniform in T. Hence the solution of (5.26) exists for $(T, \delta, \lambda, z, d)$ in an open neighborhood of $(\infty, 0, \lambda_0, 0, 0)$ and depends C^k smoothly on (δ, λ, z, d). Proposition 4 and Lemma 4 also imply that the derivatives of the solution (X_+, X_-) with respect to (λ, z, d) are uniformly bounded in T. We will show in the sequel that when d is given by (5.27) then the derivatives of (X_+, X_-) with respect to δ are also uniformly bounded in T.

Note that the above argument could be repeated using maximum norms. This shows the uniqueness of the solution.

It remains to understand the effect of varying T. We claim that the solutions of (5.26) depend C^k smoothly on T. The argument is analogous as in [20, p. 97]. We rescale the time letting $s = (1 + \omega T)$. This way (3.5) becomes

$$\dot{X} = (1 + \omega)\hat{F}(X). \tag{5.35}$$

We define an operator $\tilde{\mathcal{L}}$ analogous to $\mathcal{L}$ and consider the fixed point problem

$$\tilde{\mathcal{G}}(\omega, \lambda, z, \delta, X_+, X_-) = (X_+, X_-) \tag{5.36}$$

defined analogously as (5.26). Let $(\tilde{X}_+, \tilde{X}_-)$ be the solution of (5.36). By the uniqueness of solutions of (5.26) and (5.36) $\tilde{X}_\pm(T_0, \omega, \lambda, z, \delta) = X_\pm((1 + \omega)T_0, \lambda, z, \delta)$. Differentiating we obtain

$$T_0 D_T X_\pm((1 + \omega)T_0) = \frac{d}{d\omega} X_\pm((1 + \omega)T_0) = D_\omega \tilde{X}_\pm(T_0, \omega).$$

Here D_T and D_ω denote partial derivatives and $\frac{d}{d\omega}$ denotes the total derivative. For $\tilde{X}_\pm$ we can prove differentiability with respect to ω analogously as we showed differntiability of $X_\pm$ with respect to (δ, λ, z). Only the exponential rates must be multiplied by $(1+\omega)$, i.e. we must introduce $\tilde{\alpha}_s^\gamma = (1+\omega)\alpha_s^\gamma$, $\tilde{\alpha}_u^\gamma = (1+\omega)\alpha_u^\gamma$ and proceed as in (5.7). Note that we can consider ω in a very small interval. Hence, possibly slightly increasing γ, we conclude that $D_T X_\pm(T, \delta, \lambda, z)$ exists and is uniformly bounded in T. $\square$

Remark 9 Let $X = (X_+, X_-)$ be the solution of (5.26). It follows from (5.34) that $\|D_\delta X\| = \mathcal{O}(T)$. Similarily a higher order partial derivative is of the order $\mathcal{O}(T^l)$ if it involves differentiating l times with respect to δ.

Let $X^* = (X_+^*, X_-^*)$ be the solution of (5.26) satisfying (5.27). Recall that the pair (X_+^*, X_-^*) generates a solution of (3.5) given by (5.18). We have the following estimates.

Lemma 6

$$\begin{aligned}
&\|X_+^{*2}\|_+^u = \mathcal{O}(e^{\beta_u^\gamma(-T)}) \\
&|X_+^*| = \mathcal{O}(e^{-\alpha^* T}) \\
&\|X_-^{*1}\|_-^s = \mathcal{O}(e^{\beta_s^\gamma(T)}) \\
&|X_-^*| = \mathcal{O}(e^{-\alpha^* T})
\end{aligned}$$

Proof. We first consider X_+^*. Lemma 2 implies that $\|X_+^{*2}\|_+^u = \mathcal{O}(\|P_+^u(\cdot)X_+^*(\cdot)\|)$. We have

$$P_+^u(t)X_+^*(t) = \Phi_+^u(t,T)A^2 e_2 + \int_T^t \Phi_+^u(t,\sigma)G(X_+^*)(\sigma)d\sigma.$$

It follows from (5.29) and Lemma 2 that

$$\|P_+^u X_+^*\|_+^u = \|\tilde{\Phi}_+^u(\cdot,T)A^2\|_+^u + \mathcal{O}(\|P_+^u X_+^*\|_+^u |X_+^*|).$$

Since $|X_+^*|$ is small it follows that

$$\|X_+^{*2}\| = \mathcal{O}(\|\tilde{\Phi}_+^u(\cdot,T)A^2\|_+^u) = \mathcal{O}(|A^2|).$$

Lemma 1 and Lemma 4 imply that $|A^2| = \mathcal{O}(e^{\beta_u^{\gamma}(-T)})$. For the other inequality consider the projection $\tilde{P} = Id - P_+^u$. It follows from Lemma 2 that

$$\tilde{P}(t)(x,y,z) = (x,0,z) + \mathcal{O}(e^{\beta^{\gamma}(T)}).$$

Also, using the estimates (5.29) and the already obtained estimates on $|X_+^{*2}|$ we obtain

$$|\tilde{P}X_+^*| \leq |\int_0^t \Phi_+^c(t,\sigma)G(X_+^*)(\sigma)d\sigma| + |\int_0^t \Phi_+^s(t,\sigma)G(X_+^*)(\sigma)d\sigma| =$$
$$\mathcal{O}(e^{\beta^{\gamma}(T)}) + \mathcal{O}((K\Delta + \|X_+^{*1}\|_+^s)|(X_+^{*1},0,X_+^{*3})|),$$

where K is the same constant as in (5.11). By choosing Δ small enough we may conclude that $|(X_+^{*1},0,X_+^{*3})| = \mathcal{O}(e^{\beta^{\gamma}(T)})$. In view of the estimate on $|X_+^{*2}|$ we obtain the required result. For X_-^* note that (5.29) and (5.25) imply that $C = \mathcal{O}(e^{\beta^{\gamma}(T)})$ (see (5.21) for the definition of C). The remaining part of the argument is similar to the X_+^* case. $\square$

Proof of Theorem 4. Note that

$$\begin{aligned} X_+^*(0) &= \Phi_+^u(0,T)A^2e_2 + \int_T^0 \Phi_+^u(0,t)G(X_+^*)(t)dt \\ X_-^*(0) &= \Phi_-^s(0,-T)A^1e_1 + \int_{-T}^0 \Phi^s(0,t)G(X_+^*)(t)dt + C. \end{aligned} \tag{5.37}$$

Using Lemma 6 and the estimate (5.29) we obtain

$$\begin{aligned} X_+^{*2}(0) &= \tilde{\Phi}_+^u(0,T)A^2 + \mathcal{O}(e^{\beta_u^{\gamma}(-T,T)-\alpha^*T}) \\ X_-^{*1}(0) &= \tilde{\Phi}_-^s(0,-T)A^1 + \mathcal{O}(e^{\beta_s^{\gamma}(T,-T)-\alpha^*T}) \\ X_-^{*3}(0) &= \mathcal{O}(e^{-\alpha^*T}). \end{aligned} \tag{5.38}$$

Differentiating (5.37), using Remark 9 and Lemma 6 we obtain the identity

$$\begin{aligned} D_{\nu_1\ldots\nu_l}X_+^{*2}(0) &= D_{\nu_1\ldots\nu_l}(\tilde{\Phi}_+^u(0,T)A^2) + \mathcal{O}(e^{\beta_u^{\gamma}(-T,T)-\alpha^*T}) \\ D_{\nu_1\ldots\nu_l}X_-^{*1}(0) &= D_{\nu_1\ldots\nu_l}(\tilde{\Phi}_-^s(0,-T)A^1) + \mathcal{O}(e^{\beta_s^{\gamma}(T,-T)-\alpha^*T}) \\ D_{\nu_1\ldots\nu_l}X_-^{*3}(0) &= \mathcal{O}(e^{-\alpha^*T}), \end{aligned} \tag{5.39}$$

where $\nu_j = T, \lambda, z$ or δ, $l \in \{1,\ldots,k\}$ and $j = 1,\ldots,l$. For the derivatives with respect to δ we may have to slightly decrease α^* due to the factor T possibly appearing in the estimate of $\|D_\delta G(X^*)\|$ and in $\|D_\delta X^*\|$. To illustrate the derivation of (5.39) we estimate $D_\delta X_+^*$. Since $P^u(0)$ is the projection onto the second component (see Lemma 2) we have $X_+^*(0) = (0, X_+^{*2}(0), \underline{0})$ with

$$X_+^{*2}(0) = \tilde{\Phi}_+^u(0,T)A^2 + \int_T^0 \tilde{\Phi}_+^u(0,t)G^2(X_+^*)(t)dt.$$

We claim that

$$\frac{d}{d\delta}\tilde{\Phi}_+^u(0,t)G^2(X_+^*)(t) = \mathcal{O}(e^{\beta_u^\gamma(-T,T)-\alpha^* T}), \tag{5.40}$$

with α^* slightly decreased. By differentiating (5.15) with respect to δ and using the properties of $\tilde{\Phi}_+$ we obtain

$$|\frac{d}{d\delta}\tilde{\Phi}_+^u(0,t)y_0| \leq \tilde{K}e^{\beta_u^\gamma(0,t)}y_0.$$

with γ possibly slightly larger. Hence, using (5.29), we obtain,

$$(\frac{d}{d\delta}\tilde{\Phi}^u(0,t))G^2(X_+^*)(t) = \mathcal{O}(e^{\beta_u^\gamma(-T,T)-\alpha^* T}).$$

Using Lemma 6 and similar estimates as outlined in the proof of Lemma 5 we obtain

$$\|\frac{d}{d\delta}G^2(\delta, X_+^*)\|_+^u = \mathcal{O}(Te^{\beta_u^\gamma(-T)-\alpha^* T}).$$

By slightly decreasing α^* we obtain the estimate (5.40). It follows that $D_\delta X_+^*(0)$ satisfies (5.39).

We are now ready to derive asymptotic expansions given in the statement of Theorem 4. Let $\alpha_u^* = \frac{1}{2T}\beta_u^0(T,-T)$ and $\alpha_s^* = \frac{1}{2T}\beta_s^0(-T,T)$. Using Lemma 1 and estimates analogous as in the proof of Lemma 1 we obtain the following expressions:

$$\begin{aligned}\tilde{\Phi}_+^u(0,T)A^2 &= a_1 e^{-2\alpha_u^* T} + \mathcal{O}(e^{-2\alpha_u^* T-\alpha^* T})\\ \tilde{\Phi}_-^s(0,-T)A^1 &= a_2 e^{-2\alpha_s^* T} + \mathcal{O}(e^{-2\alpha_s^* T-\alpha^* T}),\end{aligned} \tag{5.41}$$

where a_1 and a_2 are smooth functions of (λ, z, δ). Moreover

$$\begin{aligned}D_{\nu_1\ldots\nu_l}\tilde{\Phi}^u(0,T)A^2 &= D_{\nu_1\ldots\nu_l}(a_1 e^{-2\alpha_u^* T}) + \mathcal{O}(e^{-2\alpha_u^* T-\alpha^* T})\\ D_{\nu_1\ldots\nu_l}\tilde{\Phi}^s(0,T)A^1 &= D_{\nu_1\ldots\nu_l}(a_2 e^{-2\alpha_s^* T}) + \mathcal{O}(e^{-2\alpha_s^* T-\alpha^* T}),\end{aligned} \tag{5.42}$$

where $\nu_j = T, \lambda, z$ or δ, $l \in \{1,\ldots,k\}$ and $j = 1,\ldots,l$. By combining the estimates (5.38), (5.39), (5.41) and (5.42) and replacing $2T$ by T we obtain the statement of Theorem 4. $\square$

6 The existence of an inclination-flip point

In this section we prove Theorem 5. In the proof we use the following results.

Proposition 5 *The homoclinic orbits Γ_f are non-twisted.*

Proposition 6 *For every $0 < a < \frac{1}{2}$ there exists $\gamma > 0$ such that for any sufficiently small c and δ the corresponding slow wave homoclinic orbit Γ_s is twisted.*

Proof of Theorem 5 Let $0 < a < \frac{1}{2}$, $\gamma > 0$ be such that Γ_s is twisted. By Proposition 5 Γ_f is non-twisted. Hence there exists an inclination flip point along any path in the parameter space joining Γ_s to Γ_f. The existence of such paths follows from Theorem 3. □

Proof of Proposition 5

Consider the adjoint equation

$$\dot{\psi} = -DF(\gamma_f(t))^T\psi, \tag{6.1}$$

where F is the right hand side of the (1.2) in the original coordinates (u, v, w) and $\Gamma_f = \{\gamma_f(t) : t \in \mathbb{R}\}$ is a homoclinic orbit corresponding to a fast wave. The equation (6.1) has a unique, up to multiplication by a constant, bounded solution $\zeta(t)$. This solution is normal to the stable manifold $W^s(p_0)$. For $t \approx \pm\infty$ (6.1) is close to

$$\dot{\psi} = -A^T\psi, \tag{6.2}$$

where $A = DF(0)$. The matrix $-A^T$ has eigenvalues $-\lambda_{ss}$, $-\lambda_s$, $-\lambda_u$ with normed eigenvectors e^*_{ss}, e^*_s, e^*_u. Let e_1, e_2, e_3 denote the standard basis vectors and let e_s, e_{ss}, e_u denote the normed eigenvectors of A corresponding to λ_s, λ_{ss} and λ_u respectively. We orient the eigenvectors of A so that $e_1 \cdot e_{ss} > 0$, $e_2 \cdot e_u > 0$ and $e_3 \cdot e_s > 0$ and the eigenvectors of $-A^T$ so that $e_{ss} \cdot e^*_{ss} > 0$, $e_s \cdot e^*_s > 0$, $e_u \cdot e^*_u > 0$. Since $\zeta(t)$ is bounded it follows that

$$\lim_{t\to\infty} \frac{\zeta(t)}{|\zeta(t)|} = \pm e^*_u$$

$$\lim_{t\to-\infty} \frac{\zeta(t)}{|\zeta(t)|} \in \text{span}\{e^*_s, e^*_{ss}\}$$

We choose $\zeta(t)$ so that $\lim_{t\to\infty} \frac{\zeta(t)}{|\zeta(t)|} = e_u^*$. We will show that

$$\lim_{t\to-\infty} \frac{\zeta(t)}{|\zeta(t)|} = e_s^*. \tag{6.3}$$

In view of the definition of an inclination-flip point given in Section 3 this implies the assertion of the proposition. Let $\zeta_s(t)$ be a solution of (6.1) with the property that

$$\lim_{t\to-\infty} \frac{\zeta_s(t)}{|\zeta_s(t)|} = e_{ss}^*.$$

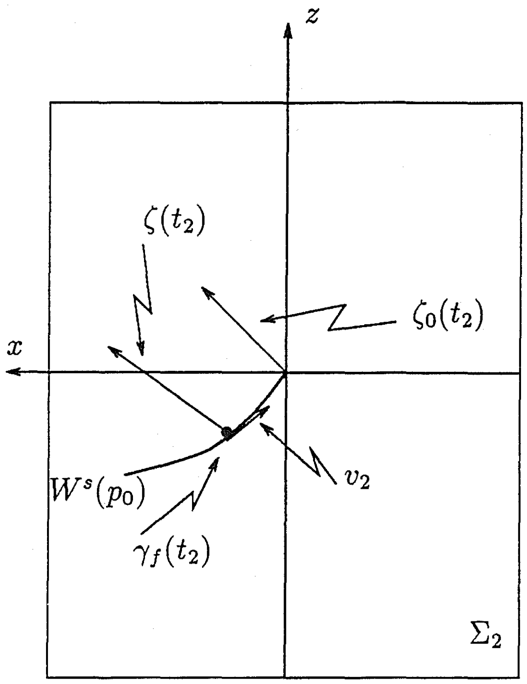
This solution is unique up to multiplication by a constant. Let N_t denote the orthogonal complement of $\dot{\gamma}_f(t)$ at $\gamma_f(t)$. The bundle $\{N_t\}_{t\in\mathbb{R}}$ is invariant for (6.1), i.e. if $v(t)$ is a solution of (6.1) and $v(t_0) \in N_{t_0}$ then $v(t) \in N_t$ for all $t \in \mathbb{R}$. Moreover $\zeta(t)$ and $\zeta_s(t)$ are contained in N_t. The line with direction vector $\zeta_s(t)$, $t \in \mathbb{R}$, divides N_t into two halfplanes $N_t^\pm$ which are invariant in the following sense: if $v(t)$ is a solution of (6.1) and $v(t_0) \in N_{t_0}^\pm$ for some $t_0 \in \mathbb{R}$ then $v(t) \in N_t^\pm$ for all $t \in \mathbb{R}$. The sign $\pm$ in the definition of $N_t^\pm$ is chosen in such a way that if $v(t_0) \in N_{t_0}^\pm$ then

$$\lim_{t\to-\infty} \frac{v(t)}{|v(t)|} = \pm e_s^*.$$

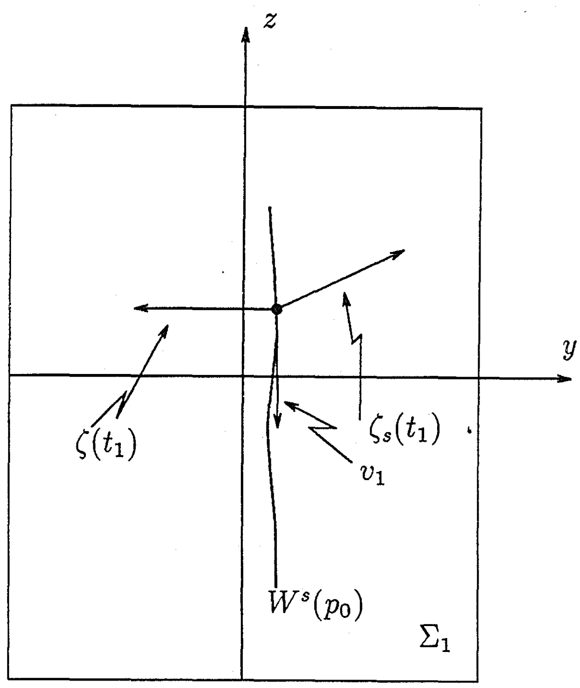
Note that N_t^+ consists of the vectors v such that the angle from $\zeta_s(t)$ to $v(t)$ measured counterclockwise is less than π.

Let $(x,\, y,\, z)$ be the coordinates introduced in Section 4. Let t_j, $j = 1,2$ be such that $\gamma_f(t_j) \in \Sigma_j$. We will show that $\zeta(t_1)$ is close to a positive multiple of the vector $(0,-1,0)$. First observe that the Exchange Lemma (Theorem 2) or Theorem 4 applied in backward time imply that near $\gamma(t_2)$ the manifold $W^s(p_0)$ is C^1 $\mathcal{O}(e^{-\frac{\mathrm{const}}{\delta}})$ close to the plane $\{y = 0\}$. Hence $\frac{\zeta(t_1)}{|\zeta(t_1)|}$ is $\mathcal{O}(e^{-\frac{\mathrm{const}}{\delta}})$ close to the vector $(0,\pm1,0)$. We need to determine the orientation of $\zeta(t)$ relative to the y-axis. We proceed as follows. For each $t \in [t_1, t_2]$ we define a section $\Sigma(t)$ in such a way that

- $\Sigma(t_j) = \Sigma_j$, $j = 1,2$,
- $\Sigma(t)$ intersects Γ_f transversely at $\gamma_f(t)$,
- $\{\Sigma(t)\}_{t\in[t_1,t_2]}$ is a continuous family of planes.



(a) Position of $W^s(p_0) \cap \Sigma_2$ and $\zeta(t_2)$.



(b) Position of $\zeta(t_1)$ and $\zeta_s(t_1)$.

Figure 6: The behavior of $\zeta(t)$ and $\zeta_s(t)$ in the sections Σ_1 and Σ_2.

We now define a family of vectors tangent to $W^s(p_0) \cap \Sigma(t)$. Let $v_2 \in T_{\gamma_f(t_2)} W^s(p_0) \cap \Sigma_2$. Let Π_t be the first hit map from $\Sigma(t)$ to Σ_2 and let $v(t) = \Pi_t^{-1}(v_2)$. The vectors $\{v(t), \dot{\gamma}_f(t)\}_{t \in [t_1, t_2]}$ define an orientation of the piece of $W^s(p_0)$ bounded by Σ_1 and Σ_2. The vector $\zeta(t)$ also defines an orientation of this manifold. It follows that the frames $\{(\zeta(t_j), v(t_j), \dot{\gamma}_f(t_j))\}_{j=1,2}$ have the same orientation. Hence the direction of $\zeta(t_1)$ can be deduced from the position of $\zeta(t_2)$ and $v(t_j)$, $j = 1, 2$.

Recall that for $\delta = 0$ there is a connection $\Gamma^l = \{\gamma^l(t) : t \in \mathbb{R}\}$ from an equilibrium $(0, 0, z^*) \in S_R$ to an equilibrium in S_L. There exists $T_0 \gg t_2$ such that for $t \in [t_2, T_0]$ the vector $\zeta(t)$ is close to a positive multiple of the vector $\zeta_0(t) = (\dot{\gamma}^{l2}(t), -\dot{\gamma}^{l1}(t), \gamma^{l1}(t))$ which is a bounded solution of the adjoint equation around Γ^l with the property that $\lim_{t \to \infty} \zeta_0(t) = (0, 0, 0)$. When $t \geq T_0$ the vector $\zeta(t)$ gradually becomes alined with e_u^*. Note that $\zeta_0(t_2) \in \Sigma_2$, which is not the case for $\zeta(t)$. The Melnikov analysis for $\delta = 0$ implies that $\zeta(t_2)$ is as shown in Fig. 6a.

We now analyze the action of Π^{-1} $(= \Pi_t^{-1})$ on v_2. The passage time from Σ_1 to Σ_2 for Γ_f equals $\frac{\omega(\delta)}{\delta}$ for some function $\omega(\delta)$, see (4.20). Recall that for (c, δ)

corresponding to a fast wave $c = \mathcal{O}(\delta)$ and $W^s(p_0) \cap \Sigma_2$ has the form

$$W^s(p_0) \cap \Sigma_2 = (-\mu_4(z - z^*), \Delta, z) + \mathcal{O}(\delta).$$

We parametrize $W^s(p_0) \cap \Sigma_2$ as follows.

$$W^s(p_0) \cap \Sigma_2 = \{(a_2 e^{-\alpha_s^* \frac{\omega}{\delta}}, \Delta, h(e^{-\alpha_s^* \frac{\omega}{\delta}})) \,:\, \omega \in [-\eta + \omega(\delta), \eta + \omega(\delta)]\},$$

where $\eta > 0$ is a constant and $h(z) = -\frac{1}{\mu_4(0)} z + \mathcal{O}(z^2)$. Recall that Π is determined by the solutions of (1.2) given in Theorem 4. We have

$$\Pi^{-1}(W^s(p_0) \cap \Sigma_2) = \{(\Delta, 0, -\omega) + \mathcal{O}(e^{-\alpha^* \frac{\omega}{\delta}}) \,:\, \omega \in [-\eta + \omega(\delta), \eta + \omega(\delta)]\}.$$

We choose v_2 as follows:

$$v_2 = v(t_2) = \frac{d}{d\omega}(a_2 e^{-\alpha_s^* \frac{\omega}{\delta}}, \Delta, h(e^{-\alpha_s^* \frac{\omega}{\delta}}))|_{\omega=0},$$

see Fig. 6a. Define $v_1 = v(t_1)$. Clearly v_1 is $\mathcal{O}(e^{-\frac{const}{\delta}})$ close to the vector $(0, 0, -1)$. Moreover $\dot{\gamma}_f(t_j)$, $j = 1, 2$ are $\mathcal{O}(\delta)$ close to positive multiples of $(-1, 0, 0)$ and $(0, 1, 0)$ respectively. It follows that $\zeta(t_1)$ is $\mathcal{O}(\delta)$ close to a positive multiple of $(0, -1, 0)$.

We now analyze the behaviour of $\zeta_s(t)$, $t \in [-\infty, t_1]$. Recall that for $\delta = 0$ there is a connecting orbit $\Gamma^r = \{\gamma^r(t) \,:\, t \in \mathbb{R}\}$ from p_0 to p_1. Let $\zeta_{s0}(t) = (\dot{\gamma}^{r2}(t), -\dot{\gamma}^{r1}(t), \gamma^{r1}(t))$. Note that $\zeta_{s0}(t)$ is a bounded solution of the adjoint equation around Γ^r normal to $W^s(S_L)$ and that $\frac{\zeta_s(t)}{|\zeta_s(t)|}$ remains close to $\frac{\zeta_{s0}(t)}{|\zeta_{s0}(t)|}$ for $t \leq t_2$. The Melnikov computations done in Section 4, namely the fact that $\eta_4 < 0$, and the closeness of ζ_s to ζ_{0s} imply that $\zeta_s(t_1)$ is as shown in Fig. 6b. In particular the angle from $\zeta_s(t_1)$ to $(0, -1, 0)$ measured in the counter clockwise direction is less than π. By continuity the same holds for the angle between $\zeta_s(t_1)$ and $\zeta(t_1)$, implying $\zeta(t_1) \in N_{t_1}^+$. □

Proof of Proposition 6 Fix $0 < a < \frac{1}{2}$. Let Γ_s be the homoclinic orbit corresponding to the slow wave. Recall that Γ_s is close to a homoclinic orbit Γ_{s0} existing for $c = \delta = 0$. Recall also that for $c = \delta = 0$ the system (1.2) is hamiltonian and Γ_{s0} is planar, i.e. $\Gamma_{s0} = \{(\gamma^1(t), \gamma^2(t), 0) \,:\, t \in \mathbb{R}\}$. An explicit computation shows that $\zeta_0(t) = (\dot{\gamma}^2(t), -\dot{\gamma}^1(t), \gamma_1(t))$ is a bounded solution of the adjoint equation around Γ_{s0} satisfying the condition $\lim_{t \to \pm\infty} \zeta_0(t) = 0$. Note that $\lim_{t \to -\infty} \dfrac{\zeta_0(t)}{|\zeta_0(t)|} = e_{ss}^*$. Consider the adjoint equation around Γ_s

$$\dot{\psi} = -DF(\gamma_s(t))^T \psi \tag{6.4}$$

and let $\zeta(t)$ be a non-zero bounded solution of (6.4) ($\zeta(t)$ is unique up to multiplication by a constant). We will show that for some $a \in (0, \frac{1}{2})$, $\gamma > 0$ c, δ positive and small

$$\lim_{t\to-\infty} \frac{\zeta(t)}{|\zeta(t)|} = -e_s^*, \tag{6.5}$$

which implies that the corresponding Γ_s is twisted.

Consider a solution of (6.4) $\zeta_s(t)$ with the property that

$$\lim_{t\to-\infty} \frac{\zeta_s(t)}{|\zeta_s(t)|} = e_{ss}^*.$$

This solution is unique up to multiplication by a constant. Consider the plane bundle N_t and the halfplanes $N_t^\pm$ defined for Γ_s analogously as for Γ_f. We will show that for suitable values of a, c and δ $\zeta(0) \in N_0^-$.

Note that $\zeta_0(0) = (\dot\gamma^2(0), 0, \gamma^1(0))$ with $\gamma^1(0) > 0$ and $\dot\gamma_2(0) < 0$. Given the form of $\zeta_0(t)$ and Γ_{s0} it is easy to see that $\zeta(0) \in N_0^+$ if $\zeta^1(0) > \zeta_s^1(0)$. Consider the quantity dist $= \zeta^1(0) - \zeta_s^1(0)$. Note that the sign of $\frac{d}{dc}\text{dist}|_{c=0}$ determines whether $\zeta(0)$ is in N_0^+ or in N_0^-. The equation (6.4) has the form

$$\dot\psi = \begin{pmatrix} 0 & f'(\gamma^1(t) & -\delta \\ -1 & -c & 0 \\ 0 & -1 & \delta\gamma \end{pmatrix} \psi \tag{6.6}$$

The functions $\zeta'(t) = \frac{d}{dc}\zeta(t)|_{c=0}$ and $\zeta_s'(t) = \frac{d}{dc}\zeta(t)|_{c=0}$ satisfy the equation

$$\dot\psi' = \begin{pmatrix} 0 & f'(\gamma^1(t) & -\delta \\ -1 & -c & 0 \\ 0 & -1 & \frac{\epsilon}{c}\gamma \end{pmatrix} \psi' + \zeta_0^2(t) \begin{pmatrix} f''(\gamma^1(t))\gamma^{1\prime}(t) \\ -1 \\ 0 \end{pmatrix} + \zeta_0^3(t) g(a) \begin{pmatrix} -1 \\ 0 \\ \gamma \end{pmatrix} \tag{6.7}$$

for $t \geq 0$ and $t \leq 0$ respectively. The function g is defined as $g(a) = \frac{d}{dc}(\delta)\Big|_{c=0}$. It follows from Melnikov analysis that $g(a) = -\frac{\hat{M}_\delta}{\hat{M}_c}$, where

$$\hat{M}_c = \int_{-\infty}^{\infty} (\gamma^2(t))^2 dt$$
$$\hat{M}_\delta = -\int_{-\infty}^{\infty} (\gamma^1(t))^2 dt. \tag{6.8}$$

Hence $g(a) > 0$.

We now express $\zeta'(t)$ and $\zeta_s'(t)$ using the variation of constants formula. Let $\xi_0(t)$ be a solution of (6.4) independent of ζ_0 and the constant solution identically equal to e_s^*. Consider the projections $P_{\zeta_0}(t)$, $P_{\xi_0}(t)$, $P_{e_s^*}(t)$ onto $\zeta_0(t)$, $\xi_0(t)$ and e_s^* such that the kernel of each projection is the sum of the ranges of the other two. Let $\Phi(t,s)$ denote the transition matrix of (6.6) for $c = \delta = 0$ and let

$$h(s) = \zeta_0^2(t)\begin{pmatrix} f''(\gamma^1(t))\gamma^{1'}(t) \\ -1 \\ 0 \end{pmatrix} + \zeta_0^3(t)g(a)\begin{pmatrix} -1 \\ 0 \\ \gamma \end{pmatrix}.$$

Using the variation of constants formula and the $t \to \pm\infty$ asymptotics of $\zeta(t)$ and $\zeta_s(t)$ we obtain the expansions

$$\begin{aligned} \zeta'(t) &= \alpha\zeta_0(t) + \int_0^t \Phi(t,s)P_{\zeta_0}(s)h(s)ds + \\ &\quad \int_\infty^t \Phi(t,s)(P_{\xi_0}(s) + P_{e_s^*}(s))h(s)ds, \quad t \geq 0 \\ \zeta_s'(t) &= \alpha_s\zeta_0(t) + \int_0^t \Phi(t,s)P_{\zeta_0}(s)h(s)ds + \\ &\quad \int_{-\infty}^t \Phi(t,s)(P_{\xi_0}(s) + P_{e_s^*}(s))h(s)ds, \quad t \leq 0 \end{aligned} \tag{6.9}$$

In particular

$$\begin{aligned} \zeta'(0) &= \alpha\zeta_0(0) + \int_\infty^0 \Phi(0,s)(P_{\xi_0}(s) + P_{e_s^*}(s))h(s)ds \\ \zeta_s'(0) &= \alpha\zeta_0(0) + \int_{-\infty}^0 \Phi(0,s)(P_{\xi_0}(s) + P_{e_s^*}(s))h(s)ds. \end{aligned}$$

We require that $|\zeta(0)| = \text{const.}$ and $|\zeta_s(0)| = \text{const.}$ This implies that $\zeta_s'(0)\cdot\zeta_0(0) = \zeta'(0)\cdot\zeta_0(0) = 0$. The last two equations determine α and α_s. When $|\gamma|$ is large

$$h(t) \approx \gamma\zeta_0^3(t)g(a)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $I(a) = g(a)\int_0^\infty \zeta_0^3(t)dt$. Since $\zeta_0^3(t) > 0$ for all $t \in \mathbb{R}$ and $\zeta_0^3(-t) = -\zeta_0^3(t)$ it follows that $I(a) > 0$ and $I(a) = g(a)\int_{-\infty}^0 \zeta_0^3(t)dt$. We obtain

$$\begin{aligned} \alpha|\zeta_0(0)|^2 &\approx \gamma I(a)\zeta_0^3(0) \\ \alpha_s|\zeta_0(0)|^2 &\approx -\gamma I(a)\zeta_0^3(0). \end{aligned}$$

It follows that

$$\zeta'(0) \approx \gamma I(a)\left(\frac{\zeta_0^3(0)\zeta_0^1(0)}{|\zeta_0(0)|^2}, 0, -1+\frac{\zeta_0^3(0)^2}{|\zeta_0(0)|^2}\right)$$
$$= \quad \gamma I(a)\frac{1}{|\zeta_0(0)|^2}(\zeta_0^3(0)\zeta_0^1(0), 0, -\zeta_0^1(0)^2).$$

Similarily

$$\zeta_s'(0) \approx \gamma I(a)\frac{1}{|\zeta_0(0)|^2}(-\zeta_0^3(0)\zeta_0^1(0), 0, \zeta_0^1(0)^2).$$

Since $\zeta^1(0) < 0$ it follows that $\frac{d}{dc}\mathrm{dist}|_{c=0} < 0$ when $\gamma \gg 0$. The proposition follows. □

7 Conclusions

In this article we analyzed the problem of the existence of 1-homoclinic orbits of the equation (1.2) near the singular orbit Γ_0 for $(\delta, c, a) \approx (0, 0, \frac{1}{2})$. In particular we proved the conjecture of Yanagida for parameter values in this region. Additionally we proved the existence of inclination-flip points for the equation (1.2), thus providing evidence for the existence of travelling waves of the FitzHugh-Nagumo equation with an arbitrary number of pulses.

Typically the existence of an inclination-flip point in the case of the eigenvalue configuration occuring for (1.2) implies the existence of n-homoclinic orbits [11]. To establish if such solutions really do occur one must check if a certain global coefficient does not vanish. This is difficult to achieve for (1.2) since the precise location of the inclination flip point it not known. Moreover, if the n-pulses exist, they are likely to be unstable as solutions of the FitzHugh-Nagumo equation. A natural question is whether complicated dynamics, in particular multiple pulse solutions, can be found in the unfolding of the singularity at $(\delta, c, a) = (0, 0, \frac{1}{2})$.

In order to investigate the existence of homoclinic orbits we constructed Shilnikov coordinates near the slow manifold S_R (Theorem 4). Our approach leads to a good understanding of the flow near a slow manifold in a similar manner as the Exchange Lemma. Additionally Theorem 4 provides means for a bifurcation analysis, since it gives an explicit expression of the flow near a slow manifold at lowest order. This lowest order approximation is given by the flow of a vector field constant in the center directions and linear in the hyperbolic directions, see Remark 2.

An assumption in Theorem 4 is that the fast stable and the fast unstable variables are one dimensional. Using the methods of this article an extension of to the case of the fast stable and fast unstable variables having arbitrary finite dimensions could be proved under the assumption that the principal eigenvalues of A^s and A^u along the relevant portion of the slow manifold remain simple. An interesting question is whether yet further extensions are possible, in particular to a generic vector field of the form (3.5) with the variables x and z having arbitrary finite dimensions. Another way of generalizing Theorem 4 would be to develop a version of the method of Lin [16], [20] suited for the singular perturbation setting.

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