Multilevel preconditioning on the refined interface and optimal boundary solvers for the Laplace equation

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May 1995

1991 Mathematics Subject Classification. 65N20, 65N30, 65P10.

Keywords. Boundary integral equations, domain decomposition, fast elliptic problem solvers, interface operators, matrix compression, multilevel preconditioning.

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†This work was supported in part by the DFG research grant 436 RUS 17/34/95 while B.N. Khoromskij was visiting the WIAS in Berlin.

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Contents

Abstract

In this paper we propose and analyze some strategies to construct asymptotically optimal algorithms for solving boundary reductions of the Laplace equation in the interior and exterior of a polygon. The interior Dirichlet or Neumann problems are, in fact, equivalent to a direct treatment of the Dirichlet-Neumann mapping or its inverse, i.e., the Poincaré-Steklov (PS) operator. To construct a fast algorithm for the treatment of the discrete PS operator in the case of polygons composed of rectangles and regular right triangles, we apply the Bramble-Pasciak-Xu (BPX) multilevel preconditioner to the equivalent interface problem in the $H^{1/2}$-setting. Furthermore, a fast matrix-vector multiplication algorithm is based on the frequency cutting techniques applied to the local Schur complements associated with the rectangular substructures specifying the nonmatching decomposition of a given polygon. The proposed compression scheme to compute the action of the discrete interior PS operator is shown to have a complexity of the order $O(N \log^q N)$, $q \in [2, 3]$ with memory needs of $O(N \log^2 N)$ where N is the number of degrees of freedom on the polygonal boundary under consideration. In the case of exterior problems we propose a modification of the standard direct BEM whose implementation is reduced to the wavelet approximation applied to either single layer or hypersingular harmonic potentials and, in addition, to the matrix-vector multiplication for the discrete interior PS operator.

1 Introduction

When numerically solving boundary or interface reductions of elliptic boundary value problems matrix compressions and preconditioning appear to be of main importance to develop efficient numerical techniques. We refer to [3, 5],[12]-[15],[22, 31, 33] for recent results on wavelet approximation in boundary element methods (BEM) which yields asymptotically optimal algorithms. Another efficient matrix compression technique in BEM based on the idea of panel clustering has been proposed in [19]. Optimal multilevel algorithms for FE discretizations of elliptic differential equations have been developed in [8, 11, 17, 29, 30, 39, 40].

We note that the multilevel methods based on BPX-type schemes [8] give rise to efficient spectrally equivalent preconditioners for a wide class of boundary/interface operators in both $H^{1/2}$- and $H^{-1/2}$-settings [30]. In this way, optimal preconditioners are implicitly incorporated into wavelet based compression schemes in BEM since the latter inherit the stability of prewavelet splittings. So far the main attention has been paid for the development of asymptotically optimal methods to solve the classical boundary integral equations with operators of the orders -1, 0 and 1.

The main topic of the present paper is the construction of efficient matrix compression and preconditioning techniques for the harmonic Poincaré-Steklov (PS) interface operator of the order 1 and for its inverse. This results in asymptotically optimal algorithms for solving boundary reductions of the Laplace equation in the interior or exterior of a polygon. Note that the proposed matrix compression technique is designed by special geometrical domain decompositions without global transformation of the original nodal basis on the boundary. Thus, a multilevel preconditioning and fast matrix-vector multiplication do a job independently (cf. the case of wavelet approximation in BEM) and they both are of a crucial importance to construct some optimal algorithm.

The *interior* Dirichlet or Neumann problems are equivalent to a direct treatment of the Dirichlet-Neumann mapping or its inverse, i.e., the PS operator. To compute the action of an interior PS operator in the case of polygons composed of rectangles and regular right triangles we propose the multilevel BPX-type scheme applied to an equivalent interface reduction associated with a nonmatching decomposition of a polygon into rectangular subdomains introduced in [23]. The underlying scheme is analyzed in the general framework of the additive Schwarz method [17] based on the stable multilevel splitting [30] of the trace space. A fast matrix-vector multiplication with arising interface operator (assembled Schur complement) is based on the frequency cutting technique applied to the local Schur complements associated with rectangular substructures specifying a skeleton. Since the resultant multilevel additive Schwarz operator gives rise to a uniformly bounded condition number we arrive at an asymptotically optimal compression scheme of the complexity $O(N \log^q N)$, $q \in [2, 3]$ with memory needs of $O(N \log^2 N)$ where N is the number of degrees of freedom on the boundary under consideration. Note that, in general, it is not possible to apply directly the wavelet approximation to the PS operators since we have no longer their explicit representation in terms of boundary integral operators.

In the case of *exterior* problems we propose a modification of the standard direct BEM whose implementation is reduced to the wavelet approximation applied to either the single layer or the hypersingular harmonic potential and, in addition, to the matrix-vector multiplication with the discrete interior PS operator as above.

Observe that direct formulations in BEM provide some explicit implementation of the Dirichlet-Neumann mapping or its inverse while the indirect (ansatz) methods operate with the integral equations over some artificial boundary potentials. We notice that in many applications the computation of a full set of Cauchy data (which usually have a physical sense) has an independent significance. In this concern we emphasize that any direct BEM as well as formulations involving some PS operators will do the job. However, the direct BEM equations always involve a pair of boundary integral operators of different kind and, thus, for each of them an appropriate wavelet based compression technique is supposed to be applied. To reduce the complexity of a direct BEM for an exterior problem we substitute in the corresponding boundary integral equation a symmetric factorization of the double layer potential operator and arrive at some equivalent equation involving only *one* symmetric and positive definite (SPD) integral operator, namely, the single layer potential V or the hypersingular operator D and the Poincaré-Steklov mapping related to the interior problem. For the approximation of the operator V or D one can apply the wavelet techniques developed in [3, 5, 12, 13, 14, 31, 32] yielding the complexity $O(N \log^q N)$, $q \in [1, 2]$. Furthermore, for a fast treatment of the interior PS operators on a polygonal boundary an asymptotically optimal multilevel BPX-type scheme developed in Section 5 may be applied.

Note that usually BEM have a certain advantage for exterior problems while FEM seem to be superior in case of bounded domains. The proposed "combined" direct formulation for the exterior problems includes one matrix-vector multiplication related to an interior PS operator and the inversion of some SPD boundary integral operator. This allows to "distribute" the complexity between exterior and interior solvers.

To solve a boundary integral equation of the second kind we introduce the Bubnov-Galerkin scheme where a new inner product may be realized with asymptotically optimal costs. The L^2-stability property of a discrete resolution operator and the quasi-optimal error estimates follow from the positive definitness of the double layer potential $E \pm K$ in a new setting. This result remains valid even for 3D Lipschitz polyhedra. We remark that the standard Galerkin methods (with respect to L^2-inner product) applied to the boundary integral equations with a resolution operator of the second kind have an advantage if the double layer potential operator admits *a priori* more efficient wavelet approximation in comparison with related first kind operators. This is the case for the biharmonic BEM [34, 26] where the double layer potential operator for the bi-Laplacian is given in terms of the Calderon projections for the Laplacian.

Though we consider here the model problem of the Laplace equation on the plane the proposed approach for fast computations with discrete PS operators may be extended to more general classes of variational elliptic BVPs, say to 3D problems and to biharmonic problems in domains with polygonal boundaries.

The remainder of the paper is organized as follows. In Section 2 we overview the boundary reductions of model BVPs for the Laplace equation based on direct and ansatz formulations. Moreover, the mapping properties of the harmonic layer potentials, the related Poincaré-Steklov operators and the operators V^2 and D^2 are collected. The transformation of the direct BEM equations for the exterior problem to a form including the only SPD operators is proposed. In Section 3, we discuss the symmetric splittings of the double layer potentials introduced in [23]. Note that similar factorizations have been considered in [28] to investigate the angular singularities of certain boundary integral operators. Section 4 is devoted to the Bubnov-Galerkin approximation of the double layer potential. The quasi-optimal error estimate for the generalized Galerkin scheme with respect to a new inner product generated by the discrete PS operator is obtained. A sharp estimate on the spectrum of the discretized double layer potential operator is given. In Section 5, we prove the uniform boundedness of the condition number for the multilevel BPX scheme on the refined skeleton. The underlying interface operator is given by the direct sum of FE approximations to the local PS operators (with the Schur complement as a stiffness matrix) associated with nonmatching decomposition of a given polygon by rectangular substructures. This leads to an efficient matrix compression technique for the discrete 'interior' PS operators in the case of polygons composed of rectangles and regular right triangles. Thus, an extension of the spectral like method from particular rectangular-type geometries to triangular and polygonal ones requires now the nonmatching domain decomposition by rectangular substructures and the multilevel BPX scheme on a related refined skeleton. In Section 6, a quasi-optimal estimate for the computing complexity of the proposed method is given. Furthermore, the results of numerical examples manifesting an asymptotically optimal performance of the proposed algorithm are provided. We conclude in Section 7 with a brief discussion of spectrally equivalent preconditioners for Galerkin approximations of the operators V and D related to some arbitrarily unstructured meshes.

2 Preliminaries

Let $\Omega_1 \subset R^2$ be a polygonal domain on the plane with the boundary $\Gamma = \cup_{j=1}^{N_0} \Gamma_j$, where Γ_j is an open edge and $\omega_j \in (0, 2\pi)$, $j = 1, \ldots, N_0$ is the interior angle at $s_j = \overline{\Gamma}_j \cap \overline{\Gamma}_{j+1}$. The exterior domain $R^2 \backslash \overline{\Omega}_1$ will be denoted by Ω_2. Let n be the unit outward normal vector on Γ.
By $H^s(\Omega_1)$ and $H^s_{loc}(R^2)$, $s \geq 0$, we denote the usual Sobolev spaces on Ω_1 and R^2, respectively, [27]. With $L^2(\Gamma)$–duality, define the trace spaces on Γ

$$H^s(\Gamma) := \begin{cases} \gamma_0 u : \; u \in H^{s+1/2}_{loc}(R^2) \; , & 0 < s < 3/2 \\ L^2(\Gamma) \; , & s = 0 \\ (H^{-s}(\Gamma))' \; , & s < 0 \; , \end{cases}$$

where the trace operator

$$\gamma_0 : H^{s+1/2}_{loc}(R^2) \to H^s(\Gamma), \quad 0 < s < 3/2$$

is continuous and has a continuous right inverse. We equip the space $H^s(\Gamma)$, $0 \leq s < 3/2$ with the canonical norm. The generalized normal derivative operator

$$\gamma_1 : H^1(\Omega_1, \Delta) \to H^{-1/2}(\Gamma)$$

is continuous (see [9]) and coincides with the operator $\gamma_1 u = \frac{\partial u}{\partial n}_{|\Gamma} = \partial_n u$ for $u \in H^2_{loc}(R^2)$. The space $H^1(\Omega_1, \Delta)$ is equipped with the usual graph norm.

Consider the interior and exterior BVPs for the Laplacian

$$\Delta\overline{u} = 0 \ \text{ in } \ \Omega_i, \quad \overline{u} \in H^1_{loc}(\Omega_i), \ i = 1,2 \tag{2.1}$$

subject to the Dirichlet

$$\gamma_0\overline{u} = u \in H^{1/2}(\Gamma) \ \text{ on } \ \Gamma \tag{2.2}$$

or Neumann

$$\gamma_1\overline{u} = v \in H^{-1/2}(\Gamma) \ \text{ on } \ \Gamma, \quad (v,1)_\Gamma = 0 \tag{2.3}$$

boundary conditions. For the exterior problem in Ω_2 we additionally require the "radiation conditions"

$$\overline{u}(x) = c_\infty + O(\frac{1}{|x|}), \quad |x| \to \infty \ .$$

The Dirichlet problems are uniquely solvable in $H^1_{loc}(\Omega_i)$, $i = 1,2$ for any $u \in H^{1/2}(\Gamma)$ while the Neumann problems have unique solution up to an arbitrary constant. Introduce the "interior" Dirichlet-Neumann mapping

$$\mathcal{T}_1 := \gamma_1 M_1 : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma) \tag{2.4}$$

which is known to be continuous for $s \in [-1/2, 1/2]$ (see [9]) where

$$M_1 u := \overline{u} : H^{1/2}(\Gamma) \to H^1(\Omega_1, \Delta)$$

is the (continuous) solution operator related to the interior Dirichlet problem (2.1), (2.2) in a weak formulation. The "exterior" Dirichlet-Neumann mapping $\mathcal{T}_2$ may be introduced along the same line. Let

$$g(x,y) := -\frac{1}{2\pi}\log|x-y|, \qquad x, y \in R^2$$

be the fundamental solution of the Laplacian. Define boundary integral operators V, K, K' and D on Γ by

$$\begin{aligned} Vu(x) &= \int_\Gamma g(x,y)u(y)dy, & Ku(x) &= \int_\Gamma \frac{\partial}{\partial n_y} g(x,y)u(y)dy, \\ K'u(x) &= \int_\Gamma \frac{\partial}{\partial n_x} g(x,y)u(y)dy, & Du(x) &= -\int_\Gamma \frac{\partial}{\partial n_x}\frac{\partial}{\partial n_y} g(x,y)u(y)dy. \end{aligned} \tag{2.5}$$

When dealing with the operator V we further assume that $cap\Gamma \neq 1$ which is valid, in particular, under the condition $diam(\Omega_1) < 1$, see [21]. The operator $\mathcal{T}_i$, $i = 1,2$ has the pseudoinverse S_i, i.e. $\mathcal{T}_i S_i \mathcal{T}_i = \mathcal{T}_i$, which is called the Poincaré-Steklov operator. One can give an explicit form of the interior and exterior Poincaré-Steklov operators by

$$\begin{aligned} S_1 &:= (\tfrac{1}{2}E + K)^{-1}V : \quad H_1^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\,, \\ S_2 &:= (\tfrac{1}{2}E - K)^{-1}V : \quad H_1^{-1/2}(\Gamma) \to H^{1/2}_{g_0}(\Gamma)\,, \quad for \quad c_\infty = 0 \end{aligned}$$

where for $s \in [-1,1]$

$$H^s_f(\Gamma) := \{u \in H^s(\Gamma) : \ (u,f)_{L^2} = 0\}, \quad f \in (H^s(\Gamma))' \ .$$

Here $g_0 \in H^{-1/2}(\Gamma)$ is the Robin potential defined as the eigen-solution $\frac{1}{2}g_0 + K'g_0 = 0$. In particular, the operator $\mathcal{T}_1$ defined by (2.4) has the explicit representation

$$\mathcal{T}_1 := \gamma_1 M_1 = V^{-1}(\frac{1}{2}E + K) \quad \text{and} \quad Ker\mathcal{T}_1 = span\{1\}\,. \tag{2.6}$$

Following [9, 2], we summarize the mapping properties of above introduced operators.

Lemma 2.1 *For all $s \in [-1/2, 1/2]$, the following operators are continuous*

$$\begin{array}{ll} V : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), & \\ K : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma), & K' : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \\ D : H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma), & Ker D = span\{1\}, \\ S_1 : H_1^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), & \\ \mathcal{T}_1 : H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma), & Ker \mathcal{T}_1 = span\{1\}. \end{array}$$

The operators V, D, S_1 and $\mathcal{T}_1$ are symmetric. Moreover, D and $\mathcal{T}_1$ are positive definite on $H_1^{1/2}(\Gamma)$ while V and S_1 are positive definite on $H_1^{-1/2}(\Gamma)$. The relations $\frac{1}{2} + K \cdot 1 = 0$ and $\frac{1}{2} g_0 + K' g_0 = 0$ hold.

As a consequence of the above lemma one can easily derive the mapping properties of the operators V^2 and D^2.

Lemma 2.2 *The operators V^2 and D^2 are both continuous*

$$\begin{array}{lll} V^2 : H^{-1}(\Gamma) & \to & H^1(\Gamma)\,, \\ D^2 : H^1(\Gamma) & \to & H^{-1}(\Gamma) \end{array}$$

and SPD on $H^{-1}(\Gamma)$ and $H_1^1(\Gamma)$, respectively. The estimates

$$(V^{-2}u, u) \cong \|u\|^2_{H^1(\Gamma)}, \quad \forall u \in H^1(\Gamma)$$

$$(D^2 u, u) \cong (u', u')_{L^2(\Gamma)}, \quad \forall u \in H^1(\Gamma)$$

hold where $u' = \frac{d}{ds}u$, $s \in \Gamma$, and $(\cdot, \cdot)$ is the L^2-inner product.

Proof. Consider the operator V^2. Applying Lemma 2.1 successively with $s = -1/2$ and $s = 1/2$ one obtains the continuity of V^2. Under the condition $diam \Omega_1 < 1$ both the operators $V : H^{-1}(\Gamma) \to L^2(\Gamma)$ and $V^{-1} : H^1(\Gamma) \to L^2(\Gamma)$ are bijective mappings [9] yielding

$$(V^{2p}u, u) = \|V^p u\|^2_{L^2(\Gamma)} \cong \|u\|^2_{H^{-p}(\Gamma)}, \quad p = 1, -1\,.$$

Thus the assertions for V^2 follow. The same holds for the operator D taking into account that $Ker\, D = span\{1\}$. ■

Introduce the natural SPD boundary reductions of the *interior Dirichlet problem*

$$given \;\; u \in H^{1/2}(\Gamma), \;\; find \quad v = \gamma_1 M_1 u = \mathcal{T}_1 u \in H^{-1/2}(\Gamma) \tag{2.7}$$

and of the *interior Neumann problem*

$$given \;\; v \in H_1^{-1/2}(\Gamma), \;\; find \quad u = S_1 v \in H_1^{1/2}(\Gamma). \tag{2.8}$$

Their fast resolution for polygonal boundaries is based on asymptotically optimal algorithms for computations with the interior PS operators S_1 and $\mathcal{T}_1$ related to a right triangular domain proposed in [23]. In Section 5 we develop an alternative approach (with optimal costs) based on multilevel preconditioning on the refined interface which appears to be well suited for both serial and parallel implementation.

In the case of exterior problems we consider boundary integral equations corresponding

to the direct symmetric formulations.
Exterior Dirichlet problem: given $u \in H^{1/2}(\Gamma)$,

$$\textit{find} \quad v = \partial_n u \in H^{-1/2}(\Gamma) \;\; \textit{with} \quad Vv = (-\frac{1}{2}E + K)u \quad \textit{on} \;\; \Gamma. \tag{2.9}$$

Exterior Neumann problem: given $v = \partial_n u \in H^{-1/2}(\Gamma)$

$$\textit{find} \;\; u \in H_1^{1/2}(\Gamma) \;\; \textit{with} \;\; Du = -(\frac{1}{2}E + K')v \quad \textit{on} \quad \Gamma. \tag{2.10}$$

Of course, the equations (2.9), (2.10) may be also regarded as the second kind boundary integral equations with respect to u and v, correspondingly.
We reformulate the equations (2.9) and (2.10) of the direct method in a form which involves the SPD operators $\mathcal{T}_1$ and S_1 instead of the double layer potentials. To that end, substitute the representation $\frac{1}{2}E + K = V\,\mathcal{T}_1$ on $L^2(\Gamma)$ and $\frac{1}{2}E - K' = D\,S_1$ on $H_1^{-1/2}(\Gamma)$ (see Corollary 3.1) into the right hand sides of (2.9) and (2.10), respectively, and obtain the equivalent equations

$$V(v + \mathcal{T}_1 u) = -u \quad , \tag{2.11}$$

$$D(u - S_1 v) = -v \; . \tag{2.12}$$

For the solution of (2.11) and (2.12) one can apply the wavelet techniques for the inversion of V and D combined with fast computations of the terms $\mathcal{T}_1 u$ and $S_1 v$, respectively, involving the interior PS operator, see Section 5.
Alternatively, the indirect formulations which involve only one boundary integral operator but contain an artificial potential may be applied. Some disadvantages of such an approach may be expected in the framework of coupled FEM–BEM methods. Besides, the computation of unknown Cauchy data on the boundary needs some additional boundary integral operator to be applied as well. More specifically, the double layer ansatz

$$U(x) = \int_\Gamma \frac{\partial}{\partial n_y} g(x,y)\psi(y)dy, \quad \psi \in H^{1/2}(\Gamma) \tag{2.13}$$

leads to the equations

$$\begin{aligned} &(\tfrac{1}{2}E - K)\psi = \gamma_0 u \; , \\ &D\,\psi = \partial_n u \end{aligned} \tag{2.14}$$

on Γ for the Dirichlet and Neumann problems, respectively. On the other hand, the single layer ansatz

$$U(x) = \int_\Gamma g(x,y)\sigma(y)dy, \qquad \sigma \in H^{-1/2}(\Gamma) \tag{2.15}$$

yields (for the same problems) the equations

$$\begin{aligned} &V\,\sigma \;=\; \gamma_0 u \; , \\ &(\tfrac{1}{2}E + K')\sigma = \partial_n u \; . \end{aligned} \tag{2.16}$$

To estimate an expected computing complexity of the above formulations (2.7)-(2.16) we indicate that the interior problems (2.7), (2.8) involve only the SPD Poincaré-Steklov operators admitting a FE approximation of the complexity $O(N \log^q N)$, $q \in$

[1, 3] up to a discretization error [23, 25], see also Section 5. For the exterior problems in the form (2.9)-(2.10) one should take care of a matrix compression procedure for the double layer potential operator as well as for one of the operators V or D. Since we deal with the operators of the orders 0, -1 and 1 the corresponding asymptotically optimal algorithms based on the wavelet approximation are fashioned by rather different ways, see [12, 13, 31, 32, 33]. Due to the above arguments, the "combined" formulations (2.11) and (2.12) look as less time consuming since in that cases the wavelet based compression techniques should be applied to either the operator V or D only.
Note that the problem of L^2-stability for the discretized operator $\frac{1}{2}E \pm K$ on a Lipschitz boundary (in 2D and 3D cases) may be addressed to the formulations (2.14) and (2.16). In Section 4 we prove some stability results for Bubnov-Galerkin schemes on polygonal boundaries which remain also valid in the case of a 3D Lipschitz polyhedra.

3 On symmetrization of the double layer potential

In this section we consider the integral equations (2.13) and (2.16) of the second kind. Following [23] we apply symmetric factorizations for the operators $\frac{1}{2}E - K$ and $\frac{1}{2}E + K'$ which reduce the corresponding equations to some SPD form with respect to a new inner product admitting an asymptotically optimal implementation.
Introduce the splittings of the spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ into direct sums of two subspaces

$$H^{1/2}(\Gamma) = span\{1\} + H^{1/2}_{g_0}(\Gamma) \,, \tag{3.1}$$

$$H^{-1/2}(\Gamma) = span\{g_0\} + H^{-1/2}_{1}(\Gamma) \,. \tag{3.2}$$

Lemma 3.1 *The splittings (3.1) and (3.2) are invariant with respect to the operators* $\frac{1}{2}E \pm K$ *and* $\frac{1}{2}E \pm K'$*, correspondingly.*

Proof. Consider the operator $\frac{1}{2}E - K$. Let $u \in H^{1/2}_{g_0}(\Gamma)$, then one obtains

$$(\frac{1}{2}u - Ku, g_0) = (u, \frac{1}{2}g_0 - K'g_0) = (u, g_0) = 0 \,,$$

i.e., $(\frac{1}{2}E - K)u \in H^{1/2}_{g_0}(\Gamma)$. The same holds for the other operators under consideration.■

Remark 3.1 *The splittings (3.1) and (3.2) remain valid for* $L^2(\Gamma)$ *with corresponding* L^2*-orthogonal sets.*

Corollary 3.1 *There holds*

$$\begin{array}{lll}
\frac{1}{2}E + K = V\,\mathcal{T}_1 = E - S_1\,D & & on\ L^2(\Gamma) \\
\frac{1}{2}E - K = S_1\,D & & on\ H^0_{g_0}(\Gamma) \subset L^2(\Gamma) \\
\frac{1}{2}E + K' = \mathcal{T}_1\,V & & on\ H^{-1/2}(\Gamma) \\
\frac{1}{2}E - K' = D\,S_1 = E - \mathcal{T}_1\,V & & on\ H^{-1/2}_1(\Gamma) \subset H^{-1/2}(\Gamma).
\end{array}$$

Corollary 3.1 shows that the operators $E \pm K$ and $E \pm K'$ allow the Bubnov-Galerkin approximations with respect to the properly chosen inner products. Such discretizations immediately imply the L^2-stability of the corresponding discrete operators yielding quasi-optimal error estimates in the energy norm.
The following simple lemma gives lower estimates for the corresponding stability constants. Since all arguments are, in fact, dimension-independent the results on the L^2-stability and quasi-optimal error estimates for the Bubnov-Galerkin schemes under consideration remain valid in the case of a 3D Lipschitz polyhedra providing some quasi-regular triangulations. We refer to [18] on L^2-stability results for standard Galerkin and collocation schemes. Define

$$\lambda(\frac{1}{2}E \pm K) := \{\lambda \in R : (\frac{1}{2}E \pm K)u = \lambda u, \text{ with some } u \in H^{1/2}(\Gamma)\}.$$

Lemma 3.2 *Let the estimates*

$$a_0(S_1 u, u) \leq (Vu, u) \leq a_1(S_1 u, u), \qquad \forall u \in H_1^{-1/2}(\Gamma)\ , \tag{3.3}$$

$$b_0(\mathcal{T}_1 u, u) \leq (Du, u) \leq b_1(\mathcal{T}_1 u, u), \qquad \forall u \in H^{1/2}(\Gamma) \tag{3.4}$$

hold with given constants $a_0,\ a_1,\ b_0,\ b_1 > 0$. Then the inequalities

$$\min_{\lambda \neq 0} \lambda(\frac{1}{2}E + K) \ = \ \min_{\lambda \neq 0} \lambda(\frac{1}{2}E + K') \geq a_0\ , \tag{3.5}$$

$$\min \lambda(\frac{1}{2}E - K) \ = \ \min \lambda(\frac{1}{2}E - K') \geq b_0 \tag{3.6}$$

are valid.

Proof. With some $u \in H_1^{1/2}(\Gamma)$, let

$$V\ \mathcal{T}_1 u = \lambda u, \quad \lambda \in R. \tag{3.7}$$

Then $w = \mathcal{T}_1 u \in H_1^{-1/2}(\Gamma)$ due to Lemma 2.1 and thus $Vw = \lambda S_1 w$. The assertion for $\frac{1}{2}E + K$ now follows from Lemma 3.1, from the first assertion of Corollary 3.1 and (3.3). The same holds for the operators $\frac{1}{2}E - K$ and $\frac{1}{2}E \pm K'$. ■

Corollary 3.2 *The estimates (3.5) and (3.6) remain valid on the spaces $L^2(\Gamma)$ and $H^{-1}(\Gamma)$, respectively.*

Proof. Examining the proof of Lemma 3.2 we conclude that (3.7) with $u \in L^2(\Gamma)$ and $(u, 1) = 0$ actually implies $Vw = \lambda S_1 w$ with some $w \in H_1^{-1}(\Gamma)$ (see [10] for corresponding regularity results). Then the assertion follows from Lemma 2.1 and the estimate $\|S_1 w\|_{L^2(\Gamma)} \geq c_1 \|w\|_{H_1^{-1}(\Gamma)}$ since $\mathcal{T}_1\ S_1 w = w$ for all $w \in H_1^{-1}(\Gamma)$. ■

4 Bubnov-Galerkin approximation of the double layer potential

Consider first the operator $\frac{1}{2}E + K' = \mathcal{T}_1 V$ defined on $H^{-1/2}(\Gamma)$. Let $W_h \subset H^1(\Omega_1)$ be the subspace of piecewise linear C^0-finite elements related to the quasi-uniform triangulation $\tau = \{\tau_n\}$ of $\overline{\Omega}_1 = \cup_n \overline{\tau}_n$ with mesh parameter $h > 0$. Denote $X_h = W_{h_{|\Gamma}} \subset H^1(\Gamma)$, $W_{h0} = W_h \cap H_0^1(\Omega_1)$ and let X_h' be the space of piecewise constant functions with respect to $\Gamma_h = \{\tau_h\}_{|\Gamma}$. Introduce the Galerkin approximations $V_h : X_h' \to X_h$ and $S_h^G : X_h' \to X_h$ of the operators V and S_1, respectively, related to X_h' and defined by

$$(V_h u, v) = (Vu, v), \qquad (S_h^G u, v) = (S_1 u, v) \qquad \forall u, v \in X_h'.$$

In general one has $(S_h^G)^{-1} \neq \mathcal{T}_h^G$ where the latter is given with respect to X_h.
By a standard way, define the FE approximation $\mathcal{T}_h : X_h \to X_h'$ of the operator $\mathcal{T}_1$ by

$$(\mathcal{T}_h u, v)_\Gamma = \int_{\Omega_1} \nabla \overline{u} \nabla v dx, \quad \forall v \in W_h \tag{4.1}$$

where $\overline{u} \in W_h$ satisfying $\gamma_0 \overline{u} = u \in X_h$ and

$$\int_{\Omega_1} \nabla \overline{u} \nabla z dx = 0 \qquad \forall z \in W_{h0}. \tag{4.2}$$

Note that the FE approximation S_h of S_1 satisfies $S_h\ \mathcal{T}_h = \mathcal{T}_h\ S_h = E$ on $Ker\mathcal{T}_1^\perp$. Due to Lemma 2.1 both S_h and $\mathcal{T}_h$ are SPD operators on $Ker\mathcal{T}_1^\perp$.
Consider the Galerkin equation: *find $u_h \in X_h'$ such that*

$$\Big[(\frac{1}{2}E + K')u_h, v\Big] = [f, v], \qquad \forall v \in X_h' \tag{4.3}$$

where

$$[u, v] = (S_1 u, v). \tag{4.4}$$

Since the operator $\frac{1}{2}E + K'$ is continuous and SPD with respect to the new inner product (4.4) we obtain the quasioptimal convergence of u_h to the exact solution u of the equation

$$(\frac{1}{2}E + K')u = f \in H_1^{-1/2}(\Gamma).$$

Lemma 4.1 *There exist unique solutions of (4.3) for small enough $h > 0$ which converge quasioptimally, i.e.,*

$$\|u - u_h\|_{H^{-1/2}(\Gamma)} \leq c \inf_{v \in X_h'} \|u - v\|_{H^{-1/2}(\Gamma)}. \tag{4.5}$$

The problem (4.3) is stable in the sense that

$$\lambda_{min}\Big((S_h^G)^{-1}\ V_h\Big) \geq a_0 > 0 \tag{4.6}$$

uniformly with respect to $h > 0$ where a_0 is defined in Lemma 3.2.

We now perturb the ideal Galerkin equation (4.3) replacing S_h^G by the FE approximate operator S_h. Without loss of generality we assume $f \in X_h'$. Since the operators S_h and S_h^G are spectrally equivalent [20] then the perturbed equation

$$\mathcal{T}_h \, V_h \widetilde{u}_h = Q_h \, f \in X_h' \tag{4.7}$$

is uniquely solvable and stable, that is

$$\inf_h \lambda_{min}(\mathcal{T}_h \, V_h) \geq \lambda_0 > 0 \, .$$

Here Q_h is the L^2-orthogonal projection onto X_h'. The error estimate for the solutions $\widetilde{u}_h$ follows from the approximation properties of the FE discretization S_h to the operator S_1 investigated in [1].

Theorem 4.1 *For a small enough $h > 0$ the estimate*

$$\|u - \widetilde{u}_h\|_{H^{-1/2}(\Gamma)} \leq c_0 \inf_{v \in X_h'} \|u - v\|_{H^{-1/2}(\Gamma)} + c_1 \inf_{z \in W_h} \|M_1 \, S_1 \, f - z\|_{H^1(\Omega)}$$

holds with the solution operator M_1 defined in Section 2.

Proof. Rewrite the equations (4.3) and (4.7) in the forms

$$V_h u_h = S_h^G P_h \, f \quad \text{and} \quad V_h \widetilde{u}_h = S_h Q_h \, f \, ,$$

respectively, where P_h is the $[\cdot,\cdot]$-orthogonal projection onto X_h'. Substituting the above equations and taking into account that $f \in X_h'$ we now obtain

$$\|u - \widetilde{u}_h\|_{H^{-1/2}(\Gamma)} \leq c\|u - u_h\|_{H^{-1/2}(\Gamma)} + c\|S_h f - S_1 f\|_{H^{1/2}(\Gamma)}. \tag{4.8}$$

The first term in the right hand side of (4.8) is estimated by (4.5) while for the second one we apply the estimate from [1]

$$\|S_h f - S_1 f\|_{H^{1/2}(\Gamma)} \leq c \inf_{z \in W_h} \|M_1 \, S_1 \, f - z\|_{H^1(\Omega)} \, .$$

This completes the proof. ■

Note that the Bubnov-Galerkin approximation of the operators $\frac{1}{2}E - K'$ and $\frac{1}{2}E \pm K$ may be derived along the same line yielding quasi-optimal error estimates.

Remark 4.1 The solution complexity (which appears to be quasi-optimal) for the equation (4.7) is estimated by those related to the wavelet based approximation of the operator V_h (respectively, D_h) as well as by the efficiency of a matrix compression technique developed for the Poincaré-Steklov operators, see Section 5. The advantage of the above introduced Bubnov-Galerkin scheme is that it admits an optimal matrix compression and may be efficiently realized in an appropriate SPD setting.

5 On matrix compression for the Poincaré-Steklov operators

The Galerkin discretizations V_h and D_h of the operators V and D on polygons with spline wavelets as basis functions have been recently developed in [31]. The inversion of the compressed operators with an accuracy of the discretization error $\varepsilon = N^{-\alpha}$, $\alpha > 0$ was shown to have the complexity $O(N \log^2 N)$ with memory needs $O(N \log N)$. The Galerkin subspaces of piecewise constant and continuous, piecewise linear functions related to quasiuniform meshes have been used in case of the operators V and D, respectively.

From now on we consider the compression techniques for the interior Poincaré-Steklov operators S_h and $\mathcal{T}_h = S_h^{-1}$ over polygonal boundaries. In the case of rectangular domains the corresponding compression schemes considered in [25] depend on the idea of [4] and they have been shown to have the complexity $O(N \log^2 N)$ to achieve the discretization error $O(N^{-\alpha})$ with some fixed $\alpha > 0$. This approach was extended in [23] to the case of right triangles and special polygons based on nonmatching domain decomposition and iterative substructuring techniques exhibiting the complexity $O(N \log^3 N)$.

Here we introduce a new elegant approach for fast computations with discrete PS operators (more precisely, with the Schur complement matrix $\mathcal{T}$ defined by $\langle \mathcal{T} U, V \rangle = (\mathcal{T}_h u, v)$ where U and V are the vector representations of u and v, respectively, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product) based on the multilevel splitting of the trace space on the refined interface related to nonmatching domain decomposition and on the corresponding BPX [8] interface preconditioner. This again yields the overall complexity of order $O(N \log^3 N)$ in the case of polygons composed of regular right triangles and rectangles. Contrary to [23], the proposed compression scheme is well suited for both serial and parallel implementations.

As the principal ingradient of the underlying technique we first consider the case of a right triangle. For a given right triangle Ω of the size $a \times b$ introduce the uniform triangulation $\{\tau_k\}$ with a meshsize $h \equiv \frac{a}{N}$, $N = 2^p$, $p \in \mathbb{N}$. Let $W_p \subset H^1(\Omega)$ be a space of piecewise linear finite elements defined on $\{\tau_k\}$ and let $W_p^0 \subset W_p$ be the corresponding subspace of functions from W_p with zero traces on the hypothenuse γ_3 of Ω. Following [23] introduce the sequence D_j, $j = 0, 1, \ldots, q \leq p$ of decompositions of Ω, as shown in Fig.1 , by successively diadical breaking of triangular pieces belonging to D_{j-1} when visiting the level j. For notational convenience we set $D_0 = \Omega$.

With a fixed subdomain pattern we associate the sequence $\{T_k\}$, $k \in I^T := \{k = 1, \ldots, 2^q\}$ of $\frac{a}{N} \times \frac{b}{N}$ -right triangles adjacent to the hypothenuse γ_3 of Ω and the sequence of rectangles $\{R_{ik}\}$, $i, k \in I^R := \{i, k : i = 1, \ldots, q;\ k = 1, \ldots 2^{i-1}\}$ which produce the resultant nonconformal and nonquasiuniform decomposition of Ω. For given $q \leq p$ define the skeleton

$$\Gamma_q := (\cup_{i,k \in I^R} \partial R_{ik}) \backslash \gamma_3 \tag{5.1}$$

which alignes with the mesh lines by definition. Note that $\Gamma_0 = \partial\Omega \backslash \gamma_3$, see Fig. 1.

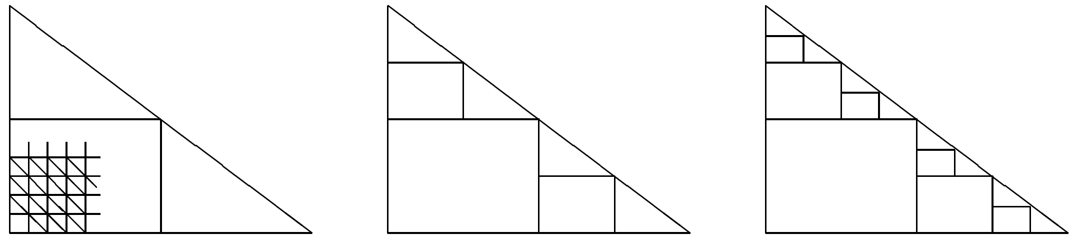

Figure 1: Decompositions D_1, D_2 and D_3 of Ω.

Let $\gamma : W_p \to C(\Gamma_q)$ be the trace operator and γ_{ik} be its restriction to R_{ik}. Introduce the trace space on Γ_q

$$Y_{\Gamma_q} := \gamma W_p^0$$

equipped with the norm

$$\|u\|_{Y_{\Gamma_q}} := \inf_{\overline{u} \in W_p^0 : \gamma \overline{u} = u} \|\overline{u}\|_{H^1(\Omega)} \ , \tag{5.2}$$

providing an $H^{1/2}$-setting. Denote by $(\cdot,\cdot)_{L^2(\Gamma_q)}$ the usual L^2 - inner product on Γ_q generating the corresponding duality $\langle \cdot,\cdot \rangle_{\Gamma_q}$. Following [24, 23], introduce the SPD interface operator $A_{\Gamma_q} : Y_{\Gamma_q} \to Y'_{\Gamma_q}$ by

$$\langle A_{\Gamma_q} u, v \rangle_{\Gamma_q} := \sum_{i,k \in I^R} (\mathcal{T}_{ik} u_{ik}, v_{ik}) \ + \ \sum_{k=1}^{2^q} (\mathcal{T}_k u_k, v_k) \tag{5.3}$$

for all $u, v \in Y_{\Gamma_q}$ where $\mathcal{T}_{ik}$ and $\mathcal{T}_k$ are defined by (2.6) for rectangles R_{ik} and triangles T_k, correspondingly. Here $u_{ik} = \gamma_{ik} u$ with the same notations for v_{ik}, u_k and v_k.
From now on we set for simplicity $q = p$. The implementation of the discrete Poincaré-Steklov operator on $\partial\Omega$ (with the Schur complement as a stiffness matrix) is reduced to the inversion of A_{Γ_p}. Note that the equivalent H^α-norm on the skeleton may be defined by

$$\|u\|^2_{Y_{\Gamma_p}} := \sum_{i,k \in I^R} \|u_{ik}\|^2_{H^\alpha(\Gamma_{ik})} + \sum_{k=1}^{2^p} \|u_k\|^2_{H^\alpha(\Gamma_k)} \tag{5.4}$$

where

$$\|u\|^2_{H^\alpha(\Gamma_{ik})} = |u|^2_{H^\alpha(\Gamma_{ik})} + \frac{1}{H_i} \int_{\Gamma_{ik}} u^2 dx$$

with $H_i = 2^{-i} a$ and with the seminorm

$$|u|^2_{H^\alpha(\Gamma_{ik})} = \int_{\Gamma_{ik}} \int_{\Gamma_{ik}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2\alpha}} dx dy, \quad 0 < \alpha \le 1 \ .$$

With the corresponding norm for $\alpha \in (0, 3/2)$, one can introduce the trace space $Y_{\Gamma_p,\alpha} := \{u = \gamma \overline{u} \ : \overline{u} \in H^{1/2+\alpha}(\Omega)\}$. Note that (5.2) now corresponds to $\alpha = 1/2$.

Consider the splitting $V_p = \sum_{j=0}^{p} V_j$ with respect to a hierarchy of nested spaces

$$V_0 \subset V_1 \subset \ldots \subset V_p = Y_{\Gamma_p}$$

defined by $V_j = \gamma W_j^0$ where $dim V_j = O(j\,2^j),\ j = 0, 1, \ldots, p$. Let $\{\varphi_{j,m}\} = \{\gamma\Phi_{j,m}\}$ be the nodal basis functions of V_j where $\Phi_{j,m}$ are the basis functions of W_j^0 such that $supp\,\Phi_{j,m} \cap \Gamma_p \neq \emptyset$. Introduce the L^2 -orthogonal projection by

$$Q_j : V_p \to V_j, \qquad ((Q_j v - v), u)_{L^2(\Gamma_p)} = 0 \quad \forall u \in V_j,\ v \in V_p$$

and define the subspaces $\mathcal{W}_j = (Q_j - Q_{j-1})V_p$ with $\mathcal{W}_0 = V_0$, $Q_{-1} = 0$. Then $V_{j+1} = V_j \oplus \mathcal{W}_{j+1}, j = 0, ..., p-1$ and we obtain the multilevel orthogonal splitting

$$V_p = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_p \tag{5.5}$$

yielding the unique decomposition

$$u = \sum_{k=0}^{p} w_k, \quad w_k \in \mathcal{W}_k, \quad u \in V_p \quad . \tag{5.6}$$

The decomposition (5.6) gives a stable prewavelet splitting of Y_{Γ_p}.

Lemma 5.1 *For every $u \in Y_{\Gamma_p}$ the estimates*

$$c_1 \|u\|^2_{Y_{\Gamma_p,s}} \leq \sum_{j=0}^{p} 2^{2sj} \|w_j\|^2_{L^2(\Gamma_p)} \leq c_2 \|u\|^2_{Y_{\Gamma_p,s}} \tag{5.7}$$

hold for any $0 \leq s < \frac{3}{2}$ where $Y_{\Gamma_p,s}$ is equipped with the norm (5.4). Moreover, there holds

$$\|u\|^2_{Y_{\Gamma_p}} \cong \inf_{u=\sum_{j=0}^{p} v_j,\, v_j \in V_j} \Big(\sum_{j=0}^{p} 2^j \|v_j\|^2_{L^2(\Gamma_p)}\Big), \qquad u \in V_p\,. \tag{5.8}$$

Proof. Let $s = \frac{1}{2}$. Adapting the arguments of Proposition 2 from [29] based on the minimizing property of the norm (5.2) and applying the simple estimate

$$\inf_{g \in W_j^0 : \gamma_{ik} g = h} \|g\|^2_{L^2(R_{ik})} \cong 2^{-j}\, \|h\|_{L^2(\partial R_{ik})}$$

we now derive

$$\|u\|^2_{Y_{\Gamma_p}} \cong \inf_{\overline{u} \in W_p^0 : \overline{u}_{|\Gamma} = u} \Big(\inf_{\overline{u} = \sum_{j=0}^{p} u_j,\, u_j \in W_j^0} \sum_{(i,k) \in I^R \cup I^T} \sum_{j=0}^{p} 2^{2j} \|u_j\|^2_{L^2(R_{ik})} \Big) \tag{5.9}$$

$$\cong \inf_{u=\sum_{j=0}^{p} v_j,\, v_j \in V_j} \Big(\sum_{j=0}^{p} 2^j \|v_j\|^2_{L^2(\Gamma_p)}\Big), \qquad u \in V_p$$

which implies (5.8). Then (5.7) follows from the orthogonality of the decomposition (5.6) along the same line as usual arguments used for justifying the stability of BPX splitting. In the general case of $s \in (0, \frac{3}{2})$ we again apply the minimizing property of the trace norm to the scale $H^{s+\frac{1}{2}}(\Gamma_{ik})$ as well as the L_2- stability of the domain space splitting $W_p^0 = \sum_{j=0}^{p} W_j^0$, i.e.,

$$\|u\|^2_{H^{s+\frac{1}{2}}(\Omega)} \cong \inf_{u=\sum_{j=0}^{p} u_j,\, u_j \in W_j^0} \Big(\sum_{j=0}^{p} 2^{2(s+\frac{1}{2})j} \|u_j\|^2_{L^2(\Omega)}\Big), \qquad u \in W_p^0\,, \tag{5.10}$$

see [29, 30] for more details. Similar to (5.9) we then obtain

$$\|u\|^2_{Y_{\Gamma_p,s}} \cong \inf_{u=\sum_{j=0}^{p} v_j,\, v_j \in V_j} \Big(\sum_{j=0}^{p} 2^{sj} \|v_j\|^2_{L^2(\Gamma_p)}\Big), \qquad u \in V_p \tag{5.11}$$

which yields (5.7) for any $s \in [0, 3/2)$. ∎

Corollary 5.1 *For any $u \in V_p$ the estimates*

$$c_1 \sum_{j=0}^{p} 2^j \|w_j\|^2_{L^2(\Gamma_p)} \le \langle A_{\Gamma_p} u, u\rangle_{\Gamma_p} \le c_2 \sum_{j=0}^{p} 2^j\, \|w_j\|^2_{L^2(\Gamma_p)} \tag{5.12}$$

and

$$c_1 \langle A_{\Gamma_p} u, u\rangle_{\Gamma_p} \le \inf_{u=\sum_{j=0}^{p} u_j,\, u_j \in V_j} \Big(\sum_{j=0}^{p} 2^j \|u_j\|^2_{L^2(\Gamma_p)}\Big) \le c_2 \langle A_{\Gamma_p} u, u\rangle_{\Gamma_p} \tag{5.13}$$

hold with constants c_1, c_2 independent of N and p .

Proof. For any $u \in V_p$ and any subdomain R_{ik} we have

$$(\mathcal{T}_{ik} u_{ik}, u_{ik}) \cong |u_{ik}|^2_{H^{1/2}(\Gamma_{ik})} \cong \|u_{ik}\|^2_{H^{1/2}(\Gamma_{ik})}$$

since $u(\xi_{ik}) = 0$ where $\xi_{ik} = \gamma_3 \cap \partial R_{ik}$. Then (5.12) follows from (5.7) and (5.4). In turn, (5.13) is a consequence of (5.8). This completes our proof. ∎

Note that (5.8) yields the norm equivalence for the approximation space $A_2^{1/2}(\{V_j\})$ on the skeleton

$$\|u\|^2_{Y_{\Gamma_p}} \cong \inf_{u=\sum_{j=0}^{p} v_j,\, v_j \in V_j} \sum_{j=0}^{p} 2^j \|v_j\|^2_{L^2(\Gamma_p)} \cong \|u\|^2_{A^{1/2}} := \|u\|^2_{L^2(\Gamma_p)} + \sum_{j=0}^{p} 2^j s_j(u)^2_{L^2(\Gamma_p)}$$

with the sequence of best approximations defined by

$$s_j(u)_{L^2(\Gamma_p)} = \inf_{v \in V_j} \|u - v\|_{L^2(\Gamma_p)}, \qquad u \in V_p.$$

Now we are in a position to design the multilevel BPX [8] or multilevel diagonal scaling (MDS) [40] preconditioners based on the additive Schwarz method related to the splitting (5.6). Note that though our scheme is similar to one developed in [29] for the case of conformal decompositions the implementation of the final algorithm under consideration differs from those proposed in [29]. In fact, the definition of A_Γ (5.3) is based on assembling the local components related to the underlying domain decomposition. This admits an efficient performance of corresponding matrix-vector multiplication with asymptotically optimal cost. The proposed construction is actually done in spirit of the multilevel methods technique with locally refined grids. The main difference, however, is that in our case we use the properly nested refinement of the interface (associated with a nonuniform decomposition of Ω) in the direction orthogonal to the hypothenuse γ_3 while the finest grid on the domain remains fixed and uniform.

Starting with the decomposition $V_p = \sum\limits_{j=0}^{p} V_j$ consider a more refined splitting of V_j into one-dimensional subspaces $V_{j,m} = span\, \varphi_{j,m}$ taking into account the L^2-stability of the corresponding nodal basis $\{\varphi_{j,m}\} = \{\gamma\Phi_{j,m}\}$ of V_j, $j = 1, ..., p$. Define the resultant splitting

$$V_p = \sum_{j=0}^{p} \sum_{m=1}^{dim V_j} span \gamma\Phi_{j,m} \ .$$

For any given $u \in V_p$ specify the action of the operator $\mathcal{T}_{ik}$ with respect to the level j_0 where the basis functions $\varphi_{j_0,m}$ come from in order to optimize the computations with the interface operator A_Γ. Introduce the representation

$$(\mathcal{T}_{ik}u, \gamma\Phi_{l,m}) = \begin{cases} (\mathcal{T}_{ik}u,\ \varphi_{l,m}), & l \geq i \\ (u, \mathcal{T}_{ik}\varphi_{l,m}) = \sum\limits_{x_\gamma \notin int\{T_i\}} a_\gamma u(x_\gamma), & l < i \end{cases} \tag{5.14}$$

which indicates that the computation of the components from (5.3) with $i > l$ is a trivial procedure.

The BPX scheme may be introduced with respect to the norm in $V_{j,m}$

$$(u, v)_{V_{j,m}} = 2^j (u, v)_{L^2(\Gamma_p)}$$

and the corresponding projection $P_{V_{j,m}} : V_p \rightarrow V_{j,m}$ defined by

$$(P_{V_{j,m}} u, v_j) = (u, v_j)_{V_{j,m}} \qquad \forall v_j \in V_{j,m}.$$

The resultant preconditioned operator usually called as multilevel additive Schwarz (MAS) operator $P_{BPX} : V_p \rightarrow V_p$ takes the form

$$P_{BPX} u := \sum_{l=0}^{p} \sum_{(i,k)\in I^R \cup I^T} \sum_{m=1}^{dim \gamma_{ik} V_l} \frac{(\mathcal{T}_{ik}u, \gamma\Phi_{l,m})}{2^l (\varphi_{l,m}, \varphi_{l,m})_{L^2(\Gamma_p)}} \cdot \varphi_{l,m} \tag{5.15}$$

where we set $\mathcal{T}_{ik} = \mathcal{T}_k$ for $i = p$. The operator equation on the skeleton with the SPD operator (5.15) may be efficiently resolved by the iterative CG method applying the corresponding optimal algorithm for fast computations with $\mathcal{T}_{ik}u$ on rectangular subdomains with respect to (5.14). The treatment of the layer neighboring the hypothenuse has, obviously, the complexity $O(N)$. The MDS scheme given by [40] for FE discretizations of differential equations may be obtained if the terms $2^l(\varphi_{l,m}, \varphi_{l,m})$

in the denominator are substituted by $(\mathcal{T}_{ik}\varphi_{l,m}, \varphi_{l,m})$. However, in our case the BPX scheme looks as a primary one. In fact, the MDS algorithm needs some extra computations of the diagonal entries $(\mathcal{T}_{ik}\varphi_{l,m}, \varphi_{l,m})$ as far as the operators $\mathcal{T}_{ik}$ are not local but at the same time it does not improve the resultant condition number $\kappa(P_{BPX})$ since we deal with the constant coefficients case.

Theorem 5.1 *The operator equation*

$$P_{BPX}u = \sum_{l=0}^{p} \sum_{(i,k)\in I^R\cup I^T} \sum_{m=1}^{dim\gamma_{ik}V_l} \frac{(\Psi, \gamma\Phi_{l,m})}{2^l(\varphi_{l,m}, \varphi_{l,m})_{L^2(\Gamma_p)}} \cdot \varphi_{l,m} \tag{5.16}$$

is equivalent to the original interface problem

$$\langle A_{\Gamma_p}u, v\rangle_{\Gamma_p} = \langle \Psi, v\rangle_{L^2(\Gamma_p)} \qquad \forall v \in V_p$$

and $\kappa(P_{BPX}) = O(1)$ uniformly with respect to N and the number of levels p. The computation of $P_{BPX}u$, $u \in V_p$ has the complexity $O(N \log^3 N)$ with memory needs of the order $O(N \log^2 N)$. The solution of (5.16) by the cascadic CG method up to the approximation error $\varepsilon = \varepsilon_{tol}N^{-\alpha}$, $\alpha > 0$ has the expense $\log \varepsilon_{tol}^{-1} \cdot O(N \log^3 N)$ where $\varepsilon_{tol} > 0$ is some a priori fixed constant.

Proof. The uniform bound on the condition number $\kappa(P_{BPX}) = O(1)$ follows from (5.13) applied to the above introduced additive Schwarz operator P_{BPX}. The complexity of the residual computation $P_{BPX}u$, $u \in V_p$ is discussed in Section 6. From [35] we know that an optimal convergence of the CCG-method introduced in [16] is achieved if we take into account the H^2-regularity of the underlying Dirichlet problem, that means in our case

$$\|u\|_{H^{1+s}(\Omega)} \;\le\; c\,\|u_{|\Gamma}\|_{H^{1/2+s}(\Gamma)}\ , \qquad s \in (0,1],$$

see [27]. In the case of the Dirichlet problem under consideration we set $s = 1$ yielding the full regularity. When using some more general boundary conditions providing a deficient regularity one can apply the convergence results for the CCG-method based on the H^{1+s}-regularity of the underlying BVP with some $0 < s < 1$, see [6, 36]. On the other hand, one also obtains the H^{1+s}-regularity with some $s \in (0, 1)$ in the case of nonconvex polygons. This completes our proof.

Remark 5.1 *Note that a more parallel version of the MAS operator (5.15) related to the splitting*

$$V_p = \sum_{j=j_0}^{p} \sum_{m=1}^{dimV_j} span\gamma\Phi_{j,m}$$

with some $j_0 > 0$ may be introduced assuming an exact solution of the coarse mesh problem related to V_{j_0}.

6 Computing complexity and numerical examples

As our main result, we have shown in Theorem 5.1 that the multilevel BPX scheme introduced for a special interface reduction of the Laplacian leads to asymptotically optimal computations with the discrete PS operator in the case of right triangles. The point is that a matrix compression technique originally developed for rectangles has been thus extended to the case of triangular and, consequently, polygonal domains. In fact, Theorem 5.1 remains valid in a more general case of polygons $\overline{\Omega} = \overline{\Omega}_R \cup \overline{\Omega}_T$ composed of M_R rectangles and M_T right triangles Ω_i such that $\overline{\Omega}_R = \cup_{i\in I_R}\overline{\Omega}_i$ and $\overline{\Omega}_T = \cup_{i\in I_T}\overline{\Omega}_i$. The extension to the case of mixed boundary conditions is rather straightforward keeping in mind the technical assumption $\partial\Omega_T \cap \partial\Omega \subset \Gamma_D$ where $\Gamma_D \subset \partial\Omega$ is the piece of $\partial\Omega$ with the Dirichlet conditions imposed. The coarse mesh space V_0 and the skeleton Γ associate now with the chosen decomposition of Ω, see Fig. 2, where '$\bullet$' marks the coarse mesh nodes and thick lines correspond to the Neumann conditions imposed.

The resultant interface operator A_Γ is introduced by the direct sum involving the terms $A_{\Gamma_p(\Omega_i)}$ defined by (5.3) for any triangular substructure $\Omega_i \in \Omega_T$ as well as by the discrete PS operators $\mathcal{T}_{1,\partial\Omega_i}$ related to the rectangles $\Omega_i \in \Omega_R$

$$\langle A_\Gamma u, v\rangle_\Gamma := \sum_{i\in I_R} (\mathcal{T}_{1,\partial\Omega_i} u_i, v_i) + \sum_{i\in I_T} \langle A_{\Gamma_p(\Omega_i)} u_i, v_i\rangle_{\Gamma_p(\Omega_i)}. \tag{6.1}$$

The multilevel BPX preconditioner for the operator A_Γ leads to the equivalent interface

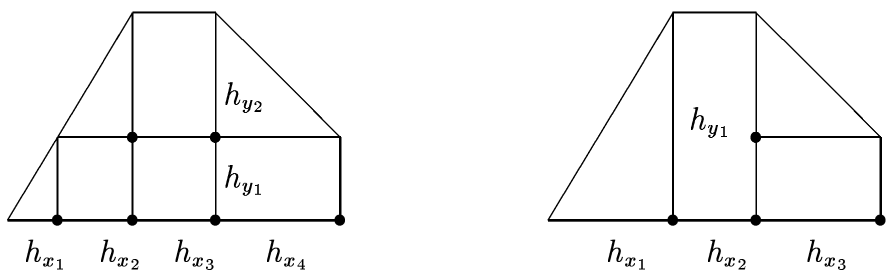

Figure 2: Coarse mesh decompositions of a polygon.

equation similar to (5.16). In turn, the solution of the underlying interface problem with the approximation error $\varepsilon = \varepsilon_{tol} N^{-\alpha}$, $\alpha > 0$ has the complexity

$$Q(A_\Gamma^{-1}) = \log \varepsilon_{tol}^{-1} \cdot O(\sum_{i\in I_R} N_i \log^2 N_i + \sum_{i\in I_T} N_i \log^3 N_i) \tag{6.2}$$

and it requires $O(\sum_{i\in I_R\cup I_T} N_i \log^2 N_i)$ memory where N_i is the number of degrees of freedom on the boundary $\Gamma_i = \partial\Omega_i$.

Consider in more details the terms in (6.2) with $i \in I_T$ related to triangular subdomains. The most laborious part in a treatment of (5.15) with a given $u \in V_p$ is the computation of the sum corresponding to the finest level $l = p$. This requires $O(N_i \log^3 N_i)$ operations for any triangle $\Omega_i \in \Omega_T$. The contributions from all remained visiting levels with $l = p-1, \ldots, 0$ may be obtained by successive extrapolation from the element $P_{V_p} u = \sum_{V_{p,m}\subset V_p} P_{V_{p,m}} u$ with respect to the explicit representation of hierarchical basis functions on the skeleton. In fact, the scalar products $(\mathcal{T}_{ik} u, \varphi_{l,m})$ are calculated by using the values $(\mathcal{T}_{ik} u, \varphi_{l+1,m})$ coming from the previous level $l+1$ by the extrapolation formulae

$$\varphi_{l,\xi} = \frac{1}{2} \sum_{\nu\in supp\varphi_{l,\xi}, \nu\neq\xi} \varphi_{l+1,\nu} + \varphi_{l+1,\xi}, \qquad l = p-1, ..., 0$$

where ξ and ν are the corresponding nodal points. Thus, with any fixed subdomain $\Omega_i \in \Omega_T$ the computation of related terms with respect to (5.15) for indices $l < p$ now costs $O(N_i \log N_i)$ operations which does not effect the resultant asymptotical performance of the algorithm. The action of the interpolation (prolongation) operator which maps the elements $P_{V_l} u = \sum_{V_{l,m} \in V_l} P_{V_{l,m}} u \in V_l$ to V_p from any visiting level $l = 0, 1, \ldots, p-1$ is also estimated by $O(N \log N)$ where $N = O(\sum_i N_i)$. Thus, the matrix-vector multiplication cost $Q(A_\Gamma)$ related to the interface operator A_Γ is estimated by

$$Q(A_\Gamma) = O(\sum_{i \in I_R} N_i \log^2 N_i + \sum_{i \in I_T} N_i \log^3 N_i) \ . \tag{6.3}$$

Finally, the cascadic variant of the CG iterative method needs $Q(A_\Gamma^{-1}) = \log \varepsilon_{tol}^{-1} \cdot Q(A_\Gamma)$ operations, where $Q(A_\Gamma)$ is given by (6.3). The above estimate is valid for any polygon composed of rectangles and regular right triangles.
We now provide some numerical examples confirming the asymptotically optimal performance of a computation with the discrete PS operator based on BPX-type interface preconditioning. The corresponding runs have been done on IBM-PC 486/8/66.
In Table 1 we give the results corresponding to the Neumann problem for the Laplace

$N_j = 2^j + 1$		BPX-cascadic CG				BPX-CG		BPS-PCG	
j		$u_0^{j+1} = U_{j-1}^j$		$u_0^{j+1} = U^j$		$u_0^j = 0$			
1	2.23	1	0.1	1	0.16	9	0.77	4	0.16
2	$8.1 \cdot 10^{-1}$	2	0.16	2	0.22	8	0.71	8	0.28
3	$2. \cdot 10^{-1}$	3	0.22	3	0.28	9	0.82	10	0.38
4	$4.8 \cdot 10^{-2}$	2	0.22	3	0.33	9	0.88	10	0.5
5	$1.14 \cdot 10^{-2}$	2	0.33	3	0.44	9	1.1	11	0.71
6	$2.8 \cdot 10^{-3}$	2	0.49	3	0.61	9	1.5	12	1.26
7	$6.8 \cdot 10^{-4}$	2	0.87	3	1.1	10	2.81	12	2.36
8	$1.68 \cdot 10^{-4}$	2	1.87	3	2.31	10	5.66	13	5.43
9	$4.16 \cdot 10^{-5}$	2	4.28	3	5.33	10	12.7	14	12.9
10	$1.04 \cdot 10^{-5}$	2	10.16	3	12.7	11	31.7	14	28.6
	$\lvert u^* - u_h \rvert_{L^2}$	IT	$t/sec.$	IT	t/sec.	IT	t/sec.	IT	t/sec.

Table 1: Iterations history for the BPX scheme applied to the Schur complement system on a rectangle.

equation on $\Omega = (0,1) \times (0,1)$ with the exact solution

$$u^*(x,y) = sink\pi x \cdot \Big[e^{-k\pi y} \cdot \frac{1}{1 - e^{-2k\pi}} - e^{ky\pi} \cdot \frac{1}{e^{2k\pi} - 1} \Big].$$

The multilevel method has been applied to the interface equation on Γ

$$given: \quad v = \partial_n u \,, \qquad \mathcal{T}_1 u \ = \ v.$$

The stopping criteria $\varepsilon_{CCG} = 5.0 \cdot h_j^2$ and $\varepsilon_{CG} = \varepsilon_{PCG} = 5.0 \cdot h_{10}^2$ have been used where the mesh parameter is defined by $h_j = 2^{-j}$, $j = 1, ..., 10$. We denote by $u_{k_j}^j$

the resultant solution on the level j obtained by k_j CG iterations. Two variants of an initial guess u_0^{j+1} for the BPX-CCG method were tested:

$$U^j_{j-1} = \frac{5}{4}\mathcal{I}_j\, u^j_{k_j} \;-\; \frac{1}{4}\mathcal{I}_j\mathcal{I}_{j-1}\, u^{j-1}_{k_{j-1}} \qquad \text{and} \qquad U^j \;= I_j\, u^j_{k_j}\,.$$

Here $I_j : V_j \to V_{j+1}$ is the linear interpolation operator while $\mathcal{I}_j : V_j \to V_{j+1}$ is an interpolation operator of the order $O(h^4)$ on the uniform mesh. The column marked by BPS corresponds to the standard Bramble-Pasciak-Schatz preconditioner with the condition number $O(1 + \log^2 N_j)$.
Note that some two grids extrapolation procedures for solving the Poisson equation with different choices of basic iterations have been considered in [41].

7 Mesh refinement and preconditioning

Recall that the proposed approach leads to asymptotically optimal schemes in the case of uniform meshes on any edge $\Gamma_j \subset \partial\Omega$ of a given polygon Ω. The interior problem is equivalent to one matrix-vector multiplication with the "interior" PS operator (or its inverse) while the exterior problem needs, in addition, the inversion of either the integral operator V or D. For such piecewise uniform meshes we apply the efficient frequency cutting and wavelet approximation to the above mentioned operators which manifest themselves the optimal matrix compression. The principal issue is a uniform bound on the condition number of the compressed or preconditioned operator resulting from a multiscale basis transformation or from a multilevel space splitting, respectively. Note that the MAS method of the complexity $O(N^2)$ for the hypersingular integral equation (in the case of quasi-uniform meshes) has been developed in [37]. The multilevel preconditioning in BEM was also discussed in [30]. In turn, examining the proof of Corollary 5.1, we find that the BPX scheme in the $H^{\frac{1}{2}}$- setting appears to be well suited for inversion of the operator D on a closed curve. In the case of the operator V it requires some additional duality arguments [30].

From now on, we assume some mesh refinement near the corner point $w_j \in \partial\Omega$. Locally refined meshes are commonly used for accurately modelling angular singularities. In general, the matrix compression techniques we are concerned with can not be extended straightforwardly to the case of non-quasiuniform partitions. However, applying some special geometrical refinement and nested selection strategy on the skeleton one can construct quasi-optimal algorithms. We now briefly discuss the specific issues arising in presence of locally refined meshes.

It turns out that in the case of the multilevel BPX scheme defined by (5.15), (5.16) the nested selection strategy possesses uniform $O(1)$ condition number estimates due to the results in [7, 8, 29, 30] developed for the FE discretizations of elliptic differential equations. These results apply verbatim to the case of interface equations if one uses the geometrical refinement with hanging nodes. This moderately deteriorates the complexity of the Schur complement computations up to $O(N_j(\log N_j)^4) + Q(A_\Gamma)$ where $Q(A_\Gamma)$ is defined by (6.2). The above estimate indicates a relatively tolerant complexity growth of the underlying BPX scheme if the corresponding geometrical refinement is obtained by successively scaling $O(\log N_j)$ times (say by factor 2) of the given master domain. Since the number of the degrees of freedom on an edge with mesh refinement is of the order $N_{ref} = O(N_j \log N_j)$ we again arrive at the complexity $O(N_{ref} \log^3 N_{ref})$. This approach will be considered in a forthcoming paper.

Consider the problem of iterative inversion of the operators V and D arising from the equations (2.11), (2.12) and (4.7). We further presume no restrictions concerning the refinement strategy and allow an arbitrary unstructured mesh on $\partial\Omega$. With a corresponding triangulation $\{\tau_n\}_{|\partial\Omega}$, let D_h and V_h be the Galerkin approximations of D and V related to the subspaces X_h and X'_h, respectively (see Section 4). The complexity of a matrix-vector multiplication is expected to be of the order $O(N^2)$. Lemma 3.3 now implies that the operator $\mathcal{P}_h = \delta_h + a\,E$ (here $a = 0$ in the case of D_h and $a = 1$ in the case of V_h) where $\delta_h : X_h \to X'_h$ is defined by

$$(\delta_h u, v) = (\frac{d}{ds}u, \frac{d}{ds}v) \;, \qquad \forall u, v \in X_h$$

gives a spectrally equivalent preconditioner to both D_h^2 and V_h^{-2} uniformly with respect to the particular refinement chosen. Thus, the equivalent equations of the form

$$D_h^2 u = D_h f_D, \qquad V_h^2 u = V_h f_V$$

may be efficiently resolved by the PCG method with the preconditioner $\mathcal{P}_h$ resp. $\mathcal{P}_h^{-1}$. Moreover, the operator $\mathcal{P}_h$ having the SPD three-diagonal stiffness matrix may be inverted by a direct method with $O(N)$ operations. The solution of the transformed equations with both the operator D_h^2 and V_h^2 by the PCG method up to the fixed error $\varepsilon_0 > 0$ has the complexity $O(N^2)$ uniformly with respect to an arbitrarily unstructured mesh on $\partial\Omega$.

Note that the proposed preconditioning technique may be extended to the case of Galerkin approximation of the operators V and D related to a properly nested refined (selected) mesh on a 3D closed surface. In such a way the BPX preconditioner should be applied to the Laplace-Beltrami operator on the surface under consideration. The underlying approach provides also an optimal preconditioner for the operator D_h defined on a nonclosed curve.

Acknowledgement. The authors want to thank Dr. E. Nikonov from JINR, Dubna for his assistance with computations.

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