THE QUALOCATION METHOD FOR SYMM'S INTEGRAL EQUATION ON A POLYGON

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ABSTRACT. This paper discusses the convergence of the qualocation method for Symm's integral equation on closed polygonal boundaries in $\mathbb{R}^2$. Qualocation is a Petrov–Galerkin method in which the outer integrals are performed numerically by special quadrature rules. Before discretisation a nonlinear parametrisation of the polygon is introduced which varies more slowly than arc–length near each corner and leads to a transformed integral equation with a regular solution. We prove that the qualocation method using smoothest splines of any order k on a uniform mesh (with respect to the new parameter) converges with optimal order $O(h^k)$. Furthermore, the method is shown to produce superconvergent approximations to linear functionals, retaining the same high convergence rates as in the case of a smooth curve.

1. INTRODUCTION

Symm's integral equation for a closed curve Γ,

$$-\frac{1}{\pi}\int_{\Gamma}\log|\mathbf{x}-\boldsymbol{\xi}|u(\boldsymbol{\xi})d\Gamma(\boldsymbol{\xi})=f(\mathbf{x})\,,\ \mathbf{x}\in\Gamma\,, \tag{1.1}$$

is a boundary integral equation of central importance for elliptic boundary value problems in the plane. Here $f:\Gamma\to\mathbb{R}$ is given, and the problem is to find $u:\Gamma\to\mathbb{R}$, or perhaps certain functionals of u. Equation (1.1) is closely related to the singular integral equation with Hilbert kernel for which the L_2–theory has been developed by S.G. Mikhlin in his fundamental paper [15]. We assume throughout the paper that the transfinite diameter of Γ is not equal to 1, so that (1.1) is uniquely solvable.

Many numerical methods have been proposed, but only for the Galerkin method is the theory wholly satisfactory. A number of numerical methods, among them the qualocation method [24, 26, 5], have aimed to achieve the high rate of convergence of the Galerkin method but with less computational effort. However, until now rigorous results for the qualocation method have been available only for a smooth curve Γ. The aim of this paper is to give a convergence theory for the qualocation method on a polygon. (The arguments can be extended without difficulty to the case of a curved polygon without cusps.)

The convergence theory will be established by appealing to a theory recently presented by Elschner and Graham [9] for the collocation method for Symm's integral equation on a polygon. In this approach the first step is to introduce a parametrisation of the curve Γ which has the effect of smoothing out the singularities at the corners, and then to apply the collocation method on a mesh which is uniform with respect to the new parameter. From the point of view of the curve Γ, the effect of this is to squeeze the mesh at the corners. The Elschner and Graham results will be described and extended to the qualocation method in Section 3.

In Section 4 we prove a superconvergence result for the error in approximating linear functionals of the solution to (1.1), showing that in most cases the qualocation method for the polygon achieves the same order of convergence as it does on a smooth curve. Section 5 contains some auxiliary spline approximation results, which are also of independent interest.

The present work arose from the realisation that the arguments of [9] are not restricted to the collocation method, but extend also to other methods expressible as projection methods with appropriate properties. In the next section we shall see that the qualocation method is a projection method in this sense.

It should be mentioned that various fully discrete versions of the qualocation method have been proposed in recent times [25, 21, 19, 14]. These are not projection methods, so the arguments used in the present paper are not directly applicable.

There is one unfortunate aspect of the analysis of Elschner and Graham [9], shared with many other recent papers on boundary integral equations [3, 4, 7, 8, 6, 13, 17, 12], and now extended to this paper. It is that the stability of the method can only be proved if the possibility is allowed of modifying the approximate solution over some number of intervals near each corner. In practice such modifications have so far never been needed, but the possibility remains that they will be found to be needed in some situations in the future. The superconvergence results in Section 4 generally require that stability holds without any corner modifications.

2. The qualocation method

The first step in implementing the qualocation method, and any of the other methods mentioned here, is to introduce a parametrisation $\gamma : [\pi, \pi] \to \Gamma$ of the curve Γ, so that (1.1) then becomes

$$-\frac{1}{\pi}\int_{-\pi}^{\pi} \log|\gamma(s)-\gamma(\sigma)|w(\sigma)d\sigma = g(s)\,,\ s\in[-\pi,\pi]\,, \tag{2.1}$$

or

$$Kw = g \tag{2.2}$$

where

$$w(\sigma) = |\gamma'(\sigma)|u(\gamma(\sigma))\,,\ g(s) = f(\gamma(s))\,, \tag{2.3}$$

so that the Jacobian of the transformation has been absorbed into the new unknown function w. If the curve Γ is smooth then γ should be chosen to be smooth, and to be such that $|\gamma'| > 0$ on Γ. We defer until the next section the choice of γ if Γ has corners.

For n a natural number, we now introduce a uniform mesh on $[-\pi, \pi]$, defined by

$$s_i = -\pi + ih\,,\ i = 0, ..., n \quad \text{with } h = 2\pi/n\,.$$

The qualocation method is like the Petrov–Galerkin method, in that it employs both a trial space V_h (the space in which the approximate solution is sought), and a test space V_h'. We take these to be spline spaces of orders k and k' respectively. Thus for $k \geq 1$, let $V_h = V_h^k$ be the space of 2π–periodic (smoothest) splines of order k on the mesh $\{s_i\}$. That is, $v \in V_h$ if and only if v is 2π–periodic, is a polynomial of degree at most $k-1$ on each subinterval $[s_{i-1}, s_i]$, and has $k-2$ continuous derivatives. Similarly, for $k' \geq 1$ let $V_h' = V_h^{k'}$ be the space of 2π–periodic (smoothest) splines of order k' on the same mesh.

It is convenient to define first the Petrov–Galerkin method for this pair of spaces. Letting (u, v) denote the L_2 inner product

$$(u,v) := \int_{-\pi}^{\pi} u(s)\overline{v(s)}ds \,, \tag{2.4}$$

the Petrov–Galerkin method for (2.2) is: find $w_h \in V_h$ such that

$$(Kw_h, \chi) = (g, \chi) \quad \forall \chi \in V_h' . \tag{2.5}$$

The qualocation method differs from the Petrov–Galerkin method only to the extent that the inner product (2.4) is replaced by a discrete equivalent $(u, v)_h$,

$$(u,v)_h := Q_h(u\overline{v}) \,, \tag{2.6}$$

where

$$Q_h(g) = h \sum_{i=0}^{n-1} \sum_{j=1}^{J} w_j g(s_i + h\xi_j) \tag{2.7}$$

with

$$0 \leq \xi_1 < \xi_2 < \ldots < \xi_j < 1 \,,\; \sum_{j=1}^{J} w_j = 1 \,,\; w_j > 0 \,. \tag{2.8}$$

Note that Q_h is just the composition of the simple J–point rule

$$Q(g) := \sum_{j=1}^{J} w_j g(\xi_j) \,. \tag{2.9}$$

However, we shall see that the recommended rules are not any of the familiar quadrature rules (Gaussian, Simpson, etc.). The reason is that the integrand $(Kw_h)\chi$ on the left of (2.5) is not smooth on each subinterval, even if Γ is a smooth curve.

Once the points and weights of the J–point quadrature rule (2.9) are determined, the qualocation method for (2.2) is defined by: find $w_h \in V_h$ such that

$$(Kw_h, \chi)_h = (g, \chi)_h \quad \forall \chi \in V_h' . \tag{2.10}$$

An important special case is the collocation method. Suppose that we take $V_h' = V_h$, and for some number $\varepsilon \in [0, 1)$, choose the rule Q in (2.9) to be the 1–point rule

$$Qg = g(\varepsilon) \,.$$

Then it is easily seen, by introducing a basis of $V_h' = V_h$ in (2.10), that (2.10) is in this case equivalent to the ε–collocation method

$$Kw_h(t_i) = g(t_i) \,,\; i = 0, \ldots, n-1 \,,$$

where

$$t_i = s_i + \varepsilon h \,,\; i = 0, \ldots, n-1 \,.$$

For the case of a smooth curve it is well known [23] that this method is stable for k even provided $\varepsilon \neq 1/2$, and for k odd provided $\varepsilon \neq 0$.

Since the ε–collocation method is known to be unstable for the two exceptional cases indicated above, it is natural for us to exclude them in what follows. We shall also insist that k and k' (the orders of V_h and V_h') have the same parity, because it is only in this case that stability results are known for either the qualocation or Petrov–Galerkin methods.

Assumption (A). We assume that k and k' (the orders of V_h and V_h' respectively) are either both even, or both odd.

Assumption (B). In the rule Q, with $J \geq 1$ and points and weights satisfying (2.8), the following two cases are excluded:

i) $J = 1$, k and k' even, $\xi_1 = 1/2$,

ii) $J = 1$, k and k' odd, $\xi_1 = 0$.

Under these assumptions the next result asserts that the qualocation solution exists, and has convergence properties at least as good as those of the basic collocation method ([1] for the case k even with $\varepsilon = 0$, [22] for the case k odd with $\varepsilon = 1/2$). Of course the interesting versions of the qualocation method, as we have indicated already, have faster convergence than the collocation method, but it is useful to establish first that at least nothing is lost in going to the more general qualocation method.

Here and in what follows H^t, $t \in \mathbb{R}$, refers to the periodic Sobolev space of order t on $[-\pi, \pi]$, with norm given by

$$||v||_t^2 := |\hat{v}(0)|^2 + \sum_{m \neq 0} |m|^{2t} |\hat{v}(m)|^2 \,, \tag{2.11}$$

where the Fourier coefficients of v are defined by

$$\hat{v}(m) = (v, \exp(\iota m s))/(2\pi)^{1/2} \,.$$

It is well known (see e.g. [27]) that, for smooth Γ, the operator K defined in (2.2) takes the form $K = A + T$, where

$$\begin{aligned} Av(s) := & -\frac{1}{\pi} \int_{-\pi}^{\pi} \log |2e^{-1/2} \sin(s-\sigma)/2| v(\sigma) d\sigma \\ = & \ (2\pi)^{-1/2} \left(\sum_{m \neq 0} \hat{v}(m) |m|^{-1} \exp(\iota m s) + \hat{v}(0) \right) \end{aligned} \tag{2.12}$$

is an isometry from H^t to H^{t+1} for any $t \in \mathbb{R}$, and T is an integral operator with smooth (periodic) kernel. Note that if Γ is the circle given by the parametrisation $\gamma(s) = r \exp(\iota s)$ then T is simply the linear functional $v \to -(1+2\log r)\hat{v}(0)/(2\pi)^{1/2}$.

The following result incorporates both the stability theorem of [5, Theorem 3], and a simple version of the convergence theorem of [5, Theorem 2]. (The fact that for Γ a circle the result holds for all h, not just for h sufficiently small, is clear from the proof of [5, Theorem 2]: the restriction to h sufficiently small enters the argument only when we consider perturbations from the case of a circle.)

Theorem 2.1. *Let assumptions (A) and (B) hold, and assume that Γ is smooth. Then, given $g \in H^{t+1}$ for $t > -1/2$, a unique solution $w_h \in V_h$ of (2.10) exists for all h sufficiently small. If Γ is a circle of radius not equal to 1 and $|\gamma'| =$ constant then w_h exists and is unique for all h. For all s, t satisfying*

$$s < k - 1/2\,,\ t > -1/2\,,\ -1 \leq s \leq t \leq k\,,$$

we have

$$||w - w_h||_s \leq c h^{t-s} ||w||_t\,.$$

In particular, the maximal order of convergence given by Theorem 2.1 is

$$||w - w_h||_{-1} \leq c h^{k+1} ||w||_k\,.$$

Saranen [20] established that for k odd the convergence rate of the mid–point collocation method (i.e., ε–collocation with $\varepsilon = 1/2$) is generally faster than the $O(h^{k+1})$ rate allowed by Theorem 2.1 (it can reach $O(h^{k+2})$ if w is sufficiently smooth). From our present perspective it is convenient to consider the mid–point collocation method as a special case of the qualocation method: according to [5], Saranen's result is recovered whenever the quadrature rule Q in (2.9) is symmetric. The explicit qualocation methods for k considered later have this property, but achieve still higher orders of convergence than the mid–point collocation method.

In [5] it is shown that faster convergence can be achieved for certain special choices of the points $\{\xi_j\}$ and weights $\{w_j\}$, the crucial consideration being the behaviour near zero of a certain function $E : [-1/2, 1/2] \to \mathbb{R}$,

$$E(y) := \sum_j w_j \Omega(\xi_j, y)(1 + \overline{\Delta'(\xi_j, y)})\,, \tag{2.13}$$

where

$$\begin{aligned} \Omega(\xi, y) &= |y|^{k+1} F^{\pm}_{k+1}(\xi, y)\,, \\ \Delta'(\xi, y) &= y^{k'} F^{\pm}_{k'}(\xi, y)\,, \\ F^{+}_{m}(\xi, y) &= \sum_{l \neq 0} \frac{1}{|l + y|^m} \exp(\iota l \xi)\,, \\ F^{-}_{m}(\xi, y) &= \sum_{l \neq 0} \frac{\operatorname{sign} l}{|l + y|^m} \exp(\iota l \xi)\,, \end{aligned} \tag{2.14}$$

with the $+$ sign holding in (2.14) if k and k' are even, and the $-$ sign if k and k' are odd.

Definition. The qualocation method (2.10) is of *order* $k + 1 + b$ if b (the *additional order*) is the largest non–negative integer such that

$$E(y) = O(|y|^{k+1+b})\,,\ y \in [-1/2, 1/2]\,.$$

We see from (2.13) and (2.14) that the method (2.10) is of order $\geq k+1$ without any special choice of the qualocation rule. Some simple rules of order $> k+1$ are shown in Table 1, extracted from [5].

k	k'	ξ_1	w_1	ξ_2	w_2	b	order
1	1	1/2	1	–	–	1	3
1	1	0	3/7	1/2	4/7	3	5
1	3	0.2308296503	1/2	$1-\xi_1$	1/2	3	5
2	2	0	1	–	–	0	3
2	2	0	3/7	1/2	4/7	2	5
2	2	0.2308296503	1/2	$1-\xi_1$	1/2	2	5

TABLE 1. Some interesting qualocation methods

Note that the first entry in the table is the mid–point collocation method for piecewise constant basis functions, which (as shown by Saranen [20]), achieves an order of 3. The next item in the table, however, is a qualocation method with piecewise constant trial and test functions that achieves an order of 5, two higher than the mid–point collocation method. The fourth entry in the table is again a collocation method, this time collocation at the breakpoints with piecewise linear functions. It too is followed by higher order qualocation methods based on piecewise linear trial functions, again with test functions of the same degree.

The significance of the order is seen in the following theorem, also taken from [5].

Theorem 2.2. *If the qualocation method (2.10) is of order $k+1+b$ with $b \geq 0$, and if the assumptions of Theorem 2.1 hold, then for all s,t satisfying*

$$s < k - 1/2\,,\ t > -1/2\,, -1-b \leq s \leq t \leq k$$

we have

$$||w - w_h||_s \leq ch^{t-s}||w||_{t+\max(-1-s,0)}\,.$$

In particular, the fastest order of convergence is seen by setting $s = -1-b$ and $t = k$, to give

$$||w - w_h||_{-1-b} \leq ch^{k+1+b}||w||_{k+b}\,.$$

Thus the "order", as defined above, is the fastest order of convergence obtainable with the particular qualocation method. As a particular example, we find for the second entry in Table 1 the result

$$||w - w_h||_{-4} \leq ch^5||w||_4\,.$$

For the application later in this paper we need to write the qualocation approximation as a projection method. Thus we define:

$$\Pi_h : H^1 \to V_h : \quad (\Pi_h v, \chi)_h = (v, \chi)_h \quad \forall v \in H^1\,,\ \forall \chi \in V_h'\,. \tag{2.15}$$

Proposition 2.1. *If Assumptions (A) and (B) hold then Π_h is a well defined projection operator with range V_h.*

Proof. To show that $\Pi_h v$ is uniquely determined by (2.15) it is only necessary to introduce bases $\{v_i\}$ and $\{v_i'\}$ for V_h and V_h', and then to show that the matrix $((v_i, v_j')_h)$ is non–singular. But this follows immediately from Theorem 3 of [5] (which proves stability of the qualocation method for an operator L_0), on taking the legitimate special case $L_0 = I$, the identity. If $v \in V_h$ then $\Pi_h v = v$ satisfies (2.15), thus Π_h is a projection with range V_h. □

It then follows, if Assumptions (A) and (B) hold, that the qualocation approximation (2.10) can be written as: find $w_h \in V_h$ such that

$$\Pi_h K w_h = \Pi_h g\,. \tag{2.16}$$

Next we introduce R_h, a solution operator for the qualocation equation. Writing $w = K^{-1} g$, so that w is the exact solution of the equation $Kw = g$, the solution of the qualocation equation (2.16) may be written as $w_h = R_h w$, where R_h is a linear operator. As a special case of Theorem 2.2 we obtain the following result, needed in the subsequent arguments.

Proposition 2.2. *If Γ is a circle of radius not equal to 1 and $|\gamma'| =$ constant then R_h exists as an operator from H^0 to H^0, and satisfies*

$$||(I - R_h)w||_s \le c h^{t-s} ||w||_{t+\max(-1-s,0)} \tag{2.17}$$

for all s, t such that $s < k - 1/2$, $t > -1/2$ and $-1 - b \le s \le t \le k$, with c independent of h.

3. The qualocation method for polygonal Γ

Let Γ be a closed polygon enclosing a simply connected bounded domain in $\mathbb{R}^2$. Suppose that Γ has corners $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{r-1}$ and that, for each j, the interior angle at $\mathbf{x}_j$ is $(1 - \chi_j)\pi$, $0 < |\chi_j| < 1$. The side joining $\mathbf{x}_j$ to $\mathbf{x}_{j+1}$ is denoted Γ_j and $|\Gamma_j|$ denotes its length. $|\Gamma|$ is the length of Γ.

We first introduce a nonlinear parametrisation $\gamma : [-\pi, \pi] \to \Gamma$ which varies more slowly than arc–length parametrisation in the vicinity of each corner of Γ. By forcing γ to vary slowly enough near each corner, the solution w of the transformed equation (2.2) then can be made as regular as desired on $[-\pi, \pi]$ (provided f is smooth), and hence w can be optimally approximated by splines of any order k on the uniform grid

$$s_i = -\pi + ih\,,\ i = 0, \ldots, n\,,\ h := 2\pi/n\,. \tag{3.1}$$

To define the parametrisation γ, choose a grading exponent $q \in \mathbb{N}$ and introduce $r + 1$ points given by:

$$-\pi < S_0 < S_1 < \ldots < S_{r-1} < \pi\,,\ S_r = S_0 + 2\pi\,,$$

with their differences having the values

$$S_{j+1} - S_j = 2\pi |\Gamma_j|^{1/q} \Big/ \sum_{m=0}^{r-1} |\Gamma_m|^{1/q}\,,\ j = 0, \ldots, r-1\,. \tag{3.2}$$

These will be preimages of the corner points $\mathbf{x}_j$ under γ. For notational convenience we extend s_i, S_j and $\mathbf{x}_j$ to $i, j \in \mathbb{Z}$ by requiring $\mathbf{x}_j$ to be r–periodic in j and by defining $S_{rm+j} = S_j + 2m\pi$, $j = 0, ..., r-1$, $s_{nm+i} = s_i + 2m\pi$, $i = 0, ..., n-1$, $m \in \mathbb{Z}$. Then we will be concerned with parametrisations $\gamma : [-\pi, \pi] \to \Gamma$ which (for all j) satisfy the assumptions:

(A1) $\gamma(S_j) = \mathbf{x}_j$;

(A2) $(s - S_j)^{-q}(\gamma(s) - \mathbf{x}_j)$, $(S_{j+1} - s)^{-q}(\gamma(s) - \mathbf{x}_{j+1}) \in \mathcal{C}^\infty[S_j, S_{j+1}]$;

(A3) $|\gamma'(s)| > 0$, $s \in (S_j, S_{j+1})$;

(A4) $\lim_{s \to S_j} |\gamma(s) - \mathbf{x}_j| / |s - S_j|^q = |\Gamma_j| / (S_{j+1} - S_j)^q$.

Note that by (3.2) the limit in (A4) does not depend on j. Furthermore, the image of the mesh (3.1) under γ is graded with exponent q to the corner points $\mathbf{x}_j$, but the corner points are not necessarily images of mesh points under γ.

Example 3.1. Following [9], choose any δ in the range

$$0 < \delta < (1/2) \min\{S_{j+1} - S_j : j = 0, ..., r-1\}.$$

Then, for $j = 0, ..., r-1$, set

$$\gamma(s) = \begin{cases} \mathbf{x}_j - \left(\dfrac{S_j - s}{S_j - S_{j-1}} \right)^q (\mathbf{x}_j - \mathbf{x}_{j-1}), \ s \in [S_j - \delta, S_j], \\ \\ \mathbf{x}_j + \left(\dfrac{s - S_j}{S_{j+1} - S_j} \right)^q (\mathbf{x}_{j+1} - \mathbf{x}_j), \ s \in [S_j, S_j + \delta]. \end{cases}$$

The gaps on $[-\pi, \pi]$ can be filled , in principle, by introducing monotonically increasing $\mathcal{C}^\infty$ connecting functions.

The next example gives a more practical construction, following [13, 6].

Example 3.2. For $j = 0, ..., r-1$, define

$$\gamma(s) = \mathbf{x}_j + \frac{(s - S_j)^q}{(s - S_j)^q + (S_{j+1} - s)^q} (\mathbf{x}_{j+1} - \mathbf{x}_j), \ s \in [S_j, S_{j+1}],$$

where the usual periodicity convention $\gamma(s + 2\pi) = \gamma(s)$ is adopted. If $q = 1$ we have

$$\gamma(s) = \mathbf{x}_j + \frac{s - S_j}{S_{j+1} - S_j} (\mathbf{x}_{j+1} - \mathbf{x}_j), \ s \in [S_j, S_{j+1}],$$

and condition (3.2) means that $(S_{j+1} - S_j)/|\Gamma_j| = 2\pi/|\Gamma|$ for all j, so the parametrisation is then proportional to arc–length.

More general constructions of γ, allowing also different grading exponents at the corners, can be found in [9, 10, 11].

Following [27], we rewrite (2.2) as the second kind equation

$$(I + M)w = e, \quad \text{with} \quad M = A^{-1}(K - A), \ e = A^{-1}g, \tag{3.3}$$

where $A : H^0 \to H^1$ is the isometric isomorphism defined in (2.12). Recall that A coincides with K when Γ is the circle of radius $e^{-1/2}$. Since it is a standard result [15, 18] that $A^{-1} = -HD + J$, where D is the (periodic) differentiation operator, H is the Hilbert transform

$$Hv(s) = -\frac{1}{2\pi} p.v. \int_{-\pi}^{\pi} \cot\left(\frac{s - \sigma}{2}\right) v(\sigma) d\sigma$$

and J is the linear functional $v \to (v,1)/2\pi$, we further have

$$M = HD(A-K) + J(K-A)\,. \tag{3.4}$$

It turns out that M is a Mellin convolution operator local to each corner; see [27] for $q=1$ and [9] in the general case.

We now recall some analytical results on Equations (2.2) and (3.3) which are needed in the convergence analysis of the qualocation method. The first theorem follows from [9, Theorem 2 and Lemma 7] when the parametrisation γ takes the simple form of Example 3.1. Combining this with the perturbation arguments in [10], one obtains the result for parametrisations satisfying (A1)–(A4).

Theorem 3.1. *The operators*

$$I+M : H^0 \to H^0 \quad \textit{and} \quad K : H^0 \to H^1$$

are continuously invertible, and we have the strong ellipticity estimate

$$Re((I+M+T)v,v) \geq c||v||_0^2 \quad \forall v \in H^0\,,$$

where T is some compact operator on H^0.

The next result, which follows from [9, Corollary 5], shows that the unique solution w of (2.2) is smooth provided the right side f of (1.1) is smooth and the grading exponent q is large enough. For $l>0$, $H^l(\Gamma)$ is defined as the restriction of the usual Sobolev space $H^{l+1/2}(\mathbb{R}^2)$ to Γ.

Theorem 3.2. *Let $l \in \mathbb{N}$, $q > (l+1/2)\max_j(1+|\chi_j|)$, and suppose $f \in H^{l+5/2}(\Gamma)$. Then the unique solution of (2.2) satisfies $w \in H^l$ and, for all j,*

$$D^m w(s) = O(|s-S_j|^{l-m-1/2}) \quad \textit{as} \quad s \to S_j\,,\ m=0,...,l\,. \tag{3.5}$$

The following result, taken from [11], describes the properties of the kernel function

$$\kappa(s,\sigma) := \frac{1}{\pi}\log\left|\frac{\gamma(s)-\gamma(\sigma)}{2e^{-1/2}\sin(s-\sigma)/2}\right| \tag{3.6}$$

of the integral operator $A-K$. Note that less precise kernel estimates have been given in [9, 10].

Theorem 3.3. *On each compact subset of $\mathbb{R}\times\mathbb{R}\backslash\{(S_j,S_j) : j \in \mathbb{Z}\}$, the derivatives $D_s^i D_\sigma^m \kappa(s,\sigma)$ of order $i+m \leq q$ are bounded and 2π-periodic. Moreover, for each j and sufficiently small $\delta>0$, for $s,\sigma \in [S_j-\delta, S_j+\delta]\backslash\{S_j\}$ we have the estimates*

$$|\kappa(s,\sigma)| \leq c|\log(|s-S_j|+|\sigma-S_j|)|\,,$$

$$|D_s^i D_\sigma^m \kappa(s,\sigma)| \leq c(|s-S_j|+|\sigma-S_j|)^{-i-m}\,,\ 1 \leq i+m \leq q\,.$$

We now consider the qualocation method (2.10) for the approximate solution of Equation (2.2) with right side $g \in H^1$ assuming throughout that Assumptions (A) and (B) hold. Define the projection operator $R_h : H^0 \to V_h^k$ by letting $R_h v \in V_h^k$ solve the qualocation equation $\Pi_h A(R_h v) = \Pi_h A v$. That is to say, R_h is the solution operator of the qualocation method for the particular case of a circle of radius $e^{-1/2}$. Using (3.3) and Proposition 2.1, it is easily seen that (2.10) may be written $\Pi_h A(I+M)w_h = \Pi_h A e$. Hence w_h solves (2.10) if and only if $\Pi_h A w_h = \Pi_h A(e - M w_h)$, and by the definition of R_h, this is equivalent to $w_h = R_h(e - Mw_h)$. Hence (2.10) is equivalent to the following non–standard projection method for the second kind equation (3.3):

$$(I + R_h M)w_h = R_h e\,. \tag{3.7}$$

As is usual for Mellin convolution equations, we are only able to prove stability for a slightly modified method. Introduce, for τ sufficiently small, the truncation operator

$$T^\tau v(s) = \begin{cases} 0\,, & s \in [S_j - \tau, S_j + \tau]\,,\ j = 0, \dots, r-1\,, \\ v(s)\,, & \text{otherwise.} \end{cases}$$

Then for any fixed natural number i^* and for n sufficiently large, define

$$K^{i^*h} = A + (K - A)T^{i^*h}\,,$$

and consider the modified qualocation method

$$\Pi_h K^{i^*h} w_h = \Pi_h g\,. \tag{3.8}$$

If $i^* = 0$ then (3.8) is equivalent to (2.10) (or (2.16)). Otherwise, (3.8) can be obtained from (2.5) by a slight change to the coefficient matrix of the corresponding linear system. By mimicking the derivation of (3.7) from (2.10), it is easily seen that (3.8) is equivalent to

$$(I + R_h M T^{i^*h})w_h = R_h e\,. \tag{3.9}$$

The following theorem, which is the main result of this section, establishes the convergence of the (modified) qualocation method with optimal order in the L_2 norm.

Theorem 3.4. *Suppose that $q > (k+1/2)\max_j(1+|\chi_j|)$ and $f \in H^{k+5/2}(\Gamma)$. Then there exists i^* such that (3.8) has a unique solution for all h sufficiently small and*

$$||w - w_h||_0 \le ch^k\,, \tag{3.10}$$

where c is a constant which depends on w and i^* but is independent of h.

Proof. Following [9, Theorem 9] we first verify the stability of (3.9), that is the estimate

$$||(I + R_h M T^{i^*h})v_h||_0 \ge c||v_h||_0 \quad \forall v_h \in V_h^k \tag{3.11}$$

for all h sufficiently small, where i^* is large enough and c does not depend on h. Since, by Theorem 3.1, $I + M$ is invertible and strongly elliptic, we obtain stability of the finite section operators $T^\tau(I+M)T^\tau$, $\tau \to 0$ (see e.g. [16] or [18, page 33]), which implies the estimate (cf. [9, Theorem 6])

$$||(I + MT^\tau)v||_0 \ge c||v||_0 \quad \forall v \in H^0\,,\ \forall \tau \le \tau_0\,. \tag{3.12}$$

Now (3.11) is obtained with the aid of (3.12) and the following perturbation result:

For each $\delta > 0$, there exists $i^* \ge 1$ such that for all h sufficiently small

$$||(I - R_h)MT^{i^*h}||_0 < \delta\,. \tag{3.13}$$

A proof of this is given in [9, Lemma 8], for the case of the basic collocation method. The arguments there use quasi–interpolants and are based on kernel estimates for M and on the bounds

$$||R_h||_0 \leq c\,, \; ||I - R_h||_{H^1 \to H^0} \leq ch \quad \forall h > 0\,, \tag{3.14}$$

following from Proposition 2.2. Thus the assertion extends immediately to the general case of the qualocation method.

A simpler proof of (3.13), which employs Theorem 3.3 and (3.14) but avoids the use of quasi–interpolants, can be found in [11].

To prove the error estimate (3.10), we observe that

$$||w - w_h||_0 \leq ||(I - R_h)w||_0 + ||w_h - R_h w||_0\,,$$

where the first term is of order h^k by Proposition 2.2 (with $s = 0$, $t = k$) and Theorem 3.2 (with $l = k$).

Furthermore, using (3.11) and then (3.9) with (3.3) and the first inequality of (3.14), we obtain

$$\begin{aligned} ||w_h - R_h w||_0 &\leq c||(I + R_h M T^{i^*h})(w_h - R_h w)||_0 \\ &= ||R_h[(I + M)w - (I + MT^{i^*h})R_h w]||_0 \\ &\leq c||(I + MT^{i^*h})(I - R_h)w + M(I - T^{i^*h})w||_0 \\ &\leq c||(I - R_h)w||_0 + c||(I - T^{i^*h})w||_0\,. \end{aligned}$$

It remains to verify that the last term is of order h^k. Now by the choice of q stated in the hypothesis we have from (3.5)

$$w(s) = O(|s - S_j|^{k-1/2})\,, \; s \to S_j\,,$$

for all j, which yields the assertion. □

The approximation w_h to w defined in (3.9) may be used to construct a corresponding approximation u_h to the solution u of the original boundary integral equation (1.1):

$$u_h(\gamma(\sigma)) = |\gamma'(\sigma)|^{-1} w_h(\sigma)\,.$$

Then, under the assumptions of the preceding theorem, this approximation converges to u with order $O(h^k)$ in a certain weighted L_2 norm, where the weight vanishes with order $O(|s - S_j|^{1-1/q})$ as $s \to S_j$ for any j; see [9].

In other situations integral functionals of u may be required, such as those representing the solutions of boundary value problems by interior potentials. These may be written as smooth linear functionals of the solution w of (2.2):

$$\int_\Gamma u\tilde{v}d\Gamma = \int_{-\pi}^{\pi} w(\sigma)v(\sigma)d\sigma = (w, v)\,, \tag{3.15}$$

where $v = \tilde{v} \circ \gamma$ and $\tilde{v} \in \mathcal{C}^\infty(\Gamma)$, $\tilde{v}$ real. Since

$$|(w, v) - (w_h, v)| \leq ||w - w_h||_{-1}||v||_1\,,$$

the following corollary is then of interest. Its proof is entirely analogous to that of Theorem 8 in [12].

Corollary 3.1. *Under the hypotheses of Theorem 3.4,*

$$||w - w_h||_{-1} \leq ch^{k+\beta},$$

where $\beta = 1$ *if* $i^* = 0$, *and* $\beta = 1/2$ *if* $i^* \geq 1$.

In the next section we shall obtain faster convergence rates for the approximation (w_h, v) to (3.15), using certain special qualocation methods, under the assumption that the method is stable with $i^* = 0$.

4. Superconvergence results for linear functionals

Let Γ be a simple closed polygon as in the preceding section, and suppose that the qualocation method (2.10) satisfies Assumptions (A) and (B) and is of order $k+1+b$, $b \geq 0$. We further assume that (2.10), or equivalently (3.9) with $i^* = 0$, is stable in H^0 so that, given $g \in H^1$, a unique solution $w_h \in V_h^k$ of (2.10) exists for all h sufficiently small.

The following theorem establishes superconvergence of the qualocation approximation to the functional (3.15).

Theorem 4.1. *Suppose the hypothesis of Theorem 3.2 holds with* $l = \min(2k, k+b)$, *and that* $\tilde{v} := v \circ \gamma^{-1} \in \mathcal{C}^\infty(\Gamma)$. *Suppose also that Theorem 3.4 holds with* $i^* = 0$. *Then we have the error estimate*

$$|(w - w_h, v)| = O(h^{l+1}) \quad as \quad h \to 0. \tag{4.1}$$

In particular, Theorem 4.1 shows that linear functionals of the mid–point collocation method with splines of odd order k can achieve an order of $k+2$, as shown by Saranen [20] for smooth Γ. This confirms the $O(h^3)$ convergence of the piecewise constant collocation observed in the numerical experiments of [9]. More interestingly, we see that the last two qualocation methods in Table 1 can yield an order of 5 in the polygonal case, just as for smooth Γ. The order is only 3 for the second and the third methods in the table, since the convergence rate established in (4.1) is never better than the $O(h^{2k+1})$ rate achieved by the corresponding Galerkin method. Finally, we note that all other higher order qualocation methods contained in Tables 1 and 4 of [5] achieve the same orders of convergence as in the smooth case.

Proof of Theorem 4.1. Let z be the unique solution of $Kz = v$. Since by assumption $v \circ \gamma^{-1} \in \mathcal{C}^\infty(\Gamma)$, Theorem 3.2 implies $z \in H^l$. Furthermore, since $K = A(I+M)$ and A and K are self–adjoint with respect to the scalar product (2.4), we obtain

$$\begin{aligned}(w - w_h, v) &= (w - w_h, Kz) = ((I+M)(w-w_h), Az) \\ &= ((I-R_h)(I+M)(w-w_h), Az) \\ &= ((I-R_h)w, Az) + ((I-R_h)M(w-w_h), Az),\end{aligned} \tag{4.2}$$

where we used (3.3) and (3.7) to obtain the third equality.

We now estimate the first term on the right side of (4.2). Setting

$$k_1 = \min(k, b) = l - k,$$

Proposition 2.2 (with $s = -k_1 - 1$, $t = k$) gives

$$\begin{aligned} |((I-R_h)w, Az)| &\leq ||(I-R_h)w||_{-1-k_1}||Az||_{1+k_1} \\ &\leq ||(I-R_h)w||_{-1-k_1}||z||_l \leq ch^{k+k_1+1}||w||_{k+k_1} \leq ch^{l+1} \end{aligned}$$

since $w \in H^l$ by Theorem 3.2. It remains to find an analogous bound for the last term in (4.2). By Proposition 2.2 (with $s = -1$, $t = 0$) and by duality, we have

$$\begin{aligned} &|(I-R_h)M(w-w_h), Az)| = |(M(w-w_h), (I-R_h^*)Az)| \\ &\leq ||M(w-w_h)||_0||(I-R_h^*)Az||_0 \leq ch||Az||_1||M(w-w_h)||_0 \\ &\leq ch||M(w-w_h)||_0\,. \end{aligned}$$

So it suffices to establish the estimate

$$||M(w-w_h)||_0 = O(h^l)\,.$$

Comparing (3.3) and (3.7) again, we get $R_hM(w-w_h) = w_h - R_hw$, hence

$$(I+MR_h)M(w-w_h) = M(w-w_h) + M(w_h - R_hw) = M(I-R_h)w\,.$$

Together with the stability of (3.7), this implies the estimate

$$||M(w-w_h)||_0 \leq c||M(I-R_h)w||_0\,.$$

To complete the proof of (4.1), it now remains to show

$$||M(I-R_h)w||_0 = O(h^l)\,.$$

In order to do so, we shall prove that

$$||M(I-P_h)w||_0 = O(h^l) \tag{4.3}$$

and

$$||(P_h - R_h)v||_0 \leq ch^l||v||_l \quad \forall v \in H^l\,, \tag{4.4}$$

where P_h denotes the orthogonal projection of H^0 onto V_h^k with respect to the L_2 inner product (2.4). The proof of (4.4) is postponed to the next section; see Corollary 5.1 with $\beta = -1$.

Since M takes the form (3.4), relation (4.3) follows from the estimate

$$||D(A-K)(I-P_h)w||_0 + ||(A-K)(I-P_h)w||_0 = O(h^l)\,. \tag{4.5}$$

To verify this, we use the following localisation procedure. Choose $\delta > 0$ sufficiently small and let ψ_j be 2π–periodic non–negative $\mathcal{C}^\infty$ cut–off functions such that $\psi_j \equiv 1$ in some neighbourhood of S_j and supp $\psi_j \subset [S_j - \delta, S_j + \delta]$. Then we have

$$D(A-K)w = \sum_j \psi_j D(A-K)\psi_j w + Tw \tag{4.6}$$

where, in view of Theorem 3.3, the kernel functions of the integral operator T and its L_2 adjoint T^* have bounded derivatives of order $\leq q-1$ on $[-\pi,\pi] \times [-\pi,\pi]$. Since by assumption $q > (l+1/2)\max_j(1+|\chi_j|)$, and hence $q \geq k+1$, T^* is a bounded operator of H^0 into H^k. Therefore, its L_2 adjoint T is a bounded map of H^{-k} into H^0 and we obtain

$$||T(I-P_h)w||_0 \leq c||(I-P_h)w||_{-k} \leq ch^{2k}||w||_k \leq ch^{2k}\,, \tag{4.7}$$

using a standard spline approximation result; see e.g. [18, Corollary 1.36].

Now we look at the jth term in the sum (4.6) representing $D(A-K)$ local to the jth corner. Without loss of generality we can assume that this is situated at $S_j = 0$ and write ψ instead of ψ_j for convenience. By Theorem 3.3 the kernel function $b(s,\sigma)$ of the integral operator $Bv := \psi D(A-K)\psi v$ satisfies the estimates

$$\begin{aligned} &|D_s^i D_\sigma^m b(s,\sigma)| \le c(|s|+|\sigma|)^{-i-m-1}\,,\ i+m \le l-k\,,\\ &s,\sigma \in [-\pi,\pi]\backslash\{0\}\,. \end{aligned} \tag{4.8}$$

Furthermore, Theorem 3.2 implies that the exact solution of (3.2) multiplied by a suitable cut–off function satisfies

$$s^{m-l}D^m w \in H^0\,,\ m = 0,\ldots,k\,. \tag{4.9}$$

Noting that the same type of arguments (with even better kernel estimates) applies to the operator $A-K$, we finally obtain (4.5) with the aid of (4.6)–(4.9) and the theorem below. □

Theorem 4.2. *Suppose that the kernel function of the operator*

$$Bv(s) = \int_{-\pi}^{\pi} b(s,\sigma)v(\sigma)d\sigma$$

satisfies (4.8), and assume that (4.9) holds. Then we have

$$||B(I-P_h)w||_0 = O(h^l)\,.$$

In the case of rather general Mellin convolution operators B, approximation results of this type have been obtained in [4, 7] for (discontinuous) piecewise polynomials, whereas [8] contains a partial result for smoothest splines which, however, does not cover the above result. Modifying the approach of [8] slightly, we are able to give the

Proof of Theorem 4.2. Following [2] and [8], we first introduce suitable quasi–interpolants, leading to local spline approximation results. Let μ be the $(k-1)$ fold convolution of k copies of the characteristic function of $(0,1)$, and define the B–spline $B_i(s)$, $s \in \mathbb{R}$, $i \in \mathbb{Z}$, as the 2π–periodic extension of $\mu(h^{-1}(s+\pi)-i)$. Note that $\{B_i : i = 0,\ldots,n-1\}$ is a basis of V_h^k if $n \ge k$, and for any element $v = \sum_i \zeta_i B_i \in V_h^k$, the inequalities

$$c^{-1}h\sum_i |\zeta_i|^2 \le ||v||_0^2 \le ch\sum_i |\zeta_i|^2 \tag{4.10}$$

hold (see e.g. [2, Chap. 4, Theorem 2.5]), where here and in what follows c is some positive constant independent of h.

Let $\{s_i\}$ be the uniform mesh introduced in the preceding section. Furthermore, let $I_i = (s_i, s_{i+1})$ and $\tilde{I}_i = (s_{i+1-k}, s_{i+k})$, and for n sufficiently large introduce the set

$$\mathcal{J} = \{i \in \mathbb{Z} : \tilde{I}_i \cap (-h,h) = \emptyset\}\,.$$

For any $v \in H^0$ we now define the quasi–interpolant $\tilde{P}_h v \in V_h^k$ by

$$\begin{aligned}\tilde{P}_h v(s) &:= \sum_{i\in\mathcal{J}} \{h^{-1}\int_{\mathbb{R}} v(t)\lambda(h^{-1}(t+\pi)-i)dt\}\mu(h^{-1}(s+\pi)-i) \\ &= \sum_{i\in\mathcal{J},0\le i<n} \{h^{-1}\int_{I_i} v(t)\lambda(h^{-1}(t+\pi)-i)dt\}B_i(s)\,, \end{aligned} \tag{4.11}$$

where λ is a bounded function on $\mathbb{R}$ satisfying supp $\lambda = [0,1]$, $\int_{\mathbb{R}} \lambda(s)ds = 1$, and if $k > 1$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \lambda(s)\mu(\sigma)(s-\sigma)^j ds d\sigma = 0\,,\; 1 \le j \le k-1\,.$$

For instance, λ can be chosen as the product of the characteristic function of $(0,1)$ with a uniquely determined polynomial of degree $k-1$ (see [2, Chap. 4, proof of Theorem 2.4]). Note that the equality in (4.11) is clear from the 2π–periodicity of v. Thus (4.11) is a slight modification of the spline approximations considered in [2, Chap. 4], ensuring that $\tilde{P}_h v = 0$ on $(-h,h)$. Moreover, (4.11) reproduces polynomials locally in the sense that if v is a polynomial of degree $\le k-1$ on an interval $\tilde{I}_i$ then $\tilde{P}_h v(s) = v(s)$ for all $s \in I_i$, any $i \in \mathcal{J}$ with $0 \le i \le n-1$ (see [2, Chap. 4, Remark 3.1]), and for these i we have the local error estimates (see [2, Chap. 4, Theorem 3.1])

$$\int_{I_i} |v - \tilde{P}_h v|^2 ds \le ch^{2m} \int_{\tilde{I}_i} |D^m v|^2 ds \quad \forall v \in H^m\,,\; m = 1, ..., k\,. \tag{4.12}$$

By virtue of (4.10), the estimate

$$\int_{I_i} |v - \tilde{P}_h v|^2 ds \le c \int_{\tilde{I}_i} |v|^2 ds \quad \forall v \in H^0 \tag{4.13}$$

is valid for any $0 \le i \le n-1$.

Since we may write

$$B(I-P_h)w = B(I-P_h)s^{l-k}s^{k-l}(I-\tilde{P}_h)w\,,$$

Theorem 4.2 is now a consequence of the estimates

$$||s^{k-l}(I-\tilde{P}_h)w||_0 \;=\; O(h^k)\,, \tag{4.14}$$

$$||B(I-P_h)s^{l-k}v||_0 \;\le\; ch^{l-k}||v||_0 \quad \forall v \in H^0\,,$$

where by duality the latter is equivalent to

$$||s^{l-k}(I-P_h)B^*v||_0 \le ch^{l-k}||v||_0 \quad \forall v \in H^0\,, \tag{4.15}$$

B^* being the integral operator with kernel $\overline{b(\sigma,s)}$.

To establish (4.14), we observe that (4.12) (with $m = k$) implies

$$\int_{I_i} |s^{k-l}(I-\tilde{P}_h)w|^2 ds \le |s_i|^{2(k-l)} ch^{2k} \int_{\tilde{I}_i} |D^k w|^2 ds$$
$$\le |(s_{i+k}/s_i)|^{2(l-k)} ch^{2k} \int_{\tilde{I}_i} |s^{k-l} D^k w|^2 ds \le ch^{2k} \int_{\tilde{I}_i} |s^{k-l} D^k w|^2 ds \tag{4.16}$$

for all $i \in \mathcal{J}$ with $0 \le i \le n-1$. Analogously, by (4.13) we have for any i satisfying $I_i \cap \{(-(k+1)h, -3h/2) \cup (3h/2, (k+1)h)\} \ne \emptyset$

$$\int_{I_i} |s^{k-l}(I-\tilde{P}_h)w|^2 ds \le c \int_{\tilde{I}_i} |s^{k-l} w|^2 ds \le ch^{2k} \int_{\tilde{I}_i} |s^{-l} w|^2 ds \,. \tag{4.17}$$

Finally, combining the estimate

$$\int_{-h}^{h} |s^{k-l}(I-\tilde{P}_h)w|^2 ds = \int_{-h}^{h} |s^{k-l} w|^2 ds \le ch^{2k} \int_{-h}^{h} |s^{-l} w|^2 ds \,,$$

with (4.16) and (4.17), we obtain

$$||s^{k-l}(I-\tilde{P}_h)w||_0^2 \le ch^{2k} \{ ||s^{k-l} D^k w||_0^2 + ||s^{-l} w||_0^2 \}$$

which with the aid of (4.9) gives (4.14).

To prove (4.15), we write

$$s^{l-k}(I-P_h)B^*v = s^{l-k}(I-P_h)(1-\chi_h)(I-\tilde{P}_h)B^*v + s^{l-k}(I-P_h)\chi_h B^*v \,,$$

where χ_h denotes the characteristic function of $(-h,h)$; recall that $\chi_h \tilde{P}_h v = 0$ for any $v \in H^0$. Now Lemma 4.1 below (with $\varrho = l-k$) implies the estimates

$$||s^{l-k}(I-P_h)\chi_h B^*v||_0 \le ch^{l-k} ||B^*v||_0 \le ch^{l-k} ||v||_0 \,,$$
$$||s^{l-k}(I-P_h)(1-\chi_h)(I-\tilde{P}_h)B^*v|| \le c ||s^{l-k}(I-\tilde{P}_h)B^*v||_0$$

for all $v \in H^0$. Thus it remains to verify the inequality

$$||s^{l-k}(I-\tilde{P}_h)B^*v||_0 \le ch^{l-k} ||v||_0 \quad \forall v \in H^0 \,. \tag{4.18}$$

Using the facts that (4.8) is also valid for the kernel of B^* and that an integral operator with Mellin convolution kernel $|s|^{\varrho}(|s|+|\sigma|)^{-\varrho-1}$, $\varrho \ge 0$, is bounded on $L_2(-\pi,\pi)$ (see e.g. [4, 7]), we now obtain that $s^m D^m B^*$ are bounded operators on H^0 for $m = 0, ..., l-k$. Using (4.12) (with $m = l-k$) and arguing as in the proof of (4.16), we get

$$\int_{I_i} |s^{l-k}(I-\tilde{P}_h)B^*v|^2 ds$$
$$\le |(s_{i+1}/s_{i+1-k})|^{2(l-k)} ch^{2(l-k)} \int_{\tilde{I}_i} |s^{l-k} D^{l-k} B^*v|^2 ds \tag{4.19}$$
$$\le ch^{2(l-k)} \int_{\tilde{I}_i} |s^{l-k} D^{l-k} B^*v|^2 ds$$

for any $i \in \mathcal{J}$ with $0 \le i \le n-1$. Furthermore, we have the obvious estimate

$$\int_{-(1+k)h}^{(1+k)h} |s^{l-k}(I-\tilde{P}_h)B^*v|^2 ds \le ch^{2(l-k)} ||(I-\tilde{P}_h)B^*v||_0^2 \le ch^{2(l-k)} ||B^*v||_0^2 \,,$$

and combining this with (4.19) yields

$$\begin{aligned} ||s^{l-k}(I-\tilde{P}_h)B^*v||_0^2 &\le ch^{2(l-k)}\{||s^{l-k}D^{l-k}B^*v||_0^2+||B^*v||_0^2\} \\ &\le ch^{2(l-k)}||v||_0^2 \end{aligned}$$

which completes the proof of (4.18). □

To complete the proof of Theorem 4.2, we need the following lemma. Its proof is based on the technique introduced in [8, Lemma 3.3].

Lemma 4.1. *If $\varrho \ge 0$ then, for all $v \in H^0$ and $h > 0$,*

(i) $||\,|s|^\varrho P_h(1-\chi_h)v||_0 \le c||\,|s|^\varrho v||_0$,

(ii) $||\,|s|^\varrho P_h\chi_h v||_0 \le ch^\varrho ||v||_0$.

Proof. (i) Let $\{B_i : i = 0, ..., n-1\}$ be the basis of V_h^k defined above, and let $G_n^{-1} = (g_{ij})_{i,j=0}^{n-1}$ be the inverse of the Gram matrix $G_n = ((B_j, B_i))_{i,j=0}^{n-1}$. Then the orthogonal projection P_h onto V_h^k takes the form

$$P_h v(s) = \sum_i \left\{ \sum_j g_{ij}(v, B_j) \right\} B_i(s)\,. \tag{4.20}$$

We now fix an integer $i_0 \ge 1$ which will be chosen sufficiently large later on, and set $t_i = |s_i|$ when supp $B_i \cap (-i_0 h, i_0 h) = \emptyset$ and $t_i = i_0 h$ otherwise. Observe that (4.20) can be written

$$P_h v = F_n H_n M_n v\,, \tag{4.21}$$

where the mappings $M_n : L_2^\varrho \to \mathbf{C}^n$, $H_n : \mathbf{C}^n \to \mathbf{C}^n$ and $F_n : \mathbf{C}^n \to L_2^\varrho$ are given by

$$\begin{aligned} M_n v &= (t_i^\varrho h^{-1/2}(v, B_i))_0^{n-1}\,, \\ H_n(\zeta_i)_0^{n-1} &= \left(h t_i^\varrho \sum_j g_{ij} t_j^{-\varrho} \zeta_j \right)_{i=0}^{n-1}\,, \\ F_n(\zeta_i)_0^{n-1} &= \sum_i t_i^{-\varrho} h^{-1/2} \zeta_i B_i\,. \end{aligned}$$

Here L_2^ϱ denotes the weighted L_2 space with norm $||\,|s|^\varrho v||_0$, and $\mathbf{C}^n$ refers to the n–dimensional Euclidean space equipped with the standard scalar product $\langle\cdot,\cdot\rangle$ and the corresponding norm $|\cdot|$.

To prove (i), we check that the operators F_n, $M_n(1-\chi_h)$ and H_n are uniformly bounded in n provided i_0 is appropriately chosen. Since the second estimate of (4.10) easily gives

$$||\,|s|^\varrho \sum_i \zeta_i B_i||_0^2 \le ch \sum_i t_i^{2\varrho}|\zeta_i|^2\,,$$

we first obtain, for all n and $(\zeta_i)_0^{n-1} \in \mathbf{C}^n$,

$$||\,|s|^\varrho F_n(\zeta_i)_0^{n-1}||_0^2 \le c \quad \sum_i |\zeta_i|^2\,, \tag{4.22}$$

hence the result for F_n. To verify the uniform boundedness of $M_n(1-\chi_h) : L_2^\varrho \to \mathbf{C}^n$, we note that for any $v \in L_2^\varrho$, $0 \le i \le n-1$ and $i_0 \ge 1$

$$\begin{aligned} |t_i^\varrho h^{-1/2}((1-\chi_h)v, B_i)|^2 &\le t_i^{2\varrho} h^{-1} \int_{s_i}^{s_{i+k}} |s|^{2\varrho}|v|^2 ds \int_{s_i}^{s_{i+k}} |s|^{-2\varrho}|(1-\chi_h)B_i|^2 ds \\ &\le ch^{-1}||B_i||_0^2 \int_{s_i}^{s_{i+k}} |s|^{2\varrho}|v|^2 ds \end{aligned}$$

which gives

$$|M_n(1-\chi_h)v|^2 = \sum_i |t_i^\varrho h^{-1/2}((1-\chi_h)v, B_i)|^2 \le c||\,|s|^\varrho v||_0^2 . \tag{4.23}$$

Thus it remains to prove

$$||H_n||_{\mathbf{C}^n \to \mathbf{C}^n} \le c \quad \forall n \ge i_0 \tag{4.24}$$

for i_0 large enough. Defining the diagonal matrix

$$D_n = \text{diag}\,\{t_i^\varrho\,,\ i = 0, ..., n-1\}$$

and setting

$$J_n = h^{-1} G_n\,,\ K_n = h^{-1} D_n G_n D_n^{-1} - J_n\,,$$

we observe that

$$H_n = h D_n G_n^{-1} D_n^{-1} = (J_n + K_n)^{-1}\,.$$

Therefore, (4.24) follows from the relations

$$||J_n||_{\mathbf{C}^n \to \mathbf{C}^n} \le c\,,\ ||J_n^{-1}||_{\mathbf{C}^n \to \mathbf{C}^n} \le c \quad \forall n \in \mathbb{N} \tag{4.25}$$

and

$$\forall \delta > 0 \quad \exists i_0 : ||K_n||_{\mathbf{C}^n \to \mathbf{C}^n} \le \delta \quad \forall n \ge i_0\,. \tag{4.26}$$

To prove (4.25), we note that (4.10) implies

$$c^{-1}\langle G_n(\zeta_i), (\zeta_i)\rangle \le h|(\zeta_i)|^2 \le c\langle G_n(\zeta_i), (\zeta_i)\rangle \quad \forall (\zeta_i) \in \mathbf{C}^n\,,$$

and hence

$$c^{-1}|(\zeta_i)|^2 \le \langle J_n(\zeta_i), (\zeta_i)\rangle \le c|(\zeta_i)|^2\,,$$

which gives the result. It remains to verify (4.26). The elements k_{ij} of K_n take the form

$$k_{ij} = h^{-1} t_i^\varrho G_{ij} t_j^{-\varrho} - h^{-1} G_{ij}\,,\ G_{ij} = (B_j, B_i)\,.$$

Thus we have for all i, j

$$|k_{ij}| \le h^{-1} G_{ij} \sup |1 - (t_l/t_m)^\varrho|\,,$$

where the supremum is taken over all indices l, m satisfying $|l-m| \le k-1$. Consequently, by the definition of t_i, this supremum can be made as small as desired for all $n \ge i_0$ if i_0 is chosen sufficiently large, and we now obtain (4.26) with the aid of the first inequality in (4.25).

(ii) By virtue of (4.21), (4.22) and (4.24), it suffices to show the estimate

$$|M_n \chi_h v|^2 = \sum_i t_i^{2\varrho} h^{-1} |(\chi_h v, B_i)|^2 \le c h^{2\varrho} ||v||_0^2 \quad \forall v \in H^0\,,\ \forall n \in \mathbb{N}\,.$$

The latter is true because $t_i \le ch$ for all i satisfying supp $B_i \cap (-h, h) \ne \emptyset$. $\square$

The superconvergence results depend on proving the estimate (4.4) for the operator $P_h - R_h$, where P_h is the orthogonal projection of H^0 onto the set V_h of smoothest 2π–periodic splines of order k on a uniform mesh, with mesh spacing $h = 2\pi/n$, and R_h denotes the solution operator of the qualocation method (2.10) for the circle. This will now be established, in a more general setting, as a corollary of two spline approximation results which also seem to be of independent interest.

With $\varphi_m(s) := e^{\imath ms}/(2\pi)^{1/2}$, let

$$T_h := \{\varphi_m : -n/2 < m \le n/2\}\,.$$

Furthermore, let $p_h : L_2 = H^0 \to V_h$ be the projection defined by

$$p_h g \in V_h\,,\ (p_h g, \chi) = (g, \chi) \quad \forall \chi \in T_h\,. \tag{5.1}$$

Note that the orthogonal projection P_h is defined by

$$P_h g \in V_h\,,\ (P_h g, \chi) = (g, \chi) \quad \forall \chi \in V_h\,. \tag{5.2}$$

Theorem 5.1. *For $0 \le t \le 2k$,*

$$||P_h g - p_h g||_0 \le ch^t ||g||_t\,,$$

if $g \in H^t$.

Proof. As usual, define

$$\Lambda_h = \{m \in \mathbb{Z} : -n/2 < m \le n/2\}\,,\ \Lambda_h^* = \Lambda_h \backslash \{0\}\,,$$

and

$$\psi_\mu = \begin{cases} \varphi_0 & \text{if } \mu = 0\,, \\ \sum_{m \equiv \mu} \left(\frac{\mu}{m}\right)^k \varphi_m & \text{if } \mu \in \Lambda_h^*\,, \end{cases}$$

so that $\{\psi_\mu : \mu \in \Lambda_h\}$ is a basis for V_h. Here and elsewhere $m \equiv \mu$ means $m \equiv \mu$ (mod n). Then $p_h g$ has the explicit formula

$$p_h g = \sum_{\mu \in \Lambda_h} \hat{g}(\mu) \psi_\mu\,, \tag{5.3}$$

since we easily verify that (5.1) is then satisfied, using the easily proved relation

$$(\psi_\mu, \varphi_\nu) = \delta_{\mu\nu}\,, \quad \text{for } \mu, \nu \in \Lambda_h\,.$$

On the other hand $P_h g$ has the explicit formula

$$P_h g = \hat{g}(0)\psi_0 + \sum_{\mu \in \Lambda_h^*} \frac{\sum_{m\equiv\mu} \left(\frac{\mu}{m}\right)^k \hat{g}(m)}{(\psi_\mu, \psi_\mu)} \psi_\mu\,, \tag{5.4}$$

since we then have $(P_h g, \psi_0) = \hat{g}(0)\psi_0 = (g, \psi_0)$, and for $\mu \in \Lambda_h^*$,

$$(P_h g, \psi_\mu) = \sum_{m \equiv \mu} \left(\frac{\mu}{m}\right)^k \hat{g}(m) = (g, \psi_\mu)\,,$$

so that (5.2) is satisfied. The denominator (ψ_μ, ψ_μ) in (5.4) can be written, for $\mu \in \Lambda_h^*$, as

$$
\begin{aligned}
(\psi_\mu,\psi_\mu) &= \left(\sum_{m\equiv\mu}\left(\frac{\mu}{m}\right)^k\varphi_m\,,\sum_{m\equiv\mu}\left(\frac{\mu}{m}\right)^k\varphi_m\right)=\sum_{m\equiv\mu}\left(\frac{\mu}{m}\right)^{2k}\\
&= \sum_{l=-\infty}^{\infty}\left(\frac{\mu/n}{l+\mu/n}\right)^{2k}=D^*(\mu/n)\,,
\end{aligned}
\tag{5.5}
$$

where for $|y|\le 1/2$

$$
D^*(y):=\sum_{l=-\infty}^{\infty}\left(\frac{y}{l+y}\right)^{2k}=1+E^*(y) \tag{5.6}
$$

and

$$
E^*(y):=y^{2k}\sum_{l\neq 0}\frac{1}{(l+y)^{2k}}\le cy^{2k}\,. \tag{5.7}
$$

Thus

$$
\begin{aligned}
P_hg-p_hg &= \sum_{\mu\in\Lambda_h^*}\frac{1}{D^*(\mu/n)}\left[\sum_{m\equiv\mu}\left(\frac{\mu}{m}\right)^k\hat g(m)-\left(1+E^*\left(\frac{\mu}{n}\right)\right)\hat g(\mu)\right]\psi_\mu\\
&= \sum_{\mu\in\Lambda_h^*}\frac{1}{D^*(\mu/n)}\left[-E^*\left(\frac{\mu}{n}\right)\hat g(\mu)+\sum_{m\equiv\mu}{}'\left(\frac{\mu}{m}\right)^k\hat g(m)\right]\sum_{p\equiv\mu}\left(\frac{\mu}{p}\right)^k\varphi_p\,,
\end{aligned}
$$

giving

$$
\begin{aligned}
||P_hg-p_hg||_0^2 &= \sum_{\mu\in\Lambda_h^*}\sum_{p\equiv\mu}\frac{1}{D^*(\mu/n)^2}\left|-E^*\left(\frac{\mu}{n}\right)\hat g(\mu)+\sum_{m\equiv\mu}{}'\left(\frac{\mu}{m}\right)^k\hat g(m)\right|^2\left(\frac{\mu}{p}\right)^{2k}\\
&= \sum_{\mu\in\Lambda_h^*}\frac{1}{D^*(\mu/n)}\left|-E^*\left(\frac{\mu}{n}\right)\hat g(\mu)+\sum_{m\equiv\mu}{}'\left(\frac{\mu}{m}\right)^k\hat g(m)\right|^2\\
&\le 2\sum_{\mu\in\Lambda_h^*}\left|E^*\left(\frac{\mu}{n}\right)\hat g(\mu)\right|^2+2\sum_{\mu\in\Lambda_h^*}\left|\sum_{m\equiv\mu}{}'\left(\frac{\mu}{m}\right)^k\hat g(m)\right|^2=:A+B\,,
\end{aligned}
$$

where we have used $D^*(y)\ge 1$, $(a+b)^2\le 2(a^2+b^2)$ and the notation

$$
\sum_{m\equiv\mu}{}'=\sum_{\substack{m\equiv\mu\\ m\neq\mu}}\,.
$$

Now for $0\le t\le 2k$, by (5.7)

$$
\begin{aligned}
A &\le c\sum_{\mu\in\Lambda_h^*}\left|\frac{\mu}{n}\right|^{4k}|\hat g(\mu)|^2\le c\sum_{\mu\in\Lambda_h^*}\left|\frac{\mu}{n}\right|^{2t}|\hat g(\mu)|^2\\
&= ch^{2t}\sum_{\mu\in\Lambda_h^*}|\mu|^{2t}|\hat g(\mu)|^2\le ch^{2t}||g||_t^2\,,
\end{aligned}
$$

and

$$\begin{aligned}
B &\leq 2 \sum_{\mu \in \Lambda_h^*} |\mu|^{2k} \left(\sum_{m \equiv \mu}{}' \frac{1}{|m|^k} |\hat{g}(m)| \right)^2 \\
&\leq cn^{2k} \sum_{\mu \in \Lambda_h^*} \left(\sum_{m \equiv \mu}{}' \frac{1}{|m|^{k+t}} |m|^t |\hat{g}(m)| \right)^2 \\
&\leq cn^{2k} \sum_{\mu \in \Lambda_h^*} \left(\sum_{m \equiv \mu}{}' \frac{1}{|m|^{2(k+t)}} \right) \left(\sum_{p \equiv \mu}{}' |p|^{2t} |\hat{g}(p)|^2 \right) .
\end{aligned}$$

Because $2(k+t) > 1$ we have

$$\sum_{m \equiv \mu}{}' \frac{1}{|m|^{2(k+t)}} = \sum_{l \neq 0} \frac{1}{|\mu + ln|^{2(k+t)}} = \frac{1}{n^{2(k+t)}} \sum_{l \neq 0} \frac{1}{|l + \mu/n|^{2(k+t)}} \leq \frac{c}{n^{2(k+t)}} ,$$

so

$$B \leq ch^{2t} \sum_{\mu \in \Lambda_h^*} \sum_{m \equiv \mu}{}' |m|^{2t} |\hat{g}(m)|^2 \leq ch^{2t} ||g||_t^2 .$$

Putting the results together, we obtain

$$||P_h g - p_h g||_0^2 \leq ch^{2t} ||g||_t^2 ,$$

for $0 \leq t \leq 2k$. □

Now let L be the (periodic) pseudo–differential operator of real order β defined by

$$Lv = \hat{v}(0)\varphi_0 + \sum_{m \neq 0} |m|^\beta \hat{v}(m) \varphi_m ,$$

or by

$$Lv = \hat{v}(0)\varphi_0 + \sum_{m \neq 0} \operatorname{sign} m |m|^\beta \hat{v}(m) \varphi_m .$$

In the former case L is "even", in the latter case it is "odd". With V_h' denoting the set of smoothest splines of order k' on the same mesh as above, we define $g_h = R_h g \in V_h$ to be the solution of the qualocation equation (cf. (2.10):

$$(Lg_h, \chi)_h = (Lg, \chi)_h \quad \forall \chi \in V_h' , \tag{5.8}$$

where the qualocation method is assumed to be (in the sense of [5]) both stable and of order $k - \beta + b$: that is to say, the "additional order of convergence" is $b \geq 0$. We also need to assume that the qualocation method is "well defined", i.e. (see [5, (2.12), (2.13)]) either $k > \beta + 1$, or $k > \beta + 1/2$ and the breakpoints are not quadrature points.

Theorem 5.2. *If $g_h \in V_h$ is the solution of the well defined qualocation method (5.8), assumed to be stable and to have additional order of convergence b, then for $0 \leq t \leq k - \beta + b$ and $t > \beta + 1/2$*

$$||g_h - p_h g||_0 \leq ch^t ||g||_{t + \max(\beta, 0)}$$

if $g \in H^{t+\max(\beta,0)}$.

Proof. Since $g_h \in V_h$ we have

$$g_h = \sum_{\mu \in \Lambda_h} \hat{g}_h(\mu)\psi_\mu \,,$$

which together with (5.3) gives

$$\begin{aligned} g_h - p_h g &= \sum_{\mu \in \Lambda_h} (\hat{g}_h(\mu) - \hat{g}(\mu))\psi_\mu \\ &= (\hat{g}_h(0) - \hat{g}(0))\psi_0 + \sum_{\mu \in \Lambda_h^*} (\hat{g}_h(\mu) - \hat{g}(\mu)) \sum_{m \equiv \mu} \left(\frac{\mu}{m}\right)^k \varphi_m \,, \end{aligned}$$

and hence

$$\begin{aligned} ||g_h - p_h g||_0^2 &= |\hat{g}_h(0) - \hat{g}(0)|^2 + \sum_{\mu \in \Lambda_h^*} \sum_{m \equiv \mu} |\hat{g}_h(\mu) - \hat{g}(\mu)|^2 \left|\frac{\mu}{m}\right|^{2k} \\ &\leq |\hat{g}_h(0) - \hat{g}(0)|^2 + c \sum_{\mu \in \Lambda_h^*} |\hat{g}_h(\mu) - \hat{g}(\mu)|^2 \,. \end{aligned}$$

Now [5, (3.4)] gives

$$\hat{g}_h(0) - \hat{g}(0) = \sum_j w_j \sum_{m \equiv 0} {}' [m]_\beta \hat{g}(m) \varphi_{m/n}(\xi_j) \,,$$

with (for $m \neq 0$)

$$[m]_\beta = \begin{cases} |m|^\beta & \text{if } L \text{ is even} \\ \operatorname{sign} m |m|^\beta & \text{if } L \text{ is odd.} \end{cases}$$

Thus with $\tau := t + \max(\beta, 0)$

$$\begin{aligned} |\hat{g}_h(0) - \hat{g}(0)|^2 &\leq \left(\sum_{m \equiv 0} {}' |m|^\beta |\hat{g}(m)|\right)^2 = \left(\sum_{m \equiv 0} {}' |m|^{\beta - \tau} |m|^\tau |\hat{g}(m)|\right)^2 \\ &\leq \sum_{m \equiv 0} {}' |m|^{2(\beta - \tau)} \sum_{m \equiv 0} {}' |m|^{2\tau} |\hat{g}(m)|^2 \\ &\leq c h^{2(\tau - \beta)} ||g||_\tau^2 \leq c h^{2t} ||g||_\tau^2 \,, \end{aligned}$$

because $\tau - \beta \geq t - \beta > 1/2$ and $\tau - \beta \geq t$. And also from [5, (3.4)], for $\mu \in \Lambda_h^*$,

$$\hat{g}_h(\mu) - \hat{g}(\mu) = -\frac{E(\mu/n)}{D(\mu/n)} \hat{g}(\mu) + R_n(\mu) \,,$$

where because the method is stable $\inf |D(y)| > 0$, and because the method is of additional order b,

$$|E(y)| \leq c|y|^{k - \beta + b} \quad \text{for } |y| \leq 1/2 \,;$$

and

$$R_n(\mu) = D\left(\frac{\mu}{n}\right)^{-1} \sum_j w_j \sum_{m \equiv \mu} {}' \left[\frac{m}{\mu}\right]_\beta \hat{g}(m) \varphi_{\frac{m-\mu}{n}}(\xi_j) (1 + \overline{\Delta'\left(\xi_j, \left(\frac{\mu}{n}\right)\right)} \,,$$

where from [5, Lemma 1 (iv)] $|\Delta'(x, y)| \leq c$ for $x \in [0, 1]$ and $|y| \leq 1/2$, giving

$$\sum_{\mu \in \Lambda_h^*} |\hat{g}_h(\mu) - \hat{g}(\mu)|^2 \leq c \sum_{\mu \in \Lambda_h^*} \left|\frac{\mu}{n}\right|^{2(k - \beta + b)} |\hat{g}(\mu)|^2 + c \sum_{\mu \in \Lambda_h^*} |R_n(\mu)|^2 =: Y + Z \,.$$

Because $t \le k - \beta + b$ and $t \le \tau$,

$$\begin{aligned} Y &= c \sum_{\mu \in \Lambda_h^*} \left| \frac{\mu}{n} \right|^{2(k-\beta+b)} |\hat{g}(\mu)|^2 \le c \sum_{\mu \in \Lambda_h^*} \left| \frac{\mu}{n} \right|^{2t} |\hat{g}(\mu)|^2 \\ &= ch^{2t} \sum_{\mu \in \Lambda_h^*} |\mu|^{2t} |\hat{g}(\mu)|^2 \le ch^{2t} ||g||_t^2 \le ch^{2t} ||g||_\tau^2 , \end{aligned}$$

and

$$\begin{aligned} Z &= c \sum_{\mu \in \Lambda_h^*} |R_n(\mu)|^2 \le c \sum_{\mu \in \Lambda_h^*} \left(\sum_{m \equiv \mu}{}' \left| \frac{m}{\mu} \right|^\beta |\hat{g}(m)| \right)^2 \\ &\le c \sum_{\mu \in \Lambda_h^*} |\mu|^{-2\beta} \left(\sum_{m \equiv \mu}{}' |m|^{\beta-\tau} |m|^\tau |\hat{g}(m)| \right)^2 \\ &\le c \sum_{\mu \in \Lambda_h^*} |\mu|^{-2\beta} \sum_{m \equiv \mu}{}' |m|^{2(\beta-\tau)} \sum_{p \equiv \mu}{}' |p|^{2\tau} |\hat{g}(p)|^2 \\ &\le ch^{2(\tau-\beta)} \sum_{\mu \in \Lambda_h^*} |\mu|^{-2\beta} \sum_{m \equiv \mu}{}' |m|^{2\tau} |\hat{g}(m)|^2 . \end{aligned}$$

Now

$$\begin{aligned} |\mu|^{-\beta} &\le \begin{cases} 1 & \text{if } \beta > 0 \\ n^{-\beta} & \text{if } \beta \le 0 \end{cases} \\ &= n^{-\beta + \max(\beta,0)} . \end{aligned}$$

Thus

$$Z \le ch^{2t+2(\max(\beta,0)-\beta)} n^{2(\max(\beta,0)-\beta)} \sum_{\mu \in \Lambda_h^*} \sum_{m \equiv \mu}{}' |m|^{2\tau} |\hat{g}(m)|^2 \le ch^{2t} ||g||_\tau^2 .$$

Thus on combining terms, we find

$$||g_h - p_h g||_0^2 \le ch^{2t} ||g||_\tau^2 = ch^{2t} ||g||_{t+\max(\beta,0)}^2 .$$

$\square$

Corollary 5.1. *Let g_h be as in Theorem 5.2, with $\beta \le 0$. Then for $0 \le t \le \min(2k, k - \beta + b)$, $t > \beta + 1/2$,*

$$||P_h g - g_h||_0 \le ch^t ||g||_t .$$

In particular, applying the last result to the pseudodifferential operator (2.12) which is of order -1, we obtain an estimate which implies (4.4).

References

[1] D.N. Arnold, W.L. Wendland, The convergence of spline collocation for strongly elliptic equations on curves, Numer. Math. **47**, 317–341 (1985).

[2] J.–P. Aubin, Approximation of Elliptic Boundary Value Problems, Wiley, New York (1972).

[3] G.A. Chandler, I.G. Graham, High order methods for linear functionals of solutions of second kind integral equations, SIAM J. Numer. Anal. **25**, 1118–1137 (1988).

[4] G.A. Chandler, I.G. Graham, Product integration–collocation methods for non–compact integral operator equations, Math. Comp. **49**, 467–478 (1988).

[5] G.A. Chandler, I.H. Sloan, Spline qualocation methods for boundary integral equations, Numer. Math. **58**, 537–567 (1990).

[6] D. Elliott, S. Prößdorf, An algorithm for the approximate solution of integral equations of Mellin type, Numer. Math. (to appear).

[7] J. Elschner, On spline approximation for a class of integral equations. I: Galerkin and collocation methods with piecewise polynomials, Mathem. Meth. Appl. Sci. **10**, 543–559 (1988).

[8] J. Elschner, On spline approximation for a class of integral equations. II: Galerkin's method with smooth splines, Math. Nachr. **140**, 273–283 (1989).

[9] J. Elschner, I.G. Graham, An optimal order collocation method for first kind boundary integral equations on polygons, Numer. Math. (to appear).

[10] J. Elschner, I.G. Graham, Parametrization methods for first kind integral equations on non–smooth boundaries, In: M. Costabel et al., eds., Boundary Value Problems and Integral Equations in Nonsmooth Domains, Marcel Dekker, New York (to appear).

[11] J. Elschner, I.G. Graham, Quadrature methods for Symm's integral equation on polygons (in preparation).

[12] I.G. Graham, Y. Yan, Piecewise constant collocation for first kind boundary integral equations, J. Austral. Math. Soc. **B 33**, 39–64 (1991).

[13] R. Kress, A Nyström method for boundary integral equations in domains with corners, Numer. Math. **58**, 145–161 (1990).

[14] W. McLean, I.H. Sloan, A fully discrete and symmetric boundary element method, IMA J. Numer. Anal. **14**, 311–346 (1994).

[15] S.G. Mikhlin, Singular integral equations, Uspehi Mat. Nauk **3**:3(25), 29–112 (1948) (Russian).

[16] S.G. Mikhlin, Variationsmethoden der Mathematischen Physik, Akademie–Verlag, Berlin (1962).

[17] S. Prößdorf, A. Rathsfeld, Quadrature methods for strongly elliptic Cauchy singular integral equations on an interval, In: H. Dym et al., eds., The Gohberg Anniversary Collection, **2**, 435–471, Birkhäuser, Basel (1989).

[18] S. Prößdorf, B. Silbermann, Numerical Analysis for Integral and Related Operator Equations, Akademie–Verlag, Berlin (1991).

[19] J. Saranen, The modified quadrature method for logarithmic–kernel integral equations on closed curves, J. Integral Eq. Appl. **3**, 575–600 (1991).

[20] J. Saranen, The convergence of even degree spline collocation solution for potential problems in smooth domains of the plane, Numer. Math. **53**, 499–512 (1988).

[21] J. Saranen, I.H. Sloan, Quadrature methods for logarithmic–kernel integral equations on closed curves, IMA J. Numer. Anal. **12**, 167–187 (1992).

[22] J. Saranen, W.L. Wendland, On the asymptotic convergence of collocation methods with spline functions of even degree, Math. Comp. **45**, 91–108 (1985).

[23] G. Schmidt, On spline collocation methods for boundary integral equations in the plane, Mathem. Meth. Appl. Sci. **7**, 74–89 (1985).

[24] I.H. Sloan, A quadrature–based approach to improving the collocation method, Numer. Math. **54**, 41–56 (1988).

[25] I.H. Sloan, B.J. Burn, An unconventional quadrature method for logarithmic–kernel integral equations on closed curves, J. Integral Eq. Appl. **4**, 117–151 (1992).

[26] I.H. Sloan, W.L. Wendland, A quadrature–based approach to improving the collocation method for splines of even degree, Z. Anal. Anw. **8**, 361–376 (1989).

[27] Y. Yan, I.H. Sloan, On integral equations of the first kind with logarithmic kernels, J. Integral Eq. Appl. **1**, 549–579 (1988).

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