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Recurrence of ancestral lines and offspring trees in time stationary branching populations

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0. Introduction and summary

We are studying time-discrete branching populations or "systems of independent branching Markov chains" on some complete, separable metric space of sites A : each n-th generation individual branches, independently of the others, into a daughter population whose distribution $\kappa_{(a)}$ depends on the mother individual's site a only, and given the history of the population up to time n, the (n+1)-st generation is the independent superposition of these daughter populations. In particular, we will focus on *time-stationary* processes $\underline{\Phi} = (\Phi_n)_{n\in Z}$ taking their values in the locally finite populations on A and following a branching dynamics κ of the described kind. By a suitable marking of the individuals (cf. the appendix) one can keep track of their genealogical relationships; therefore one is enabled to speak of an individual's *offspring tree* and an individual's *ancestral path*, and of the subsystems of mutually related individuals in $\underline{\Phi}$, which we call the *clans* of $\underline{\Phi}$.

Let us first have a look at a particular case, namely that of *pure* Markovian *migration* : For time sationary "systems of independent Markov chains" on a *discrete space* of sites, it turns out (cf. Thm. 7.6) that, a.s., each individual path is forward recurrent (i.e. will visit some site infinitely often in the future) if and only if it is backward recurrent (i.e. has visited some site infinitely often in the past).

In the presence of branching, the future history of an individual is no more given simply by a forward path, but by its offspring tree. Turnig first to the case of a *discrete* space of sites, let us call an offspring tree *recurrent* if it hits some site infinitely often. In a time stationary κ-branching process $\underline{\Phi}$, with probability one every clan which contains a recurrent offspring tree also contains a unique both forward and backward recurrent "immortal trunk", and the clans with this property are just those which ever hit the region

$$A_{\text{rec}} := \{ a\in A : T_a \text{ is almost surely recurrent}\}.$$

In this sense (similar to the pure migration case), a *spatial separation* of the "individually forward recurrent" clans (i.e. those which contain some recurrent offspring tree) and the "individually forward transient" clans hold true. Moreover, the individually forward recurrent clans altogether constiute a time stationary κ-branching subsystem of $\underline{\Phi}$.

While, as just stated, every "individual forward recurrent" clan is *hermetic* in the sense that all its individual ancestral paths are backward recurrent, the converse need not be true (cf.

example 5.4 in [MSW I]. Also, there exist equilibrium situations (even with a discrete space of sites!) in which the hermetic and the *immigrative* clans (i.e. those with transient ancestral paths) cannot be spatially separated (Ex. 7.3).

One central aim of the paper is to investigate the possibilities and the difficulties in extending results of the just mentioned kind to the case of *continuous* spaces of sites, and to clarify the role of "individual positive recurrence" therein. As it turns out, the latter exibits a remarkable robustness (both concerning the "non-discreteness" of the space of sites and also towards the "direction of time"); before we discuss this, let us briefly tell what happens with "forward and backward individual recurrence" in *continuous models* (as described at the beginning of the introduction):
We say that an ancestral path (respectively an offspring tree) is *recurrent* if it hits some bounded set at infinitely many time points; otherwise we call it *transient*.. Like in the discrete case, the hermetic clans (i.e. those with recurrent ancestral paths) form a time stationary κ-branching subsystem. But, in contrast to the discrete case, individual forward recurrence does not imply individual backward recurrence any more; and this may be so even in the absence of branching: we give an example (1.3) of a purely migrating system on $A = \mathbf{R}_+$ in which all individuals have transient lines of descent but follow forward recurrent paths. Also, another nice feature of the discrete situation does not carry over to the continuous case: the subsystem of individually forward transient clans need not necessarily follow the branching dynamics κ any more, cf. Example 1.9. So we are led to define the *transient type part* $\underline{\Phi}_{\text{trans}}$ of a time stationary κ-process $\underline{\Phi}$ as the system of those clans which actually never had a chance to develop a recurrent offspring tree, i.e. which never hit the set

$$A_{\text{REC}} := \{ a \in A : \mathrm{T}_a \text{ is with positive probability recurrent}\}.$$

Then both $\underline{\Phi}_{\text{trans}}$ and its complement in $\underline{\Phi}$ *are* time stationary κ-process; all ancestral lines of the latter grow out of A_{REC} (but, like in Example 1.9, some of them might indeed miss the chance to develop recurrent offspring trees).

Remarkably, as far as positive recurrence is concerned, the forward and backward concepts turn out to be equivalent also in the general class of models. Call an ancestral line (respectively an offspring tree) *positive recurrent* if it visits some bounded set with a positive lower frequency. Let $\underline{\Phi}$ be a time stationary κ-process, then, almost surely, for each clan Ψ in $\underline{\Phi}$ the following assertions are equivalent:

i) Ψ has a positive recurrent ancestral line

ii) Ψ contains some positive recurrent offspring tree

iii) Ψ hits the set

$$A_{\text{PREC}} := \{ a \in A : \mathrm{T}_a \text{ is with positive probability positive recurrent}\}.$$

iv) Ψ contains a unique both forward and backward positive recurrent genealogical line (which we call its *trunk*).

Moreover (see 4.5 and 4.9), there exists an at most countable partition (S_k) of A_{PREC} depending on the distribution of $\underline{\Phi}$) with the following properties:

a) For all k, there are finitely many clans in $\underline{\Phi}$ that ever hit S_k

b) the subsystem $\underline{\Phi}^k$ consisting of those clans is again a stationary κ-process

c) the positive recurrent part of $\underline{\Phi}$ is the (not necessarily independent) superposition of the $\underline{\Phi}^k$.

The processes $\underline{\Phi}^k$ consitute finitely fed equilibria in the sense of [LMW], section 2.10. : in S_k there is a possibly random finite number of immortal individuals following a time stationary Markov chain, and these individuals may generate emigrants outside of S_k (in fact only outside of A_{PREC}). Each of the processes $\underline{\Phi}^k$ consist of finitely many clans having positive recurrent trunks.

Another main aim of the paper is to explore the gap beween "positive recurrence on the clan level" and "individual positive recurrence". The former gets a rigorous meaning within the "family dynamics" introduced in [LMW] (which corresponds to the "clan process" in the measure-valued and "historical" scenario of [DP], sect. 6): there, the "equilibrium clans" constitute a system of independent Markov chains, whose paths possibly break off. The stationary κ-processes $\underline{\Phi}$ *with a finite number of clans* can be characterized as those ones for which there exists a sequence of regions $X_n \subseteq A$ whose random population numbers $\Phi_n(X_n)$ are tight and which capture (at least in probability) asymptotically all the ancestral lines of present time individuals encountered in a bounded region (Thm. 6.1). As example 2.8.4 in [LMW] shows, with a completely transient behavior on the individual level, there may still occur positive recurrence on the clan level. This is, however, impossible under certain restriction on the fertility of the individuals (Prop. 6.8), e.g. if for all sites $a \in A$ the mean number of children of δ_a does not exceed 1. Even without any "supercriticalities", however, the recurrence behaviour on the clan level may be different from that on the individual level: we conjecture that, for the simple branching random walk on $\mathbf{Z}^d$, $d \geq 3$, the equilibrium clans are recurrent (in the sense that they visit each site infinitely often) if and only if $d < 5$. (The corresponding result for branching Brownian motion has recently been proved in [SW]).

Section 7, which continues studies initiated in [MSW], is devoted to the case of a discrete space of sites A. Like with Markov chains, the concepts discussed above can then be localized to sites, and A_{REC} and A_{PREC} can be replaced by their subsets

$$A_{rec} := \{ a \in A : T_a \text{ hits } a \text{ almost surely} \}$$

and

$$A_{\mathrm{prec}} := \{ a \in A : \mathrm{T}_a \text{ hits } a \text{ in a time with finite expectation} \},$$

which, for any time stationary κ-process, turn out to be essentially equal to A_{REC} and A_{PREC}, respectively (7.5 and 7.15). "Mutual reachability" under the branching dynamics induces a partition (X_k) of A_{rec} which now renders a decomposition $(\underline{\Phi}^k)$ of the recurrent part of any time stationary κ-process $\underline{\Phi}$ in much the same way as described above; the population numbers $\Phi_0{}^k(X_k)$ are finite exactly for those $\underline{\Phi}^k$ which belong to the positive recurrent part of $\underline{\Phi}$, or equivalently, for those X_k which are contained in A_{prec} (7.9, 7.13). Also, in the discrete case it turns out that a time stationary κ-process with finitely many clans having recurrent ancestral lines in fact is finitely fed, thus growing out of positive recurrent trunks (see 7.18, and 2.5 b) for a non-discrete counterexample). A tool for proving these results is the *age dependent Galton Watson dynamics embedded at a site* as defined in [MSW 1], which models the branching structure of excursion time lengths from site a in the offspring tree T_a.

The additional assumption of *finite first moments* would forbid certain pathologies like that of Example 1.3 also in the case of general spaces of sites. We have refrained from a special treatment of the first order case in this paper in order not to make it longer as it already is; let us just indicate that, also in the first order case, the "backward recurrence concepts" turn out to be more robust than the "forward recurrence concepts"; for example, the question whether a time stationary κ-process $\underline{\Phi}$ of first order consists of hermetic clans only can be decided from the intensity measure of Φ_0 and the intensity (or mean) kernel of κ only, whereas the corresponding question with "hermetic" replaced by "individual forward recurrent" needs a knowledge both on the intensity measure Φ_0 and on κ itself.

1. Individual backward and forward recurrence and transience in equilibrium branching populations

In our study of the recurrence and transience behavior of the individuals' ancestral paths and offspring trees in equilibrium branching populations, we will always restrict to the following class of models:

The individuals live in some complete separable metric space (A, ρ_A), which can be viewed as a space of *sites* or space of *types*. The collection of Borel-measurable subsets of A (which we simply address as *regions*) is denoted by $\mathcal{A}$. At each fixed time, the number of individuals in each bounded region is a.s. finite. The random populations are thus modeled as locally finite random counting measures Φ on A and develop in discrete time in a Markovian way: Given the prehistory $(\Phi_0 \ldots, \Phi_{n-1}, \Phi_n)$ up to time n, where $\Phi_n = \sum_{i \in I} \delta_{a_i}$, the random

population Φ_{n+1} arises as the independent superposition $\sum_{i \in I} \chi_i$ of the daughter populations χ_i of the mother individuals δ_{a_i}; the distribution of the random population χ_i is assumed to depend on its mother's site a_i only .
The distribution of the daughter population $\chi^{(a)}$ of a mother indivdual δ_a at site a will be denoted by $\kappa_{(a)}$; the family $\kappa = (\kappa_{(a)})_{a \in A}$, is thus *the* basic datum of the random dynamics under consideration.

In the whole paper we will focus our attention to *equilibrium situations*, i.e. to situations in which the distributions of Φ_n and Φ_{n+1} coincide. In such equilibrium situations, Kolmogorov's extension theorem renders a doubly infinite Markovian sequence $(\Phi_n)_{n \in Z}$ of random populations whose distribution is time stationary. We will refer to $(\Phi_n)_{n \in Z}$ as a *stationary κ–process*, and to Φ_0 as a random *κ-equilibrium population.*
Since Φ_n is the independent superposition of the n-th generation offspring populations of the individuals in Φ_0 , the assumed a.s. local finiteness of Φ_n enforces that the n-th generation offspring of an individual δ_a is a.s. locally finite for $E\Phi_0$-almost all a, or in the language of [LMW], that the *clustering powers* $\kappa_{(a)}^{[n]}$ of $\kappa_{(a)}$ exist for $E\Phi_0$-almost all a.
We will require once and for all that the random n-th generation offspring population $\chi_n^{(a)}$ of an individual δ_a is a.s. locally finite for *all* $a \in A$ and $n \geq 1$; beside the measurable dependence $a \rightarrow \kappa_{(a)}$ this will be our only requirement on the branching dynamics κ.

By a suitable marking of the individuals (cf. Appendix), one is able to read off the genealogical relationships between individuals of different generations.
We say that the *ancestral path* $(\ldots, a_{n-1}, a_n)$ of an individual δ_{a_n} in the population Φ_n is *backward transient* if it enters from infinity in the sense that for all bounded subsets B of A, a_m is outside of B for all sufficiently early times m.

Let us consider for a moment the special case of independent migration in which each daughter population consists of exactly one individual - we are then in the scenario of independent Markov chains ([L], [LP]). One could guess that in an equilibrium situation of independent Markov chains, backward transience of all individuals' paths implies forward transience of all individuals' paths. The following example, however, shows that things may be totally different from this even in the case of a deterministic migration dynamics. In this example, the space of sites will be $\mathbf{R}_+ := [0, \infty)$ with the usual metric.

1.1. Proposition. There exists a mapping $\vartheta: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ with $\vartheta(x) = x-1 \;\; \forall\, x \geq 1$, admitting a sequence $(\Phi_n)_{n \in Z}$ of random populations with the following properties:
i) $\Phi_{n+1} = \vartheta(\Phi_n)$ (where, for $\Phi = \sum_{i \in I} \delta_{a_i}$, $\vartheta(\Phi)$ denotes the "shifted" population $\sum_{i \in I} \delta_{\vartheta(a_i)}$)

ii) $(\Phi_n)_{n\in \mathbf{Z}}$ has a time stationary distribution

iii) the ancestral paths $(\ldots, x_{n-1}, x_n)$ of all individuals in Φ_n obey $x_{n-m} \to \infty$ as $m \to \infty$, and therefore are backward transient.

1.2. Remark. Intuitively, this scenario can be described as follows: Each individual migrates in a deterministic way to the left until it reaches the unit interval. Since the migration step is one, its "code" $y = x-[x]$ is not changed during this migration. After having reached the unit interval $[0,1)$ at the location y, the individual jumps to the location $\vartheta(y)$ in the next step. This makes already clear that the situation is forward recurrent: all paths will visit the unit interval infinitely often. However, the equilibrium population will be constructed in such a way that initially all paths enter from infinity !

The **proof of Proposition 1.1** is provided by the following example together with the mapping $f(x) := ([x]+1, g(x-[x]))$ from $\mathbf{R}_+$ onto $\mathbf{N} \times \{0,1\} \times K$, where g is some bijective, bi-measurable mapping from $[0,1)$ onto $\{0, 1\} \times K$, and K denotes the set of those 0-1 sequences $(k_i)_{i\in \mathbf{N}}$ which have infinitely many ones. (Note that, as a measurable subset of the standard Borel space $\{0,1\}^{\mathbf{N}_0}$, the set $\{0, 1\} \times K$ is σ-isomorphic to $[0,1)$.)

1.3. Example. As indicated in the preceding remark, we put $A := \mathbf{N} \times \{0,1\} \times K$, and equip A with the metric inherited from the Euclidian one under the mapping f. An element $a \in A$ consists of three components (q, v, s), where $q \in \mathbf{N}$ stands for the "position" of the individual, $v \in \{0,1\}$ is an index which will indicate whether the individual, in the course of its prehistory, has ever visited position 1, and $s \in K$ constitutes, together with v, the individual's code. The position of the individual can be interpreted as its remaining lifetime: an individual arriving at position 1 undergoes a renewal in the next time unit.

A mapping T on A is defined as follows:

$T((q, v, s)) := (q-1, v, s)$ if $q > 1$

$T((1, v, s)) := (q(s), 1, t(s))$

where , for $s = (k_i)_{i\in \mathbf{N}}$, $q(s) := \min\{k_i \mid i\neq 0\}$, and $t(s) := (k_{i+q(s)})_{i\in \mathbf{N}}$.

The equilibrium population Φ_0 is now constructed as the outcome from a stream of immigrants as follows: First we assign to each $n \in \mathbf{Z}$ a random element $s_n = (k_{n,i})_{i\in \mathbf{N}}$ in K, specifying the construction of (s_n) only below. For each position $n \geq 1$, we put $b_n := (n, 0, s_n)$, thus obtaining those individuals δ_{b_n} in Φ_0 which are presently immigrating from infinity and have not yet undergone a renewal. In oder to bring in also the actual effect of all the "already renewed" individuals, we put for $n = 0, -1, -2, \ldots$

$$a_n := T^{1-n}((1, 0, s_n)).$$

We will show in a minute that $(s_n)_{n\in \mathbf{Z}}$ can be chosen such that

1. the distribution of (s_n) is invariant with respect to a shift in n

2. $\sum_{n\geq 1} k_{-n,n} < \infty$ almost surely.

We now put $\Phi_0 := \sum_{n<1} \delta_{a_n} + \sum_{n\geq 1} \delta_{b_n}$. Because of 1., $T(\Phi_0)$ has the same distribution as Φ_0. Noting that the position component of a_n equals 1 if and only if $k_{-n,n} = 1$, we conclude from 2. that Φ_0 is locally finite.

It remains to ensure the existence of a random sequence $(s_n)_{n\in \mathbf{Z}} = (k_{n,i})_{n\in \mathbf{Z},\, i\in \mathbf{N}}$ in K which obeys 1. and 2. To this purpose, let $(\eta_j)_{j\in \mathbf{Z}}$ be a two-sided i.i.d. sequence of random natural numbers with infinite expectation. For all $n\in Z$, $i \in \mathbf{N}$, we set

$$k_{n,i} := \begin{cases} 1 \text{ if } \eta_{i+n} \geq i \\ 0 \text{ otherwise} \end{cases}$$

Then obviously $(k_{n,i})$ obeys 1. and 2.; moreover, by the Borel-Cantelli lemma there holds $\sum_{i\geq 1} k_{0,i} = \infty$ a.s., and therefore s_0 belongs to K a.s. ♦

1.4. Notation: For all time points n, m $\in$ **Z**, and all regions $X \subseteq A$, $\Phi_{m\,|\,n,X}$ denotes the subpopulation of all those individuals in Φ_m which are related "in the direct line" to an X-individual in Φ_n. In other words, $\Phi_{m\,|\,n,X}$ is the subpopulation of those individuals in Φ_m which, in the case $n < m$, have an ancestor at time n in X, in the case $n > m$ have a descendant at time n in X, and in the case $m = n$ are situated in X.

1.5 Definition. We say that a stationary κ-process $(\Phi_n)_{n\in \mathbf{Z}}$ is *immigrative* if, with probability one, all its ancestral paths are backward transient, and that $(\Phi_n)_{n\in \mathbf{Z}}$ is *of transient type* if, with probability one, all individuals in (Φ_n) have only finitely many descendants in each $B\in \mathcal{B}$ (the symbol $\mathcal{B}$ denotes the bounded, measurable subsets of A, which we will also simply call "bounded regions").

Thus, with the notation introduced in 1.4, $(\Phi_n)_{n\in \mathbf{Z}}$ is immigrative iff, for all $B\in \mathcal{B}$ and $k\in \mathbf{Z}$,

$\sum_{n\geq 1} \Phi_{k-n\,|\,k,B}(B) < \infty$ a.s,

and $(\Phi_n)_{n\in \mathbf{Z}}$ is of transient type iff, for all $B\in \mathcal{B}$ and $k\in \mathbf{Z}$, $\sum_{n\geq 1} \Phi_{k+n\,|\,k,B}(B) < \infty$ a.s.

(Note that, due to the assumed stationarity of $(\Phi_n)_{n\in \mathbf{Z}}$, it is enough to require these properties only for $k = 0$.)

Example 1.3 provides an immigrative equilibrium population of non-transient type even in a pure migration case, and example 5.4 in [MSW I] shows that even in the discrete case there may exist first order equilibrium populations of transient type which are not immigrative.

We call a bounded region B *immigrative* for $(\Phi_n)_{n\in \mathbf{Z}}$ if $\sum_{n\geq 1} \Phi_{-n\,|\,0,B}(B) < \infty$ a.s, and *transient* for $(\Phi_n)_{n\in \mathbf{Z}}$ if $\sum_{n\geq 1} \Phi_{n\,|\,0,B}(B) < \infty$ a.s. It is easy to see that the union of two immigrative bounded regions B_1 and B_2 is again immigrative. (Indeed, if an individual in B_1 would have infinitely many ancestors in B_2, then with positive probability there would also exist individuals in B_2 having infinitely many ancestors in B_2.) However, the union of two transient bounded regions need not be transient, as the following modification of Example 1.3 shows:

1.6. Example. Let A be as in Example 1.3, and consider the following dynamics, which is now not any more a pure migration, but still is deterministic: an individual at site (q, v, s) steps to (q-1, v, s) as long as its position q is larger than one, from (1, 0, s) , where $s = (k_i)$, it splits into the population $\Psi(s) := \sum_{i\geq 1} k_i \delta_{(i,1,s)}$, and at (1,1,s) it simply dies. Let b_n be as in Example 1.3, and put, for n<1, $\gamma_n := D^{1-n}((1,0,s_n))$, where $D^m(a)$ denotes the n-th generation offspring population arising from an individual δ_a . Then $\Phi_0 := \sum_{n<1} \gamma_n + \sum_{n\geq 1} \delta_{b_n}$ is an equilibrium population, for which both $B_1 := \{1\}\times\{0\}\times K$ and $B_2 := \{1\}\times\{1\}\times K$ are transient, but $B_1 \cup B_2$ is not - in fact, an infinite offspring of each B_1-individual reaches B_2 in the course of time.♦

1.7. Definition. For a stationary κ-process $\underline{\Phi} = (\Phi_n)_{n\in Z}$ we denote by $\Phi_{n,imm}$ the subpopulation of Φ_n consisting of all those individuals whose ancestral paths pass through any $B \in \mathcal{B}$ only finitely often, and by $\Phi_{n,herm} := \Phi_n - \Phi_{n,imm}$ the subpopulation of Φ_n consisting of all those individuals whose ancestral paths pass through some $B \in \mathcal{B}$ infinitely often. Due to the following lemma, which will be used several times in the course of the paper, both $(\Phi_{n,imm})_{n\in Z}$ and $(\Phi_{n,herm})_{n\in Z}$ are stationary κ-processes; they will be called the *immigrative* and the *hermetic* part of $(\Phi_n)_{n\in Z}$, respectively.

1.8. Lemma. Consider a subset Y of A^{Z_-} which is measurable and invariant with respect to the shift $(\ldots,c_{-1}, c_0) \rightarrow (\ldots,c_{-1})$ in the sense that $(\ldots,c_{-1}, c_0)\in Y$ iff $(\ldots,c_{-1})\in Y$. Let $\underline{\Phi} = (\Phi_n)_{n\in Z}$ be a stationary κ-process, and let for all $n \in Z$, γ_n be the subpopulation consisting of all those individuals in Φ_n whose ancestral paths $(\ldots,a_{-1}, a_0)$ belong to Y. Then also $\underline{\gamma} = (\gamma_n)_{n\in Z}$ is a stationary κ-process.

Proof. Writing $\gamma_t' := \Phi_t - \gamma_t$, $t \in Z$, we observe that the distribution of (γ_n, γ_n'), given $(\gamma_t,\gamma'_t)_{t\leq n-1}$, is, due to independence of the branching, the product of the κ-offspring distributions of γ_{n-1} and γ_{n-1}' ; hence $(\gamma_n)_{n\in Z}$ is a Markovian sequence following the dynamics κ. On the other hand, the distribution of γ_n equals that of γ_{n-1} due to the stationarity of $\underline{\Phi}$ (also in a refined model, in which individuals are "marked" with their ancestral lines). ♦

Let us now turn to "forward" concepts: The fact that a stationary κ-process $(\Phi_n)_{n\in Z}$ is of transient type can be reexpressed as follows:

For almost all Φ_0 and all individuals δ_a in Φ_0, the total number of δ_a 's random offspring in a bounded region B, that is, $\sum_{i>0} \chi_i^{(a)}(B)$, is finite a.s. ; in other words, a.s. the population Φ_0 does not charge

$$A_{REC} := \{ a\in A : \text{for some bounded region B}, \sum_{i>0} \chi_i^{(a)}(B) = \infty \text{ with positive probability}\}.$$

Clearly by its definition, immigration into A_{REC} is prohibited in the sense that

$$E\chi^{(a)}(A_{REC}) = 0 \quad \text{for all } a \notin A_{REC} .$$

Since a prohibited immigration in the forward time direction implies a prohibited emigration of ancestral paths in the backward time direction, we infer that, in a stationary κ-process $(\Phi_n)_{n\in Z}$, the random population Φ_n is the sum of two subpopulations:
the population $\Phi_{n,rec}$ consisting of all those individuals of Φ_n whose ancestral paths originally come from A_{REC},
and the population $\Phi_{n,trans}$ consisting of all those individuals of Φ_n whose ancestral paths never visited A_{REC}.
By Lemma 1.8, both $(\Phi_{n,rec})_{n\in Z}$ and $(\Phi_{n,trans})_{n\in Z}$ are stationary κ-processes, and will be called the *recurrent type* and the *transien type part* of $(\Phi_n)_{n\in Z}$, respectively.
For some time we conjectured that the recurrent type part $(\Phi_{n,rec})_{n\in Z}$ as defined above just always consists of those individuals which are related to some individual in (Φ_n) having a "recurrent offspring tree". But alas, having a positive chance to generate a recurrent offspring tree is not the same as actually doing it, and in fact the conjecture turned out to be wrong:

1.9.Example. Modify example 1.3 as follows: an individual at site (1, 0, s) , that is, a "fresh immigrant" at position 1, follows the dynamics of 1.3 only with probability 1–p, and dies with probability p, where p $\in$ (0,1) is a fixed parameter. Like in 1.3, a steady stream of immigrants gives rise to a stationary process $(\Phi_{n,})$. For this branching dynamics one has $A_{REC} = A$, and therefore $(\Phi_{n,rec}) = (\Phi_{n,})$, since trivially all ancestral lines come from A, but still there are also "transient individuals" in $\underline{\Phi}$, whose line of descendants is extinct after hitting $\{1\}\times\{0\}\times K$ for the first time. This example also shows that it would be of no advantage to replace "with positive probability" by "with probability one" in the definition of A_{REC}, since this would result in $A_{REC} = N\times\{1\}\times K$, which is eventually left by all ancestral lines backward in time. ♦

2. Finitely fed equilibrium distributions

Let $\underline{\Phi} = (\Phi_n)_{n\in Z}$ be a stationary κ-process with equilibrium distribution P. Recall that in 1.4 we defined $\Phi_{m|n,X}$ as the subpopulation of those individuals in Φ_m which are related "in the direct line" to an X-individual in Φ_n. Now a region X may have the property $\Phi_{0|-n,X} \xrightarrow{d} \Phi_0$ (or equivalently, $\Phi_{0|-n,A\setminus X} \xrightarrow{d} o$) as $n \to \infty$ (where $\xrightarrow{d}$ denotes convergence in distributions of random populations w.r.to the vague topology, cf. the Appendix, and o stands for the void population). In this case we call X a P-*feeding region* . In other words: X is a P-feeding region iff, for each bounded region B the number of present time individuals in B, whose ancestral lines did not pass through X at past time $-n$ converges in probability to zero as $n \to \infty$.

An equivalent formulation (in the language of the monograph [LMW]) is the following: Let ${}_X P$ be the distribution of the restriction of Φ_0 to X. Then X is a feeding region for $(\Phi_n)_{n\in Z}$ if and only if the "clustered distributions" $({}_X P)_{\kappa[n]}$ converge weakly to P as $n \to \infty$.

The system $\mathcal{F}(P)$ of P-feeding regions is ascending and closed under finite intersections. Writing $\Lambda_P = E\Phi_0$ for the intensity measure of P, we note that from

$$X \in \mathcal{F}(P)\,,\ X' \in \mathcal{A}\,,\ \Lambda_P(X \Delta X') = 0$$

it follows that $X' \in \mathcal{F}(P)$. A P-feeding region X is said to be *essentially minimal* if all $X' \in \mathcal{F}(P)$ obey $\Lambda_P(X \setminus X') = 0$. Then, if it exists at all, an essentially minimal P-feeding region S is uniquely determined by P up to an arbitrary Λ_P–null set in $\mathcal{A}$.

We will prove that any essentially minimal P-feeding region S is *P-autonomous* in the following sense: with probability one, there is no S-immigrant (i.e. an S-individual with some ancestor outside of S) in $\underline{\Phi}$. By the way, an equivalent way to express this property is the following (cf. [LMW]):

A region X is P-autonomous iff $J(a,X) = 0$ for Λ_P-almost all sites $a \in A$,

where $J(a,.) := E\chi^{(a)}(.)$ defines the *intensity kernel* J of the branching dynamics κ.

2.1. Proposition. Each essentially minimal P-feeding region is P-autonomous.

Proof. We will show that each non-P-autonomous P-feeding region S can be "essentially decreased", still remaining P-feeding. To this end, we fix a P-feeding region S for which $\Phi_{0|-1,A\setminus S}$ (S) is not a.s. equal to zero. We also fix a bounded region $X \subseteq S$ such that $\Phi_{0|-1,A\setminus S}(X) > 0$ with positive probability, and we write $\Psi := \Phi_{0|-1,A\setminus S}$.

Let us fix some other bounded region V. We first want to get some bound on the random number of individuals entering X at time zero from outside S and having an n-th generation offspring which charges V. For all k = 1, 2,… there holds

$$E\,[\int_X \kappa^{[n]}_{(b)}\,(\chi(V)>0)\,\Psi(db)\,;\,\Psi(X)=k] \;\leq k\,\mathrm{Prob}\,[\Phi_{n|-1,A\setminus S}(V) > 0,\,\Psi(X)=k].$$

Defining the measure q on X by

$$q(L) := \sum_{k>0} k^{-1}\, E(\Psi(L);\Psi(X)=k)\,, \quad L \in \mathcal{A}, L \subseteq X,$$

we obtain

$$(o) \qquad \int_X \kappa^{[n]}_{(b)}(\chi(V)>0)\, q(db) \leq \text{Prob}\,[\Phi_{n|-1,\mathcal{A}\setminus S}(V) > 0] \to 0 \quad \text{as } n \to \infty.$$

For all $\mathcal{A}$-measurable subsets L of X we put

$$p(L) := \sum_{k>0} k^{-1}\, E(\Phi_0(L);\Phi_0(X)=k)$$

Due to our assumption, q(X) is nonzero; moreover, since Ψ is a subpopulation of Φ_0, q is absolutely continuous w.r. to p. Let f be a version of the Radon-Nikodym derivative dq/dp, and choose c>0 small enough such that $L := f^{-1}([c,\infty))$ has positive p-measure (and therefore also positive intensity $\Lambda_P(L)$).

For all natural numbers m there holds

$$\text{Prob}[\Phi_{n|0,L}(V) > 0] \leq \text{Prob}[\Phi_0(X) > m] + \sum_{i=1}^{m} \text{Prob}[\Phi_{n|0,L}(V) > 0\,;\ \Phi_0(X) = i\,]$$

$$= \text{Prob}[\Phi_0(X) > m] + \sum_{i=1}^{m} E[\,\text{Prob}[\Phi_{n|0,L}(V) > 0 \mid \Phi_0\,]\,;\Phi_0(X) = i]$$

$$\leq \text{Prob}[\Phi_0(X) > m] + \sum_{i=1}^{m} E[\int_L \kappa^{[n]}_{(b)}(\chi(V)>0)\,\Phi_0(db)\ ;\ \Phi_0(X) = i]$$

$$\leq \text{Prob}[\Phi_0(X) > m] + m \int_L \kappa^{[n]}_{(b)}(\chi(V)>0)\, p(db).$$

Since, when restricted to L, p is absolutely continuous w.r. to q with a bounded density, we thus obtain from (o) :

$$\limsup_{n\to\infty} \text{Prob}[\Phi_{n|0,L}(V) > 0] \leq \text{Prob}[\Phi_0(X) > m] + 0.$$

Since the r.h.s. can be made arbitrarily small for large enough m, we infer that $\lim_{n\to\infty} \text{Prob}[\Phi_{n|0,L}(V) > 0] = 0$, revealing that $\mathcal{A} \setminus L$ is a P-feeding region. Hence also S\L, which is essentially smaller than S, is a P-feeding region, and thus S cannot be essentially minimal. ♦

In section 2.9 of [LMW] the *finitely fed* equilibrium distributions were introduced as those κ-equilibria P for which there exists a P-autonomous, P-feeding region X with the property $P(\Phi(X)<\infty) = 1$. From [LMW], theorem 2.9.17, proposition 2.9.18 and theorem 2.9.23 there results directly the following

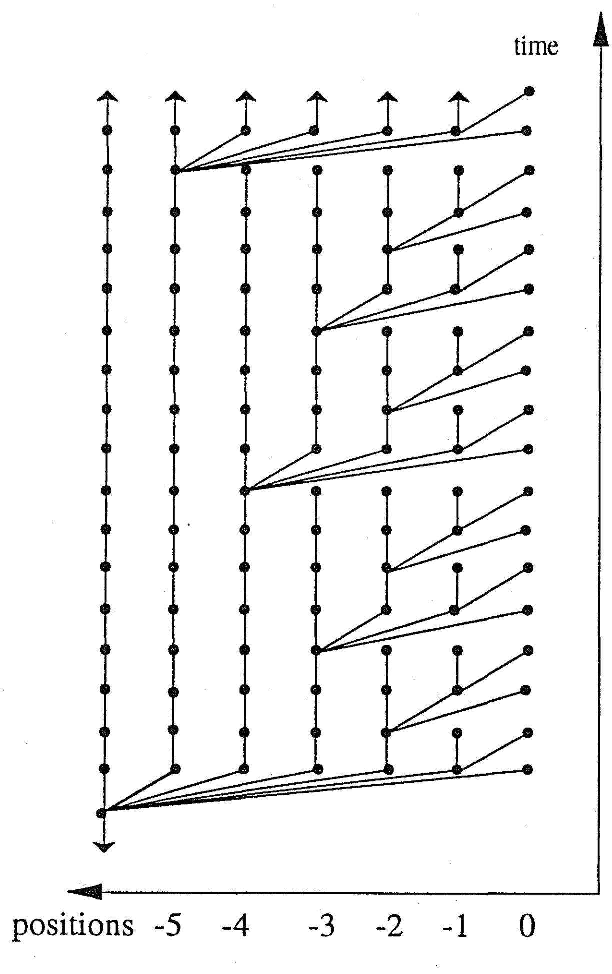
2.2. Proposition. An equilibrium distribution P is finitely fed iff there exists a P-feeding region X with the property $P(\Phi(X)<\infty) = 1$. In this case there exists an essentially minimal P-feeding region S_P with the property

$$\chi^{(a)}(S_P) = 1 \text{ a.s.} \quad \text{for } \Lambda_P\text{-almost all sites } a \in S_P$$

(recall that $\chi^{(a)}$ stands for the random daughter population of a mother individual at site *a*).

2.3. Remark. The genealogical picture of finitely fed equilibrium distributions resp. their associated stationary κ-process $\underline{\Phi}$ thus is the following: With probability one, backwards in time the ancestral lines of all individuals are eventually trapped in S_P; restricted to S_P, the dynamics is that of a (possibly random) finite number of independently migrating individuals (which may well generate also "emigrants" outside of S_P).

Let $\underline{\Phi}$ be a stationary κ-process, and denote by r_n the number of equivalence classes of mutually related individuals in Φ_n. Clearly, r_n is a.s. nonincreasing in n; due to the stationarity of (Φ_n), $r_n = r_0$ a.s. The number $r{:=}r_0$ of *clans* in $\underline{\Phi}$, which may be infinite, is random in the sense that it depends on the realization of $\underline{\Phi}$. We say that $\underline{\Phi}$ (resp. the corresponding equilibrium distribution of Φ_0) is *of finite type* if r is a.s. finite. The above remark states in particular that each finitely fed equilibrium distribution is of finite type; the converse, however, is not true.



Indeed, [LMW], 2.8.4, provides the example of a stationary κ-process $\underline{\Phi}$ with r = 1, i.e. consisting of one single clan, where the deterministic branching dynamics is such that the total offspring number $\sum_{n=1}^{\infty} \chi_n^{(a)}(A)$ of each individual δ_a is finite - and therefore $\underline{\Phi}$ is certainly not finitely fed. Let us recall that in this example the space of sites is $A = \bigcup_{i\geq 0} \{-i\}\times\{1,\ldots, 2^i\}$, where each site consist of a "position" and and "age" component. The branching dynamics is such that an individual at position i steadily ages, until it dies at the age $j = 2^i$. At midlife $j = 2^{i-1}$, it gives rise to one daughter at each of the positions $-i+1, -i+2,\ldots, 0$ and age 1. The picture displays (part of) the projection of a realization of the random time-stationary clan $\underline{\Phi}$ onto the position components.

With probability one, every ancestral line in $\underline{\Phi}$ eventually leaves each of the countably many sites in A forever; hence, if A is equipped, as usual, with a metric ρ_A making just the finite subsets of A bounded, then $\underline{\Phi}$ is clearly immigrative. We will show:

2.4. Proposition. With a suitable metric ρ on the (countable) space of sites A, the time-stationary κ-process (Φ_n) constructed in [LMW], Example 2.8.4, (which is of finite type) is hermetic and of transient type.

Proof. We start from the observation that the region $X := \{(-i, 1) : i > 0\}$ is visited by the ancestral path of each individual in (Φ_n) infinitely often. We will construct a suitable metric ρ which makes X a bounded set. To this purpose, first put, for $a,b \in A$,

$$\sigma(a,b) := \begin{cases} 1 \text{ if } a,b \in X \text{ and } a \neq b \\ \text{the Euclidian distance of } a \text{ and } b \text{ otherwise} \end{cases}$$

For $x, y \in A$, call any finite sequence $a_1, \ldots, a_n$ with $a_1 = x$ and $a_n = y$ a *path* from x to y. We define the *length* of a path w by $|w| := \sum_{1 \leq i < n} \sigma(a_i, a_{i+1})$, and put $\rho(x,y) := \min\{|w| : w \text{ is a path from x to y}\}$. The metric ρ generates the discrete topology; a subset B of A is ρ-bounded if the Euclidean distance of the elements $b \in B$ to X is bounded. The intensity measure ν of Φ_0, which (cf. [LMW] 2.8.4) is given by $\nu(\{(-i,j)\}) = 2^{-i}$, is also locally finite w.r. to ρ, and since for every site $a \in A$ the random n-th generation offspring $\chi_n^{(a)}$ is finite, also the "a.s. local finiteness requirement" for the offspring populations formulated in section 1 is met. As stated at the beginning of the proof, each ancestral line in $\underline{\Phi}$ visits the (ρ-bounded) region X infinitely often, and therefore $\underline{\Phi}$ is hermetic. On the other hand, since the total offspring number $\sum_{n=1}^{\infty} \chi_n^{(a)}(A)$ of each individual δ_a is finite, $\underline{\Phi}$ is of transient type. ♦

2.5. Remark. a) Proposition 2.4 gives an example of a hermetic, finite type stationary κ-process which is not finitely fed. We will show in section 7 that, whenever a space of sites A is countable and the metric ρ_A is such that just the finite subsets of A are bounded, then every hermetic time stationary κ-process of finite type *is* finitely fed. Thus, with the usual concept of boundedness, on a countable space of sites peculiarities like the mentioned ones cannot happen.

b) Note that, in the scenario constructed in Proposition 2.4, the bounded region X, which is not immigrative, is the countable union of the immigrative singletons $\{x\}$, $x \in X$.

3. Equilibrium populations with frequent ancestors in a bounded region

The program of this section is to prove the following result which is the basic tool for the decomposition theorems 4.5 and 5.3 in the next sections:

3.1. Theorem. A time stationary κ-process is finitely fed provided there is a bounded region X and a constant c>0 such that, with probability one, all ancestral lines visit X with an upper frequency not less than c.

The **proof** of this theorem will be divided into several steps, some of them interesting in their own right. To begin with, let us turn to simple zero-one sequences (having in the back of our mind the time points at which a fixed ancestral line visits a fixed bounded region, and planning to count these time points as "successes").

For a zero-one sequence $y = (y_i)_{i \geq 0}$, we call an index i a *success* if $y_i = 1$. For $0 \leq m < n$, let us call $f_y(m,n] := \frac{1}{n-m} \sum_{i=m+1}^{n} y_i$ the *success frequency* of y *between m and n*, and $\limsup_{n \to \infty} f_y(0,n]$ the *upper success frequency* of y.

3.2. Lemma. Let $y = (y_i)_{i \geq 0}$ be a zero-one sequence with upper success frequency $c > 0$, and $(t_j)_{j \geq 1}$ be a strictly increasing sequence of natural numbers such that $f_y(0, t_j] > c/2$ for all $j \geq 1$. Then, for an arbitrary fixed natural number n the following hold true:

a) For all suitably large j, there are at least $(c/8)\ t_j$ indices $k \in \{n, \ldots, t_j\}$ such that for all $m = 1, \ldots, n$ there holds $f_y(k-m, k] \geq c/8$.

b) For all suitably large j, there are at least $(c/8)\ t_j$ indices $k \in \{1, \ldots, t_j\}$ such that for all $m = 1, \ldots, n$ there holds $f_y(k, k+m] \geq c/8$.

Proof. a) Let j be a fixed number, and denote by r the number of sucesses in $\{1, \ldots, t_j\}$. Call a success i in $\{n, \ldots, t_j\}$ *bad* if $f_y(i-m_i, i] < c/8$ for some $m_i \in \{1, \ldots, n\}$. Thus, the set B of bad successes is covered by the union of the "intervals" $\{i-m_i+1, \ldots, i\}$, $i \in B$. It is easy to check that we find a B-covering subsystem of $\{\{i-m_i+1, \ldots, i\} : i \in B\}$ such that each index $k \in \{1, \ldots, t_j\}$ is covered at most twice. Indeed, for each index which is covered three times by intervals I_1, I_2, I_3, we may drop one of these intervals without any loss. Hence the total number b of covered indices is at least one half of the sum of all interval lengths of this subcollection, and the total number of covered successes is less than $2b \cdot c/8 \leq 2\, t_j \cdot c/8$. There are at least $t_j \cdot c/2 - n$ successes in $\{n, \ldots, t_j\}$, and at most $t_j \cdot c/4$ of them are bad. Now take j large enough that $c/2 - c/4 - n/t_j > c/8$.
b) is proved in a similar way. ♦

3.3. Proposition. Let $\underline{\Phi} = (\Phi_t)_{t\in Z}$ be a stationary κ-process such that Φ_0 is a.s. non-void. Further, let X be a bounded region and c be a positive constant. Assume that with probability one the ancestral line of an arbitrary individual in $(\Phi_t)_{t\in Z}$ visits X with an upper frequency not less than c. Then the following hold true:

a) With probability not less than c/8, the realization (Φ_t) has the following property (F): there is an X-individual δ_{a_0} in Φ_0 which, for all $n > 1$, has some line of descendants $a_0, a_1^{(n)}, \ldots, a_n^{(n)}$ visiting X with a frequency not less than c/8 in each of the time intervals $\{0,1,\ldots,m\}$, $0\leq m\leq n$.

b) With probability not less than c/8, there is an X-individual δ_{a_0} in Φ_0 whose line of ancestors $(\ldots a_{-1},a_0)$ visits X with a frequency not less than c/8 in each of the time intervals $\{-m,-m+1,\ldots, 0\}$, $m\geq 0$.

Proof. a) For each $n\geq 1$, say that the realization (Φ_t) has property F_n if there is an X-individual δ_{a_0} in Φ_0 which has some line of descendants $a_0, a_1^{(n)}, \ldots, a_n^{(n)}$ visiting X with a frequency not less than c/8 in each of the time intervals $\{0,1,\ldots,m\}$, $0\leq m\leq n$. Since obviously property F_n is weaker than F_{n+1}, and (F) is equivalent to the validity of all F_n, all we have to show is that Prob[(Φ_t) has property F_n] $\geq c/8$ for all n. Thus let n be a fixed natural number, and let us, for a fixed realization (Φ_t), estimate how large a percentage of the time-shifted realizations (Φ_{t-k}), $k\geq 0$, have property F_n. To this end, we take a fixed ancestral line $(\ldots a_{-1},a_0)$ in (Φ_t), and write $y_i := 1_X(a_{-i})$. By assumption, the upper success frequency of $y = (y_i)$ is not less than c, hence by Lemma 1 there exists a strictly increasing sequence of natural numbers (t_j) (depending on the sequence y and thus on the realization (Φ_t)) which provides, for any large enough j, at least $(c/8)\, t_j$ indices $k \in \{n,\ldots, t_j\}$ such that for all $m=1,\ldots, n$ there holds $f_y(k-m, k]) \geq c/8$. But for each index k of this kind, the shifted realization $(\Phi_{t-k})_{n\in Z}$ certainly has property F_n. Therefore we obtain

$$\liminf_{j\to\infty} \frac{1}{t_j} \sum_{0<k\leq t_j}^{\infty} 1\{(\Phi_{t-k})_{t\in Z} \text{ has property } F_n\} \geq c/8 .$$

On the other hand the individual ergodic theorem guarantees that $\lim_{T\to\infty} \frac{1}{T} \sum_{0<k\leq T} 1\{(\Phi_{t-k})_{t\in Z}$ has property $F_n\}$ exists for a.all realizations (Φ_n) and has Prob[(Φ_t) has property F_n] as its expectation, which thus cannot be smaller than c/8.

b) This is proved in a completely analogous way. ♦

3.4. Corollary. Under the assumptions of proposition 3.3, with probability 1 all ancestral lines visit the region X with a lower frequency not less than c/8.

Proof. By cancelling all clans in (Φ_t) whose ancestral lines do not have the stated property we arrive at a stationary κ-process (Ψ_t); cf. Lemma 1.8. Assume that (Ψ_t) is non-void with positive probability. Then the stationary κ-process which arises by conditioning (Ψ_t) to

$\{\Psi_0 \neq o\}$ would meet the assumptions of Proposition 1, but violate its conclusion b). Since this is a contradiction, (Ψ_t) is a.s. void. ♦

3.5. Proposition. Under the assumptions of Proposition 3.3, with probability one there exists a genealogical line $(a_t)_{\in \mathbf{Z}}$ in $(\Phi_t)_{t\in \mathbf{Z}}$ visiting X both forward and backward in time with a lower frequency not less than c/8.

Proof. 1. Let (Φ_t) be a fixed realization, and denote, for all $r \in \mathbf{Z}_+$, by $\gamma_{0,r}$ the subpopulation of all those individuals δ_{a_0} in Φ_0 which have a line of descendants $a_0, a_1, \ldots, a_r$ that visits X with frequency at least c/8 in all time intervals $\{0,\ldots,k\}$, $0 \leq k \leq r$.
(Putting k=0 in this condition ensures that $\gamma_{0,r}$ consists only of X-individuals)
Moreover, for all $s \in \mathbf{Z}_+$, we denote by $\gamma_{s,r}$ the population of all individuals δ_{a_s} in Φ_s which have a line of descendants $a_s, a_{s+1}, \ldots, a_{s+r}$ such that $a_0, a_1, \ldots, a_s, a_{s+1}, \ldots, a_{s+r}$ visits X with frequency at least c/8 in all time intervals $\{0,\ldots,k\}$, $0 \leq k \leq s+r$, where $a_0, a_1, \ldots, a_s$ is a part of the ancestral line of δ_{a_s}. Since $\gamma_{s,r+1}$ obviously is a subpopulation of $\gamma_{s,r}$, we conclude that for all $s \geq 0$ there exists the limit population $\gamma_{s,\infty}$.
2. What we are aiming to prove in step 3 is that each individual in $\gamma_{s,\infty}$ has a descendant in $\gamma_{s+1,\infty}$. To this purpose, it is helpful to check first that, for fixed s, the population $\gamma_{s,r}$ is finite provided r is large enough. Indeed, for large enough r, each ancestral line $a_0, a_1, \ldots, a_s, a_{s+1}, \ldots, a_{s+r}$ visiting X with a frequency not less than c/8 must visit X at least once during $\{s+1,\ldots, s+r\}$. So, an infinity of individuals in $\gamma_{s,r}$ would lead to an infinity of X-individuals between times s+1 and s+r, which is prohibited by a.s. local finiteness.
3. Now take a fixed individual $\mathcal{J}$ in $\gamma_{s,\infty}$. Necessarily, for all $r \in \mathbf{Z}_+$, it has an offspring in $\gamma_{s+1,r}$. For large enough r, by step 3, $\gamma_{s+1,r}$ is finite, so our individual $\mathcal{J}$ has a finite collection $\mathcal{J}_1, \ldots, \mathcal{J}_m$, say, of daughters in $\gamma_{s+1,r}$. On the other hand, for all $t > r$, $\mathcal{J}$ has a "good" line of descendants up to time s+1+t which has to pass through one of its daughters; this daughter therefore belongs to $\gamma_{s+1,t}$, and so at least one of the $\mathcal{J}_j$ belongs to $\gamma_{s+1,\infty}$.
4. Proposition 3.3a) guarantees that $\gamma_{0,\infty}$ is with positive probability non-void. Part 3 together with Corollary 3.4 thus ensures that the event E := {(in $(\Phi_t)_{t\in \mathbf{Z}}$ there exists a genealogical line $(a_t)_{\in \mathbf{Z}}$ visiting X both forward and backward in time with a lower frequency not less than c/8} has a positive probability.
5. The event E is invariant with respect to time shift. Assume that the probability of E were less than one, then we could condition the distribution of $(\Phi_t)_{t\in \mathbf{Z}}$ to the complement of E. By a well-known argument from the general theory of time stationary Markov chains (cf. e.g. [LMW] 5.1.2), the resulting distribution is again that of a stationary κ-process. It still meets the assumptions of Proposition 3.3, but now assigns probability zero to the event E, which is a contradiction to what was proved in steps 1 to 4. ♦

3.6. Proposition., Under the assumptions of Proposition 3.3, $(\Phi_t)_{t\in Z}$ is of finite type (i.e. consists a.s. of only finitely many clans).

Proof. For any natural numbers k and n we put $h_{k,n} := \frac{1}{n} \sum_{0<i\leq n} 1\{\Phi_{-i}(X) > k\}$.

By the individual ergodic theorem, $h_{k,n}$ converges as $n \to \infty$, a.s. towards a random variable σ_k , the expectation of which is $\mathrm{Prob}[\Phi_0(X) \geq k]$. For all bounded regions B, denote by $r_B = r_B((\Phi_t))$ the number of those clans in (Φ_t) which have a representative in the restriction of the population Φ_0 to B. For a fixed, non-void realization (Φ_t), take B large enough such that $\Phi_0(B) > 0$. We choose from each of the clans which contribute to $r_B((\Phi_t))$ one representative, thus obtaining a non-void, finite collection $\delta_{a_1},\ldots,$ $\delta_{a_{r_B}}$ of B-individuals in Φ_0. For abbreviation we set $v := c/8$. By assumption, there exists (with probabability one) a natural number n_0 such that for all $j \in \{1,\ldots, r_B\}$ and all $n \geq n_0$ the individual δ_{a_j} has at least $nv/2$ X-ancestors in the time interval $\{-n,\ldots, 1\}$. For all natural numbers m and n , denote by $s_{m,n} = s_{m,n}((\Phi_t))$ the total number of those time points in $\{-n,\ldots,-1\}$ at which there are exactly m of the representative individuals' ancestors in X. Obviously there holds

$$(*) \quad w_{k,n} := \frac{1}{n} \sum_{m\geq k} s_{m,n} \leq h_{k,n} .$$

The collection of those time points in $\{-n,\ldots,-1\}$ which carry a multiplicity of at least k X-ancestors can be charged by a total of at most $n\cdot w_{k,n}\cdot r_B$ X-ancestors, whereas the rest can carry a total of at most $n\,(1-w)\,(k-1)$. For $n \geq n_0$, however, at least $n\,(v/2)\,r_B$ ancestors have to be placed somewhere into $\{-n,\ldots,-1\}$, which implies

$$w_{k,n}\, r_B + (1-w_{k,n})\,(k-1) \geq (v/2)\, r_B .$$

This implies together with (*) (provided that $r_B \geq k$):

$$(**) \quad \frac{(v/2)\, r_B-(k-1)}{r_B-(k-1)} \leq \liminf_{n\to\infty} h_{k,n} .$$

Denote by p the probability that (Φ_t) has an infinite number of clans. Then choose $B = B_k$ large enough that , with probability not less than p/2, $(v/2)\, r_B$ exceeds 2k and r_B exceeds k. For each (Φ_t) such that $(v/2)\, r_B \geq 2k$, we infer from (**) that $\limsup_{n\to\infty} h_{k,n} \geq v/4$. Therefore, $\mathrm{Prob}[\Phi_0(X) \geq k] \geq p/2\cdot v/4$, which, since k was arbitrary, enforces that $p = 0$. ♦

Each time stationary κ-process $\underline{\Phi} = (\Phi_t)_{t\in Z}$ of finite type consists of a random r number of clans $\underline{\Psi}_i = (\Psi_{i,t})$, $1\leq i\leq r$. Let $\underline{\Psi} := \underline{\Psi}_\iota$ arise by a random sampling of ι from $\{1,\ldots,r\}$ provided that $r\neq 0$, and define $\underline{\Psi}$ to be void if $r = 0$, that is, if the realization of $\underline{\Phi}$ is void. Then $\underline{\Psi}$ again constitutes a time-stationary κ-process, which we call the *typical clan* in the κ-process $\underline{\Phi}$.

3.7. Proposition. A time stationary κ-process $(\Phi_t)_{t\in Z}$ of finite type is finitely fed if and only if its typical clan has this property.

Proof. The interesting part is the "if" direction. Let S be a feeding region for the typical clan $\underline{\Psi}$ such that $\mathrm{Prob}[\Psi_0(S) < \infty] = 1$. Then clearly also $\mathrm{Prob}[\Phi_0(S) < \infty] = 1$. Moreover, for each $k > 0$ such that $\mathrm{Prob}[r=k] > 0$, conditioned on $\{r=k\}$ each of the k clans $\underline{\Psi}_i$, $1\le i\le k$, must have the property that $\Psi_{0\,|\,-n,A\setminus S}(B)$ converges to zero in probability as $n\to\infty$, since otherwise $\underline{\Psi}$ would not have this property. Therefore, for each natural number k satisfying $\mathrm{Prob}[r=k] > 0$, conditoned on $\{r=k\}$, $\Phi_{0\,|\,-n,A\setminus S}(B) = \sum_{i=1}^{k} \Psi_{i,0\,|\,-n,A\setminus S}(B)$ converges to zero in probability as $n\to\infty$. Since r is a.s. finite, this reveals that (Φ_t) is finitely fed. ♦

3.8. Proposition. Let $(\Phi_t)_{t\in Z}$ be a time stationary κ-process consisting of exactly one clan, and assume that with probability one the ancestral line of one individual (and therefore of all individuals) in (Φ_t) visits some bounded region X with a positive upper frequency. Then there exists with probability one exactly one two- sided genealogical line $(a_t)_{t\in Z}$ in (Φ_t) visiting X both forward and backward in time with a positive lower frequency. Moreover, $(\Phi_t)_{t\in Z}$ is finitely fed.

Proof. 1. For the first statement, in view of Proposition 3.5 it remains to show that with probability 1 there exist no more than one two-sided genealogical line with the stated property. Assume the contrary, i.e. there exists a constant $v>0$ such that, with postive probability, an ancestral line splits into two lines of descendants, each line visiting X with a lower frequency not less than v. By time stationarity, this splitting happens with positive probability at time 0, and because of the ergodic theorem, with positive probability there are an infinity of such splittings along the negative time axis. A counting argument very similar to that in the proof of Proposition 3.6 now shows that this is impossible, since it would lead to an "overflow", i.e. an infinity of X-individuals in a bounded time interval.

2. Put $S := \{a\in A$: with positive probability, in the offspring $(\chi_1^{(a)}, \chi_2^{(a)}, \ldots)$ there is an infinite line of descendants visiting X with a positive lower frequency$\}$. Obviously, immigration into S is a.s. impossible, and therefore S is autonomous for (Φ_t). Proposition 3.5 guarantees, that with probability one Φ_0 has at least one individual in S. On the other hand, it is a.s. impossible that Φ_0 has more than one individual in S, since this would contradict step 1. Therefore $\mathrm{Prob}[\Phi_0(S)=1] = 1$, and, due to the autonomy of S, the ancestral line of this S-individual never leaves S backwards in time. Since (Φ_t) consists of one single clan, every ancestral line in (Φ_t) eventually gets trapped in S, revealing that S is in fact a feeding region. ♦

The **proof** of Theorem 3.1 now is immediate from Propositions 3.6, 3.7 and 3.8.

3.9. Remark. i) The converse of Theorem 3.1 false, as the following simple example shows: Put $A := \bigcup_{k>0} \{(k,1),\ldots,(k,k)\}$, and consider deterministic migration from (k,j) to (k,j+1) as long as j<k, and from (k,k) to (k,1). Placing *one* individual randomly into A, namely with probability $2^{-k}\ 1/k$ at site (k,j), we obtain a (singleton) equilibrium population which is certainly finitely fed but violates the condition in Theorem 3.1.

ii) If we place, in the setting of i), one individual at each site $a \in A$, we obtain an equilibrium population which obviously is not finitely fed, but in which each individual has an ancestral line visiting a suitable bounded region (depending on the individual) with positive frequency. It is this type of equilibrium populations whose structure we will investigate in more generality in the next section.

4. Equilibrium distributions of positive recurrent type

The results of the preceding chapter also serve to show the following:

4. 1. Theorem. Let $\underline{\Phi} = (\Phi_t)_{t\in Z}$ be a stationary κ-process and B be a bounded region. Then, with probability one, $\lim_{n\to\infty}\frac{1}{n}\sum_{0<i\leq n} 1_B(a_{t-i})$, i.e. the *asymptotic frequency of visits* in B, exists for each ancestral line $(\ldots, a_{t-1}, a_t)$ in $\underline{\Phi}$.

Before proving this, we introduce the

4. 2. Notation. Let B be a bounded region and c a positive constant. For a stationary κ-process $\underline{\Phi}$, we denote by $\underline{\Phi}^{B,c}$ the subsystem of all those individuals whose ancestral lines visit B with an upper frequency not less than c.

By Lemma 1.8, $\underline{\Phi}^{B,c}$ is again a stationary κ-process, and by Theorem 3.1, $\underline{\Phi}^{B,c}$ is finitely fed.

Proof of Theorem 4. 1. For a fixed n, take the typical clan $\underline{\Psi}$ of the finitely fed stationary κ-process $\underline{\Phi}^{B,1/n}$, and denote its essentially minimal feeding region by S. Conditioned that $\underline{\Phi}^{B,1/n}$ be non-void, there is a.s. exactly one S-individual δ_{a_0} in Ψ_0; it is part of a genealogical line $(\ldots, a_{-1}, a_0, a_1, \ldots)$ which constitutes a time stationary Markov chain on S and visits $B\cap S$ with a positive lower frequency (cf. Proposition 3.8). By the ergodic theorem, a.s. the asymptotic frequency of the backward path's visits in $B\cap S$ exists (and is positive due to the assumption on the lower frequency of visits); since all ancestral paths in the clan $\underline{\Psi}$ eventually coalesce with $(a_0, a_{-1},\ldots)$, we have proved the assertion for $\underline{\Psi}$ and thus also for $\underline{\Phi}^{B,1/n}$. Since those individuals in $\underline{\Phi}$ which do not belong to any of the $\underline{\Phi}^{B,1/n}$ have, by definition, ancestral paths visiting B with asymptotic frequency zero, the theorem is proved.

◆

Let here and in the rest of the section X_n , n = 1, 2,... , be a sequence of bounded regions which *increases towards A* in the sense that $X_n \subseteq X_{n+1}$ for all n, and that each bounded region B is included in some X_n. For a stationary κ-process $\underline{\Phi} = (\Phi_n)_{n\in \mathbb{Z}}$, the sequence $\underline{\Phi}^{X_n,1/n}$ obviously increases towards the subsystem of individuals in $\underline{\Phi}$ whose ancestral line visits *some* bounded region with a positive upper frequency (or, equivalently, with a positive frequency). This subsystem constitutes again a stationary κ-process; we denote it by $\underline{\Phi}_{\mathrm{prec}}$ and call it the *positive recurrent part* of $\underline{\Phi}$. We say that $\underline{\Phi}$ is *of positive recurrent type* if $\underline{\Phi} = \underline{\Phi}_{\mathrm{prec}}$ a.s., i.e. if with probability one each ancestral line in $\underline{\Phi}$ visits some bounded region with a positive frequency.

4. 3. Remark. Every finitely fed stationary κ-process $\underline{\Phi}$ is of positive recurrent type. Indeed, if S is an essentially minimal $\underline{\Phi}$-feeding region, then the typical clan $\underline{\Psi}$ of $\underline{\Phi}$ has a.s. exactly one two-sided genealogical line $(\dots, a_{-1}, a_0, a_1, \dots)$ in S, and this constitutes the path of a Markov chain in equilibrium; therefore, by the ergodic theorem, with probability one the weak limit of $\lim_{n\to\infty} \frac{1}{n} \sum_{0<i\le n} \delta_{a_{-i}}$ exists, and it assigns positive measure to a suitably large bounded region (which may, however, depend on the particular realization). In addition, by Proposition 3.8 with probability one $(\dots, a_{-1}, a_0, a_1, \dots)$ is the only two-sided genealogical line in $\underline{\Psi}$ which has both forwards and backwards a positive asymptotic frequency in a suitable bounded region.

4. 4. Proposition. A stationary κ-process of finite type is finitely fed if and only if it is of positive recurrent type.

Proof. By the previous remark, every finitely fed stationary κ-process is of positive recurrent type. Conversely, assume that $\underline{\Phi}$ is of finite type and of positive recurrent type as well. Then also its typical clan process $\underline{\Psi}$ is of positive recurrent type, and by Proposition 3.7 it suffices to show that $\underline{\Psi}$ is finitely fed. We put

$S_n := \{ a\in A$: with positive probability, $(\chi_n^{(a)})$ contains an infinite line of descendants visiting X_n with a positive lower frequency$\}$.

Also, let E_n denote the event

$\{\underline{\Psi}$ is non void and its ancestral line visits X_n with an upper frequency not less than $1/n\}$.

From Proposition 3.8 we infer that $\underline{\Psi}$, when conditioned to E_n, has with probability one exactly one two-sided genealogical line in S_m for each $m\geq n$, and therefore also in $\bigcup_{m\geq n} S_m =$

$\bigcup_{m\geq 1} S_m =: S$. This reveals that S is a feeding region for $\underline{\Psi}$, and since $\Psi_0(S) \leq 1$ with probability 1, $\underline{\Psi}$ is finitely fed. ◆

The following result clarifies the structure of the equilibrium distributions of positive recurrent type.

4. 5. Theorem. Each stationary κ-process $\underline{\Phi}$ of positive recurrent type can be represented as a (not necessarily independent) superposition $\underline{\Phi} = \sum_{0<k<K} \underline{\Phi}^k$ (with $K \in \{1,2,\ldots,\infty\}$) of finitely fed stationary κ-processes $\underline{\Phi}^k$ having essentially minimal feeding regions S_k with the following properties:

i) the S_k are pairwise disjoint

ii) $\chi^{(a)}(\bigcup_{j>0} S_j) = \chi^{(a)}(S_k) = 1$ a.s. for all $a \in S_k, 0<k<K$

iii) each S_k is $\underline{\Phi}$-autonomous

iv) $\underline{\Phi}^k(S_k) = \underline{\Phi}(S_k) < \infty$ a.s. , $0<k<K$

v) $\bigcup_{k>0} S_k$ is an essentially minimal feeding region for $\underline{\Phi}$.

vi) $\mathrm{Prob}[\Phi_0^k \neq o] > 0$, $0<k<K$.

Proof. 1. Let us partition the clans of $\underline{\Phi}$ according to a recurrence behaviour of their ancestral lines. To this end, put $B_i := X_i \setminus X_{i-1}$, where $X_0 := \varnothing$ and $(X_i)_{i=1,2,\ldots}$ increases towards A , and denote , for all n and m $\in$ N, by $C_{n,m}$ the set of all those clans whose ancestral path visits B_n with an asymptotic frequency $f \in (1/(m+1), 1/m]$. Next, take an arbitrary numbering $(n_k,m_k)_{k\in N}$ of N×N and put

$$Y_0 := \varnothing; \quad Y_k := C_{n_k,m_k} \setminus Y_{k-1} \quad \text{for } k = 1,2,\ldots,$$

thus obtaining a sequence of pairwise disjoint sets whose union is $Y := \bigcup_{m,n>0} C_{n,m}$. Note that by assumption, with probability one each clan in $\underline{\Phi}$ belongs to Y, and therefore to exactly one of the Y_k. Moreover, we denote , for all n and m $\in$ N, by $D_{n,m}$ the set of all those forward genealogical lines $(a_0, a_1,\ldots)$ which visit B_n with an asymptotic frequency $f \in (1/(m+1), 1/m]$, and put

$$Z_0 := \varnothing; \quad Z_k := D_{n_k,m_k} \setminus Z_{k-1} \quad \text{for } k = 1,2,\ldots.$$

2. For all k = 1,2,..., denote by $\underline{\Phi}^k$ the sum of all those clans which belong to Y_k . By Lemma 1.8, $\underline{\Phi}^k$ is a stationary κ-process, and by Theorem 3.1, $\underline{\Phi}^k$ is finitely fed. Let U_k be an essentially minimal $\underline{\Phi}^k$-feeding region. Because of Remark 4.3 we can choose the regions U_k such that for all $a \in U_k$, $(\chi_t^a)_{t\geq 0}$ contains with probability one exactly one positive recurrent genealogical line which, moreover, belongs to Z_k . Since the Z_k are pairwise

disjoint, it follows that with probability one no individual in $\underline{\Phi}^k$ can have a child in some U_j for $j \neq k$, in other words,

(*) $\chi^{(a)}(U_j) = 0$ a.s. for $E\Phi_0^k$-almost all $a \in A$, where j is different from k

By an inessential shrinking of U_k we can achieve that (*) holds true for all $a \in U_k$. Thus, from now on we may and do assume that

(o) $\chi^{(a)}(\bigcup_{j \neq k} U_j)) = 0$ a.s. for all $a \in U_k$, $k = 1,2,\ldots$

3. Since the essentially minimal $\underline{\Phi}^k$-feeding region U_k is uniquely determined up to $E\Phi_0^k$-null sets, we conclude by means of Proposition 2.2 that

$$\chi^{(a)}(U_k) = 1 \quad \text{a.s.} \qquad \text{for } E\Phi_0^k\text{-almost all } a \in U_k \ , \quad k = 1,2,\ldots$$

Put

$$S_k^{(1)} := \{ a \in U_k : \chi^{(a)}(U_k) = 1 \quad \text{a.s.} \} \ ,$$

and define recursivley

$$S_k^{(m+1)} := \{ a \in S_k^{(m)} : \chi^{(a)}(S_k^{(m)}) = 1 \quad \text{a.s.} \} \quad , \quad m=1,2,\ldots ;$$

then $S_k := \bigcap_{m>0} S_k^{(m)}$ is an essentially minimal $\underline{\Phi}^k$-feeding region such that

(oo) $\chi^{(a)}(S_k) = 1$ a.s. for all $a \in S_k$

Since the S_k arise by decreasing the U_k , the relation (o) holds for them as well. From (o) and (oo) it is obvious that the S_k are pairwise disjoint.

4. Since S_k is an essentially minimal feeding region for $\underline{\Phi}^k$, with probability one there happens no immigration into S_k in the whole of $\underline{\Phi}^k$. On the other hand, because of step 2, with probability one nobody in $\underline{\Phi} - \underline{\Phi}^k$ ever has a child in S_k. Hence S_k is $\underline{\Phi}$-autonomous.

5. Since with probability one, nobody in $\underline{\Phi} - \underline{\Phi}^k$ stems from S_k (recall that the ancestral line of an individual in $\underline{\Phi}^j$ gets trapped in S_j backwards in time), we infer from 4. that , a.s., $\underline{\Phi} - \underline{\Phi}^k$ has no individuals in S_k at all, which, together with 3., proves iv).

6. From the remark in 5. it is clear that $S := \bigcup_{k>0} S_k$ is a feeding region for $\underline{\Phi}$. Then since any other $\underline{\Phi}$-feeding region L is $\underline{\Phi}^k$-feeding for all k, there holds $E\Phi_0^k(S_k \backslash L) = 0$ due to the essential minimality of S_k, and therefore also $E\Phi_0(S \backslash L) = \sum_{k>0} E\Phi_0^k(S \backslash L) = \sum_{k>0} E\Phi_0^k(S_k \backslash L) = 0$, proving v).

7. Property vi) is simply achieved by cancelling those of the $\underline{\Phi}^k$ which are a.s. void, and renumbering the rest. ♦

4.6. **Remark.** In the decomposition of $\underline{\Phi}$ provided by Theorem 4.5, the finitely fed component $\underline{\Phi}^k$ consists of all those clans in $\underline{\Phi}$ which ever hit the region S_k; in this sense the components of the positive recurrent type $\underline{\Phi}$ are spatially separated, at least concerning their "trunks" or "lines of survival" in the distinct essentially minimal feeding regions. In

Corollary 5.3 we will see that there holds a similar spatial separation of the positive recurrent part from its complement in a general stationary κ-process, and we set immediately out to prepare this result.

4.7. Notation. We put

$$A_{\mathrm{PREC}} :=$$

$$\{ a \in A : \text{there exists a bounded region B such that Prob } [\liminf_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} 1_N(\chi_i^{(a)}(B)) > 0] > 0\}$$

in other words, a site a belongs to A_{PREC} if the random offspring $(\chi_1^{(a)}, \chi_2^{(a)}, \ldots)$ of an individual δ_a visits with positive probability some bounded region with a positive lower frequency.

4.8. Remark. a) Clearly, $\chi^{(a)}(A_{\mathrm{PREC}}) = 0$ a.s. for all sites $a \notin A_{\mathrm{PREC}}$.

b) A similar reasoning as in Remark 4.3 shows that $E\Phi_0$-almost every site in an essentially minimal feeding region S of a finitely fed stationary κ-process $\underline{\Phi}$ belongs to A_{PREC}. Indeed, let $\underline{\Psi}$ the restriction of the typical clan of $\underline{\Phi}$ to S ; if $\underline{\Psi}$ is non-void, then the line $(a_0, a_1, \ldots)$ of descendants of the unique individual δ_{a_0} in Ψ_0 constitutes the path of a Markov chain in equilibrium; therefore, by the ergodic theorem, with probability one the weak limit of $\lim_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} \delta_{a_i}$ exists, and it assigns positive measure to a suitably large bounded region B (which, however, may depend on the ergodic component to which a_0 belongs). Thus, $E\Psi_0$- (and therefore also $E\Phi_0$-) every site in S belong to A_{PREC}.

c) We conclude from b) that A_{PREC} is a feeding region for every finitely fed stationary κ-process $\underline{\Phi}$.

d) Let $\underline{\Phi}$ be a stationary κ-process of positive recurrent type, and $(\underline{\Phi}^k)$ be a decomposition of $\underline{\Phi}$ into finitely fed components according to Theorem 4.5. For all k, according to a) and c), A_{PREC} is a $\underline{\Phi}^k$-autonomous, $\underline{\Phi}^k$-feeding region, hence with probability one, all ancestral lines in $\underline{\Phi}^k$, and therefore those in the whole of $\underline{\Phi}$, get eventually trapped in A_{PREC}.

4.9. Theorem. For each stationary κ-process of positive recurrent type, A_{PREC} is an essentially minimal feeding region.

Proof. In view of the previous remark, we only have to show the essential minimality of A_{PREC}. Assume we could take away from A_{PREC} a subset L of positive Φ_0-intensity measure, so that $A_{\mathrm{PREC}} \setminus L$ is still $\underline{\Phi}$-feeding, or in other words,

(*) $\quad \lim_{n\to\infty} \mathrm{Prob}[\Phi_{n|0,L}(X) > 0] = 0 \quad$ for all bounded regions X.

Without loss of generality we may assume that there exists a $B \in \mathcal{B}$ and a positive real number s such that all sites $a \in L$ obey

$$\mathrm{Prob}[\liminf_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} 1_N(\chi_i^{(a)}(B)) > 0] \geq s.$$

This implies

$$\mathrm{Prob}[\liminf_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} 1_N(\Phi_{i|0,L}(B))] \geq s\, \mathrm{Prob}[\Phi_0(L) > 0] > 0.$$

By Fatou's lemma we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} \mathrm{Prob}[\Phi_{i|0,L}(B))>0] \; > \; 0$$

which contradicts (*). ♦

4.10. Corollary. From Theorem 4.9 and Remark 4.3 we obtain that each finitely fed stationary κ-process $\underline{\Phi}$ obeys $\Phi_0(A_{PREC}) < \infty$ a.s. ♦

In the proof of the characterization theorem 4.12 we will need the following property of stationary κ-processes with a trivial positive recurrent part:

4.11. Lemma. Let $\underline{\Phi}$ be a stationary κ-process whose positive recurrent part is a.s. void. Then there holds for all bounded regions B:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} \mathrm{Prob}[\Phi_{i|0,B}\,(B) > 0] \; = \; 0 \; .$$

Proof. By assumption, with probability one all B-individuals in Φ_0 have ancestral lines visiting B with asymptotic frequency 0. Now consider the event $Y_j := \{\Phi_{-j|0,B}\,(B) > 0\} = \{\Phi_{0|-j,B}\,(B) > 0\}$, in other words, at least one B-individual in Φ_0 has a B-ancestor in Φ_{-j} . Since with probability 1 there are only finitely many B-individuals in Φ_0 , there holds $\lim_{n\to\infty} \frac{1}{n} \sum_{0<j\leq n} 1_{Y_j} = 0$ a.s. Passing to expectations, we infer that $\lim_{n\to\infty} \frac{1}{n} \sum_{0<j\leq n} \mathrm{Prob}[\Phi_{0|-j,B}\,(B)>0]$ $= 0$, which amounts to c) by stationarity of $\underline{\Phi}$. ♦

4.12. Theorem. For any stationary κ-process $\underline{\Phi}$ the following assertions are equivalent:

a) $\underline{\Phi}$ is of positive recurrent type

b) $\inf_{L\in\mathcal{B}} \sup_{n\geq 0} \mathrm{Prob}[\Phi_{0|-n,A\setminus L}(B) > 0] \; = \; 0 \qquad (B\in\mathcal{B})$

c) $\inf_{L\in\mathcal{B}} \sup_{n\geq 0} \sup_{Y\in\mathcal{M}} \left| \mathrm{Prob}[{}_B\Phi_{0|-n,L}\in Y] - \mathrm{Prob}[{}_B\Phi_0\in Y] \right| \; = \; 0 \quad (B\in\mathcal{B})$

(where $\mathcal{M}$ denotes the collection of Borel-measurable sets of populations on A, and ${}_B\Phi := \Phi((.)\cap B)$ stands for the restriction of the population Φ to the region B).

In words, b) says that, for all bounded regions B, the probability that there are present time B-individuals stemming from spatially remote ancestors tends to zero with increasing remoteness uniformly in the degree of the ancestors, and c) states that the distribution of that present time B-subpopulation which stems from n-th degree ancestors in a bounded region L tends in variation distance to the distribution of the whole present time B-population, uniformly in n as L increases towards A.

Proof. 1. Assume that $\underline{\Phi}$ is of positive recurrent type. Let $\underline{\Phi} = \sum_{k>0} \underline{\Phi}^k$ be a decomposition of $\underline{\Phi}$ into finitely fed components $\underline{\Phi}^k$ with essentially minimal feeding regions S_k as in Theorem 4.5. (For simplicity of notation, we let the index k run through the natural numbers, at the price that some of the $\underline{\Phi}^k$ and consequently also some of the S_k may be void, which does no harm.) Put $S := \bigcup_{k>0} S_k$ and $R_m := \bigcup_{0<k\leq m} S_k$, m = 1,2,... Also. let L_k be a sequence of bounded regions with union S.

We will estimate $\Phi_{0|-n,\mathcal{A}\setminus L_k}(B)$ for arbitrary $B \in \mathcal{B}$. To this end, first take an arbitrary bounded subregion U of S, and observe that for all k, m > 0

$$\begin{aligned}\Phi_{0|-n,\mathcal{A}\setminus L_k}(U) &\leq \Phi_{0|-n,\mathcal{A}\setminus R_m}(U) + \Phi_{0|-n,R_m\setminus L_k}(U)\\ &\underset{\text{a.s.}}{=} \Phi_{0|-n,\mathcal{A}\setminus R_m}(U\cap(A\setminus R_m)) + \Phi_{0|-n,R_m\setminus L_k}(U)\\ &\leq \Phi_0(U\setminus R_m) + \Phi_{0|-n,R_m\setminus L_k}(U),\end{aligned}$$

where we used the $\underline{\Phi}$ - autonomy of R_m in the second equality. This entails that

$$\text{Prob}[\Phi_{0|-n,\mathcal{A}\setminus L_k}(U) > 0] \leq \text{Prob}[\Phi_0(U\setminus R_m) > 0] + \text{Prob}[\Phi_{-n}(R_m\setminus L_k) > 0]$$

Since both summands on the right hand side in fact do not depend on n and can be made arbitrarily small for large enough m and k (depending on m, recall that $\Phi_0(R_m)$ is a.s. finite), we infer that

$$(*) \qquad \inf_{L\in\mathcal{B}} \sup_{n\geq 0} \text{Prob}[\Phi_{0|-n,\mathcal{A}\setminus L}(U) > 0] = 0.$$

Next, fix an arbitrary bounded subregion V of the complement of S, and observe that for all k, m > 0 there holds

$$(+) \qquad \Phi_{0|-n,\mathcal{A}\setminus L_k}(V) \leq \Phi_{0|-n,\mathcal{A}\setminus S}(V) + \Phi_{0|-n,S\setminus R_m}(V) + \Phi_{0|-n,R_m\setminus L_k}(V)$$

The first summand on the r.h.s. tends to 0 in probability as $n \to \infty$, since S is a $\underline{\Phi}$-feeding region. For the second summand there holds

$$\text{Prob}\,[\Phi_{0|-n,S\setminus R_m}(V) > 0] \leq \text{Prob}\,[\sum_{i>m} \Phi_0^i(V) > 0] \;;$$

since $\sum_{i>m} \Phi_0^i(V)$ is finite a.s. due to the boundedness of V, the expression on the r.h.s. tends to 0 as $m\to\infty$.

Finally, as to the third summand in the r.h.s of (+), for every fixed m, $\text{Prob}[\Phi_{-n}(R_m\setminus L_k) > 0] = \text{Prob}[\Phi_0(R_m\setminus L_k) > 0]$ tends to 0 as $k\to\infty$. So, what we have proved up to now is

$$\inf_{L\in\mathcal{B}} \limsup_{n\geq 0} \text{Prob}[\Phi_{0|-n,\mathcal{A}\setminus L}(V) > 0] = 0.$$

Together with (*) this yields

$$\inf_{L\in\mathcal{B}} \limsup_{n\geq 0} \text{Prob}[\Phi_{0|-n,\mathcal{A}\setminus L}(B) > 0] = 0 \qquad (B\in\mathcal{B}).$$

Thus, for given $B\in\mathcal{B}$ and $\varepsilon > 0$, there exists a set $C_\varepsilon \in \mathcal{B}$ and a natural number n_ε such that for all $n > n_\varepsilon$ and all $C \in \mathcal{B}$ which cover C_ε there holds

$$\text{Prob}[\Phi_{0|-n,\mathcal{A}\setminus C}(B) > 0] \leq \varepsilon.$$

Now let (D_i) be a sequence of bounded regions which increases towards $\mathcal{A}$ and starts with $D_1 := C_\varepsilon$. Choose, for all $n \leq n_\varepsilon$, $i = i(n)$ so large that

$$\text{Prob}[\Phi_{0|-n,\mathcal{A}\setminus D_i}(B) > 0] \leq \varepsilon\,.$$

Put $D := D_{i^*}$, where $i^* := \max\,\{i(n) : 1\leq n\leq n_\varepsilon\}$. Then there holds for *all* $n \geq 1$

$$\mathrm{Prob}[\Phi_{0|-n,\mathcal{A}\setminus D}(B) > 0] \quad \leq \quad \varepsilon.$$

We have thus proved that a) implies b).

2. For all $n \in \mathbf{N}$, B and $L \in \mathcal{B}$, and $Y \in \mathcal{M}$ we have

$$\begin{aligned} &|\,\mathrm{Prob}[{}_B\Phi_{0|-n,L} \in Y] - \mathrm{Prob}[{}_B\Phi_0 \in Y]\,| \\ &= |\,\mathrm{Prob}[{}_B\Phi_{0|-n,L} \in Y] - \mathrm{Prob}[({}_B\Phi_{0|-n,L} + {}_B\Phi_{0|-n,\mathcal{A}\setminus L}) \in Y]\,| \\ &\leq 2\,\mathrm{Prob}[\Phi_{0|-n,\mathcal{A}\setminus L}(B) > 0]. \end{aligned}$$

This shows immediately that c) is a consequence of b).

3. Now assume the validity of c). Then, for any $B \in \mathcal{B}$, any sequence L_k of bounded regions which increases towards $\mathcal{A}$, any sequence of nonnegative integers n_k and each $m \in \mathbf{Z}$ there holds:

$$|\,\mathrm{Prob}[\Phi_{0|-n_k,L_k}(B) = m] - \mathrm{Prob}[\Phi_0(B) = m]\,| \to 0 \quad \text{as } k \to \infty;$$

in other words, the sequence ζ_k has the same limit in distribution as the sequence $\zeta_k + \eta_k$ as $k \to \infty$, where $\zeta_k := \Phi_{0|-n_k,L_k}(B)$ and $\eta_k := \Phi_{0|-n_k,\mathcal{A}\setminus L_k}(B)$. An elementary argument (cf. LMW 2.7.37) shows that then $\Phi_{0|-n_k,\mathcal{A}\setminus L_k}(B)$ must converge to zero in probability as $k \to \infty$, which, since (n_k) was arbitrary, amounts to b).

4. Finally we set out to prove the implication b) $\Rightarrow$ a). We have to show that $\underline{\gamma} := \underline{\Phi} - \underline{\Phi}_{\mathrm{prec}}$ is a.s. void. Let us fix a bounded region B. Together with $\underline{\Phi}$, also $\underline{\gamma}$ has property b). Since we already know that b) and c) are equivalent, we can choose, for given $\varepsilon > 0$, an $L \in \mathcal{B}$ so large that for all $i = 1,2,\ldots$

$$(*) \qquad |\,\mathrm{Prob}[\gamma_{0|-i,L}(B) > 0] - \mathrm{Prob}[\gamma_0(B) > 0]\,| < \varepsilon.$$

By Lemma 4.11 there holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} \mathrm{Prob}[\gamma_{i|0,L}(B) > 0] \;=\; 0,$$

leading together with (*) to

$$\mathrm{Prob}[\gamma_0(B) > 0]\,| \leq \varepsilon.$$

Since ε and B, were arbitrary, γ_0 must vanish a.s., and therefore $\underline{\Phi}$ is of positive recurrent type. ♦

5. Equilibrium distributions with a trivial positive recurrent part

We start this section with a "forward characterization" of the equilibrium distributions with a trivial, i.e. almost surely void, positive recurrent part:

5.1. Theorem. For all stationary κ-processes $\underline{\Phi}$, the following assertions are equivalent:

a) $\underline{\Phi}_{\mathrm{prec}}$ is void a.s.

b) $\liminf\limits_{n\to\infty} \mathrm{Prob}[\Phi_{n|0,B}(B) > 0] = 0 \qquad (B \in \mathcal{B})$

c) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{0<i\leq n} \mathrm{Prob}[\Phi_{i|0,B}(B) > 0] = 0 \qquad (B \in \mathcal{B})$

Proof. By Lemma 4.11, a) implies c). The implication c) $\Rightarrow$ b) is obvious. To show that b) implies a), we proceed indirectly, assuming that $\underline{\Phi}_{\mathrm{prec}}$ is non-void with positive probability. Then according to Theorem 4.5 there is a finitely fed κ-stationary processes $\underline{\gamma}$ which is a subsystem of $\underline{\Phi}$ and which is non-void with positive probability. Condition $\underline{\gamma}$ to be non-void, and take its typical clan $\underline{\Psi}$. The (unique) two-sided infinite genealogical line $(\ldots a_{-1}, a_0, a_1, \ldots)$ in $\underline{\Psi}$ follows a time-stationary Markov chain. For large enough $B \in \mathcal{B}$,

$$\mathrm{Prob}[a_i \in B] > 3/4 \qquad (i \in \mathbf{Z})$$

and hence

$$\mathrm{Prob}[a_0 \in B, a_n \in B] > 1/2 \quad (n = 1,2\ldots)$$

Thus $\liminf\limits_{n\to\infty} \mathrm{Prob}[\Psi_{n|0,B}(B) > 0] > 0$. ♦

5.2. Theorem. Let $\underline{\Phi}$ be a stationary κ-process. Then $\underline{\Phi}_{\mathrm{prec}}$ is void a.s. if and only if $E\Phi_0(A_{\mathrm{PREC}}) = 0$, in other words, if $\underline{\Phi}$ contains with probability one no individuals in A_{PREC}.

Proof. 1. Assume $E\Phi_0(A_{\mathrm{PREC}}) > 0$. Similar as in the proof of the previous theorem, consider, for some $B \in \mathcal{B}$ and all natural numbers i, the event $Z_i := \{\Phi_{i|0,B}(B) > 0\}$. From our assumption there results, for B large enough, that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} 1_{Z_i} > 0 \quad \text{with positive probability.}$$

By Fatou's lemma, this implies

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{0<i\leq n} \mathrm{Prob}[\Phi_{i|0,B}(B) > 0] > 0,$$

hence, by Theorem 5.1, $\underline{\Phi}_{\mathrm{prec}}$ is not a.s. void.

2. Assume that $E\Phi_0(A_{\mathrm{PREC}}) = 0$. Then $\underline{\gamma} := \underline{\Phi}_{\mathrm{prec}}$ is a.s. void, since according to Theorem 4.9, A_{PREC} is a feeding region for $\underline{\Phi}_{\mathrm{prec}}$. ♦

5. 3. Corollary. For each stationary κ-process $\underline{\Phi}$, with probability one its positive recurrent part $\underline{\Phi}_{\mathrm{prec}}$ consists of all those clans in $\underline{\Phi}$ which ever hit the region A_{PREC} (in the sense that they have at least one individual in A_{PREC}). ♦

5. 4. Remark. a) We do not know whether the conclusions of Theorems 4.9 and 5.2 (and thus also of Corollary 5.3) would still hold true if one replaces, in the definition of A_{PREC}, $\liminf\limits_{n\to\infty}$ by $\limsup\limits_{n\to\infty}$.

b) The following example shows that in condition b) of Theorem 5.1, $\liminf_{n\to\infty}$ cannot be replaced by $\lim_{n\to\infty}$, in other words, there exists a stationary κ-process $\underline{\Phi}$ with a trivial positive recurrent part and such that, for some $B\in\mathcal{B}$, $\limsup_{n\to\infty} \mathrm{Prob}[\Phi_{n|0,B}(B)>0] > 0$.

5. 5. Example. 1. Put A:= the set of non-zero counting measures τ on $\mathbf{Z}$. Following [MKM], 3.3.2, A can be equipped with a complete metric ρ_A which generates the restriction of the vague topology to A. (this is just the topology of pointwise convergence, when τ is identified with the sequence $(\tau(\{j\}))_{j\in Z}$) and which makes exactly the subsets of finite unions of the sets $A^{[j]}$, $j\in \mathbf{Z}$, bounded, where $A^{[j]}$ is the set of all counting measures charging j. Consider the deterministic dynamics κ which takes an individual δ_τ into $\delta_{\theta\tau}$, where $\theta\tau$ denotes the shifted counting measure $\tau((.)+1)$.

2. We focus on the bounded region $B := A^{[0]}$, and aim to construct a Poisson equilibrium population $\Phi_0 = \sum_{i\in I} \delta_{\tau_i}$ such that the associated stationary κ-process $\underline{\Phi}$ has a trivial positive recurrent part and obeys

(*) $\qquad \limsup_{n\to\infty} \mathrm{Prob}[\Phi_{n|0,B}(B)>0]$.

Denote the intensity measure of Φ_0 by W; then the requirement that the Poisson population Φ_0 be in κ-equilibrium boils down to stationarity of W with respect to θ, and (*) is certainly implied by the condition

(**) $\qquad \limsup_{n\to\infty} W(\tau(\{0\})>0, \tau(\{n\})>0) > 0.$

3. We claim that $\underline{\Phi}$ has a trivial positive recurrent part provided the process (Φ_n) is ergodic with respect to the shift $\Theta: \sum_{i\in I} \delta_{\tau_i} \to \sum_{i\in I} \delta_{\theta\tau_i}$. Indeed, assume that $\underline{\Phi}_{\mathrm{prec}}$ were non-void with positive probability. Then there exists some finitely fed subsystem $\underline{\gamma}$ of $\underline{\Phi}$, also non-void with positive probability. Due to the simple nature of the dynamics, γ_0 is an a.s. finite population. Put $\underline{\gamma}^{(k)} := \underline{\gamma}$ if $\gamma_0(A) \le k$, and $\gamma_t^{(k)} := o$, $t\in \mathbf{Z}$, if $\gamma_0(A) > k$. For large enough k, $\gamma_0^{(k)}$ is a *nontrivial* subpopulation of Φ_0, the distribution of $\gamma_0^{(k)}$ is Θ-stationary and the expectation of $\gamma_0^{(k)}(A)$ is finite. By [MKM], 6.5.3, this would contradict the Θ-ergodicity of (Φ_n).

4. The remaining task is to construct a locally finite, θ-invariant measure W obeying (**) and such that the Poisson distribution Π_W with intensity measure W is Θ-ergodic. Under the additional assumption that $\Lambda_W(\{0\}) := \int \tau(\{0\})\, W(d\tau) = 1$, both of these conditions can be reexpressed in terms of the Palm distribution W_0 of W (cf. 9.2.1 in [MKM]); namely, the ergodicity of Π_W amounts to

a) $\qquad W_0(\lim_{n\to\infty} \frac{1}{2n+1} \sum_{-n\le j\le n} \tau(\{j\}) = 0) = 1,$

and (**) is equivalent to

b) $\qquad \limsup_{n\to\infty} W_0(\tau(\{n\})>0) > 0.$

We will construct W as a vague limit of a sequence $W^{(m)}$ of stationary measures on A, all of them obeying $\Lambda_{W^{(m)}}(\{0\}) = 1$. To this end, define inductively a sequence σ_m , m=1, 2,... of elements of A by

$$\sigma_1 := \delta_0 + \delta_1$$

$$\sigma_{m+1} := \sigma_m + \theta^{-10^m} \sigma_m , \qquad m = 1,2,\ldots$$

Note that σ_m consists of a total of 2^m - extremely widespread- atoms.
The "typical cluster" σ_m will now be placed in a shift-invariant way:

$$W^{(m)} := 2^{-m} \sum_{i \in \mathbf{Z}} \delta_{\theta^i \sigma_m} \quad .$$

Note that $\Lambda_{W^{(m)}}(\{0\}) = 1$, and that (cf. [MKM] section 11.4) the Palm distribution of $W^{(m)}$ is

$$W_0^{(m)} = 2^{-m} \int \delta_{\theta^i \sigma_m} \sigma_m(di) .$$

In order to see that the sequence $W_0^{(m)}$, m=1,2..., converges weakly (as distributions on the counting measures on **Z**) , we use a representation of $W_0^{(m)}$ in terms of a {-1,1}-valued fair coin tossing sequence $\xi = (\xi_m)_{m=1,2,\ldots}$. That is, we define inductively a sequence ρ_m , m=1, 2,..., of random elements of A by

$$\rho_1(\xi) := \delta_0 + \theta^{\xi_1} \delta_0$$

$$\rho_{m+1}(\xi) := \rho_m(\xi) + \theta^{10^m \xi_{m+1}} \rho_m(\xi) , \qquad m = 1,2,\ldots$$

It is easy to check (by induction) that the distribution of $\rho_m(\xi)$ equals $W_0^{(m)}$. Obviously, $\rho_m(\xi)$ increases, as m→∞, to a random counting measure $\rho_\infty(\xi)$, the distribution of which we denote by S.
The fact that $\Lambda_{W^{(m)}}(\{0\}) \equiv 1$ together with the just proved weak convergence of $W_0^{(m)}$ towards S implies (cf. 10.3.8 in [MKM]) that $W^{(m)}$ converges, as m→∞, vaguely towards a measure W which obeys $\Lambda_W(\{0\}) = 1$ and whose Palm measure W_0 equals S. The desired properties a) and b) can now be read off directly from the properties of $\rho_\infty(\xi)$: property a) is clear since the atoms of $\rho_\infty(\xi)$ are so terribly widespread, and property b) is clear since, for all k = 1,2,..., $\rho_\infty(\xi)$ charges 10^k with probability 1/2.♦

5.6. Remark. a) Though the stationary κ-process $\underline{\Phi}$ constructed in the above example has a trivial positive recurrent part, it is both hermetic and of recurrent type. Indeed, the support of W_0-almost all (and thus also of W-almost all) clusters τ is unbounded both from above and from below, and therefore, with probability one, all the individual paths $(\theta^n \tau)_{n \in \mathbf{Z}}$ in $\underline{\Phi}$ hit the bounded set $A^{[0]}$ infinitely often both backward and forward in time.

b) Let us now show that the entire space of sites A is an essentially minimal feeding region for $\underline{\Phi}$, by proving that for all j∈ **Z** and all regions $Y \subseteq A^{[j]}$ with $E\Phi_0(Y) = W(Y) > 0$, the sequence Prob[$\Phi_{0|-n,Y}(A^{[0]})$] does not converge to 0 as n→∞ . By shift-invariance of W, we can assume without loss of generality that j =0, and we are certainly done if we manage to show that $c_n := W(\{\theta^{-n}\tau \in Y, \tau(\{0\}) > 0\}) = W(\{\tau \in Y, \tau(\{n\}) > 0\})$ does not tend to 0 as $n \to \infty$. In view of $\Lambda_W(\{0\}) = 1$ we have $W_0 = \sum_{k>0}^{\infty} k\, W(\tau(\{0\}) = k)\, W((.) \mid \tau(\{0\}) = k)$. Hence from the assumption $Y \subseteq A^{[0]}$, there results $W_0(Y) > 0$, and $c_n \to 0$ is equivalent to

$$c_n' := W_0(\{\tau \in Y, \tau(\{n\}) > 0\}) \to 0 .$$

Coming back to the $\{-1,1\}$-valued fair coin tossing sequence ξ, we note that $\text{Prob}[\rho_\infty(\xi) \in Y] > 0$. Conditioned on $\rho_\infty(\xi) \in Y$, $(\xi_m)_{m=1,2,\ldots}$ cannot tend to -1 in probability due to the strong law of large numbers, and therefore, $\text{Prob}[\rho_\infty(\xi) \in Y, \xi_m = 1]$ cannot tend to 0 as $m \to \infty$. Since $\xi_m = 1$ implies $\rho_\infty(\xi)(10^{m-1}) > 0$, we infer that $W_0(\{\tau \in Y, \tau(\{10^{m-1}\}) > 0\})$ does not converge to 0 as $m \to \infty$.♦

6. Equilibrium distributions with a finite number of clans

Recall (cf. Proposition 2.2) that the finitely fed stationary κ-process $\underline{\Phi}$ are those for which there exists a region X such that i) $\Phi_{0|-n,X} \xrightarrow{d} \Phi_0$ and ii) $\Phi_0(X) < \infty$ a.s. Moreover, X can be chosen such that, for all $n \in \mathbf{Z}$, $\Phi_n(X)$ equals a.s. to the random number r of clans in $\underline{\Phi}$.

The more general class of *finite type* stationary κ-processes $\underline{\Phi}$, i.e. those with an a.s. finite number of clans, can be characterized in a quite similar way:

6.1. Theorem. A stationary κ-process $\underline{\Phi}$ is of finite type if and only if there exists a sequence of regions X_n such that

a) $\Phi_{0|-n,X_n} \xrightarrow{d} \Phi_0$

and b) $(\Phi_{-n}(X_n))$ is tight (which is equivalent to the tightness of $(\Phi_0(X_n))$ due to the stationarity of $\underline{\Phi}$).

If $\underline{\Phi}$ is of finite type and (X_n) is any sequence of regions with the property a), then there exists a sequence of regions (S_n) satisfying a) as well as

b') $\Phi_{-n}(S_n)$ converges in probability to the random number r of clans in $\underline{\Phi}$

and c) $S_n \subseteq X_n$.

Proof of Theorem 6.1. I. Assume that there is a sequence (X_n) having property a) and b). Let (B_m) be a sequence of bounded regions increasing towards A, and denote by $r_{0,m}$ the number of those clans in $\underline{\Phi}$ which charge the region B_m at time point zero. Then there holds for all natural numbers k and m :

$$(*) \qquad \text{Prob}\,[r_{0,m} \geq k] \leq \sup_{n>0} \text{Prob}\,[\Phi_{-n}(X_n) \geq k].$$

Indeed, choose for given $\varepsilon > 0$, the number n so large that $\text{Prob}[\Phi_{0|-n,A\setminus X_n}(B_m) > 0] < \varepsilon$. Since $r_{0,m} \geq k$ together with $\Phi_{0|-n,A\setminus X_n}(B_m) = 0$ implies that $\Phi_{-n}(X_n) \geq k$, we infer that

$$\text{Prob}\,[r_{0,m} \geq k] \;\leq\; \text{Prob}\,[\Phi_{-n}(X_n) \geq k] - \varepsilon.$$

Using (*), we obtain for all $k > 0$:

$$\text{Prob}[r \geq k] \;=\; \text{Prob}[\sup_{m>0} r_{0,m} \geq k] \;=\; \sup_{m>0} \text{Prob}\,[r_{0,m} \geq k] \;\leq\; \sup_{n>0} \text{Prob}\,[\Phi_{-n}(X_n) \geq k].$$

Since, due to the tightness assumption b) the r.h.s. tends to zero as $k \to \infty$, the total number r of clans must be finite a.s.

II. Conversely, assume that $\underline{\Phi}$ is of finite type. Without loss of generality we assume that $\underline{\Phi}$ is a.s. non-void, and denote its typical clan by $\underline{\Psi}$.

1. We will construct a sequence (R_n) of regions obeying

$$\text{a) } \Psi_{0|-n,R_n} \xrightarrow{d} \Psi_0$$

$$\text{b') } \Psi_{-n}(R_n) \to 1 \quad \text{in probability as } n \to \infty;$$

this sequence then obviously also has the desired properties with respect to $\underline{\Phi}$.

2. Let L_m, $m = 1,2,\ldots$, be a sequence of bounded regions increasing towards A, and choose for each $m \in \{1,2,\ldots\}$ a natural number t_m such that for all $n \geq t_m$, $\text{Prob}[\Psi_{-n|0,L_m}(A) = 1] > 1-1/m$.

3. Now fix m and $n \geq t_m$, and write $Y := \{\Psi_{-n|0,L_m}(A) = 1\}$, describing the event that there is exactly one individual in Ψ_{-n} whose offspring charges the region L_m at time 0. The conditional probabilities $\text{Prob}[Y|\Psi_{-n}]$ satisfy according to 2:

$$E[1 - \text{Prob}[Y|\Psi_{-n}]] < 1/m;$$

by Markov's inequality this implies that

$$\text{Prob}[Y|\Psi_{-n}] > 1-1/\sqrt{m} \quad \text{with probability not less than } 1-1/\sqrt{m}\,.$$

4. We define $B_{n,m} := \{a \in A \,|\, \text{Prob}[\chi_n^{(a)}(L_m) > 0] > 1-1/\sqrt{m}\,\}$, and we claim that almost all realizations Ψ_{-n} for which $\text{Prob}[Y|\Psi_{-n}] \geq 1-1/\sqrt{m}$ obey $\Psi_{-n}(B_{n,m}) = 1$, at least for $m \geq 7$. Indeed, let Ψ_{-n} consist of the individuals $\delta_{a_1}, \delta_{a_2}, \ldots$ and put $Z_i = 1_{\{\chi_n^i(L_m) > 0\}}$, where χ_n^i denotes the random n-th generation offspring of an individual at site a_i. The Z_i are independent and sum to one with a probability larger than $1-1/\sqrt{m}$. Then it follows from elementary considerations (cf. e.g. LMW 2.8.5) that, provided $m \geq 7$, there is exactly one index i_0 for which $\text{Prob}[Z_{i_0}=1]$ is larger than $1-1/\sqrt{m}$.

5. Combining steps 3 and 4 we infer that $\text{Prob}[\Psi_{-n}(B_{n,m}) = 1] \geq 1-1/\sqrt{m}$ for all $m \geq 7$ and all $n \geq t_m$.

6. In order to construct the sequence R_n, take without loss of generality the sequence (t_m) strictly increasing, and for all $n \geq t_7$, let m(n) be such that $t_{m(n)} \leq n < t_{m(n)+1}$, and put

$$R_n := B_{n,m(n)}\,.$$

By step 5, the sequence (R_n) obeys $\mathrm{Prob}[\Psi_{-n}(R_n) = 1] \geq 1-1/\sqrt{m(n)}$; this shows already that (R_n) has property b'). In order to check that it has also property a), we reason as follows: Putting $m := m(n)$, on one hand, from step 5 and the definition of $B_{n,m}$, the probability that there is at least one R_n-individual in Ψ_{-n} having an offspring in Ψ_0 which charges L_m, is not less than $(1-1/\sqrt{m})^2$. On the other hand, due to step 2, the probability that *all* L_m-individuals have one common ancestor in Ψ_{-n} is not less than $1-1/m$. Hence the probability that all $L_{m(n)}$-individuals in Ψ_0 have one common R_n–ancestor in Ψ_{-n} tends to one as $n \to \infty$.

7. To construct an S_n which also meets condition c), assume that (X_n) obeys a), take a sequence (R_n) obeying a) and b'), and put $S_n := X_n \cap R_n$. For a given $\varepsilon > 0$, choose a bounded region L and a natural number n_0, both so large that, for all $n \geq n_0$, $\mathrm{Prob}[\,\Psi_{-n|0,L}(A) = \Psi_{-n|0,L}(R_n) = \Psi_{-n}(R_n) = 1] > 1-\varepsilon$, and moreover $\mathrm{Prob}[\Psi_{0|-n,\mathcal{A}\setminus X_n}(L) = 0] > 1-\varepsilon$. The intersection of these two events implies that all L-individuals in Ψ_0 have one common ancestor in Ψ_{-n} which belongs to both R_n and X_n, and that, apart from this ancestor, there are no other R_n - individuals in Ψ_{-n}. Consequently,

$$\mathrm{Prob}[\,\Psi_{0|-n,\mathcal{A}\setminus S_n}(L) = 0;\ \Psi_{-n}(S_n) = 1] > 1-2\varepsilon\,. \ \blacklozenge$$

6.2. Definition. Let $\underline{\Phi}$ be a stationary κ-process of finite type. A sequence of regions (S_n) obeying

a) $\Phi_{0|-n,S_n} \xrightarrow{d} \Phi_0$

and b') $\Phi_{-n}(S_n) \to r(\underline{\Phi})$ in probability

will be called an *asymptotically minimal feeding sequence of regions* for $\underline{\Phi}$.

(Note that b') is equivalent to "$\Phi_0(S_n) \to r(\underline{\Phi})$ in probability", since the random number $r(\underline{\Phi})$ is a.s. shift-invariant.)

According to Proposition 2.2, a stationary κ-process $\underline{\Phi}$ of finite type is finitely fed if and only if it has a constant asymptotically minimal sequence $S_n = S$; in this case, S is an essentially minimal feeding region for $\underline{\Phi}$.

Theorem 6.1 guarantees that every stationary κ-process with a finite number of clans has an asymptotically minimal feeding sequence of regions; the next proposition states that it is "asymptotically unique" in a probabilistic sense:

6.3. Proposition. Let $\underline{\Phi}$ be a stationary κ-process of finite type and (R_n) be an asymptotically minimal feeding sequence of regions for $\underline{\Phi}$. Then another sequence of regions X_n is asymptotically minimal feeding for $\underline{\Phi}$ iff $\Phi_{-n}(R_n \Delta X_n) \to 0$ in probability.

Proof. 1. Assume that (X_n) is another asymptotically minimal feeding sequence of regions. Then, according to the last step in the proof of Theorem 6.1, also $S_n := X_n \cap R_n$ has this property. Since both $\Phi_{-n}(R_n)$ and $\Phi_{-n}(X_n \cap R_n)$ converge towards $r(\underline{\Phi})$ in probability, it follows that $\Phi_{-n}(R_n \setminus X_n)$ converges to 0 in probability. Exchanging the roles of X_n and R_n then completes the "only if" direction of the proof.

2. Conversely, assume that $\Phi_{-n}(R_n \Delta X_n) \to 0$ in probability. Then clearly also (X_n) meets condition b'). On the other hand, for all bounded regions B the event $\{\Phi_{0|-n,\mathcal{A}\setminus X_n}(B)>0\}$ is included in the event $\{\Phi_{0|-n,\mathcal{A}\setminus R_n}(B)>0\} \cup \{\Phi_{-n}(R_n \Delta X_n)>0\}$, and therefore (X_n) also has property a). ♦

The next proposition states, loosely speaking, that any asymptotically minimal feeding sequence of regions R_n is "slowly varying" for large n in a probabilistic sense:

6.4. Proposition. Let $\underline{\Phi}$ be a stationary κ-process of finite type and (R_n) be an asymptotically minimal feeding sequence of regions for $\underline{\Phi}$. Then $\Phi_{-n}(R_n \Delta R_{n+1}) \to 0$ in probability.

Proof. All we have to show in view of Proposition 6.3 is that $S_n := R_{n-1}$, $S_1 := \varnothing$ is asymptotically minimal feeding as well.

To check condition a) for (S_n), note that for all bounded regions B, L there holds
$\{\Phi_{0|-n,R_{n-1}}(B)=\Phi_0(B)\} \supseteq \{\Phi_{-1|-n,R_{n-1}}(L) = \Phi_{-1}(L)\} \cap \{\Phi_{0|-1,L}(B) = \Phi_0(B)\}$.
Now take, for fixed B and for given $\varepsilon > 0$, L so large that $\mathrm{Prob}[\Phi_{0|-1,L}(B) \neq \Phi_0(B)] < \varepsilon/2$.
Then choose n_0 so large that for all $n \geq n_0$,
$\mathrm{Prob}[\Phi_{-1|-n,R_{n-1}}(L) \neq \Phi_{-1}(L)] = \mathrm{Prob}[\Phi_{0|-(n-1),R_{n-1}}(L) \neq \Phi_0(L)] < \varepsilon/2$.
Then, for all $n \geq n_0$, $\mathrm{Prob}[\Phi_{0|-n,S_n}(B) \neq \Phi_0(B)] < \varepsilon$.
In order to ensure that (S_n) also obeys property b'), note that $\Phi_0(R_n)$ and hence also $\Phi_0(S_n)$ converge to $r(\underline{\Phi})$ in probability as $n \to \infty$. But the latter is equivalent to property b'), cf. the remark after Definition 2.2. ♦

6.5. Example. Let us revert to the example 2.8.4 from [LMW] which we sketched in section 2, and describe an asymptotically mimimal feeding sequence. For all natural numbers n we put

$$X_n := \{ a \in \mathcal{A} : \kappa^{[n]}_{(a)}(\chi(\{(0,1)\}) = 1 \}.$$

Since the number of clans in $\underline{\Phi}$ a.s. equals 1, and since on the other hand by construction of $\underline{\Phi}$ there holds a.s. $\Phi_0(\{(0,1)\} = 1$, the sequence (X_n) must be asymptotically minimal feeding for $\underline{\Phi}$, and morover there must hold a.s.

$$\Phi_0(X_n) = 1 \qquad (n = 1, 2, \ldots)$$

As one checks easily, a site $(-i,j) \in \mathcal{A}$ belongs to X_n if and only if there holds

$$(*) \quad 1 \leq j \leq 2^{i-1} \quad ; \quad 2^{i-1} - n + 1 \leq j \leq 2^i - n.$$

In particular one has $n < 2^i$. If $n < 2^{i-1}$, then (*) is equivalent to

$$2^{i-1} - n + 1 \leq j \leq 2^{i-1}.$$

If $2^{i-1} \leq n < 2^i$, then (*) takes the form

$$1 \leq j \leq 2^i - n .$$

Denoting by i(n) the least natural number i with the property $n < 2^i$, we can rewrite

$$X_n = \{-i(n)\} \times \{1, \ldots, 2^{i(n)} - n\} \cup \bigcup_{i > i(n)} \{-i\} \times \{2^{i-1} - n + 1, \ldots, 2^{i-1}\}.$$

By some straightforward calculations which exploit the fact that

$$E\, \Phi_0(\{(-i, j)\}) = 2^{-j} \qquad ((-i, j) \in A),$$

one can show that

$$E\, \Phi_0(X_{n+1} \Delta X_n) < 2/n \qquad (n = 1, 2, \ldots). \ \blacklozenge$$

Note that in the preceding example the asymptotically minimal feeding sequence (X_n) "diverges" in the sense that for any bounded region B and all suitably large n the region X_n is disjoint from B. As a general fact there holds

6.6. Proposition. Let (X_n) be an asymptotically minimal feeding sequence of a stationary κ-process $\underline{\Phi}$ of finite type. Then the following assertions are equivalent:

a) the positive recurrent part $\underline{\omega}$ of $\underline{\Phi}$ is a.s. void

b) $\liminf_{n \to \infty} \mathrm{Prob}[\Phi_0(X_n \cap B) > 0] = 0 \qquad (B \in \mathcal{B}).$

If, in addition, $\underline{\Phi}$ is immigrative, then in b) the lower limit can be replaced by the limit.

Proof. For abbreviation we put

$$S := A_{\mathrm{PREC}} , \quad c := \mathrm{Prob}[\underline{\omega} \text{ non-void}].$$

1. Assume c > 0. Then there holds by Theorem 4.9

$$0 < c = \mathrm{Prob}[\omega_0(S) > 0] ,$$

hence we can select a bounded region B with the two properties

$$B \subseteq S \ ; \quad \mathrm{Prob}[\omega_0(B) > 0] > 0.$$

Let us now show that property b) fails for this region B.

Obviously there holds

$$\liminf_{n \to \infty} \mathrm{Prob}[\Phi_0(X_n \cap B) > 0] \geq \liminf_{n \to \infty} \mathrm{Prob}[\omega_0(X_n \cap S \cap B) > 0] .$$

Since (X_n) obeys property a) in Theorem 6.1 also with respect to $\underline{\omega}$, we can continue the preceding inequality, using Theorems 4.9 and 6.1 and Proposition 6.3:

$$\ldots = \mathrm{Prob}[\omega_0(S \cap B) > 0] = \mathrm{Prob}[\omega_0(B) > 0] > 0.$$

Therefore a) is a consequence of b).

2. Conversely, let us assume that condition b) is violated for some bounded region B_0. For all $B \in \mathcal{B}$ which contain B_0 there results

$$\begin{aligned} \liminf_{n \to \infty} \mathrm{Prob}[\Phi_{n \mid 0, B}(B) > 0] &= \liminf_{n \to \infty} \mathrm{Prob}[\Phi_{0 \mid -n, B}(B) > 0] \\ &= \liminf_{n \to \infty} \mathrm{Prob}[\Phi_{0 \mid -n, X_n \cap B}(B) > 0] \end{aligned}$$

$$\geq \liminf_{n \to \infty} \mathrm{Prob}[\Phi_{-n}(X_n \cap B) > 0]$$

$$- \limsup_{n\to\infty} \mathrm{Prob}[\Phi_{-n}(X_n\cap B) > 0;\ \Phi_0(B) = 0]\,.$$

If we choose B sufficiently large, then the right hand side of this chain of inequalities becomes positive, and we conclude by means of Theorem 5.1 that $\underline{\omega}$ is not a.s. void.

3. Now assume that $\underline{\Phi}$ is immigrative and of finite type.

For all bounded regions B, the sequence $(\Phi_{0\,|\,-n,B}(B))_{n>0}$ tends to 0 a.s. by assumption. Therefore, for any $B_0 \in \mathcal{B}$, we obtain like in the second step of the proof:

$$0 = \lim_{n\to\infty} \mathrm{Prob}[\Phi_{0\,|\,-n,B}(B) > 0]$$

$$\geq \limsup_{n\to\infty} \mathrm{Prob}[\Phi_0(X_n\cap B) > 0] \ - \ \mathrm{Prob}[\Phi_0 \neq o\,;\ \Phi_0(B) = 0]\,.$$

For sufficiently large B the last term becomes arbitrarily small, and there results

$$\limsup_{n\to\infty} \mathrm{Prob}[\Phi_0(X_n\cap B) > 0] \ = \ 0 \qquad (B_0 \in \mathcal{B}).$$ ♦

In case of a "substochastic intensity kernel", all stationary κ-processes of finite type just consist of a finite number of independently migrating particles only:

6.7. Proposition. Assume that $E\chi^{(a)}(A) \leq 1$ for all $a \in A$, and let $\underline{\Phi}$ be a stationary κ-processes with finitely many clans. Then with probability one there is no branching in $\underline{\Phi}$; in other words, is $\underline{\Phi}$ is finitely fed and the region A is essentially minimal $\underline{\Phi}$-feeding.

Proof. It suffices to show the assertion for the typical clan $\underline{\Psi}$ of $\underline{\Phi}$, which in addition is conditioned to be non-void. Let (R_n) be an asymptotically minimal feeding sequence of regions. For given $\varepsilon > 0$, take a bounded region L so large that

$$\mathrm{Prob}[\Psi_0(L) > 0] \ > 1-\varepsilon\,.$$

Then choose n so large that

$$\mathrm{Prob}[\Psi_{0|-n,R_n}(L) = \Psi_0(L)] > 1-\varepsilon$$

and $\quad \mathrm{Prob}[\Psi_{-n}(R_n) = 1] \ > 1-\varepsilon\,.$

We then have

(*) $\quad \mathrm{Prob}[\Psi_{0|-n,R_n}(L) \geq 1\,;\ \Psi_{-n}(R_n) = 1] \ > 1-3\varepsilon\,;$

on the other hand there holds

(**) $\quad \mathrm{Prob}[\Psi_{0|-n,R_n}(L) = 1\,;\ \Psi_{-n}(R_n) = 1] + 2\,\mathrm{Prob}[\Psi_{0|-n,R_n}(L) > 1\,;\ \Psi_{-n}(R_n) = 1]$

$$\leq E[\Psi_{0|-n,R_n}(L);\ \Psi_{-n}(R_n) = 1] \ \leq \ 1,$$

the last inequality coming from our assumption that $E\chi^{(a)}(A) \leq 1$ for all $a \in A$..

From (*) and (**) there results

$$\mathrm{Prob}[\Psi_{0|-n,R_n}(L) > 1\,;\ \Psi_{-n}(R_n) = 1] < 3\varepsilon,$$

and thus

$$\mathrm{Prob}[\Psi_0(L) > 1] \ < 5\varepsilon\,.$$

Since L was arbitrarily large, we obtain that $\mathrm{Prob}[\Psi_0(A) = 1] \ = 1.$ ♦

Even a weaker restriction on the fertility than that of Proposition 6.7 enforces all stationary κ-processes of finite type to be finitely fed:

6.8. Proposition. Let $\underline{\Phi}$ be a stationary κ-process with finitely many clans and assume that there exists a feeding region X for $\underline{\Phi}$ such that the familiy of random variables $\chi_t^{(a)}(X)$, $a \in$ X,
$t = 1, 2, \ldots$, is tight. Then $\Phi_0(X)$ is a.s. finite.

Proof. Like in the previous proposition, it suffices to show the assertion for the typical clan $\underline{\Psi}$ of $\underline{\Phi}$, which in addition is conditioned to be non-void. Let (R_n) be an asymptotically minimal feeding sequence of regions for $\underline{\Phi}$ such that $R_n \subseteq X$ for all $n \geq 1$. For all bounded regions $L \subseteq X$ and all natural numbers n, k there holds

$$\mathrm{Prob}[\Psi_{0|-n,R_n}(L) \geq k\, ;\, \Psi_{-n}(R_n) = 1] \leq \sup_{a \in X,\, t>0} \mathrm{Prob}\,[\chi_t^{(a)}(X) \geq k] := \delta(k)\,.$$

Therefore there holds

$$\begin{aligned} &\mathrm{Prob}[\Psi_0(L) \geq k\,] \\ &\quad \leq \limsup_{n\to\infty} \mathrm{Prob}[\Psi_{0|-n,R_n}(L) \neq \Psi_0(L)] + \limsup_{n\to\infty} \mathrm{Prob}[\Psi_{-n}(R_n) \neq 1] + \delta(k) \\ &\quad = \delta(k). \end{aligned}$$

Hence, for all natural numbers k, $\mathrm{Prob}[\Psi_0(X) \geq k\,] \leq \delta(k)$. Since $\delta(k)$ tends to 0 as $k \to \infty$ due to the assumed tightness, $\Psi_0(X)$ must be finite a.s. ♦

7. The discrete case

In the whole section we will assume that the space of sites A is at most countable, and is equipped with a metric ρ_A which makes just the finite subsets of A bounded. Thus, the symbol $\mathcal{B}$ will simply stand for the collection of finite subsets of A. The *mean* (or *intensity*) *matrix* J of the branching dynamics κ under consideration is given by

$$J(x,y) := E\chi^{(x)}(\{y\}), \quad x,y \in A.$$

Like in [MSW] we say that two sites $x,y \in A$ are J-*equivalent* if

$$\sum_{n \geq 0} J^{[n]}(x,y) > 0 \quad \text{and} \quad \sum_{n \geq 0} J^{[n]}(y,x) > 0$$

i.e. if the total progeny of an x-individual including itself charges y with poitive probability, and vice versa. (Recall that the entries $J^{[n]}(x,y)$ of the intensity matrix J are the expected numbers of the n-th generation offspring at site y of a mother individual at site x.)
For each site $a \in A$ we denote the corresponding J-*equivalence class* by $A_{[a]}$.
Now let $\underline{\Phi}$ be a stationary κ-process. We call a site a $\underline{\Phi}$*-immigrative* if, a.s., there are no ancestral paths in $\underline{\Phi}$ that visit a infinitely often, in other words, if the singleton $\{a\}$ is immigrative for $\underline{\Phi}$. In contrast, we call a site a $\underline{\Phi}$*-hermetic* if it is charged by $E\Phi_0$ and, with probability one, the ancestral line of every a-individual in $\underline{\Phi}$ visits the site a infinitely often.
In [MSW 2] we proved:
7.1. The J-equivalence class of a $\underline{\Phi}$-hermetic site in a stationary κ-process $\underline{\Phi}$ consists of $\underline{\Phi}$-hermetic sites only and is $\underline{\Phi}$-autonomous.

Since the finite or countable union of $\underline{\Phi}$-autonomous regions is again $\underline{\Phi}$-autonomous, 7.1 implies
7.2. The set of all $\underline{\Phi}$-hermetic sites (which we denote henceforth by $A_{\mathrm{herm},\underline{\Phi}}$) is $\underline{\Phi}$-autonomous.

Like in [MSW 2] we say that a stationary κ-process $\underline{\Phi}$ is *of standard type* if all sites are either $\underline{\Phi}$-hermetic or $\underline{\Phi}$-immigrative. In this case, the immigrative and the hermetic part of $\underline{\Phi}$ arise through a *spatial separation*: $\underline{\Phi}_{\mathrm{imm}}$ consists of all those clans in $\underline{\Phi}$ which never hit $A_{\mathrm{herm},\underline{\Phi}}$, and $\underline{\Phi}_{\mathrm{herm}}$ consists of all those clans whose ancestral lines emerge from $A_{\mathrm{herm},\underline{\Phi}}$. In particular (cf. section 2 for the notation) there holds with $S := A_{\mathrm{herm},\underline{\Phi}}$:

$$(*) \qquad \Phi_{0|-n,S} \xrightarrow{d} \Phi_{0,\mathrm{herm}}, \quad \Phi_{0|-n,A\setminus S} \xrightarrow{d} \Phi_{0,\mathrm{imm}}$$

The following example shows that in general (i.e. if $\underline{\Phi}$ is not of standard type) there need not exist any region S with the properties (*), and hence also in this weaker sense a spatial separation of the immigrative and the hermetic part is not always possible. Let us recall (cf. [MSW 1] that an *age dependent Galton Watson dynamics* on the space of sites (or "remaining lifetimes" $A = \mathbf{N}$ admits a nontrivial branching at most from the site 1; on all sites $n \neq 1$, it consists just in the shift $n \rightarrow n-1$.

7.3. Example. In [MSW1], section 7 , a subcritical age dependent Galton-Watson branching dynamics γ was constructed that admits a nontrivial hermetic stationary γ-process $\underline{\xi}$. On the other hand (cf. [MSW1], Prop. 6.1) there exists a nontrivial immigrative stationary γ-process $\underline{\eta}$ as well, whose construction is by far easier than that of $\underline{\xi}$. Indeed, let for $n=1,2,\ldots,$ ζ_n denote the n-th generation offspring population of an individual at site 1 (where all the ζ_n are independent) , then $\eta_0 := \sum_{n\geq 1} \delta_n + \sum_{n\geq 1} \zeta_n$ is an immigrative equilibrium population (the first summand describing the fresh immigrants from infinity, the second summand, which is locally finite due to the subcriticality of γ , describing the already renewed individuals). The independent superposition $\underline{\Phi}$ of $\underline{\xi}$ and $\underline{\eta}$ is a stationary γ-process for which the site 1 (and by the way, all other sites, too) is neither hermetic nor immigrative. We claim that there exists no set $S \subseteq \mathbf{N}$ such that (*) holds true. Indeed, assume that S has these properties. Then it must be infinite, since otherwise, due to the subcriticality of γ, $\Phi_{n|0,\,S} \xrightarrow{d} o$, contrary to the assumed convergence $\Phi_{n|0,\,S} \xrightarrow{d} \xi_0$.

Since $\underline{\xi}$ is hermetic, the probability p_m that there are 1-individuals in ξ_0 whose ancestral lines have not passed through 1 during the times $-m+1,\ldots,-1$, converges to 0 as $m \rightarrow \infty$. By stationarity, p_m equals the probability that there are 1-individuals in ξ_{m-1} whose ancestral lines have not passed through 1 during the times $0, \ldots, m-2$. On the other hand, the latter event certainly is implied by $\{\xi_0(\{m\}) > 0\}$, and therefore $\mathrm{Prob}[\xi_0(\{m\}) > 0] \rightarrow 0$ as $m \rightarrow \infty$.

We show now that the infinite set S cannot meet the second one of the two requirements in (*), in other words, that the convergence

$$(+) \qquad \Phi_{0|-n,\,\mathbf{N}\setminus S} \xrightarrow{d} \eta_0$$

is impossible.

To this end, we choose a strictly increasing sequence $n_1 < n_2 < \ldots$ in S. Let m be a natural number and $n_k > m$. We know already that, if $m \rightarrow \infty$, the individuals in Φ_{n_k-m} at position m are most probably "fresh immigrants" which had position n_k at time zero, i.e. in Φ_0. So, for any given $\varepsilon>0$, by choosing m large enough, we get for all k satisfying $n_k>m$:

$$\mathrm{Prob}[\Phi_{n_k-m\,|\,0,\,\mathbf{N}\setminus S}(\{m\}) > 0] < \varepsilon.$$

On the other hand there holds

$$\mathrm{Prob}[\eta_0(\{m\}) > 0] = 1 \qquad (m= 1, 2\ldots),$$

which is incompatible with (+). ◆

Let us now "localize" also the forward recurrent behaviour and put

$$A_{\mathrm{rec}} := \{ a \in A : \sum_{i>0} \chi_i^{(a)}(\{a\}) > 0 \text{ a.s. } \},$$

i.e. A_{rec} consists of all sites $a \in A$ which are *κ-recurrent* in the sense that a is a.s. visited by some member of the total progeny of an a-individual.

7.4. Theorem. Let $\underline{\Phi}$ be a stationary κ-process and B be a finite region. Then the following assertions are equivalent:

i) $E\Phi_0(B \cap A_{\mathrm{rec}}) = 0$

ii) $\sum_{n>0} \Phi_{n\,|\,0,B}(B) < \infty$ a.s.

iii) $\sum_{n>0} \Phi_{n\,|\,0,B}(X) < \infty$ a.s. $(X \in \mathcal{B})$.

Proof. 1. Assume $E\Phi_0(B \cap A_{\mathrm{rec}}) > 0$. We pick up some site a in $B \cap A_{\mathrm{rec}}$ which is charged by $E\Phi_0$ and observe that

$$\sum_{n>0} \Phi_{n\,|\,0,B}(B) \geq \sum_{n>0} \Phi_{n\,|\,0,\{a\}}(\{a\}) ,$$

which is infitite with positive probability. Hence ii) implies i).

2. Obviously ii) is a consequence of iii).

3. For the implication i) ⇒ iii), it is enough to show that, given any $a \in A \setminus A_{\mathrm{rec}}$ which is charged by $E\Phi_0$, all sites $b \in A$ satisfy

$$(*) \qquad \sum_{i>0} \chi_i^{(a)}(\{b\}) < \infty \quad \text{a.s.}$$

We will proceed in three steps, first proving (*) for $a=b$, then for $b \in A_{[a]} \setminus \{a\}$, and finally for $b \in A \setminus A_{[a]}$.

4. By 3.5. in [MSW 1], for all $x \in A$ which are charged by $E\Phi_0$, the embedded branching dynamics $\gamma_{[x]}$ is either subcritical or critical. The dynamics $\gamma_{[a]}$ cannot be a pure migration, since by assumption a does not belong to A_{rec}. Hence

$$\mathrm{Prob}[\sum_{i>0} \chi_i^{(a)}(\{a\}) < \infty] = U_{[a]}[\sum_{n>0} \varphi_n(\{1\}) < \infty] = 1,$$

where (cf. section 3 in [MSW 1]) $U_{[a]}$ denotes the distribution of the offspring $(\varphi_1, \varphi_2, \ldots)$ of of an individual at site 1 under the embedded branching dynamics $\gamma_{[a]}$. Hence (*) holds at least for $b = a$.

5. Now let b be any site in $A_{[a]} \setminus \{a\}$. The embedded branching dynamics $\gamma_{[b]}$ cannot be a pure migration either, since this would imply $b \in A_{\mathrm{rec}}$, and hence by 10.1 in [MSW 1] also $a \in A_{\mathrm{rec}}$. Hence almost surely a b-individual has at most finitely many b-descendants, i.e.

$$(+) \qquad U_{[b]}[\sum_{n>0} \varphi_n(\{1\}) < \infty] = 1.$$

Assume for a moment

$$\sum_{i>0} \chi_i^{(a)}(\{b\}) = \infty \quad \text{with positive probability.}$$

Then (+) would imply that with positive probability an a-individual $\mathcal{J}$ should have infinitely many *direct* b-descendants , i.e. b-descendants without b-ancestors on the ancestral line back to $\mathcal{J}$. Since by assumption there is some natural number m such that $J^{[m]}(b,a) > 0$, the probability for an a-individual to have infinitely many a-descendants would be positive, contradicting step 4.

6. Finally, take some $b \in A \backslash A_{[a]}$. Since (*) is clearly satisfied if $J^{[m]}(a,b) = 0$ for all $m = 1,2,\ldots$, we may assume that $J^{[m]}(b,a) = 0$ for all $m = 1,2,\ldots$

The idea is now to show that in each offspring tree of an a-individual that contains *exactly* k a-descendants, there can be a.s. no more than finitely many b-descendants.

Thus, let k be any nonnegative integer such that

$$\sum_{i>0} \chi_i^{(a)}(\{a\}) = k \quad \text{with positive probability}$$

Writing ξ_n for the number of those a-individuals in Φ_n which have exactly k a-descendants in the realization $\underline{\Phi}$, we obtain a shift-invariant random population $\psi := \sum_{n \in \mathbf{Z}} \xi_n \delta_n$ on the time axis $\mathbf{Z}$.

The random number of b-descendants at time n+j of an a-individual that contributes to ψ at time n is given by the distribution of $\chi_j^{(a)}(\{b\})$ conditioned under $\{\sum_{i>0} \chi_i^{(a)}(\{a\}) = k\}$. For all $n \in \mathbf{Z}$, let, on the time axis $\mathbf{Z}$, $\eta^{(n)}$ be a random daughter population (of a mother individual δ_n) whose distribution is that of $\sum_{j>0} \chi_j^{(a)}(\{b\})\, \delta_{n+j}$ conditioned under $\{\sum_{i>0} \chi_i^{(a)}(\{a\}) = k\}$.

Taking ψ as random mother population, we infer that its daughter population ρ is a.s. locally finite on $\mathbf{Z}$, since $\rho(\{n\})$ has the same distribution as the random number of those b-individuals in Φ_n whose ancestral lines contain an a-individual that has exactly k a-descendants. That is, we have a shift invariant , non-trivial distribution Q on the counting measures on $\mathbf{Z}$ (namely that of ψ) and a stationary clustering field on $\mathbf{Z}$ (given by the distribution of $\eta^{(0)}$) such that "the clustered distribution exists". Hence, by the same argument as in the proof of 4.8. in [MSW 1] we conclude, using a result of K. Hermann (cf. [MKM] 11.2.4), that $\sum_{j>0} \mathrm{Prob}[\eta^{(0)}\{j\}>0]$ must be finite. By construction, this latter expression equals the sum of the conditional probabilities

$$\sum_{j>0} \mathrm{Prob}[\chi_j^{(a)}(\{b\})>0 \mid \sum_{i>0} \chi_i^{(a)}(\{a\}) = k].$$

Hence, by the Borel-Cantelli lemma,

$$\mathrm{Prob}[\sum_{j>0} \chi_j^{(a)}(\{b\}) < \infty \mid \sum_{i>0} \chi_i^{(a)}(\{a\}) = k] = 1.$$

Since this relation holds for all $k \geq 0$ obeying $\mathrm{Prob}[\sum_{i>0} \chi_i^{(a)}(\{a\}) = k] > 0$, we obtain the validity of (*). ♦

Theorem 7.4 shows in particular that for any site $a \notin A_{rec}$ and all bounded regions B there holds

$$\mathrm{Prob}[\sum_{n>0} \Phi_{n \mid 0,\{a\}}(B) = \infty] = 0 .$$

If in addition $E\Phi_0(\{a\}) > 0$, this implies $\mathrm{Prob}[\sum_{n>0} \chi_n^{(a)}(B) = \infty] = 0$, i.e. $a \notin A_{REC}$.

Therefore we have

7.5. For each stationary κ-process $\underline{\Phi}$, $E\Phi_0(A_{REC} \setminus A_{rec}) = 0$.

Also, note that A_{rec} is a subset of A_{REC} , because an an individual δ_a having a.s. at least one a-descendant has a.s. at least infinitely many of them.

We have seen in Example 1.3 that in the general situation there exist examples of immigrative stationary κ-processes that are not of transient type. For the discrete case considered in the present section, however, we read off from [MSW1], Thm 8.1, and [MSW2], Thm 15.2, respectively:

7.6. Theorem. Every immigrative stationary κ-process is of transient type. If $\chi^{(a)}(A) \leq 1$ a.s. for all sites $a \in A$ (i.e. if the dynamics κ is a "substochastic shift"), then also the converse is true.

An immediate consequence is:

7.7. Corollary. For each stationary κ-process $\underline{\Phi}$, its immigrative part $\underline{\Phi}_{imm}$ is a.s. a subsystem of its transient part $\underline{\Phi}_{trans}$, and its recurrent part $\underline{\Phi}_{imm}$ is a.s. a subsystem of its hermetic part $\underline{\Phi}_{herm}$. If κ is a substochastic shift, then $\underline{\Phi}_{imm} = \underline{\Phi}_{trans}$ a.s., and $\underline{\Phi}_{herm} = \underline{\Phi}_{rec}$ a.s.

As was shown at the beginning of section 10 in [MSW1], there holds

7.8. Every κ-recurrent site which is charged by the intensity measure $E\Phi_0$ of a stationary κ-process $\underline{\Phi}$, is also $\underline{\Phi}$-hermetic.

From [MSW1], 10.1, one reads off:

7.9. Every J-equivalence class $X \subseteq A$ which is charged by the intensity measure $E\Phi_0$ of a stationary κ-process $\underline{\Phi}$ is either contained in or disjoint from A_{rec} . The former case holds true if and only if the branching dynamics κ constitutes within X a "stochastic shift" (i. e. a pure Markovian migration) whose transition matrix $(J(x,y))_{x,y \in X}$ is irreducible and recurrent.

Now consider a stationary κ-process $\underline{\Phi}$ of recurrent type, and put

$S := \{a\in A_{\mathrm{rec}} : E\Phi_0\{a\} > 0\}$. Due to 7.5, a.s. all ancestral paths in $\underline{\Phi}$ get trapped in S backwards in time. Therefore, all $x \in A\backslash S$ must be $\underline{\Phi}$-immigrative. Since by 7.8 all $a\in S$ are $\underline{\Phi}$-hermetic, we observe:

7.10. Proposition. Every stationary κ-process $\underline{\Phi}$ of recurrent type is also of standard type, and there holds

$$A_{\mathrm{herm},\underline{\Phi}} = \{a\in A_{\mathrm{rec}} : E\Phi_0\{a\} > 0\}.$$

In contrast to this, in [MSW], 14.3 we gave an example of hermetic stationary κ-process which is not of standard type.

7.11. Remark. a) Let $\underline{\Phi}$ be a stationary κ-process. According to 7.9, the set $S := \{a\in A_{\mathrm{rec}} : E\Phi_0\{a\} > 0\}$ consists of full J-equivalence classes, and these are all $\underline{\Phi}$-autonomous by 7.1 and 7.8. As a countable union of $\underline{\Phi}$-autonomous regions, S is $\underline{\Phi}$-autonomous as well. Another application of 7.9 reveals that the branching dynamics κ constitutes within S a recurrent Markov chain with transition matrix $(J(x,y))_{x,y\in X}$. The statements in section 4 on the structure of the stationary κ-processes of positive recurrent type thus can be carried over (in the discrete case) with one exception: Now the set S need no more be a union of pairwise disjoint, $\underline{\Phi}$-autonomous, *finitely populated* regions: This is the case if and only if and only if $\Phi_0(X)$ is a.s. finite for all J-equivalence-classes X in S, i.e. if $(J(x,y))_{x,y\in S}$ is the transition matrix of a positive recurrent Markov chain.

b) If, for a J-equivalence-class $X \subseteq S$, $\Phi_0(X)$ is with positive probability infinite, i.e. if $(J(a,b))_{a,b\in X}$ is null-recurrent, then it follows from [LP], Theorem 1.4, that the restriction of Φ_0 to X is a doubly stochastic Poisson population.

7.12. Notation. For a site $a \in A$, we put

$$\tau_a := \min\{n\in N: \chi_n^{(a)}(\{a\} > 0\}\ , \text{ where } \min\varnothing = \infty .$$

In these terms we get $A_{\mathrm{rec}} := \{a\in A: \tau_a < \infty \text{ a.s.}\}$, and we define

$$A_{\mathrm{prec}} := \{a\in A: E\tau_a < \infty\}\ ;$$

note that A_{prec} clearly is a subset of A_{PREC} defined in 4.7. As a counterpart to 7.9 we have (cf. 10.2 in [MSW 1]):

7.13. For a J-equivalence class X which is charged by the intensity measure $E\Phi_0$ of a stationary κ-process $\underline{\Phi}$, the following assertions are equivalent:

i) $X\cap A_{\mathrm{prec}} \neq \varnothing$ ii) $X\subseteq A_{\mathrm{prec}}$

iii) the branching dynamics κ constitutes within X an irreducible positive recurrent Markov chain

iv) X is $\underline{\Phi}$-autonomous and $\Phi_0(X) < \infty$ a.s.

As a counterpart to 7.4 there holds

7.14. Theorem. Let $\underline{\Phi}$ be a stationary κ-process and B be a finite region. Then the following assertions are equivalent:

i) $E\Phi_0(B \cap \mathcal{A}_{\text{prec}}) = 0$

ii) $\Phi_{n\,|\,0,B}(B) \to 0$ in probability

iii) $\Phi_{n\,|\,0,B}(X) \to 0$ in probability $(X \in \mathcal{B})$.

Proof. I. Assume i) is false, i.e. there exists some site a in $B \cap \mathcal{A}_{\text{prec}}$ which is charged by $E\Phi_0$. If we denote by d the period of the irreducible Markov chain induced by κ in the J-equivalence class $\mathcal{A}_{[a]}$, then by 7.13 the sequence $\text{Prob}[\chi^{(a)}_{nd}(\{a\}) > 0]$ has a positive limit. This is in conflict with ii), and therefore ii) implies i).

II. Obviously iii) implies ii).

III. For the implication i) $\Rightarrow$ iii) , it suffices to fix a site a outside of $\mathcal{A}_{\text{prec}}$ and charged by $E\Phi_0$, and to show that for all $b \in \mathcal{A}$

(*) $\chi^{(a)}_n(\{b\}) \to 0$ in probability as $n \to \infty$.

By 3.5. in [MSW1] the embedded clustering dynamics $\gamma_{[a]}$ is subcritical or critical.

1. If $\gamma_{[a]}$ is not a stochastic shift, i.e. if the site a does not even belong to $\mathcal{A}_{\text{rec}}$, then we obtain from Theorem 7.4:

$$\sum_{n>0} \chi^{(a)}_n(\{b\}) < +\infty \quad \text{a.s.} \quad \text{for all } b \in \mathcal{A} ,$$

which implies (*).

2. For the rest of the proof we assume that $\gamma_{[a]}$ is a stochastic shift, and therefore $a \in \mathcal{A}_{\text{rec}} \setminus \mathcal{A}_{\text{prec}}$. Then by 7.9 and 7.13 the restriction of κ to $S := \mathcal{A}_{[a]}$ constitutes an irreducible recurrent Markov chain, whose transition kernel K obeys

$$K^{[n]}(a, a) \to 0 .$$

Hence (*) is true in the special case $b = a$. To show it also for all other $b \in \mathcal{A}$, let us denote by d the period and by $Z_1, Z_2, \ldots$ the cyclic subclasses of K, assuming that $a \in Z_1$.

3. As an intermediate step, we will show:

$$\text{(o)} \quad \lim_{n\to\infty} \left|\, \text{Prob}[\chi^{(a)}_n(\{b\}) > 0] - \text{Prob}[\chi^{(x)}_n(\{b\}) > 0] \,\right| = 0 \qquad (x \in Z_1).$$

To this end we note first that for all natural numbers m, n with $m < n$ there holds

$$\left|\, \text{Prob}[\chi^{(a)}_n(\{b\}) > 0] - \int \text{Prob}[\chi^{(y)}_{n-m}(\{b\}) > 0]\, K^{[m]}(a, dy) \,\right|$$

$$\leq \ \text{Prob}[\chi^{(a)}_{n\,|\,m,\mathcal{A}\setminus S}(\{b\}) > 0] =: c_{m,n} .$$

Writing $\underline{\Psi}$ for that "component" of $\underline{\Phi}$ whose ancestral lines come from S, we obtain:

$$c_{m,n} \leq \text{Prob}[\Psi_0(\{a\}) > 0]^{-1}\ \text{Prob}[\Psi_{n\,|\,m,\mathcal{A}\setminus S}(\{b\}) > 0].$$

But for fixed m, this latter expression tends to 0 as $n \to \infty$, since S is a feeding region for $\underline{\Psi}$.

The same convergence holds true if we substitute the site a by any $x \in Z_1$. Writing $\|\mu\| := \sum_{y \in C} |\mu(\{y\})|$ for any finite signed measure μ on C, we obtain for all natural numbers m, n satisfying $n > m$:

$$\left| \int \mathrm{Prob}[\chi^{(y)}_{n-m}(\{b\}) > 0]\, K^{[m]}(a, dy) - \int \mathrm{Prob}[\chi^{(y)}_{n-m}(\{b\}) > 0]\, K^{[m]}(x, dy) \right|$$
$$\leq \| K^{[m]}(a, (.)) - K^{[m]}(x, (.)) \|,$$

which tends to zero for $m \to \infty$ due to our assumptions on K, a and x. Summarizing, we obtain the asserted convergence relation (o).

4. Now assume that (*) is violated for some site b, i.e. that there exists a number $\eta > 0$ and a sequence $n_1 < n_2 < \ldots$ such that

$$\mathrm{Prob}[\chi^{(a)}_{n_m}(\{b\}) > 0] > 2\eta \qquad (m = 1,2,\ldots)$$

Our assumptions imply that $K^{[d]}$ is aperiodic and null-recurrent in each of the sets Z_1, $Z_2, \ldots, Z_d$, and $E\Phi_0(Z_1) > 0$. Thus an application of 7.13 to the case where κ is substituted by $K^{[d]}$ yields

$$c := \mathrm{Prob}[\Phi_0(Z_1) = \infty] > 0.$$

For any natural number k, choose s large enough so that

$$\beta(s,\eta)(\{k, k+1, \ldots\}) > 1/2,$$

where $\beta(s,\eta)$ stands for the binomial distribution with s trials and success probability η. Let L be any finite subset of Z such that

$$\mathrm{Prob}[\Phi_0(L) \geq s] \geq c/2$$

and choose, in virtue of (o), m large enough such that $\mathrm{Prob}[\chi^{(x)}_{n_m}(\{b\}) > 0] > \eta \qquad (x \in L)$.

Then there would result

$$\mathrm{Prob}[\Phi_{n_m}(\{b\}) \geq k] \geq \mathrm{Prob}[\Phi_0(L) \geq s]\ \beta(s,\eta)(\{k, k+1, \ldots\}) \geq c/4,$$

which leads to a contradiction since k was arbitrary.

Thus the convergence relation (*) indeed holds for all $b \in A$. ♦

Theorem 7.14 shows in particular that for any site $a \notin A_{\mathrm{prec}}$ and all bounded regions B there holds

$$\Phi_{n\,|\,0,\{a\}}(B) \to 0 \quad \text{in probability}.$$

If in addition $E\Phi_0(\{a\}) > 0$, this implies the convergence of $\mathrm{Prob}[\chi^{(a)}_n(B) > 0]$ to 0 as $n \to \infty$, and consequently $a \notin A_{\mathrm{PREC}}$. Therefore we have the following companion to 7.5:

7.15. For each stationary κ-process $\underline{\Phi}$, $E\Phi_0(A_{\mathrm{PREC}} \setminus A_{\mathrm{prec}}) = 0$.

Theorem 7.14 also yields the following complement to 5.1:

7.16. A stationary κ-process $\underline{\Phi}$ has a trivial positive recurrent part if and only if it satisfies

(*) $\quad \lim_{n \to \infty} \mathrm{Prob}[\Phi_{n|0,B}(B) > 0] = 0 \qquad (B \in \mathcal{B}).$

Proof. 1. Clearly, (*) implies condition b) of 5.1, which in turn enforces $\underline{\Phi}_{\mathrm{prec}}$ to be a.s. void.
2. Conversely, assume that $\underline{\Phi}_{\mathrm{prec}}$ is a.s. void, i.e. $E\Phi_0(A_{\mathrm{prec}}) = 0$. Then (*) results immediately from Theorem 7.13. ♦

We conclude with two specialities of the discrete case which both have counterexamples in the general setting (see 5.6.b) and 2.5 b)).

7.18. Theorem. A stationary κ-process $\underline{\Phi}$ is of positive recurrent type if and only if it has an essentially minimal feeding region. In this case $S := \{a \in A_{\mathrm{prec}} : E\Phi_0\{a\} > 0\}$ is even strictly minimal feeding, in the sense that it is covered by any other feeding region of $\underline{\Phi}$.

Proof. 1. First let us show that each feeding region X for $\underline{\Phi}$ covers S. Indeed, assume there would exist an $a \in S\setminus X$. Then, on one hand, $\Phi_{n\,|\,0,\{a\}}(\{a\})$ *does not* converge to 0 in probability as $n \to \infty$ due to Theorem 7.14, on the other hand it *does* converge to 0 in probability , since X is a feeding region; hence $S \subseteq X$.
2. Now assume that $\underline{\Phi}$ is of positive recurrent type. Then by Theorem 4.9 and 7.15, A_{prec} is essentially minimal feeding for $\underline{\Phi}$.
2. Conversely, assume that X is an essentially minimal feeding region for $\underline{\Phi}$. Assume there would exist a $b \in X\setminus S$ which is charged by $E\Phi_0$. Then, on one hand, there exists a bounded set B such that
$\Phi_{n\,|\,0,\{b\}}(B)$ *does not* converge to 0 in probability due to the essential minimality of X; on the other hand it *does* converge to 0 in probability by 7.14. Hence also S is $\underline{\Phi}$-feeding, and therefore $\underline{\Phi}$ is of positive recurrent type. ♦

The following theorem states that in the discrete case and for a stationary κ-process with finitely many clans, individual backward recurrence *implies* individual forward positive recurrence.

7.19. Theorem. A stationary κ-process $\underline{\Phi}$ is finitely fed if and only if it is hermetic and of finite type.

Proof. 1. The "only if" direction is clear from Remark 4.3.
2. For the rest of the proof we assume that $\underline{\Phi}$ is hermetic and of finite type. For all sites a , denote by $\underline{\Phi}^a$ the subsystem consisting of all those individuals in $\underline{\Phi}$ whose ancestral paths visited the site a infinitely often; by Lemma 1.8 , $\underline{\Phi}^a$ is again a stationary κ-process, and it consists of a.s. finitely many clans as well. If we can show that a.s. the ancestral line of each clan in $\underline{\Phi}^a$ visits the site a with a positive frequency, then $\underline{\Phi}$ must be of positive recurrent type, since each clan in $\underline{\Phi}$, which is by assumption hermetic, belongs to at least one of the $\underline{\Phi}^a$.
3. Fix a site $a \in A$, and assume that $\underline{\Phi}^a$ is non-void with positive probability. Let $\gamma := \gamma_{[a]}$ denote the branching dynamics of the age dependent Galton-Watson dynamics embedded at

site a, and $\underline{\xi}$ denote the corresponding stationary γ-process (whose successive populations ξ_n live on the space $\{1,2,\ldots\}$ of remaining lifetimes [of the excursions from site a]). The number of clans in $\underline{\xi}$ is the same as that in $\underline{\Phi}^a$, and in particular it is a.s. finite. Since, due to [MSW1] 3.5, $\gamma_{[a]}$ assigns to each individual an expected number of daughters less or equal to 1, we infer from Proposition 6.5 that $\underline{\xi}$ is finitely fed, and hence by 4.3 with probability one the asymptotic frequency of ancestors at site 1 in the ancestral line of each individual in $\underline{\xi}$ is positive. Equivalently, with probability one the asymptotic frequency of ancestors at site a in the ancestral line of each individual in $\underline{\Phi}^a$ is positive.

4. By 3. and 2., $\underline{\Phi}$ is of positive recurrent type. The number $\Phi_0(A_{\text{prec}})$ of clans in $\underline{\Phi}$ is by assumption a.s. finite, and therefore $\underline{\Phi}$ is even finitely fed. ♦

8. Appendix.

Throughout the paper we have freely been speaking of genealogical relationships between individuals in a process which follows a branching dynamics κ . These relationships, however, remain disguised as long as we model the subsequent generations merely as counting measures on the space of sites. Similar as in [MSW], section 1, one can, however, easily display the genealogical relationships by passing to a suitably *refined* branching model, in which every individual carries as a mark two numbers from the unit interval : every daughter inherits as her first number the second one of her mother and chooses her second number, independently of her sisters, uniformly distributed from $[0,1]$. Formally, this is achieved by defining the *refined space of sites* $\tilde{A} := A \times [0,1]^2$, equipped with the metric $\rho_{\tilde{A}}((a, x,y), (b, u,v)) := \rho_A(a,b) + |x-u| + |y-v|$. The *refined branching dynamics* $\tilde{\kappa}$ sends an individual $\delta_{(a,x,y)}$ into the random daughter population $\sum_{i \in I} \delta_{(b_i,y,v_i)}$, where $\sum_{i \in I} \delta_{b_i}$ is the random daughter population of $\delta_{(a,x,y)}$ under κ, and the v_i are independent and uniformly distributed on $[0,1]$. Clearly, every $\tilde{\kappa}$-equilibrium population renders a κ-equilibrium population by projection. Conversely, we can lift a κ-equilibrium population Φ to a $\tilde{\kappa}$-equilibrium population $\tilde{\Phi}$ in the following way: we choose independent marks uniformly from $[0,1]^2$ for all the individuals in Φ, which gives a population $\tilde{\Psi}$ on $\tilde{A}$, whose daughter population under $\tilde{\kappa}$ is the desired $\tilde{\kappa}$-equilibrium population. This lifting and projection establishes a 1-1 correspondence between the κ-equilibrium distributions and the $\tilde{\kappa}$-equilibrium populations. Now almost all realizations $\underline{\tilde{\Phi}} = (\tilde{\Phi}_n)_{n \in \mathbf{Z}}$ of a stationary $\tilde{\kappa}$-process have the following property:

$(\mathcal{T})$ All individuals in $\underline{\tilde{\Phi}}$ carry different marks, and for all $n \in \mathbf{Z}$ every individual $\delta_{(b,u,v)}$ in $\tilde{\Phi}_n$ has exactly one *ancestor* $\delta_{(a,x,y)}$ in $\tilde{\Phi}_{n-1}$ in the sense that $y = u$.

On the other hand, for each realization $\underline{\tilde{\Phi}}$ having property $(\mathcal{T})$ and each individual in $\underline{\tilde{\Phi}}$, its *ancestral path*, its *offspring tree* and its *clan* are well defined.

In this way (cf. MSW I], 1.5), we could also associate with almost every $\tilde{\Phi}_n$ the population H_n of its ancestral paths, thus obtaining a (discrete time) "historical process" in the spirit of Dawson and Perkins ([DP], ch. 6).

As mentioned above, many statements and proofs of the paper indeed require to keep track of the genealogical relationships of individuals, and therefore should (and can) be properly reformulated in a "refined" or "historical model" as described above. We have decided to omit this (in most cases rather starightforward) reformulation in order not to overburden the notation.

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