On a Penrose–Fife Model with Zero Interfacial Energy Leading to a Phase–field System of Relaxed Stefan Type

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Abstract.

In this paper we study an initial–boundary value Stefan–type problem with phase relaxation where the heat flux is proportional to the gradient of the inverse absolute temperature. This problem arises naturally as limiting case of the Penrose–Fife model for diffusive phase transitions with non–conserved order parameter if the coefficient of the interfacial energy is taken as zero. It is shown that the relaxed Stefan problem admits a weak solution which is obtained as limit of solutions to the Penrose–Fife phase–field equations. For a special boundary condition involving the heat exchange with the surrounding medium, also uniqueness of the solution is proved.

1 Introduction

In this paper, we study the initial–boundary value problem

$$c_0\,\theta_t \,-\, \lambda'(\chi)\,\chi_t \,+\, k\,\Delta\left(\frac{1}{\theta}\right) = \,g \qquad \text{in } Q\,, \tag{1.1}$$

$$\mu\,\chi_t \,+\, \beta(\chi) \,\ni\, s'(\chi) \,+\, \frac{\lambda'(\chi)}{\theta} \qquad \text{in } Q\,, \tag{1.2}$$

$$k\,\frac{\partial\,\theta}{\partial\,n} \,+\, \alpha\,(\theta \,-\, \theta_\Gamma) \,=\, 0 \qquad \text{in } \Sigma\,, \tag{1.3}$$

$$\theta(\cdot,0) = \theta_0\,, \quad \chi(\cdot,0) = \chi_0 \quad \text{in } \Omega\,. \tag{1.4}$$

Here, $\Omega \subset \mathbb{R}^3$ denotes a bounded domain with smooth boundary Γ; $T > 0$ is some final time, and $Q := \Omega \times (0,T)$, $\Sigma := \Gamma \times (0,T)$. In addition, c_0, k, μ, α denote positive physical constants.

Equations (1.1)-(1.2) may be regarded as the system of phase–field equations governing the kinetics of a phase transition with non–conserved *order parameter* χ that occurs in the three–dimensional container Ω. In this connection, the variable θ represents the absolute (Kelvin) temperature, while g and θ_Γ stand for the density of distributed heat sources and the outside temperature, respectively. Typically, χ represents a volume density of one of the phases. In an ice–water system, for instance, χ may be identified with the liquid fraction.

Concerning the nonlinearities s, λ, β occuring in (1.1)-(1.2), we make the following assumptions: s and λ are smooth, and $\beta = \partial I$, i.e. β denotes the maximal monotone graph representing the subdifferential of the indicator function I of the interval [0, 1] (cf. formula (2.1)). The variational inequality (1.2) then entails that the variable χ is forced to attain only values in the physically meaningful range [0, 1]. We should remark at this place that the whole analysis of this paper remains true (with obvious modifications) for much more general maximal monotone graphs β.

The phase–field equations (1.1)-(1.2) are closely connected to two models for phase transitions that have been the subject of intense mathematical research in recent years, namely the *Penrose–Fife model* and the *Stefan model*. Indeed, if the local free energy density $F = F(\chi,\theta)$ is assumed in the form

$$F(\chi,\theta) = -c_0\,\theta\,\ln(\theta) + \theta\Big(I(\chi) - s(\chi)\Big) - \lambda(\chi)\,, \tag{1.5}$$

then (1.1)-(1.2) coincide with the phase–field equations resulting from the Penrose–Fife approach (cf. [12]) if no interfacial energies are present. On the other hand, if we make the particular choice (cf. [4])

$$\lambda(\chi) = -L\,\chi\,, \quad s(\chi) = \frac{L}{\theta_C}\,\chi\,, \tag{1.6}$$

where L and θ_C represent latent heat and a critical temperature (of melting, say), then (1.1)-(1.2) becomes

$$c_0\,\theta_t + L\,\chi_t + k\,\Delta\left(\frac{1}{\theta}\right) = g \quad \text{in } Q\,, \tag{1.7}$$

$$\mu\,\chi_t + \beta(\chi) \ni L\left(\frac{1}{\theta_C} - \frac{1}{\theta}\right) \quad \text{in } Q\,. \tag{1.8}$$

The latter system may be considered as a Stefan–type problem with phase relaxation, where the heat flux $\vec{q}$ is given by

$$\vec{q} = k\,\nabla\left(\frac{1}{\theta}\right) \tag{1.9}$$

instead of by the usual Fourier law. This becomes more evident in the case $\mu = 0$, because then (1.8) can be equivalently written as (if $\theta > 0$, which ought to be true since θ represents the absolute temperature)

$$\chi \in H(\theta - \theta_C), \tag{1.10}$$

with the *Heaviside graph* H. Substitution of (1.10) in (1.7) indeed leads to the *enthalpy formulation* of the Stefan problem, but with the heat flux given by (1.9).

From the mathematical point of view, the phase–field equations (1.1)-(1.2) are considerably more difficult to handle than both the Stefan problem with phase relaxation and usual Fourier–type heat flux and the (usual) Penrose–Fife system. In particular, the appearance of the inverse temperature $1/\theta$ in both (1.1) and (1.2) is a possible source of singularities which is not present in the standard Stefan problem; on the other hand, in contrast to the Penrose–Fife system with non–zero interfacial energy, where the second equation has the form

$$\mu\,\chi_t + \beta(\chi) - \varepsilon\,\Delta\,\chi \ni s'(\chi) + \frac{\lambda'(\chi)}{\theta} \tag{1.11}$$

instead of (1.2), the diffusive term $-\varepsilon\,\Delta\,\chi$ is missing, which entails less spatial regularity for the order parameter field.

Our line of argumentation to overcome the above–mentioned difficulties will be the following. Assuming the function λ concave, we regard the system (1.1)-(1.4) as limiting case of the Penrose–Fife model with non–zero interfacial energy (i.e. for $\varepsilon > 0$). For the Penrose–Fife system, a general existence result (cf. Laurençot [8, 9]) is known, yielding solution pairs $(\theta_\varepsilon, \chi_\varepsilon)$ for $\varepsilon > 0$. We shall derive a priori bounds, independent of ε, for these solutions, and then use compactness arguments and a passage–to–the–limit procedure for $\varepsilon \searrow 0$ to establish the desired existence result for weak solutions to (1.1)-(1.4).

The remainder of this paper is organized as follows. In Section 2, we define our notion of a weak solution to (1.1)-(1.4), specify the general assumptions for the data of the system and introduce the approximating Penrose–Fife system. Section 3 brings the derivation of global a priori estimates for the approximating solutions, and in Section 4 the passage to the limit is performed. Finally, in Section 5, we argue on other boundary conditions than (1.3), and we study a special case, namely

$$k\,\frac{\partial\,\theta}{\partial\,n} + \alpha\,\theta\,(\theta - \theta_\Gamma) = 0 \quad \text{in } \Sigma\,. \tag{1.12}$$

If one substitutes (1.3) with (1.12), then not only existence but also uniqueness of the weak solution to the resulting problem can be established. By this uniqueness result, we can conclude that the system (1.1)-(1.2), (1.12), (1.4) is indeed the natural asymptotic limit of the analogous Penrose–Fife system (which has been investigated in [7]) when the interfacial energy tends to zero.

We should remark at this place that a corresponding analysis is possible for the system (1.7), (1.10), i.e. for the unrelaxed Stefan problem with heat flux given by (1.9). Since the employed techniques and, in particular, the obtained regularity results, are considerably different, this will be the subject of a forthcoming paper.

2 Main Result

In order to state precise assumptions on the data and to introduce a variational formulation of the problem (1.1)-(1.4), which henceforth will be called **(P)** for simplicity, we first fix some notations. Let $(\cdot,\cdot)$ represent either the scalar product in $L^2(\Omega)$ or the duality pairing between V' (the dual space of $V := H^1(\Omega)$) and V, and let $\|\cdot\|$ stand for the norms in both $L^2(\Omega)$ and $(L^2(\Omega))^3$. The trace of a function $v \in H^1(\Omega)$ on the boundary Γ is denoted by $v_{|\Gamma} \in H^{1/2}(\Gamma)$ or, if no confusion may arise, just by v. Furthermore, the notations for Sobolev spaces are the same as in [10], for instance.

Recalling that c_0, k, μ, α are positive constants and that β is the maximal monotone graph from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\beta(r) = \begin{cases} (-\infty, 0] & \text{if} \quad r = 0 \\ \{0\} & \text{if} \quad 0 < r < 1 \\ [0, +\infty) & \text{if} \quad r = 1 \end{cases} \tag{2.1}$$

with domain $[0,1]$, the problem **(P)** is analyzed under the additional assumptions

$$\lambda, s \in C^2([0,1]), \tag{2.2}$$

$$\lambda \text{ is a concave function }, \tag{2.3}$$

$$g \in L^\infty(Q), \tag{2.4}$$

$$\theta_\Gamma \in L^\infty(\Sigma), \quad \theta_\Gamma > 0 \quad \text{a.e. in } \Sigma, \quad \frac{1}{\theta_\Gamma} \in L^\infty(\Sigma), \tag{2.5}$$

$$\partial_t \theta_\Gamma \in L^\infty(\Sigma), \tag{2.6}$$

$$\theta_0 \in L^\infty(\Omega), \quad \theta_0 > 0 \quad \text{a.e. in } \Omega, \quad \frac{1}{\theta_0} \in L^\infty(\Omega), \tag{2.7}$$

$$\theta_0 \in H^1(\Omega), \tag{2.8}$$

$$\chi_0 \in L^\infty(\Omega), \quad 0 \le \chi_0 \le 1 \quad \text{a.e. in } \Omega. \tag{2.9}$$

Remark 2.1. Observe that, owing to (2.5) and (2.7),

$$\theta_\Gamma \ge c \quad \text{a.e. in } \Sigma, \qquad \theta_0 \ge c \quad \text{a.e. in } \Omega, \tag{2.10}$$

for some constant $c > 0$. Moreover, it is a standard matter to verify that (2.7) and (2.8) imply that

$$\theta_0^{\,r} \in H^1(\Omega) \qquad \forall r \in \mathbb{R}, \tag{2.11}$$

as well as

$$\theta_0^{\,r}{}_{|\Gamma} \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma) \quad \forall r \in \mathbb{R}. \tag{2.12}$$

Let us specify our notion of a weak solution to problem **(P)**.

Definition 2.2. A couple of functions (θ, χ) is called a *weak solution* to **(P)** if

$$\theta \in W^{1,\infty}(0,T;V') \cap L^\infty(0,T;L^2(\Omega)), \tag{2.13}$$

$$\chi \in W^{1,\infty}(0,T;L^2(\Omega)), \tag{2.14}$$

$$\theta > 0, \quad 0 \le \chi \le 1 \quad \text{a.e. in } Q, \tag{2.15}$$

and if there exist functions

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(Q), \tag{2.16}$$

$$\xi \in L^\infty(0,T;L^2(\Omega)), \tag{2.17}$$

satisfying

$$u = \frac{1}{\theta}, \quad \xi \in \beta(\chi) \quad \text{a.e. in } Q, \tag{2.18}$$

such that the following equations and conditions hold

$$\Big(\partial_t \big(c_0\,\theta - \lambda(\chi)\big)(\,\cdot\,,t)\,,v\Big) = k \int_\Omega \nabla u(\,\cdot\,,t)\cdot\nabla v \,+\, \alpha \int_\Gamma \big(\theta_\Gamma\, u^2 - u\big)(\,\cdot\,,t)v + (g(\,\cdot\,,t)\,,v)$$

$$\forall\, v \in H^1(\Omega), \text{ for a.e. } t \in (0,T), \tag{2.19}$$

$$\mu\,\chi_t + \xi = s'(\chi) \,+\, \lambda'(\chi)\,u \quad \text{a.e. in } Q, \tag{2.20}$$

$$\theta(\,\cdot\,,0) = \theta_0, \quad \chi(\,\cdot\,,0) = \chi_0. \tag{2.21}$$

Remark 2.3. Due to (2.14)-(2.15) and (2.2), it turns out that $\lambda(\chi) \in W^{1,\infty}(0,T;L^2(\Omega))$ and $\partial_t\lambda(\chi) = \lambda'(\chi)\,\chi_t$ a.e. in Q. From (2.16) one easily infers the following regularity property for the trace of u,

$$u_{|\Gamma} \in L^\infty(0,T;H^{1/2}(\Gamma)) \cap L^\infty(\Sigma), \tag{2.22}$$

and (2.22) also provides a meaning to the boundary integral in (2.19). By virtue of (2.13), one can check that θ is a weakly continuous function from $[0,T]$ into $L^2(\Omega)$, so that the initial conditions (2.21) make sense in the space $L^2(\Omega)$.

Remark 2.4. The conditions $\theta > 0$ and $u = 1/\theta$, holding a.e. in Q, can be rewritten in terms of maximal monotone operators. Indeed, letting ρ denote the maximal monotone graph specified by

$$\rho(r) = -\frac{1}{r} \qquad \text{if} \quad 0 < r < +\infty,$$

such conditions reduce to $-u \in \rho(\theta)$. Alternatively, one can prescribe that $-\theta \in \rho(u)$ a.e. in Q and consider θ as an auxiliary unknown (say, playing the same role as ξ). This is precisely the approach followed by Kenmochi and Niezgódka in [7].

Remark 2.5. As $\xi \in \beta(u)$ a.e. in Q and $\beta = \partial I$, it is well–known (cf., e.g., [3, p. 54]) that the variational inequality

$$\mu\, \chi_t(x\,,t)\,(\chi(x\,,t) - r) \,\leq\, \big(s'(\chi) + \lambda'(\chi)u\big)(x\,,t)\,(\chi(x\,,t) - r) \\ \forall\, r \in [0\,,1]\,, \quad \text{for a.e. } (x\,,t) \in Q\,, \tag{2.23}$$

gives an equivalent formulation of (2.20).

The main result of this paper states the existence of solutions to the problem **(P)**.

Theorem 2.6. *Assume that* (2.1)-(2.9) *hold. Then problem* **(P)** *has a weak solution.*

To prove the theorem, we approximate **(P)** by the initial boundary value problem arising from the phase–field model proposed by Penrose and Fife [12]. The method of approximation consists of mollifying the equation (2.20) by adding the term $-\varepsilon\triangle\chi$ $(\varepsilon > 0)$, supplied with homogeneous Neumann boundary conditions. Then one can use the available solutions found by Laurençot [8,9] for the resulting system, derive estimates independent of ε, and finally pass to the limit as $\varepsilon \searrow 0$. This is essentially our procedure. However, in order to exploit the results of Laurençot, we first have to regularize the data g and χ_0.

For any $\varepsilon > 0$, we introduce the function $g_\varepsilon : Q \to \mathbb{R}$ defined by

$$g_\varepsilon(x\,,t) = \frac{1}{\varepsilon}\int_0^t e^{-(t-\tau)/\varepsilon}\, g(x\,,\tau)\, d\tau\,, \quad (x\,,t) \in Q.$$

Recalling (2.4), it is not difficult to see that

$$g_\varepsilon\,,\ \partial_t g_\varepsilon \in L^\infty(Q)\,, \tag{2.24}$$

$$\|g_\varepsilon\|_{L^\infty(Q)} \,\leq\, \|g\|_{L^\infty(Q)} \quad \forall\, \varepsilon > 0\,, \tag{2.25}$$

$$g_\varepsilon \to g \quad \text{strongly in } L^2(Q) \text{ as } \varepsilon \searrow 0. \tag{2.26}$$

On the other hand, let $\chi_{0\varepsilon} \in H^1(\Omega)$ denote the solution to the elliptic variational problem

$$(\chi_{0\varepsilon}\,, v) \,+\, \varepsilon \int_\Omega \nabla\chi_{0\varepsilon} \cdot \nabla v = (\chi_0\,, v) \quad \forall\, v \in V.$$

In view of (2.9), from a (weak) maximum principle argument we deduce that

$$0 \,\leq\, \chi_{0\varepsilon} \,\leq 1 \quad \text{a.e. in } \Omega\,, \quad \forall\, \varepsilon > 0. \tag{2.27}$$

Since $-\,\varepsilon\,\triangle\chi_{0\varepsilon} = \chi_0 - \chi_{0\varepsilon}$, it is straightforward to conclude that

$$\frac{\partial\chi_{0\varepsilon}}{\partial n} = 0 \quad \text{and} \quad \chi_{0\varepsilon} \in H^2(\Omega)\,, \tag{2.28}$$

$$\varepsilon\|\nabla\chi_{0\varepsilon}\|^2 \,+\, \|\varepsilon\,\triangle\chi_{o\varepsilon}\|^2 \,\leq\, 2|\Omega|\,, \tag{2.29}$$

where $|\Omega|$ denotes the Lebesgue measure of the domain Ω. In addition, the convergence property

$$\chi_{0\varepsilon} \to \chi_0 \quad \text{strongly in } L^2(\Omega) \text{ as } \varepsilon \searrow 0\,, \tag{2.30}$$

can be shown, for instance, via singular perturbations techniques (see [11]).

Now we have all the necessary ingredients to be able to apply the existence result in [8,9].

Proposition 2.7. *Under the assumptions (2.1)-(2.2), (2.5)-(2.8), (2.24), (2.27)-(2.28), there exists a quadruple* $(\theta_\varepsilon\,, u_\varepsilon\,, \chi_\varepsilon\,, \xi_\varepsilon)$ *satisfying*

$$\theta_\varepsilon \in H^1(0\,,T; L^2(\Omega)) \cap L^2(0\,,T; H^1(\Omega)) \cap L^\infty(Q)\,, \tag{2.31}$$

$$u_\varepsilon \in H^1(0\,,T; L^2(\Omega)) \cap L^2(0\,,T; H^2(\Omega)) \cap L^\infty(Q)\,, \tag{2.32}$$

$$\chi_\varepsilon \in H^1(0\,,T; H^1(\Omega)) \cap L^\infty(0\,,T; H^2(\Omega))\,, \tag{2.33}$$

$$\xi_\varepsilon \in L^\infty(0\,,T; L^2(\Omega))\,, \tag{2.34}$$

$$\theta_\varepsilon > 0\,, \quad u_\varepsilon = \frac{1}{\theta_\varepsilon} \quad \textit{a.e. in } Q\,, \tag{2.35}$$

$$0 \le \chi_\varepsilon \le 1\,, \quad \xi_\varepsilon \in \beta(\chi_\varepsilon) \quad \textit{a.e. in } Q\,, \tag{2.36}$$

$$\partial_t\Big(c_0\theta_\varepsilon - \lambda(\chi_\varepsilon)\Big) + k\,\triangle u_\varepsilon = g_\varepsilon \quad \textit{a.e. in } Q\,, \tag{2.37}$$

$$\mu\,\partial_t\chi_\varepsilon - \varepsilon\,\triangle\chi_\varepsilon + \xi_\varepsilon = s'(\chi_\varepsilon) + \lambda'(\chi_\varepsilon)u_\varepsilon \quad \textit{a.e. in } Q\,, \tag{2.38}$$

$$k\frac{\partial u_\varepsilon}{\partial n} + \alpha\,(\theta_\Gamma\, u_\varepsilon^2 - u_\varepsilon) = 0\,, \quad \frac{\partial \chi_\varepsilon}{\partial n} = 0 \quad \textit{a.e. in } \Sigma\,, \tag{2.39}$$

$$\theta_\varepsilon(\,\cdot\,,0) = \theta_0\,, \quad \chi_\varepsilon(\,\cdot\,,0) = \chi_{0\varepsilon}\,. \tag{2.40}$$

For the proof of this theorem we refer the reader to [8,9]. Nonetheless, let us acknowledge that in his procedure Laurençot considers a suitable regularization of the problem (2.37)-(2.40) as well, and then makes use of very general results on quasilinear parabolic problems due to Amann [2].

Remark 2.8. In the above statement we have not expressed all the regularity properties of $\theta_\varepsilon\,, u_\varepsilon$ and χ_ε. For instance, the additional properties (2.13) and (2.14) follow from a comparison in (2.37) and (2.38). Moreover, $u_\varepsilon \in C^0([0\,,T]; H^1(\Omega))$ and $\chi_\varepsilon \in C^0(\overline{Q})$ because of known interpolation or embedding theorems for Sobolev spaces. Let us also observe that, since $u_\varepsilon(\,\cdot\,,t)\,,\chi_\varepsilon(\,\cdot\,,t) \in H^2(\Omega)$ $(\subset L^\infty(\Omega))$ for a.e. $t \in (0\,,T)$, the boundary conditions in (2.39) hold even in $H^{1/2}(\Gamma)$, a.e. in $(0\,,T)$ (cf. Remark 2.1).

Henceforth we shall denote the problem (2.37)-(2.40) by $(\mathbf{P}_\varepsilon)$. Multiplying (2.37) and (2.38) by a test function $v \in H^1(\Omega)$ and accounting for (2.39), we obtain the variational equalities

$$\Big(\partial_t\Big(c_0\,\theta_\varepsilon - \lambda(\chi_\varepsilon)\Big)(\,\cdot\,,t)\,,v\Big) = k\int_\Omega \nabla u_\varepsilon(\,\cdot\,,t)\cdot\nabla v + \alpha\int_\Gamma\Big(\theta_\Gamma\, u_\varepsilon^2 - u_\varepsilon\Big)(\,\cdot\,,t)\,v + (g_\varepsilon(\,\cdot\,,t)\,,v)$$

$$\forall\, v \in H^1(\Omega)\,, \quad \text{for a.e. } t \in (0\,,T)\,, \tag{2.41}$$

$$\mu\left(\partial_t \chi_\varepsilon(\,\cdot\,,t)\,,v\right) + \varepsilon \int_\Omega \nabla \chi_\varepsilon(\,\cdot\,,t)\cdot\nabla v + (\xi_\varepsilon(\,\cdot\,,t)\,,v) = \left((s'(\chi_\varepsilon) + \lambda'(\chi_\varepsilon)\,u_\varepsilon)(\,\cdot\,,t)\,,v\right)$$

$$\forall\, v \in H^1(\Omega)\,, \quad \text{for a.e. } t \in (0\,,T)\,, \tag{2.42}$$

which will be employed in the sequel.

3 Uniform Estimates

In this section, we show estimates, independent of ε , for the solution to problem $(\mathbf{P}_\varepsilon)$ determined by Proposition 2.7. We start by summarizing some inequalities satisfied by χ_ε . In fact, the next lemma is addressed to a general problem of the form

$$a\,w_t - b\,\triangle w + \eta = f \quad \text{a.e. in } Q\,, \tag{3.1}$$

$$0 \le w \le 1\,, \quad \eta \in \beta(w) \quad \text{a.e. in } Q\,, \tag{3.2}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{a.e. in } \Sigma\,, \tag{3.3}$$

$$w(\,\cdot\,,0) = w_0\,, \tag{3.4}$$

where $a > 0$, $b > 0$, and

$$f \in H^1(0\,,T; L^2(\Omega)) \cap L^2(0\,,T; H^1(\Omega))\,, \tag{3.5}$$

$$w_0 \in H^2(\Omega)\,, \quad \frac{\partial w_0}{\partial n} = 0 \quad \text{a.e. in } \Gamma\,, \quad 0 \le w_0 \le 1 \quad \text{in } \Omega. \tag{3.6}$$

Lemma 3.1. *Assume that* (2.1) *and (3.5)-(3.6) hold. Then the system (3.1)-(3.4) admits one and only one solution*

$$w \in W^{1,\infty}(0\,,T; L^2(\Omega)) \cap H^1(0\,,T; H^1(\Omega)) \cap L^\infty(0\,,T; H^2(\Omega)) \tag{3.7}$$

satisfying

$$\frac{a}{2}\|w_t(\,\cdot\,,t)\|^2 + b\int_0^t \|\nabla w_t(\,\cdot\,,\tau)\|^2 d\tau \le \frac{1}{2a}\|f(\,\cdot\,,0) + b\,\triangle w_0\|^2 + \int_0^t\int_\Omega f_t\, w_t$$

$$\textit{for a.e. } t \in (0\,,T)\,, \tag{3.8}$$

$$\|\eta(\,\cdot\,,t)\|^2 \le 2\|f(\,\cdot\,,t)\|^2 + 2\|f(\,\cdot\,,0) + b\,\triangle w_0\|^2 + 4\,a\int_0^t\int_\Omega f_t\, w_t \quad \textit{for a.e. } t \in (0\,,T), \tag{3.9}$$

$$\frac{a}{2}\|\nabla w(\,\cdot\,,t)\|^2 + b\int_0^t \|\triangle w(\,\cdot\,,\tau)\|^2 d\tau \le \frac{a}{2}\|\nabla w_0\|^2 + \int_0^t\int_\Omega \nabla f\cdot\nabla w$$

$$\forall\, t \in [0\,,T]. \tag{3.10}$$

Proof. The uniqueness of w follows easily from the monotonicity of β via a standard contradiction argument (else one can see, for instance, [3, Theorem 2.1, p. 189]). In order to prove (3.8)-(3.10) rigorously, we replace in (3.1)-(3.2) the graph β by its Yosida approximation

$$\beta_m(r) = \begin{cases} m\,r & \text{if} \quad r < 0 \\ 0 & \text{if} \;\; 0 \le r \le 1 \\ m\,(r-1) & \text{if} \quad r > 1 \end{cases}, \qquad m \in \mathbb{N}. \tag{3.11}$$

Hence, denoting by w_m the solution to

$$a\,\partial_t w_m - b\,\triangle w_m + \beta_m(w_m) = f \quad \text{a.e. in } Q\,, \tag{3.12}$$

subjected to the conditions (3.3)-(3.4), it turns out that w_m is more regular than w. More precisely, for any $m \in \mathbb{N}$ one has (cf., e.g., [7, Lemma 4.1]), in addition to (3.7),

$$w_m \in H^2(\delta\,,T;L^2(\Omega)) \cap H^1(\delta\,,T;H^2(\Omega)) \quad \forall\,\delta \in (0\,,T)\,, \tag{3.13}$$

$$w_m \in C^1([0\,,T];L^2(\Omega)) \cap C^0([0\,,T];H^2(\Omega))\,. \tag{3.14}$$

Note that also

$$\partial_t w_m(\,\cdot\,,0) = \frac{1}{a}\Big(f(\,\cdot\,,0) + b\,\triangle w_0\Big)\,, \tag{3.15}$$

because of (3.6) and (3.11). Now let us just sketch the deduction of (3.8)-(3.10) for w_m and $\eta_m = \beta_m(w_m)$. First, we differentiate (3.12) with respect to time, multiply by $\partial_t w_m$, and integrate over $\Omega \times (\delta\,,t)$ for $0 < \delta < t$ (we are allowed to do this by virtue of (3.14)). As $\beta_m' \ge 0$ a.e. in $\mathbb{R}$, the estimate

$$\frac{a}{2}\|\partial_t w_m(\,\cdot\,,t)\|^2 + b\int_\delta^t \|\nabla(\partial_t w_m)(\,\cdot\,,\tau)\|^2 d\tau \;\le\; \frac{a}{2}\|\partial_t w_m(\,\cdot\,,\delta)\|^2 + \int_\delta^t\int_\Omega f_t\, w_t$$

holds for any $\delta \in (0\,,T)$ and any $t \in (\delta\,,T)$. Then, taking the limit as $\delta \searrow 0$ and recalling (3.7), (3.14), and (3.5), the inequality (3.8) is a straightforward consequence of (3.15). To derive (3.9), it suffices to test (3.12) by η_m, integrate only in space, and use Young's inequality

$$A\,B \le \frac{\delta}{p}|A|^p + \frac{p-1}{p\,\delta^{1/(p-1)}}|B|^{p/(p-1)}\,, \qquad A\,,B \in \mathbb{R}, \quad \delta > 0\,, \quad 1 < p < \infty\,, \tag{3.16}$$

(when $p = 2$) along with (3.8). On account of (3.3)-(3.7), the inequality (3.10) can be found after multiplication of (3.12) by $-\triangle w_m$ and integration by parts in space and time. Therefore, as w_m and η_m satisfy (3.8)-(3.9), with the help of (3.3)-(3.6) and (3.12) it is not difficult to infer that

$$\|w_m\|_{W^{1,\infty}(0\,,T;L^2(\Omega))\cap H^1(0\,,T;H^1(\Omega))\cap L^\infty(0\,,T;H^2(\Omega))} + \|\eta_m\|_{L^\infty(0\,,T;L^2(\Omega))} \le C_1 \quad \forall\, m \in \mathbb{N},$$

where the constant $C_1 > 0$ depends only on $a, b, T, \|f\|_{H^1(0\,,T;L^2(\Omega))}$, and $\|w_0\|_{H^2(\Omega)}$. Hence, there are two functions $\tilde{w}\,,\tilde{\eta}$ such that, possibly extracting subsequences, $w_m \to \tilde{w}$ and $\eta_m \to \tilde{\eta}$ weakly star in the abovenamed spaces, as $m \nearrow \infty$. Moreover, by compactness we have $w_m \to \tilde{w}$ strongly, for instance, in $C^0([0,T];L^2(\Omega))$, which ensures that

$$\lim_{m\to\infty}\int_0^T \Big(\eta_m(\,\cdot\,,t)\,, w_m(\,\cdot\,,t)\Big)\,dt = \int_0^T \Big(\tilde{\eta}(\,\cdot\,,t)\;, \tilde{w}(\,\cdot\,,t)\Big)\,dt\,.$$

Then, passing to the limit in the approximating system and recalling [3, Prop. 1.1, p. 42], we conclude that $\tilde{w}\,,\tilde{\eta}$ fulfil (3.1)-(3.4), and, consequently, that $\tilde{w}$ must coincide with the unique solution w to the limit problem. Finally, the estimates (3.8)-(3.10) are satisfied by the limit functions w and η, thanks to the weak-star lower semicontinuity of norms. $\square$

Now, we work directly on the problem $(\mathbf{P}_\varepsilon)$ and derive uniform estimates for the quadruple $(\theta_\varepsilon,\, u_\varepsilon,\, \chi_\varepsilon,\, \xi_\varepsilon)$. Throughout the remainder, we let (2.1)-(2.9) and (2.24)-(2.30) hold.

Lemma 3.2. *There exists a constant C_2 such that*

$$\|\ln(u_\varepsilon)\|^2_{H^1(0,T;L^2(\Omega))} + \|u_\varepsilon\|^2_{L^\infty(0,T;H^1(\Omega))} + \|u_{\varepsilon|\Gamma}\|^3_{L^\infty(0,T;L^3(\Gamma))}$$

$$+\|\chi_\varepsilon\|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \varepsilon\,\|\chi_\varepsilon\|^2_{H^1(0,T;H^1(\Omega))} \le C_2 \quad \forall\,\varepsilon\in(0,1]. \tag{3.17}$$

Proof. We multiply (2.37) by $-\partial_t u_\varepsilon$ and integrate in space and time. On account of (2.35), (2.39)-(2.40), and (2.6)-(2.8) (cf. also Remark 2.1), a formal Green formula allows us to deduce the identity

$$c_0\int_0^t\int_\Omega\left|\frac{\partial_t u_\varepsilon}{u_\varepsilon}\right|^2 + \frac{k}{2}\|\nabla u_\varepsilon(\,\cdot\,,t)\|^2 + \frac{\alpha}{3}\int_\Gamma(\theta_\Gamma\, u_\varepsilon^3)(\,\cdot\,,t)$$

$$= -\int_0^t\int_\Omega \lambda'(\chi_\varepsilon)\,(\partial_t\chi_\varepsilon)\,(\partial_t u_\varepsilon) + \frac{k}{2}\|\nabla\theta_0^{-1}\|^2 + \frac{\alpha}{6}\int_\Gamma\frac{2\theta_\Gamma(\,\cdot\,,0)-3\theta_0}{\theta_0^3}$$

$$+\frac{\alpha}{2}\int_\Gamma u_\varepsilon^2(\,\cdot\,,t) + \frac{\alpha}{3}\int_0^t\int_\Gamma(\partial_t\theta_\Gamma)u_\varepsilon^3 - \int_0^t\int_\Omega g_\varepsilon\,\partial_t u_\varepsilon \quad \text{for a.e.}\quad t\in(0,T)\,. \tag{3.18}$$

A rigorous justification of (3.18) needs some regularization of $(\mathbf{P}_\varepsilon)$ or, at least, of (2.37) (however, concerning this matter we refer, e.g., to [13] or [8]). Let now ω denote a constant fulfilling

$$\|v\|^2_{H^1(\Omega)} \le \omega\left(\|\nabla v\|^2 + \int_\Gamma v^2\right) \quad \forall\, v\in H^1(\Omega)\,. \tag{3.19}$$

Observe that (cf. (2.10))

$$\frac{\alpha}{3}\int_\Gamma(\theta_\Gamma\, u_\varepsilon^3)(\,\cdot\,,t) \ge \frac{\alpha c}{3}\int_\Gamma u_\varepsilon^3(\,\cdot\,,t)\,,$$

and, thanks to (3.16),

$$\frac{\alpha}{2}\int_\Gamma u_\varepsilon^2(\,\cdot\,,t) \le \frac{\alpha c}{6}\int_\Gamma u_\varepsilon^3(\,\cdot\,,t) + \frac{2}{3}\frac{\alpha}{c^2}\,\mathcal{H}^2(\Gamma)\,,$$

where $\mathcal{H}^2(\Gamma)$ indicates the bi–dimensional measure of Γ. Moreover, in view of (2.25), we have

$$\left|\int_0^t\int_\Omega g_\varepsilon\,\partial_t u_\varepsilon\right| \le \frac{c_0}{2}\int_0^t\int_\Omega\left|\frac{\partial_t u_\varepsilon}{u_\varepsilon}\right|^2 + \frac{1}{2c_0}\|g\|^2_{L^\infty(Q)}\int_0^t\int_\Omega u_\varepsilon^2\,.$$

Then, recalling (2.4)-(2.8) and (2.11)-(2.12), by (3.18)-(3.19) and (3.16) it is not difficult to find a constant C_3, independent of ε, such that

$$\frac{c_0}{2}\int_0^t\int_\Omega\left|\frac{\partial_t u_\varepsilon}{u_\varepsilon}\right|^2 + \frac{k}{2}\|\nabla u_\varepsilon(\,\cdot\,,t)\|^2 + \frac{\alpha c}{6}\int_\Gamma u_\varepsilon^3(\,\cdot\,,t)$$

$$\leq C_3\Big(1 + \int_0^t \|\nabla u_\varepsilon(\cdot,\tau)\|^2 d\tau + \int_0^t\int_\Gamma u_\varepsilon^3\Big) - \int_0^t\int_\Omega \lambda'(\chi_\varepsilon)\,(\partial_t\chi_\varepsilon)\,(\partial_t u_\varepsilon)$$

$$\text{for a.e.} \quad t \in (0,T)\,. \tag{3.20}$$

On the other hand, owing to Proposition 2.7 along with (2.2) and (2.27)-(2.28), it turns out that Lemma 3.1 holds for χ_ε. Hence, from (3.8) it follows that

$$\frac{\mu}{2}\|\partial_t\chi_\varepsilon(\cdot,t)\|^2 + \varepsilon\int_0^t \|\nabla(\partial_t\chi_\varepsilon)(\cdot,\tau)\|^2\,d\tau \leq \frac{1}{2\mu}\|s'(\chi_{0\varepsilon}) + \lambda'(\chi_{0\varepsilon})\,\theta_0^{-1} + \varepsilon\,\triangle\chi_{0\varepsilon}\|^2$$

$$+ \int_0^t\int_\Omega s''(\chi_\varepsilon)|\partial_t\chi_\varepsilon|^2 + \int_0^t\int_\Omega \lambda''(\chi_\varepsilon)|\partial_t\chi_\varepsilon|^2\,u_\varepsilon + \int_0^t\int_\Omega \lambda'(\chi_\varepsilon)\,(\partial_t u_\varepsilon)\,(\partial_t\chi_\varepsilon)$$

$$\text{for a.e.} \quad t \in (0,T)\,. \tag{3.21}$$

Since

$$\lambda''(\chi_\varepsilon)|\partial_t\chi_\varepsilon|^2 u_\varepsilon \leq 0 \quad \text{a.e. in}\, Q$$

because of (2.3) and (2.35), adding (3.20) to (3.21) and accounting for (2.2), (2.7), and (2.29), we infer that the sum of the left–hand sides is bounded from above by

$$C_4\Big(1 + \int_0^t \Big(\|\nabla u_\varepsilon(\cdot,\tau)\|^2 + \int_\Gamma u_\varepsilon^3(\cdot,\tau) + \|\partial_t\chi_\varepsilon(\cdot,\tau)\|^2\Big)\,d\tau\Big),$$

where C_4 is a constant independent of ε. Therefore, applying Gronwall's lemma, it is easy to determine another constant C_5, depending only on $c_0\,, k\,, \alpha\,, c\,, \mu\,, C_4\,,$ and T, such that

$$\int_0^t\int_\Omega |\partial_t\big(\ln(u_\varepsilon)\big)|^2 + \|\nabla u_\varepsilon(\cdot,t)\|^2 + \int_\Gamma u_\varepsilon^3(\cdot,t) + \|\partial_t\,\chi_\varepsilon(\cdot,t)\|^2$$

$$+\varepsilon\int_0^t \|\nabla\,(\partial_t\,\chi_\varepsilon)\,(\cdot,\,\tau)\|^2\,d\tau \leq C_5 \quad \text{for a.e. } t \in (0,T)\,.$$

Then, as $\ln(\theta_0^{-1}) \in L^\infty(\Omega)$ (cf. Remark 2.1), the estimate (3.17) is a straightforward consequence of (3.19), (2.27) and (2.29). □

Lemma 3.3. *There is a constant C_6 such that*

$$\|\xi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon\,\|\chi_\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \leq C_6 \quad \forall\,\varepsilon \in (0,1]\,. \tag{3.22}$$

Proof. By virtue of Lemma 3.1, using now (3.9) and arguing as above, it is not difficult to show that

$$\|\xi_\varepsilon(\cdot,t)\|^2 \leq 4\,\|s'\|^2_{L^\infty(0,1)}|\Omega| + 4\,\|\lambda'\|^2_{L^\infty(0,1)}\,\|u_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$+C_7\left(1 + \|\partial_t\chi_\varepsilon\|^2_{L^2(Q)}\right) + 4\mu\int_0^t\int_\Omega \lambda'(\chi_\varepsilon)\,(\partial_t u_\varepsilon)\,(\partial_t\chi_\varepsilon) \quad \text{for a.e. } t \in (0,T)\,, \tag{3.23}$$

C_7 being a constant independent of ε. Hence, multiplying (3.20) by 4μ and adding the result to (3.23), by (3.17) one concludes that also $\|\xi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}$ is uniformly bounded with respect to ε. Next, a comparison of the terms in (2.38) allows us to control $\|\varepsilon\triangle\chi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}$, whence (3.22) follows in view of the boundary condition in (2.39). □

Lemma 3.4. *There is a constant* C_8 *such that*

$$\|\theta_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\ln(u_\varepsilon)\|^2_{L^2(0,T;H^1(\Omega))} \le C_8 \quad \forall\, \varepsilon \in (0,1]\,. \tag{3.24}$$

Proof. Choosing $v = \theta_\varepsilon$ in (2.41) and integrating in time, with the help of (2.35), (3.16), (2.5), and (2.25), we deduce that

$$\frac{c_0}{2}\,\|\theta_\varepsilon(\,\cdot\,,t)\|^2 + k\int_0^t\!\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \alpha\, t\,\mathcal{H}^2(\Gamma) \le \frac{c_0}{2}\,\|\theta_0\|^2$$

$$+\frac{\alpha}{3}\int_0^t\!\int_\Gamma u_\varepsilon^3 + \frac{2\,\alpha\, t}{3}\,\mathcal{H}^2(\Gamma)\,\|\theta_\Gamma\|^{3/2}_{L^\infty(\Sigma)} + \frac{1}{2}\,\|\lambda'\|^2_{L^\infty(0,1)}\int_0^t \|\partial_t\chi_\varepsilon(\,\cdot\,,\tau)\|^2\,d\tau$$

$$+\frac{t}{2}\,|\Omega|\|g\|^2_{L^\infty(Q)} + \int_0^t \|\theta_\varepsilon(\,\cdot\,,\tau)\|^2\,d\tau \qquad \forall\, t \in [0,T]\,.$$

Therefore, on account of (3.17), an application of Gronwall's lemma yields (3.24). □

Lemma 3.5. *There exists a constant* C_9 *such that*

$$\|u_\varepsilon\|_{L^\infty(Q)} \le C_9 \quad \forall\, \varepsilon \in (0,1]\,. \tag{3.25}$$

Proof. Due to (2.2), (3.17), and to Sobolev's embedding theorems, $\|\partial_t(\lambda(\chi_\varepsilon))\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|u_\varepsilon\|_{L^\infty(0,T;L^6(\Omega))}$ are bounded independently of ε. Then, thanks to (2.5) and (2.7) as well, we can make use of the result stated in [6, Lemma 2.3] (a more general version is given in [9, Lemma 4.1]) to obtain (3.25). We point out that the argument is based on the Moser technique and consists of testing (2.37) by u_ε^p and estimating the norms $\|u_\varepsilon\|_{L^p(Q)}$ (uniformly with respect to ε and p) for a divergent sequence of exponents p. □

Let us note that (3.25) and (2.35) entail

$$\theta_\varepsilon \ge \frac{1}{C_9} \quad \text{a.e. in } Q\,, \tag{3.26}$$

whereas (3.25) and (3.17) ensure that (cf. Remark 2.3 and especially (2.22))

$$\|u_\varepsilon|_\Gamma\|_{L^\infty(\Sigma)} \le C_9 \tag{3.27}$$

for any $\varepsilon \in (0,1]$. Owing still to (3.25), we can finally derive a bound for the time derivatives of u_ε and θ_ε.

Lemma 3.6. *There is a constant* C_{10} *such that*

$$\|u_\varepsilon\|_{H^1(0,T;L^2(\Omega))} + \|\theta_\varepsilon\|_{W^{1,\infty}(0,T;V')} \le C_{10} \quad \forall\, \varepsilon \in (0,1]\,. \tag{3.28}$$

Proof. Since $\partial_t\, u_\varepsilon = u_\varepsilon\, \partial_t(\ln(u_\varepsilon))$, by (3.17) and (3.25) we infer that $\|\partial_t\, u_\varepsilon\| \le C_9\sqrt{C_2}$. Hence, recalling also (2.2), (2.5), (3.27), and (2.25), the estimate (3.28) follows from (2.41). □

Now, we are in the position to pass to the limit, at least for a subsequence, in the problem $(\mathbf{P}_\varepsilon)$ when ε tends to 0. In the next section, we will show that any weak–star limit of $(\theta_\varepsilon, \chi_\varepsilon)$ yields a weak solution of $(\mathbf{P})$, thus proving Theorem 2.6.

4 Passage to the limit

Lemmas 3.2 to 3.6 imply the existence of functions θ, u, χ, ξ such that, possibly taking subsequences,

$$\theta_\varepsilon \to \theta \quad \text{weakly star in } W^{1,\infty}(0,T;V') \cap L^\infty(0,T;L^2(\Omega)), \tag{4.1}$$

$$u_\varepsilon \to u \quad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(Q), \tag{4.2}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly star in } W^{1,\infty}(0,T;L^2(\Omega)), \tag{4.3}$$

$$\xi_\varepsilon \to \xi \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \tag{4.4}$$

as $\varepsilon \searrow 0$. Moreover, it turns out that (cf. (3.17) and (3.22))

$$\begin{aligned} \varepsilon\, \chi_\varepsilon \to 0 \qquad & \text{strongly in} && H^1(0,T;H^1(\Omega)) \\ & \text{and weakly star in} && L^\infty(0,T;H^2(\Omega)). \end{aligned} \tag{4.5}$$

Thanks to (4.2), by standard compactness arguments, including the Aubin lemma (see, e.g., [10, p.58]), we deduce that

$$u_\varepsilon \to u \quad \text{strongly in } C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^{1-\delta}(\Omega)), \quad \text{for any } \delta > 0. \tag{4.6}$$

In order to verify that the quadruple (θ, u, χ, ξ) solves the problem $(\mathbf{P})$, we note that the initial conditions (2.21) result easily from (2.40), (2.30), (4.1), and (4.3) (cf. also Remark 2.3). In addition, due to (3.26) and (2.36), the properties (2.15) are satisfied. The relationship $u = \theta^{-1}$ holds a.e. in Q by virtue of (2.35), (4.1), and (4.6). Indeed, $\theta_\varepsilon\, u_\varepsilon = 1$ for any $\varepsilon > 0$ and $\theta_\varepsilon\, u_\varepsilon \to \theta\, u$ weakly in $L^1(Q)$ as $\varepsilon \searrow 0$. To complete the proof of (2.18) and to prove (2.19)-(2.20), we need to state some strong convergence for the sequence $\{\chi_\varepsilon\}$.

Lemma 4.1. *For $\varepsilon \searrow 0$, we have $\chi_\varepsilon \to \chi$ strongly in $C^0([0,T];L^2(\Omega))$.* (4.7)

Proof. We multiply (2.38) by $\chi_\varepsilon - \chi$ and integrate in space and time. On account of (2.39)-(2.40) and (2.21), we obtain

$$\begin{aligned} &\frac{\mu}{2}\, \|(\chi_\varepsilon - \chi)(\,\cdot\,,t)\|^2 + \varepsilon \int_0^t \|\nabla \chi_\varepsilon(\,\cdot\,,\tau)\|^2\, d\tau + \int_0^t\!\!\int_\Omega \xi_\varepsilon (\chi_\varepsilon - \chi) \\ &= R_\varepsilon(t) + \int_0^t\!\!\int_\Omega \Big(s'(\chi_\varepsilon) - s'(\chi)\Big)(\chi_\varepsilon - \chi) + \int_0^t\!\!\int_\Omega \Big(\lambda'(\chi_\varepsilon) - \lambda'(\chi)\Big)\, u_\varepsilon\, (\chi_\varepsilon - \chi), \end{aligned} \tag{4.8}$$

where

$$R_\varepsilon(t) := \frac{\mu}{2}\, \|\chi_{0\varepsilon} - \chi_0\|^2 - \int_0^t\!\!\int_\Omega (\varepsilon\, \Delta \chi_\varepsilon)\, \chi + \int_0^t\!\!\int_\Omega \Big(s'(\chi) + \lambda'(\chi) u_\varepsilon - \chi_t\Big)(\chi_\varepsilon - \chi),$$

for any $t \in [0,T]$. Observe that

$$\xi_\varepsilon\, (\chi_\varepsilon - \chi) \ge 0 \quad \text{a.e. in } Q$$

because of (2.36), (2.1), and (2.15),

$$|s'(\chi_\varepsilon) - s'(\chi)| \le \|s''\|_{L^\infty(0,1)} |\chi_\varepsilon - \chi| \quad \text{a.e. in } Q$$

because of (2.2), and

$$u_\varepsilon \Big(\lambda'(\chi_\varepsilon) - \lambda'(\chi)\Big) (\chi_\varepsilon - \chi) \le 0 \quad \text{a.e. in } Q$$

because of (2.3) and (2.35). Therefore, it follows from (4.8) that

$$\|(\chi_\varepsilon - \chi)(\,\cdot\,, t)\|^2 \le \frac{2}{\mu} R_\varepsilon(t) + C_{11} \int_0^t \|(\chi_\varepsilon - \chi)(\,\cdot\,, \tau)\|^2 \, d\tau \,, \tag{4.9}$$

with $C_{11} = 2\,\|s''\|_{L^\infty(0,1)}/\mu$. But, owing to (2.30), (4.5), (4.3), and (4.6), $R_\varepsilon(t)$ tends to zero, as $\varepsilon \searrow 0$, for any $t \in [0,T]$, and $\|R_\varepsilon\|_{W^{1,\infty}(0,T)}$ is bounded independently of ε. Then, by compactness,

$$\|R_\varepsilon\|_{C^0([0,T])} \to 0 \quad \text{as } \varepsilon \searrow 0 \,.$$

On the other hand, (4.9) and Gronwall's lemma yield

$$\|(\chi_\varepsilon - \chi)(\,\cdot\,, t)\| \le \frac{2}{\mu} \|R_\varepsilon\|_{C^0([0,T])} \exp(C_{11} t) \,,$$

for any $t \in [0,T]$. Thus (4.7) is completely proved. □

As a first consequence, (4.7) and (4.4) imply that $\xi_\varepsilon \, \chi_\varepsilon \to \xi \, \chi$ weakly in $L^1(Q)$, whence (cf. (2.36) and (2.1))

$$\xi(x,t) \Big(\chi(x,t) - r\Big) \ge 0 \quad \forall \, r \in [0,1] \,, \text{ for a.e. } (x,t) \in Q \,,$$

that is, $\xi \in \beta(\chi)$ a.e. in Q (one may see Remark 2.5). Also, using just the continuity of λ', s' in $[0,1]$ and the Lebesgue dominated convergence theorem, from (4.7) we deduce that, at least for subsequences,

$$\lambda'(\chi_\varepsilon) \to \lambda'(\chi) \quad \text{and} \quad s'(\chi_\varepsilon) \to s'(\chi) \quad \text{a.e. in } Q \text{ and strongly in } L^p(Q),$$
$$\text{for any } p \in [1, \infty) \,. \tag{4.10}$$

Thanks to (4.10) and (4.3)-(4.6), a passage to the limit in (2.42) yields (2.20). It remains to show (2.19). Note that (4.6) (with $\delta < 1/2$) and (3.27) entail

$$u_{\varepsilon_{|\Gamma}} \to u_{|\Gamma} \quad \text{strongly in } L^p(\Sigma) \,, \quad \text{for any } p \in [1, \infty) \,. \tag{4.11}$$

Now, it suffices to recall (2.41), (4.1)-(4.3), (4.10), (2.5), and (2.26) for realizing that θ, χ, u fulfil (2.19). This concludes the proof of Theorem 2.6. □

Remark 4.2. Let us point out that the assumption (2.2) can be replaced by the weaker conditions

$$\lambda \in C^1([0,1]) \,, \qquad s \in C^{1,1}([0,1]) \, \Big(\equiv W^{2,\infty}(0,1)\Big) \,, \tag{4.12}$$

without affecting the existence result. Indeed, in our argumentation we have only exploited the properties (4.12) and (2.3) of λ and s (cf., in particular, Lemma 3.2 and Lemma 4.1). However, in this setting one should take regularizing sequences $\{\lambda_\varepsilon\}$ and $\{s_\varepsilon\}$ in the approximation procedure (cf. Proposition 2.7).

Remark 4.3. In the case when the initial datum χ_0 lies in $H^1(\Omega)$, the solution component χ belongs to $L^\infty(0,T;H^1(\Omega))$, besides (2.14). This additional regularity can be proved by working on the inequality (3.10) written for χ_ε. One checks that $\|\chi_{o\varepsilon}\|_{H^1(\Omega)}$ is bounded independently of ε and makes use of (4.12), (2.3), (3.16), and (3.17), to estimate the right–hand side, finally applying Gronwall's lemma. Observe also that this further a priori bound would allow to skip the details of Lemma 4.1, the convergence (4.7) being easily established by compactness.

5 Remarks on the boundary condition

The boundary condition considered in our approach,

$$-k\,\frac{\partial\theta}{\partial n} = \alpha\,(\theta-\theta_\Gamma) \quad \text{in } \Sigma\,, \tag{5.1}$$

is quite usual in the framework of the Fourier heat flux law. In fact, if one assumes that $\vec{q} = -k\,\nabla\theta$, then (5.1) says that the heat flux is directly proportional to the difference between inside and outside temperatures at the boundary. But, if one takes another heat flux law, then the meaning of (5.1) is no longer the same. In our framework $\vec{q}$ is defined as

$$\vec{q} = k\nabla\left(\frac{1}{\theta}\right), \tag{5.2}$$

so that (5.1) reads

$$\vec{q}\cdot\vec{n} = \frac{\alpha}{\theta^2}\,(\theta-\theta_\Gamma)\,,$$

and the rate factor has become a decreasing function of the absolute temperature, namely α/θ^2. In this connection, one could think of a general boundary condition of the form

$$\vec{q}\cdot\vec{n} = \tilde{\alpha}\,(\theta)\,(\theta-\theta_\Gamma) \quad \text{in } \Sigma\,, \tag{5.3}$$

where $\vec{q}$ is prescribed once and for all by (5.2) and where $\tilde{\alpha}$ denotes some given function. Now, one expects that $\tilde{\alpha}$ is non–negative and possibly decreasing. Some existence (and regularity) results have been shown for the regularized problem $(\mathbf{P}_\varepsilon)$ with (5.1) replaced by (5.3), for alternative choices of $\tilde{\alpha}$. The case $\tilde{\alpha}(\theta) = \alpha/\theta$ has been examined by Kenmochi and Niezgódka in [7] and is particularly interesting, since it can be proved that there is a unique solution (cf. also the later Theorem 5.1). The model with the natural condition $\alpha(\theta) = \alpha$ (constant) is discussed in [5], but there the existence of solutions relies on the additional (and somehow unphysical) requirement that the source term g be non-negative. We also quote another investigation by Laurençot [9] dealing with the situation $\tilde{\alpha}(\theta) = \alpha/\theta^{m+1}$ (with $0 < m < 1$), though it came from (5.1) via the heat flux law $\vec{q} = k\,\nabla(1/\theta^m)$.

Next, taking again (5.2) into consideration, we claim that our analysis of the actual problem $(\mathbf{P})$, as well as the related existence result (i.e., Theorem 2.6), can be extended to functions $\tilde{\alpha}$ of the following type

$$\tilde{\alpha}(\theta) = \frac{\alpha}{\theta^p}\,, \qquad p \geq 1\,,$$

in the boundary condition (5.3). More precisely, arguing in terms of $u = 1/\theta$ (cf. (2.18)) and following the same technique, it is possible to treat the following set of conditions

$$-k\,\frac{\partial u}{\partial n} = \gamma\,u^{p_1} - \zeta\,u^{p_2} \quad \text{in } \Sigma\,, \tag{5.4}$$

where the data $\gamma, \zeta : \Sigma \to \mathbb{R}$ and $p_1, p_2 \in \mathbb{R}$ satisfy (cf. (2.5)-(2.6))

$$\gamma \in L^\infty(\Sigma), \quad \gamma > 0 \quad \text{a.e. in } \Sigma, \quad \frac{1}{\gamma} \in L^\infty(\Sigma), \tag{5.5}$$

$$\zeta \in L^\infty(\Sigma), \quad \zeta \ge 0 \quad \text{a.e. in } \Sigma, \tag{5.6}$$

$$\gamma_t, \zeta_t \in L^\infty(\Sigma), \tag{5.7}$$

$$p_1 \ge 1, \quad p_2 \ge 0, \quad p_1 > p_2 . \tag{5.8}$$

Note that (5.4) is a generalization of (5.1). Regarding the formulation, the variational equality (2.19) changes into

$$\Big(\partial_t\big(c_0\,\theta - \lambda(\chi)\big)(\,\cdot\,,t), v\Big) = k \int_\Omega \nabla u(\,\cdot\,,t)\cdot\nabla v + \int_\Gamma (\gamma\, u^{p_1} - \zeta\, u^{p_2})(\,\cdot\,,t)\, v$$

$$+ \big(g(\,\cdot\,,t), v\big) \qquad \forall\, v \in H^1(\Omega), \quad \text{for a.e. } t \in (0,T), \tag{5.9}$$

and the approximating solution u_ε needs to satisfy (5.4). The suitable modifications of the proof are left to the interested reader.

Instead, we want to show here that in the case $p_1 = 1$, $p_2 = 0$ a uniqueness result can be deduced for problem **(P)**. This case corresponds to the choice made in [7] and has the advantage that the boundary condition is linear with respect to u.

Theorem 5.1. *Assume that (2.1), (2.3)-(2.4), (2.7)-(2.9), (4.12), and (5.5)-(5.7) hold. Let $p_1 = 1$, $p_2 = 0$, and consider the problem* (P) *with (2.19) replaced by (5.9). Then, there exists a unique weak solution.*

Proof. Suppose that there are two solutions (θ_1, χ_1) and (θ_2, χ_2). Take u_i and ξ_i, $i = 1, 2$, as in (2.18), in order that (5.9) and (2.20) are satisfied. In view of (2.16), we set

$$M := \max\{\|u_1\|_{L^\infty(Q)}, \|u_2\|_{L^\infty(Q)}\} . \tag{5.10}$$

First we integrate the difference of the two equations (5.9) from 0 to $\tau \in [0,T]$. Thanks to (2.21) (same initial values for both solutions), we obtain

$$c_0\Big((\theta_1 - \theta_2)(\,\cdot\,,\tau), v\Big) - \Big(\big(\lambda(\chi_1) - \lambda(\chi_2)\big)(\,\cdot\,,\tau), v\Big)$$

$$= k \int_\Omega \nabla \int_0^\tau (u_1 - u_2)(\,\cdot\,,t)\,dt \cdot \nabla v + \int_\Gamma \int_0^\tau \big(\gamma(u_1 - u_2)\big)(\,\cdot\,,t)\,dt\, v, \tag{5.11}$$

for any $v \in H^1(\Omega)$. Next, we choose $v = (u_1 - u_2)(\,\cdot\,,\tau)$ as test function in (5.11). Since

$$-(\theta_1 - \theta_2)(u_1 - u_2) = \frac{|u_1 - u_2|^2}{u_1\, u_2} \ge \frac{|u_1 - u_2|^2}{M^2} \quad \text{a.e. in } Q,$$

because of (2.18), (2.15), and (5.10), accounting also for (5.5) and (4.12), from (5.11) we infer that

$$\frac{1}{M^2}\,\|(u_1 - u_2)(\,\cdot\,,\tau)\|^2 + \frac{k}{2}\int_\Omega \partial_\tau \left|\nabla \int_0^\tau (u_1 - u_2)(\,\cdot\,,t)\,dt\right|^2$$

$$+ \int_\Gamma \frac{1}{2\gamma(\,\cdot\,,\tau)}\, \partial_\tau \left| \int_0^\tau \Big(\gamma(u_1-u_2)\Big)(\,\cdot\,,t)\,dt \right|^2$$

$$\le \|\lambda'\|_{L^\infty(0,1)}\, \|(\chi_1-\chi_2)(\,\cdot\,,\tau)\|\, \|(u_1-u_2)(\,\cdot\,,\tau)\| \quad \forall\, \tau\in[0,T]\,. \tag{5.12}$$

On the other hand, due to (2.15), (2.18), and to the monotonicity of the graph β, we have

$$\mu\,(\chi_1-\chi_2)_t\,(\chi_1-\chi_2) \le \Big(s'(\chi_1)-s'(\chi_2)\Big)\,(\chi_1-\chi_2)$$

$$+\Big(\lambda'(\chi_1)-\lambda'(\chi_2)\Big)\,u_1\,(\chi_1-\chi_2) + \lambda'(\chi_2)\,(u_1-u_2)\,(\chi_1-\chi_2)\,,$$

with (cf. (2.3))

$$\Big(\lambda'(\chi_1)-\lambda'(\chi_2)\Big)\,u_1\,(\chi_1-\chi_2) \le 0$$

a.e. in Q. Hence, integrating over Ω and recalling (4.12) again, we easily find that

$$\frac{\mu}{2}\,\partial_\tau\|(\chi_1-\chi_2)(\,\cdot\,,\tau)\|^2 \le \|s''\|_{L^\infty(0,1)}\,\|(\chi_1-\chi_2)(\,\cdot\,,\tau)\|^2$$

$$+\|\lambda'\|_{L^\infty(0,1)}\,\|(u_1-u_2)(\,\cdot\,,\tau)\|\,\|(\chi_1-\chi_2)(\,\cdot\,,\tau)\| \quad \text{for a.e. } \tau\in(0,T)\,. \tag{5.13}$$

Therefore, adding (5.12) and (5.13), integrating in time, and setting

$$S(t) := \frac{1}{M^2}\int_0^t \|(u_1-u_2)(\,\cdot\,,\tau)\|^2\,d\tau + \frac{k}{2}\left\|\nabla\int_0^t (u_1-u_2)(\,\cdot\,,\tau)\,d\tau\right\|^2$$

$$+\int_\Gamma \frac{1}{2\gamma(\,\cdot\,,t)}\left|\int_0^t \Big(\gamma(u_1-u_2)\Big)(\,\cdot\,,\tau)\,d\tau\right|^2 + \frac{\mu}{2}\,\|(\chi_1-\chi_2)(\,\cdot\,,t)\|^2\,,$$

we see that

$$S(t) \le -\int_0^t\int_\Gamma \frac{\gamma_t(\,\cdot\,,\tau)}{\gamma(\,\cdot\,,\tau)}\quad \frac{1}{2\gamma(\,\cdot\,,\tau)}\left|\int_0^\tau \Big(\gamma(u_1-u_2)\Big)(\,\cdot\,,\sigma)\,d\sigma\right|^2 d\tau$$

$$+\|s''\|_{L^\infty(0,1)}\int_0^t \|(\chi_1-\chi_2)(\,\cdot\,,\tau)\|^2\,d\tau$$

$$+2\,\|\lambda'\|_{L^\infty(0,1)}\int_0^t \|(u_1-u_2)(\,\cdot\,,\tau)\|\,\|(\chi_1-\chi_2)(\,\cdot\,,\tau)\|\,d\tau\,,$$

for any $t\in[0,T]$. Using Young's inequality, with the help of (5.5) and (5.7) we deduce that

$$S(t) \le C_{12}\int_0^t S(\tau)\,d\tau \quad \forall\, t\in[0,T]\,, \tag{5.14}$$

where C_{12} depends only on $\mu\,, M\,, \|\lambda'\|_{L^\infty(0,1)}\,, \|s''\|_{L^\infty(0,1)}$ and $\|\gamma_t/\gamma\|_{L^\infty(\Sigma)}$. Now, (5.14) and Gronwall's lemma imply that $S(t)=0$ for any $t\in[0,T]$, whence $u_1=u_2$, $\chi_1=\chi_2$, and the uniqueness result is completely proved. □

Remark 5.2. Under the assumptions of Theorem 5.1. the convergence properties stated in (4.1)-(4.7) are valid for the whole sequence $\{(\theta_\varepsilon\,, u_\varepsilon\,, \chi_\varepsilon\,, \xi_\varepsilon)\}$, and not only for some subsequence. At the same time, the uniqueness result implies that there are no other solutions to the relaxed Stefan problem **(P)** besides the one which arises as limit for $\varepsilon\searrow 0$ of solutions to the Penrose–Fife system **(P$_\varepsilon$)**. In this sense, the relaxed Stefan problem **(P)** is the natural asymptotic limit of the Penrose–Fife model if the contribution of the interfacial energy density to the total free energy density tends to zero.

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