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Karmeshu[1] , Henri Schurz[2]

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[1] School of Computer and Systems Sciences
Jawaharlal Nehru University
New Delhi, 110067
India

[2] Institute of Applied Analysis
and Stochastics
Mohrenstraße 39
D – 10117 Berlin
Germany

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Edited by
Institut für Angewandte Analysis und Stochastik (IAAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2004975
e–mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e–mail (Internet): preprint@iaas-berlin.d400.de

Effects of distributed delays on the stability of structures under seismic excitation and multiplicative noise

KARMESHU
School of Computer and Systems Sciences, Jawaharlal Nehru University, New Delhi 110067, India

and

HENRI SCHURZ
Institute of Applied Analysis and Stochastics,
Mohrenstr. 39, Berlin 10117, Germany

Abstract. The effects of seismic excitation and multiplicative noise (arising from environmental fluctuations) on the stability of a single degree of freedom system with distributed delays are investigated. The system is modelled in the form of a stochastic integro–differential equation interpreted in Stratonovich sense. Both deterministic stability and stochastic moment stability are examined for the system in the absence of seismic excitation. The model is also extended to incorporate effects of symmetric nonlinearity. The simulation of stochastic linear and nonlinear systems are carried out by resorting to numerical techniques for the solution of stochastic differential equations.

Keywords. Stochastic stability; Weak and strong time delay; Random oscillations; Seismic excitation; Stochastic differential equations; Numerical methods; Implicit Euler, Mil'shtein and Balanced methods

1. Introduction

The technique of vibration suppression by active control of civil engineering structures has drawn considerable attention in recent years. The purpose of applying the control is to reduce the damage caused by unexpected disturbances like earthquakes, wind and dynamic excitations. The main emphasis of the work done on the control design concerns instability of structures due to unavoidable time delay, unless it could be compensated properly, e.g. see Agrawal, Fujino and Bhartia (1993). Time delay arises on account of a number of factors such as the time taken in data ac-

quisition from sensors located at different parts of the structure or unsynchronized application of control forces. Even if the time delay is negligible, the structure could be destabilized due to excitations by the control force.
A number of attempts have been made to treat the system as a linear optimal feedback control problem. These studies have been restricted to SDOF (single-degree of freedom) and n-DOF controlled structures with constant time delay described by linear differential equations. An important aspect which has been generally ignored in modelling of such systems is the neglect of stochastic effects. It is well-known that parametric noise can destabilize a linear system which is otherwise stable, see Kozin (1969), Arnold (1974), Karmeshu and Bansal (1975), Karmeshu (1976) or Ariaratnam and Srikantiah (1978). Moreover, the delay magnitude varies from one sensor to another and from one actuator to another, as already mentioned in Rohmann (1987). It may be useful to examine the effect of continuous and nonconstant time delay instead of fixed time lags.
The object of this paper is to investigate the effects of seismic excitations and multiplicative noise on the stability of a SDOF system with distributed delays. The paper consists of six sections. Section 2 contains the formulation of the model. The resulting stochastic differential equation (SDE) with multiplicative noise is interpreted in Stratonovich sense. Section 3 is concerned with the stability analysis of the deterministic linear system. Section 4 and 5 are devoted to the stochastic analysis of the linear system with weak and strong delay. Mean square stability of the system without seismic excitation is also investigated. Section 6 deals with the numerical simulation for transient solution of the linear stochastic model. Effects of nonlinearity in the stochastic SDOF system are contained in section 7. Finally, the last section summarizes the observed effects and ends up with concluding remarks. In an appendix numerical methods for the solution of SDEs and results on mean square stability used for robust and reliable simulations have been outlined.

2. Formulation of the models with distributed delays

The model considered here is based on distributed delays instead of fixed discrete time delay. The displacement $x(t)$ of an SDOF structure follows the integro–differential equation

$$\ddot{x} + \omega^2 x + 2\zeta\omega\dot{x} + g_1 \int_0^t K_1(t-s)x(s)\,ds + g_2 \int_0^t K_2(t-s)\dot{x}(s)\,ds \;=\; 0 \qquad (1)$$

where ω, ζ, g_1, g_2 are nonnegative real parameters. The parameters g_1 and g_2 represent feedback gains of the displacement and velocity of the oscillations, whereas ω and ζ natural frequency and damping ratio coefficients, respectively. $K_1(t)$ and $K_2(t)$ are absolutely integrable weight functions specifying the distributed delays. For simplicity, it is also supposed that these functions are normalized, i.e.

$$\int_0^\infty |K_1(u)|\,du \;=\; 1 \qquad \text{and} \qquad \int_0^\infty |K_2(u)|\,du \;=\; 1\,. \qquad (2)$$

Equation (1) can be interpreted as a distributed delay version of the SDOF system with discrete delay considered by Agrawal, Fujino and Bhartia (1993).

When the system is subjected to seismic excitations and environmental fluctuations, equation (1) modifies to the form

$$\ddot{x} + \omega^2 x + 2\zeta\omega\dot{x} + g_1 \int_0^t K_1(t-s)x(s)\,ds + g_2 \int_0^t K_2(t-s)\dot{x}(s)\,ds \;=\; F(x,\dot{x},t) \tag{3}$$

where $F(x,\dot{x},t)$ reflects stochastic perturbation forces which can be decomposed as

$$F(x,\dot{x},t) \;=\; \sigma_1 x(t)\xi_1(t) \;+\; \sigma_2\dot{x}(t)\xi_2(t) \;+\; \sigma_3\ddot{z}(t)\,. \tag{4}$$

The first two terms on the right side of (4) correspond to stochastic environmental perturbations, and the last term represents a ground level acceleration corresponding to seismic excitations. The random environmental perturbations per unit displacement and per unit velocity are modelled by the independent white noise processes $\xi_1(t)$ and $\xi_2(t)$. σ_i, $(i = 1,2)$ give the magnitude of the fluctuations incorporated in (4). The first two expressions represent state–dependent noise. Such terms are called multiplicative noise.
Several attempts to model seismic excitations have been made, e.g. Shinozuka (1967, 1972) or Kozin (1977). Following Bolotin (1960) seismic excitation can be modelled as the nonstationary process

$$\ddot{z}(t) \;=\; I(t)\,\xi_3(t) \tag{5}$$

where $\xi_3(t)$ is a white noise process being independent of $\xi_1(t)$ and $\xi_2(t)$. $I(t)$ is supposed to be of the form

$$I(t) \;=\; \exp(-\alpha t) - \exp(-\beta t)\,, \quad 0 < \alpha < \beta \tag{6}$$

with its parameters α and β.
The resulting integro–differential equation (3) can be interpreted in many different ways due the variety of stochastic integration calculus. Two major interpretations have crystallized out, namely Itô and Stratonovich calculus. These two objects are related each another in the sense that the results of one of them can be prescripted to the other via a transformation formula, see e.g. in Arnold (1974). Here we have adopted the Stratonovich interpretation as it is preferable for modelling physical phenomena, cf. Wong and Zakai (1965). Thus, from now on we make use of the termini 'Stratonovich interpretation' and 'Itô prescription'.
Additionally, when effects of symmetric nonlinearities are to be incorporated, see Moon (1987), the equation of motion takes the form

$$\ddot{x} + \omega^2 x + 2\zeta\omega\dot{x} + g_1 \int_0^t K_1(t-s)x(s)\,ds + g_2 \int_0^t K_2(t-s)\dot{x}(s)\,ds + \gamma x^3 = F(x,\dot{x},t). \tag{7}$$

Note that in the absence of external excitations, displacement and velocity feedback gains, equation (7) has the form of the well–known Duffing equation, cf. Moon (1987).
To analyze the model we need specific forms of the weight functions $K_1(t)$ and $K_2(t)$.

For the sake of simplicity we shall especially examine the case $K_1(t) = K_2(t) = K(t)$ in this paper. The main attention is drawn to the two forms

$$K(t) = \nu \exp(-\nu t) \quad \text{and} \tag{8}$$
$$K(t) = \nu^2 t \exp(-\nu t) \,. \tag{9}$$

In consonance with similar approaches in population ecology, e.g. Mac Donald (1978), the form (8) is termed as '**weak delay**' and 9 as '**strong delay**'. At first we consider the case incorporating weak delay.

Weak delay. Writing $x_1(t)$ for the displacement $x(t)$ as well as $x_2(t)$ for the velocity $\dot{x}(t)$ of the vibration, system (7) is equivalently described by the set

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(\omega^2 x_1 + 2\zeta\omega x_2 + g_1 x_3 + g_2 x_4 + \gamma x_1^3) + \sigma_1 x_1 \xi_1(t) + \sigma_2 x_2 \xi_2(t) + \sigma_3 I(t) \xi_3(t) \\
\dot{x}_3 &= -\nu x_3 + \nu x_1 \\
\dot{x}_4 &= -\nu x_4 + \nu x_2 \,.
\end{aligned} \tag{10}$$

Thereby, the stochastic integro–differential equation has been replaced by a set of four coupled differential equations.

Strong delay. In a similar way we now introduce the system with strong delay. Using the identities $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, the obtained stochastic integro–differential equation can be rewritten to

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(\omega^2 x_1 + 2\zeta\omega x_2 + g_1 x_3 + g_2 x_5 + \gamma x_1^3) + \sigma_1 x_1 \xi_1(t) + \sigma_2 x_2 \xi_2(t) + \sigma_3 I(t) \xi_3(t) \\
\dot{x}_3 &= -\nu x_3 + \nu x_4 \\
\dot{x}_4 &= -\nu x_4 + \nu x_1 \\
\dot{x}_5 &= -\nu x_5 + \nu x_6 \\
\dot{x}_6 &= -\nu x_6 + \nu x_2 \,.
\end{aligned} \tag{11}$$

Thus, the original systems with both weak and strong delay have artificially blown up to very specific sets of coupled stochastic differential equations without time delay. This appearance is due to the special kernel structure of the distributed lag.

3. Stability analysis of the deterministic linear model

When stochastic perturbation forces as well as cubic nonlinearity are absent the equation of motion simplifies to (1). Its Laplace transformation gives the characteristic equation

$$H(\Delta) := \Delta^2 + \omega^2 + 2\zeta\omega\Delta + g_1 \bar{K}_1(\Delta) + g_2 \Delta \bar{K}_2(\Delta) = 0 \tag{12}$$

where $\bar{K}_i(\Delta)$ is the Lapalce transform of $K_i(\Delta)$ $(i = 1, 2)$. The stability of the linear system described by equation (1) requires that all roots of equation (12) have

negative real parts. Assume that $K_1(t) = K_2(t) = K(t)$.
Now we examine the two cases corresponding to weak and strong delay.

Weak delay. In the case $K(t) = \nu \exp(-\nu t)$ equation (12) becomes

$$(\Delta + \nu)(\Delta^2 + 2\zeta\omega\Delta + \omega^2) + \nu(g_2\Delta + g_1) \;=\; 0\,. \tag{13}$$

Applying Routh–Hurwitz criterion, e.g. see May (1974), equation (13) possesses roots with negative real parts iff $2\zeta\omega + \nu > 0$, $\nu(\omega^2 + g_1) > 0$ and

$$(2\zeta\omega + \nu)(2\zeta\omega + g_2) + 2\frac{\zeta\omega^3}{\nu} \;>\; g_1\,. \tag{14}$$

Strong delay. Suppose $K(t) = \nu^2 t \exp(-\nu t)$. Then equation (12) reduces to

$$(\Delta^2 + 2\nu\Delta + \nu^2)(\Delta^2 + 2\zeta\omega\Delta + \omega^2) + \nu^2(g_2\Delta + g_1) \;=\; 0\,. \tag{15}$$

Proceeding as above, the stability conditions are $\zeta\omega + \nu > 0$,

$$\nu^2(\omega^2 + g_1) \;>\; 0 \qquad \text{and} \qquad \frac{\omega^2}{\nu} + 4\zeta\omega + \nu \;>\; \frac{\eta^2 + \omega^2 + g_1}{\eta} \tag{16}$$

$$\text{where} \qquad \eta \;=\; \frac{2\omega^2 + 2\nu\zeta\omega + g_2\nu}{2(\zeta\omega + \nu)}\,.$$

A cursory comparison of inequalities (14) and (16) shows that stability conditions in case of weak delay are more likely satisfied than those of strong delay.

4. Stochastic analysis of the linear system with weak delay

While neglecting effects of nonlinearity ($\nu = 0$), the system under seismic excitation and environmental fluctuations is governed by the set (11) of linear SDEs. Let the triplet $(W^1(t), W^2(t), W^3(t))$ represent three mutually independent Wiener processes driving the considered dynamics. Then the system governing the stochastic process $X(t) = (x_1(t), ..., x_d(t))^T$ with $d = 4$ can be written in the form

$$dX(t) = \underline{A}X(t)\,dt \,+\, B^1X(t) \circ dW^1(t) \,+\, B^2X(t) \circ dW^2(t) \,+\, C\,dW^3(t) \tag{17}$$

where $\underline{A}, B^1, B^2$ are the real $d \times d$ – matrices, and C the d – dimensional real vector corresponding to system (11). Here $()^T$ denotes the transposed of the inscribed vector or matrix. Equation (17) is interpreted in Stratonovich sense.
For the purpose of more convenient mathematical handling, we transform system (17) to its equivalent Itô form (cf. [18] or [27]). One encounters with its Itô prescription

$$dX(t) = AX(t)\,dt \,+\, B^1X(t)\,dW^1(t) \,+\, B^2X(t)\,dW^2(t) \,+\, C\,dW^3(t)\,. \tag{18}$$

It turns out that the only difference between the matrices $\underline{A}$ and A rests with the element $a_{2,2} = \underline{a}_{2,2} + \frac{1}{2}\sigma_2^2$. The system matrices of the drift and diffusion parts have the following form

$$A = (a_{i,j})_{i,j=1\dots4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & -2\zeta\omega + \frac{1}{2}\sigma_2^2 & -g_1 & -g_2 \\ \nu & 0 & -\nu & 0 \\ 0 & \nu & 0 & -\nu \end{pmatrix},$$

$$B^1 = (b^1_{i,j})_{i,j=1\dots4} \quad \text{with} \quad b^1_{i,j} = \begin{cases} \sigma_1 & : \; if \quad i=2, j=1 \\ 0 & : \; else \end{cases},$$

$$B^2 = (b^2_{i,j})_{i,j=1\dots4} \quad \text{with} \quad b^2_{i,j} = \begin{cases} \sigma_2 & : \; if \quad i=j=2 \\ 0 & : \; else \end{cases}$$

$$\text{and} \qquad C = (0 \quad \sigma_3 I(t) \quad 0 \quad 0)^T \; . \tag{19}$$

The solution process $X(t)$ is a diffusion process, and accordingly a detailed description of the probabilistic behaviour of the model can be obtained in terms of the probability density $p = p(x_1, x_2, x_3, x_4; t)$ solving its Fokker–Planck equation (FPE). The FPE corresponding to equation (18) has the form

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\frac{\partial}{\partial x_1}[x_2 p] - \frac{\partial}{\partial x_2}[\{-\omega^2 x_1 + (\frac{1}{2}\sigma_2^2 - 2\zeta\omega)x_2 - g_1 x_3 - g_2 x_4\}p] \\ &\quad -\frac{\partial}{\partial x_3}[\nu(x_1 - x_3)p] - \frac{\partial}{\partial x_4}[\nu(x_2 - x_4)p] + \frac{1}{2}\frac{\partial^2}{\partial x_2^2}[(\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 I^2(t))p] \; . \end{aligned} \tag{20}$$

The complete solution of this FPE being subject to appropriate boundary conditions seems to be impossible. Accordingly, one has to be content with the evolution of moments of the process $X(t)$. It is pertinent to emphasize that in the presence of seismic excitation the transient solution of process $X(t)$ is very crucial. However, before embarking on a detailed numerical analysis of the stochastic system, at first we study the stability behaviour of the system in the moment sense in the absence of seismic excitation.

Moment stability of the system without seismic excitation. While examining the asymptotical behaviour the term of seismic excitation can be neglected. Modelling with the approach of Bolotin (1960) this term is damped out exponentially fast, and hence it plays no role for the long–term behaviour of the system. When seismic excitation is absent, SDE (18) reduces to

$$dX(t) = AX(t)\,dt + B^1 X(t)\,dW^1(t) + B^2 X(t)\,dW^2(t) \; . \tag{21}$$

Taking expectation, the system of first moments $Y^i(t) = \mathbb{E}\, X^i(t)$ follows the equation

$$dY(t) := d\,\mathbb{E}\,X(t) = A\,\mathbb{E}\,X(t)\,dt = A\,Y(t)\,dt \; . \tag{22}$$

Obviously, system (21) is asymptotically stable in the mean sense iff the real parts of all eigenvalues of the matrix A are negative. The characteristic polynomial c of

A expanded in z is

$$\begin{aligned} c_A(z) &= det(A - zU_4) \quad (23) \\ &= (z+\nu)\left(z^3 + (2\zeta\omega - \frac{1}{2}\sigma_2^2 + \nu)z^2 + ((g_2 + 2\zeta\omega - \frac{1}{2}\sigma_2^2)\nu + \omega^2)z + \nu(\omega^2 + g_1)\right) \end{aligned}$$

where U_4 represents the unit matrix in $\mathbb{R}^{4X4}$.
Using Routh–Hurwitz criterion, we find that the system has stable first moments if

$$2\zeta\omega + \nu > \frac{1}{2}\sigma_2^2 \quad \text{and}$$

$$(2\zeta\omega + \nu - \frac{1}{2}\sigma_2^2)(\omega^2 + 2\zeta\omega\nu + g_2\nu - \frac{1}{2}\sigma_2^2\nu) > (\omega^2 + g_1)\nu . \quad (24)$$

It may be noted that multiplicative fluctuations characterizing the displacement have no influence on the stability of first moments. On the other hand multiplicative fluctuations effecting random velocity make the conditions for stability more stringent than those for the deterministic case, i.e. when $\sigma_2 = 0$ (cf. equation (14)). The analysis of evolution and asymptotical behaviour of the second moments is more difficult and tiring than above for the first moments. For this purpose we follow Arnold (1974). One obtains

$$\dot{Q}(t) = AQ + QA^T + B^1QB^{1^T} + B^2QB^{2^T} \quad (25)$$

for the second moment matrix $Q(t) = (\mathbb{E}\, X^i(t)X^j(t))_{i,j=1...4}$. Analyzing the deterministic system (25) one gains the desired assertions on the stability of SDE (21) in mean square sense. We introduce the notations

$$p_1 = q_{1,1},\ p_2 = q_{1,2},\ p_3 = q_{1,3},\ p_4 = q_{1,4},\ p_5 = q_{2,2},$$

$$p_6 = q_{2,3},\ p_7 = q_{2,4},\ p_8 = q_{3,3},\ p_9 = q_{3,4} \quad \text{and} \quad p_{10} = q_{4,4} .$$

With this notation one finds the equivalent vector differential equation to (25)

$$\dot{p} = Sp \quad (26)$$

where p(t) is the corresponding $d'(d'+1)/2 = 10$ – dimensional vector of second moments. The elements of matrix S are received from (25) using the symmetry of the second moment matrix $Q(t)$. For the decision on mean square stability one encounters with the characteristic polynomial $c_S(z) = det(S - zU_{10})$ with degree 10 where U_{10} represents the unit matrix in $\mathbb{R}^{10X10}$. An application of computer packages like MAPLE for symbolic computations provides unwieldy expressions running into several pages. Thus there is no hope of getting an exact analysis of roots of the polynomial as functions of system parameters. Therefore one has to resort to numerical computations with specific parameter setup as discussed in section 6.

5. Stability analysis of the linear system with strong delay

Proceeding on similar lines, one can carry out moment stability analysis of the system with strong delay. The system is governed by a set of linear SDEs (13) while assuming $\gamma = 0 = \sigma_3$. After rewriting the system to its equivalent Itô form one obtains the drift matrix

$$A = (a_{i,j})_{i,j=1\ldots6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega^2 & -2\zeta\omega + \frac{1}{2}\sigma_2^2 & -g1 & 0 & -g2 & 0 \\ 0 & 0 & -\nu & \nu & 0 & 0 \\ \nu & 0 & 0 & -\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu & \nu \\ 0 & \nu & 0 & 0 & 0 & -\nu \end{pmatrix} \tag{27}$$

which establishes the qualitative mean behaviour of the linear system. Its characteristic polynomial is

$$\begin{aligned} c_A(z) &= det(A - zU_6) \qquad (28)\\ &= (z+\nu)^2\left((z+\nu)^2(z^2 - z(-2\zeta\omega + \frac{1}{2}\sigma_2^2) + \omega^2) + z\nu^2 g_2 + \nu^2 g_1\right) \end{aligned}$$

This can be rewritten to

$$c_A(z) = (z+\nu)^2(z_4 + 2b_1 z^3 + \nu b_2 z^2 + 2\nu b_3 z + b_4)$$

where

$$b_1 = \zeta\omega + \nu - \frac{\sigma_2^2}{4}, \tag{29}$$

$$b_2 = \frac{\omega^2}{\nu} + 4\zeta\omega + \nu - \sigma_2^2, \tag{30}$$

$$b_3 = \omega^2 + \nu\zeta\omega + \frac{1}{2}g_2\nu - \frac{1}{4}\sigma_2^2\nu \tag{31}$$

and

$$b_4 = \nu^2(\omega^2 + g_1) . \tag{32}$$

Necessary and sufficient conditions for negative real parts of roots of the polynomial (28) are

$$b_1 > 0, b_3 > 0, b_4 > 0, b_2 > \frac{b_3}{b_1} + \frac{b_1}{b_3}(\omega^2 + g_1), \tag{33}$$

see May (1974). We note that in the presence of velocity–dependent noise (i.e. $\sigma_2 > 0$) conditions (29) and (30) become more stringent.

One can obtain the differential matrix equation for the second moment matrix by proceeding on similarly to the previous section. However, in order to derive conditions for mean square stability of the system, one has to resort to numerical computations, what we omit here.

6. Numerical simulation for the linear model with weak delay

In the presence of seismic excitation and environmental fluctuations the knowledge about the transient solution of the vector process $X(t)$ is of paramount importance. The carring out of transient analysis requires to simulate sample paths of the considered system. It would be of great interest to have an estimate of the frequency with which the sample paths cross the boundaries of the deterministic solution of the system. Just as well it is interesting to study the duration time for which the sample paths remain outside the boundaries or above some critical values. In view of these goals we resort to the numerical solution of stochastic systems.
Another important aspect relates to the numerical computation and visualization of the evolution of first and second moments of variables of interest. In linear systems we have two alternatives, namely the direct statistical estimation of the moments from the stochastic system or the solution of the corresponding ordinary differential equation systems. Of course, the latter one one could prefer in linear models without additive noise. However, in view of incorporating seismic excitations or nonlinear models we make use of numerical techniques and statistical estimation procedures.

6.1. Numerical simulation of sample paths and first moments

It is recommended to implement the implicit Euler method with drift implicitness 0.5 or a Balanced method with simple correction by a scalar factor depending locally on the current Wiener process increments. This enables us to integrate fairly accurate stochastically stiff systems and guarantees numerical stability of moments, particularly mean square stability, cf. Schurz (1993, 1994). These procedures provide us with very robust approximate solutions.
For gaining insight into the evolution of components of the system without seismic excitation, we resort to the implicit Euler method with implicitness $\rho = 0.5$ and equidistant time step size $\Delta = t_{n+1} - t_n = 0.001$. This method employs the scheme

$$Y_{n+1} = (U - \rho A\Delta)^{-1} \left(U + (1-\rho)A\Delta + B^1 \Delta W_n^1 + B^2 \Delta W_n^2\right) Y_n \tag{34}$$

$$(n = 0, 1, 2, \ldots)$$

starting in $Y_0 = X(0)$ at time t_0 for the linear system without additive noise. $\triangle W_n = W(t_{n+1}) - W(t_n)$ denotes the current Wiener process increment, whereas Y_{n+1} gives the value of the approximate solution at time $t_{n+1} = t_0 + n\Delta$. Once again U represents the unit matrix.
The parameter values for the simulation of the model with weak delay are

$$X(0) = (1,0,0,0);\ \omega = 5;\ \zeta = 0.1514;\ g_1 = 11.85;\ g_2 = 4.87; \tag{35}$$

$$\nu = 0.1;\ \sigma_1 = 5;\ \sigma_2 = 1.4 \quad \text{and} \quad \sigma_3 = 0\,.$$

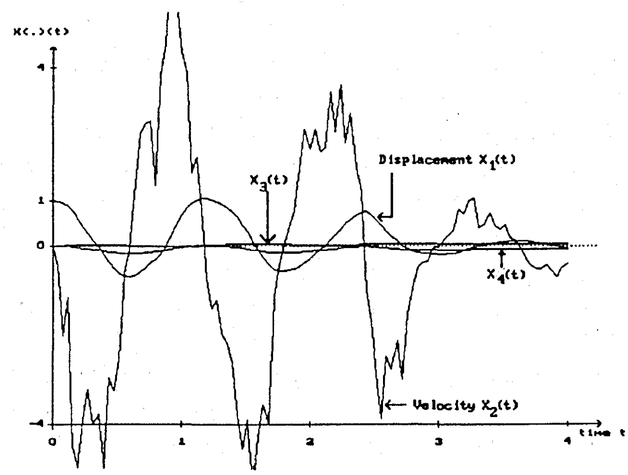

Figure 1. Pathwise evolution of components without seismic excitation.

Figure 1 visualizes a plot of the components of the random oscillator with weak delay. Obviously, the components are more or less rapidly damped out. For the corresponding first moments one observes the same pattern. The damping seems to be slightly slower than that in the deterministic model, and from a critical value of σ_2 on the dynamical damping merges into increasing amplitudes (amplification) and instability sets in (not due to numerics!). Furthermore, for this parameter choice (35) we know that the linear homogeneous equation has a mean square stable, and hence a mean stable equilibrium solution (We checked it with MAPLE). This fact is demonstrated in figure 2.

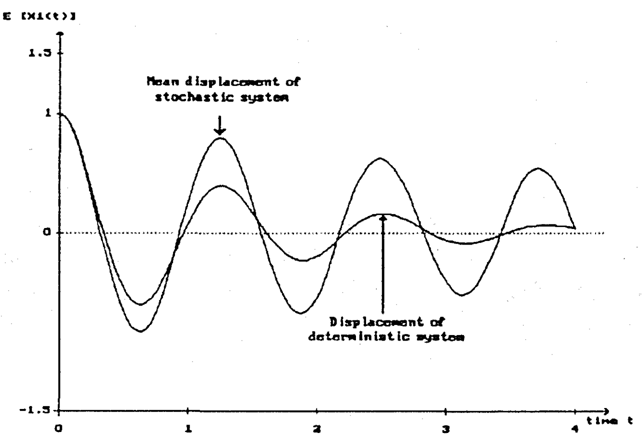

Figure 2. Comparison of displacements without seismic excitation.

The plot of this figure compares the mean displacement of the stochastic system with the displacement of the deterministic one. By figure 2 it is shown that an amplification in the mean sense is effected by multiplicative noise forces. Therein the value of parameter σ_2 plays the decisive role establishing this amplification process. However, the phase of the mean oscillations does not seem to be changed under these noise forces. For producing figure 2 we have applied the same Euler method (34) with implicitness and step size as in figure 1.
Finally, we have carried out some simulations for the system with weak delay in the presence of seismic excitation using the same setup (35). The new parameters are

$$\sigma_3 = 0.9;\ \alpha = 1 \text{ and } \beta = 2\,.$$

For further simplification, we make use of Balanced methods circumventing the costly inversion of correction matrices in the implicit Euler methods. The Balanced methods introduced by Mil'shtein et al. (1992) give an alternative means of controlling the numerical moment behaviour (for details see appendix). We suggest the Balanced method

$$Y_{n+1} = c_n^{-1} \cdot ((Uc_n + A\Delta + B^1 \triangle W_n^1 + B^2 \triangle W_n^2) Y_n + C \triangle W_n^3) \qquad (36)$$

$$(n = 0, 1, 2, \ldots)$$

for the numerical solution of the linear system. Its correction factor c_n is simply a real number and choosen with

$$c_n \ := \ 1 + ||A||\Delta + (1 + ||B^1||^2)|\triangle W_n^1| + (1 + ||B^2||^2)|\triangle W_n^2|\,.$$

Consequently, this numerical method simply represents a locally and appropriately corrected Euler method. Furthermore, it gives more numerical stability (visible with large step sizes and other parameter choices).

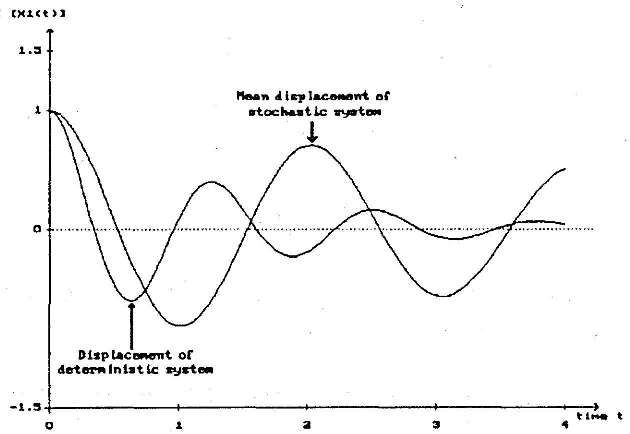

Figure 3. Evolution of deterministic and mean displacement with seismic excitation.

Results concerning the temporal evolution of estimated first moment of the displacement are viewed in figure 3, as well as the corresponding deterministic evolution. The presence of seismic excitation leads to a shift in the phase of the stochastic displacement compared with the deterministic one, as seen in this figure.
The results depicted in the figures above can be considered as reliable. For confirming the presented results, we repeated the numerical integration and estimation with smaller step sizes several times.
Corresponding numerical studies could also be executed on computers for the model with strong delay, but this is omitted to the interested reader. Note that the computational effort increases considerably here. However, costly comparison studies between the models of weak and strong delay would be of great interest. To encourage the reader, this could help to decide which model 'reflects better' the seismic reality and allows more comfortable active controls in seismic structures.

6.2. Numerical simulation of exit frequencies

In order to answere the questions pertaining to the frequency with which the sample paths cross the boundaries of the deterministic solution and duration time remaining outside the boundaries, we make use of sample pathwise simulation of the linear system with weak delay and seismic excitation. For this purpose 1000 simulations are carried out. The following figure shows the proportion of sample paths of the displacement moving outside the interval $[-0.5, +0.5]$ for increasing time t.

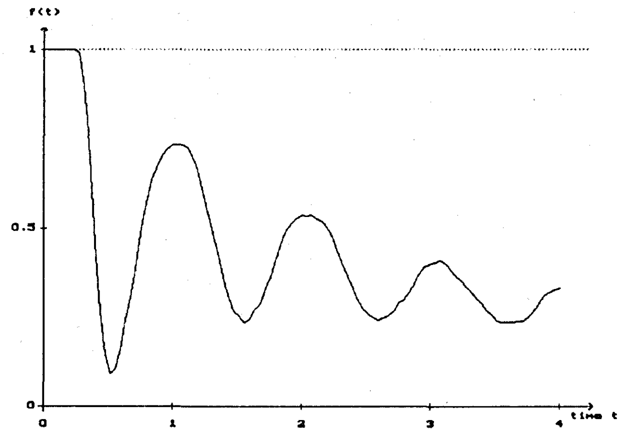

Figure 4. Temporal evolution of the relative exit frequency.

7. Effects of feedback and stochasticity under cubic nonlinearity ($\gamma > 0$)

In this section we briefly discuss some effects coming in through the nonlinear term $x_1^3(t)$ in the model system, i.e. $\gamma \neq 0$. Up to a certain level of γ we can treat this case as a perturbation of the corresponding linear system. Thus, for very small parameters γ, it would not effect decisively the dynamical behaviour. This is dramatically changing with increasing γ. For practical simulations we recommend to use only

Balanced methods with appropriate weights. To some extent they guarantee the most numerically stable behaviour which is possible. Now we draw the attention to several special cases in the presence of cubic nonlinearity.

7.1. A deterministic nonlinear oscillator with small feedback

At first we numerically examine the deterministic dynamical behaviour in the phase plane of the first two components. Under small feedback parameters (g_1, g_2) the system behaves like a stable nonlinear oscillator at the environment of the equilibrium point zero. Thus we can expect a spiralization process of the corresponding flow converging to its steady state (equilibrium zero) in the phase plane. This effect is demonstrated in figure 5. As already mentioned, the system without distributed delays undergoes the movement of the well–known Duffing oscillator. This oscillator possesses one till three steady states (equilibria) depending on the system parameters (the sign infront of ω_2 establishes its number). In our model we observe the presence of only one steady state which is stable because of the positivity of ω^2. The qualitative behaviour seems to be the same under small feedback, as seen in the following figure.

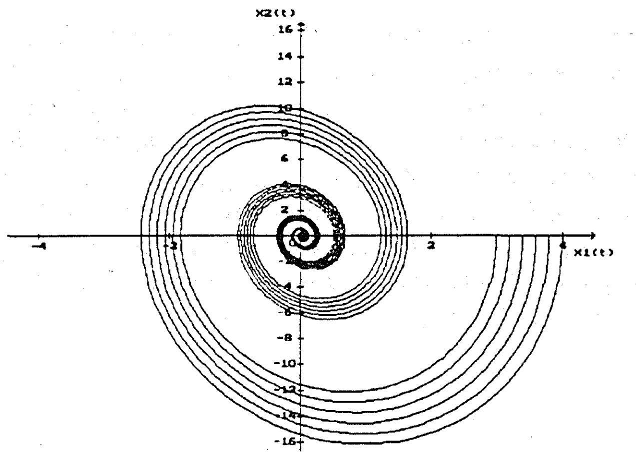

Figure 5. Phase diagram of the deterministic nonlinear system with small feedback.

7.2. A deterministic nonlinear oscillator with large feedback

A change of feedback parameters g_1 and g_2 as well as the exponential rate ν cause several effects. It turns out that the spiralization and attraction process of the obtained flow are decisively influenced by them, e.g. the clustering process, the speed and direction of attraction. This fact is viewed in figure 6. Beyond critical values for the feedback gains it even leads to a destabilization of the dynamical system, whereas the value of ν establishes more the stiffness of the system (simultaneous appearance of slowly and rapidly varying components).

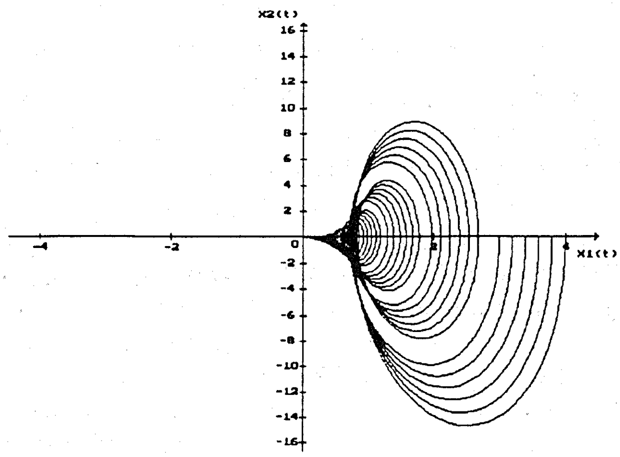

Figure 6. Phase diagram of the deterministic system with large feedback.

7.3. A stochastic nonlinear oscillator with distributed delays

Now we incorporate stochasticity. Experiments show that the speed of attraction is damped in the presence of stochastic terms. Noise as well as cubic nonlinearity effect an amplification of the system when it is moving far from the equilibrium point. Furthermore the built–in stochasticity destabilizes the spriralization process (form, duration and convergence) depending on its intensity parameters in the phase plane. However, under small noise an attraction of the flow can be kept, cf. figure 7. The following figure visualizes the dynamical behaviour in terms of a stochastic flow in the phase plane. There paths for several initial values are plotted with respect to the same random noise output.

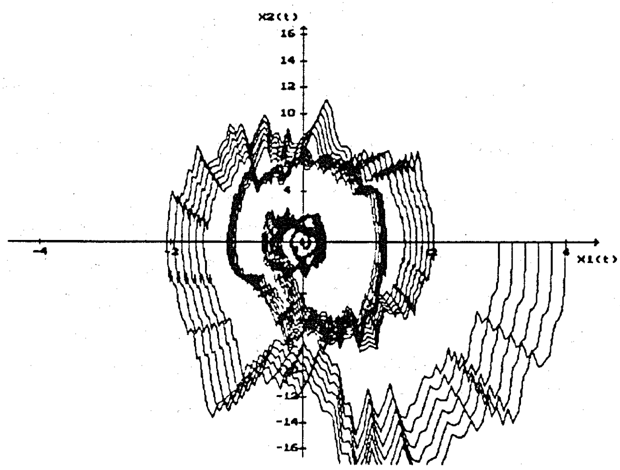

Figure 7. Phase diagram of a noisy nonlinear system with delay.

In contrast to our specific model setup, the general noisy Duffing oscillator is a

well–studied example of a bistable random oscillator with three steady states (The zero point is unstable!). This physical object leads to a chaotic movement between two attraction regions and a symmetric change of clustered trajectories (tunneling). In our model the parameter σ_1 determines the nearness to this behaviour, however processes of attraction and separation alternate. With growing σ_1 one observes a significant change between separation and attraction periods around different points, finally the movement of the oscillator becomes more and more chaotic.
For the same parameter setup as in section 6 the system has still a stable behaviour under cubic nonlinearity with parameter $\gamma = 1$. This appearance is confirmed by figure 8. There we estimated the mean square evolution of the nonlinear system. As seen the amplitudes decrease monotonically. However this process is damped with growing influence of seismic excitation and nonlinearity. Thus, then one can expect serious instabilities, both in estimation procedures for the mean evolution and in the displacement.

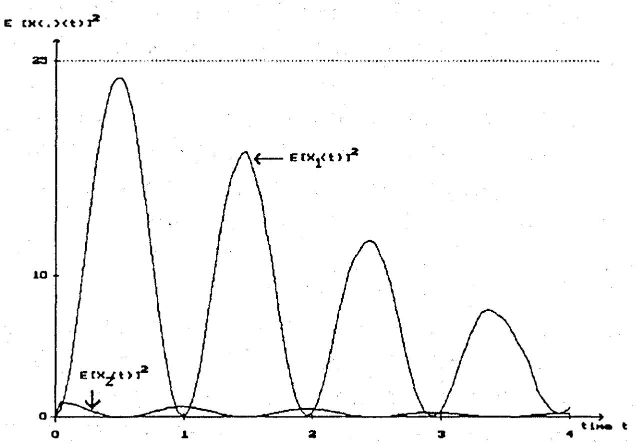

Figure 8. Mean square evolution of the components X_1 and X_2 with delay and seismic excitation ($\gamma = 1$).

As we have already seen, the nonlinearity can cause several effects. It is important to take into consideration such terms. By varying parameters σ_1 and σ_2 one could expect a significant change in the long–term behaviour of solutions, whereas by varying σ_3 in the transient behaviour. This process which is often called bifurcation undergoes a pitchfork bifurcation within the framework of the Duffing oscillator. The point where just this effect occurs seems to be of great interest. Such bifurcation points indicate some structural instability in general. Corresponding bifurcation diagrams should show that bifurcation is related to the appearance of new solutions and a change in the stability behaviour. Stable and unstable steady states (equilibria) may occur in our seismic models. Intensive studies in this field would sprinkle the bounds of possibility of this paper, hence we omit it here. For a discussion with respect to stochastic bifurcation theory see the papers arising from the school of Arnold or in Horsthemke and Lefever (1984). However, note this theory for higher–dimensional systems is still in its beginning, and reliable numerical methods to detect steady states and to follow the qualitative behaviour of stochastic continuous time dynamical systems are urgently required.

8. Conclusions

The paper has examined and visualized some effects of nonconstant weak and strong time delay on seismic structures. Additionally we have introduced stochasticity, i.e. several noise sources, in seismic models. Through this paper we suggest to examine carefully the stochastic dynamical systems governing the seismic structures. It could be useful to understand more accurately their behaviour, before one introduces active controls, particularly stochastic controls, in them. The form of stochasticity introduced herein has not only the purpose of expressing some uncertainity during modelling. It also allows an interpretation of random perturbations of system–determining parameters, e.g. such as ω^2 or $2\zeta\omega$.
The paper demonstrates that Stratonovich noise destabilizes the temporal behaviour of seismic structures, although not new in the general theory of stochastic dynamical systems. The same effect was observed in both linear and nonlinear models. Incorporating stochasticity and cubic nonlinearity leads to a significant amplification of the system components. Up to a certain level of stochasticity the damping and attraction process is delayed more or less, after that serious instabilities can even occur. Seismic excitations following the suggestion (6) due to Bolotin (1960) mainly effect a change in the transient system behaviour and a shift in the phase between displacement and velocity of the oscillations.
For solving of model equations for seismic structures, numerical techniques must be very carefully choosen. Then they are useful to indicate qualitative characteristics of these dynamical systems, e.g. stability of an equilibrium point, and provide the user with reliable results. Numerical simulations should apply implicit Euler methods, or more general some appropriately choosen Balanced methods to the system equations. They are able to control fairly well the behaviour of the simulated dynamical system, at least up to the mean square level.
Further stochastic generalizations could also be introduced and examined, e.g. some jump components or stochastic delay terms. Stochastic delay terms would lead to serious difficulties in the calculus. This requires other approaches than Itô and Stratonovich calculus, such like the Mallivian calculus. However, much work still has to be done in this subject.

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Appendix A. Numerical methods for solving SDE's

The development of stochastic numerical techniques began with some examinations to generate the simplest numerical solutions in the early 50ies. At this stage, in its infancy, one has already known about the well-known Euler scheme and its convergence to the exact solution. Meanwhile there are many contributions on this subject. Mil'shtein [33] has done a pioneer work in this field. Further contributions followed by himself in the 80ies, e.g. see the book [34]. Furthermore, we want to point to the paper of Wagner and Platen [46]. This paper established the foundations for a general calculus on numerical analysis for stochastic differential equations. They developed the stochastic Taylor formula, a counterpart of the deterministic Taylor formula, which allows to develope systematically numerical methods of higher convergence order. Finally, it started the development as an 'own field' in the 80ies. These investigations and more have merged into the monograph of Kloeden and Platen [27]. A little bit more experimentally and computer oriented, Kloeden et al. [28] have recently published a book on this field too. By the French school, e.g. Pardoux and Talay [38] or Talay [43, 44], further important contributions have brought into the subject. An alternative approach to stochastic numerical methods mentioned here has given by Kushner [32]. He constructs Markov chain approximations in order to treat efficiently stochastic control problems, such as the heavy traffic problem. This appendix shall follow the approach of Artemiev, Mil'shtein, Kloeden and Talay. For further sophisticated reviews and interesting facts see the papers of Artemiev, Clark and Cameron, Newton, Rümelin, Shkurko, Strittmatter and many, many others. However, we are only able to present some basic facts on this subject, and do not claim any completeness of this appendix.
In contrast to the deterministic calculus, the stochastic integration depends on the choice of intermediate points. By varying of this choice one gains different integral notions. Two of them turn out to be very effective. One is the Itô integral, which has more mathematical meaning due to convenient properties, whereas the other, the Stratonovich integral is more appropriate for practical modelling (cf. Wong and Zakai [47]). Both versions can be transformed each another in a natural way. Now, for the sake of simplicity, we only consider the Itô type of stochastic differential equations. These equations are often driven by a stochastic process having independent identically Gaussian distributed increments. The Wiener process $W(t)$ with zero mean and $\mathbb{E}\,[W(t)]^2 = t$ represents such a object. Although very erratic, this stochastic process is appropriate for modelling (at least in mean square sense). A plenty of applications rely on this noise source. Now, let be given a m-dimensional Wiener process $(W^j(t))_{j=1,\dots,m}$ which drives the Itô differential equation

$$dX(t) = a(t, X(t))\,dt + \sum_{j=1}^{m} b^j(t, X(t))\,dW^j(t) \tag{37}$$

starting at $X(0) = x_0 \in \mathbb{R}^d$ on the time interval $[0, T]$. Solutions $\{X(t) : t \geq 0\}$ of (37) exist and are unique under the assumptions of Lipschitz continuity and of 'appropriate' polynomial boundedness of the functions $a_i(.)$ and $b_i^j(.)$ $(i = 1...d, j = 1...m)$. The simplest method generating numerically such solutions is the Euler–

Maruyama method (or shorter Euler method). These solutions are constructed by

$$Y_{n+1} = Y_n + a(Y_n)\,\Delta_n + \sum_{j=1}^{m} b^j(Y_n)\,\Delta W_n^j; \quad (n = 0,1,2,\ldots). \tag{38}$$

Here Y_{n+1} denotes the value of the approximate solution using integration step size $\Delta_n = t_{n+1} - t_n$ at time point t_{n+1}. With $\Delta W_n^j = W^j(t_{n+1}) - W^j(t_n)$ we denote the current j–th increment of the Wiener process $W^j(t)$ which can be generated as a standard Gaussian random variable multiplied by $\sqrt{\Delta_n}$. At least for 'small enough' step sizes Δ_n, by corresponding convergence theorems the application of the method (38) to the equation (37) is justified to obtain an approximate solution depending on the practical purpose one is going to follow. In case of pathwise approximation (strong) one requires that a constant $K = K(T) > 0$ (T terminal time) exists such that

$$\forall t_n : \mathbb{E}\,||X(t_n) - Y(t_n)|| \leq K(T) \cdot \Delta^{\gamma} \tag{39}$$

where $\Delta = \sup_n \Delta_n < +\infty$. In contrast to that, for momentwise approximation (weak) it is sufficient to demand only the existence of positive constants $K = K(T, g)$ such that

$$\forall t_n : ||\,\mathbb{E}\,(g(X(t_n)) - g(Y(t_n)))|| \leq K(T, g) \cdot \Delta^{\beta} \tag{40}$$

with respect to a class of 'sufficiently smooth' functions g (often $g \in C_p^{\infty}$). The weak convergence has more practical usage because one is mostly interested in the calculation of moments only. In those cases one may even simplify the generation of the random variables ΔW_n^j in (38). For equidistant approximations, it turns out to take any independent identically distributed random variables $\xi_{j,n}$ instead of ΔW_n^j which satisfy the moment relation

$$|\,\mathbb{E}\,\xi| + |\,\mathbb{E}\,\xi^3| + |\,\mathbb{E}\,\xi^5| + |\,\mathbb{E}\,\xi^2 - \Delta| + |\,\mathbb{E}\,\xi^4 - 3\Delta^2| \leq C \cdot \Delta^3$$

for a constant $C > 0$. Thus we keep the weak convergence order β of numerical methods, at least up to the order $\beta = 2.0$. For example, this is true for three–point distributed random variables ξ with

$$\mathbb{P}(\xi = \pm\sqrt{3\Delta}) = \frac{1}{6} \quad \text{and} \quad \mathbb{P}(\xi = 0) = \frac{2}{3}.$$

Moreover, for method (38) one can take simpler random variables else than above. Choosing noise increments $\xi_{j,n}$ satisfying

$$|\,\mathbb{E}\,\xi| + |\,\mathbb{E}\,\xi^3| + |\,\mathbb{E}\,\xi^5| + |\,\mathbb{E}\,\xi^2 - \Delta| \leq C \cdot \Delta^2 \tag{41}$$

for a constant $C > 0$, e.g. two–point distributed random variables ξ with

$$\mathbb{P}(\xi = \pm\sqrt{\Delta}) = \frac{1}{2}\,,$$

the weak convergence order $\beta \leq 1$ of numerical methods is kept, cf. [27]. The mentioned simplifications in the generation of random variables save time and computational effort, but the same procedure cannot be applied to the scheme (38) approximating pathwisely the solution of (37) via the requirement (39). The method (38) possesses the strong convergence order $\gamma = 0.5$ and weak convergence order $\beta = 1.0$. For proofs, e.g. see [27].

Mil'shtein has done one of the first trials of systematic construction of numerical methods and proved the convergence of the well-known Mil'shtein methods (with $\gamma = 1.0$ and $\beta = 1.0$). In general, corresponding higher order methods are derived from the stochastic Taylor expansion, which is due to the iterative application of Itô's lemma, by appropriate truncation. This approach suggested firstly in Wagner and Platen [46] is described in Mil'shtein [34] or in Kloeden and Platen [27]. For further details, see Artemiev [20], Clark and Cameron [22], Newton [37], Pardoux and Talay [38], Talay [43, 44] or Kloeden, Platen and Schurz [28, 29].

As in deterministic analysis, asymptotical stability and the already mentioned convergence together give reasonable, robust, and hence well-behaving numerical solutions. For the sake of guaranteeing stable numerical behaviour, in our experiments we used slightly changed schemes based on the Euler method. Their stability guarantees that small initial perturbations have no effective influence on the further dynamical behaviour. Thus, initial errors can not effect increasing terminal errors. In deterministic analysis one suggests implicit methods, i.e. such methods which also involve locally the new value of the approximation. These methods ensure a numerically stable behaviour. In stochastic analysis there are such methods too. However, the variety of different stability concepts is here much larger, cf. Kozin [30]. We suggest to follow the concept of moment stability. For a brief exposition on mean square stability, i.e. the stability of second moments, see in appendix B. To some extent, stochastically stable behaviour can be guaranteed through the family of implicit Euler methods established by the scheme

$$Y_{n+1} = Y_n + \{\rho a(Y_{n+1}) + (1-\rho)a(Y_n)\}\Delta_n + \sum_{j=1}^{m} b^j(Y_n)\,\Delta W_n^j\,; \quad (n = 0,1,2,\ldots) \tag{42}$$

with fixed implicitness parameter $\rho \in [0,1]$. Of course, for the application of these methods the corresponding system (42) must be locally resolvable. An alternative means to control the stochastic stability behaviour is given by the Balanced methods. Following Mil'shtein et al. [35], these methods possess the form

$$Y_{n+1} = Y_n + a(Y_n)\,\Delta_n + \sum_{j=1}^{m} b^j(Y_n)\,\Delta W_n^j + \sum_{j=0}^{m} c^j(t_n, Y_n)(Y_n - Y_{n+1})|\Delta W_n^j| \tag{43}$$

where $c^j(.,.)$ are bounded dxd - matrices and $\Delta W_n^0 = \Delta_n$ $(n = 0,1,2,\ldots)$. For existence and convergence of them it is convenient to require that the matrices

$$M(t,x) := U + \sum_{j=0}^{m} \alpha_j c^j(t,x) \tag{44}$$

are invertible. Additionally the norms of their inverses must be uniformly bounded for all pairs $(t,x) \in [0,T] \times \mathbb{R}^d$ and sequences of nonnegative real numbers $(\alpha_j)_{j=0,1,\ldots,m}$ with $\alpha_0 \in [0,\hat{\alpha}]$. The strong convergence with order $\gamma = 0.5$ has been proven in [35]. Furthermore, one immediately concludes the weak convergence

order $\beta = 0.5$ of them applied to general models of type (37).
In particular, the methods (42) and (43) are appropriate to integrate reliably and stably systems interpreted as physically stiff. Stiffness often occurs in physical phenomena, e.g. in stochastic mechanics. One encounters with stiffness if there are two dynamical components in the system where one is moving rapidly and very erratic whereas the other relatively slow. Generally speaking, the methods (42) are recommendable to indicate moment stability and in such cases where the deterministic part plays the decisive role in the dynamics. In those cases where the stochastics is more important one should rather take some appropriately choosen Balanced methods. They can even be implemented simpler and can achieve better control on the dynamical behaviour of numerical solutions. The problem of choosing the weight matrices c^j in the Balanced methods (43) is circumvented by the simple choice of $c^j = \alpha_j ||b^j|| U$ with positive scalars α_j in case of finite norms. Note that these methods are especially constructed to treat more accurately systems with linear multiplicative noise or with diffusion vectors having only bounded linear-polynomial growth (More can not be expected in general, due to assumptions guaranteeing their strong convergence!). However, the knowledge about them is still in its infancy.
With the methods (42) and (43) we have obtained reasonable results for our random oscillators. In this paper corresponding higher order methods are deliberately avoided. There is still a lack of extensive studies concerning stability of them. Furthermore, higher order methods would not be always applicable to nonlinear models. They require 'too much smoothness, boundedness' of the drift $a^i(.)$ and diffusion coefficients $b_i^j(.)$ and 'additional information' on the noise (multiple Itô integrals with respect to the Wiener process). Their generation involves massive systems and sets of multiple integrals being generated very costly. Sometimes these methods cause explosions in their numerical solutions close to zero (due to unbounded derivatives of drift and diffusion), or they do not show stable numerical behaviour (e.g. see in [40]). Thus, up to now we can not recommend higher order methods for solving general nonlinear models, for visualizing of stochastic dynamics or for estimating of qualitative characteristics (such as Lyapunov exponents which describe the long-term behaviour) of them. However, in specific models they can be very effective. For example, if one can make use of some structural peculiarities. Thereby one should not refuse them in advance, but a careful study of the continuous time system is necessary before one numerically solves it.

Summarize: Some brief introductory words has been assembled. The stochastic Taylor expansion together with two main theorems on weak and strong convergence due to Mil'shtein (cf. [34]) form the main tools for developing numerical methods in general. We have presented implicit Euler, Mil'shtein and Balanced methods. For higher order methods and Runge-Kutta methods we point to the literature. Structural studies can give advantages and hints for choosing and implementing corresponding numerical algorithms. For example, for coloured noise see Mil'shtein and Tretjakov [36] (Higher convergence order is possible!). One should take into consideration these structural advantages. We have not enlightened the relation Itô versus Stratonovich. Stratonovich calculus is better for practical model interpretation, but Itô calculus possesses useful properties and is necessary for correct implementation on computers. Note there is no difference between these calculi with purely additive noise. The question of choice of the appropriate calculus is one of the modelling issue.

Appendix B. Mean-square stability of an equilibrium solution

In this section we present some basic results on asymptotical mean square stability of an equilibrium point of both the continuous time stochastic differential equations and corresponding discrete time methods. The results for continuous time systems of this kind go back to Khas'minskij, whereas corresponding discrete time results have been worked out by Artemiev and Schurz. The main theorems involve assertions on linear systems or the linearized system of an original nonlinear one. It is required that both the drift and diffusion part vanish at one and the same stationary point. For simplicity we suppose that the null solution effects this specific equilibrium situation. Our seismic systems possess this property, but also in a plenty of further applications one observes this occurence, e.g. in Lotka–Volterra systems describing population growth processes in Biology (cf. Gard [23]).
At first, an assertion about the exponential mean square stability of the null solution for linear continuous time systems

$$dX(t) = A(t)\,X(t)\,dt + \sum_{j=1}^{m} B^j(t)\,X(t)\,dW^j(t) \qquad (45)$$

starting in $X(0) = x_0$ at time $t_0 \geq 0$ is formulated. Before we state the main theorems below we recall the notions of (asymptotical) mean square and exponential (mean square) stability of the null solution.

Definition B.1. Let $X_t(x_0, t_0)$ denote the solution of equation (45) starting in x_0 at time t_0. Then the null solution $X \equiv 0$ is called **(asymptotically) mean square stable** iff

$$\exists \delta > 0\ \forall t_0 \geq 0\ \forall x_0 \in \mathbb{R}^d \quad ||x_0|| < \delta \quad : \quad \lim_{t\to\infty} \mathbb{E}\,||X_t(x_0, t_0)||^2 = 0, \qquad (46)$$

where $||\cdot||$ denotes the Euclidean vector norm in $\mathbb{R}^d$, and **exponentially mean square stable** iff

$$\exists c_1, c_2 > 0\ \forall t_0 \geq 0 \quad : \quad \mathbb{E}\,||X_t(x_0, t_0)||^2 \leq c_1 ||x_0||^2 \exp(-c_2(t - t_0)). \qquad (47)$$

For results on exponential stability, see the works of Sasagawa [39] and Khas'minskij [26]. Further contributions on stability and different concepts can be found in Arnold [18], Kozin [30] or Kushner [31]. For sophisticated approaches, e.g. see the papers of Baxendale (e.g. [21]) or the collection of papers in Arnold and Wihstutz [19].
The following theorem shall form the base of our further considerations for corresponding discrete systems. It also gives us the meaningfulness for which discrete systems one should require mean square stability of their null solution.

Theorem B.1. *Assume that the matrix-valued functions $A(t)$ and $B^j(t)$ in equation (45) are bounded on $[t_0, \infty)$. Then, for exponential stability of the null solution in the mean square sense it is necessary that for any, and sufficient that for a particular symmetrical, positive definite, continuous and bounded $d \times d$-matrix $C(t)$ with $x^T C(t) x \geq k_1 |x|^2$ $(k_1 > 0)$ for all $t \geq 0$ the matrix-valued differential equation*

$$\frac{dD(t)}{dt} + A^T(t)D(t) + D(t)A(t) + \sum_{j=1}^{m} {B^j}^T(t)D(t)B^j(t) = -C(t) \tag{48}$$

possesses a solution matrix $D(t)$ with the same properties as the matrix $C(t)$.

This theorem and its proof can be found in a more general form in [26]. We took the presented formulation from [18]. Similarly to the definition above one can introduce the notion of mean square stability of the null solution for numerical methods.

Definition B.2. A numerical solution $(Y_n)_{n\in\mathbb{N}}$ with fixed step size Δ starting in y_0 at time t_0 has an **(asymptotically) mean square stable** null solution iff

$$\exists \delta > 0 \; \forall t_0 \geq 0 \; \forall y_0 \in \mathbb{R}^d \quad ||y_0|| < \delta \; : \; \lim_{n\to\infty} \mathbb{E}||Y_n||^2 = 0 \tag{49}$$

where we understand the limit in (49) taken only at discrete times t_n.

In contrast to stability analysis for continuous time systems, the stability analysis for numerical methods is in its very beginning. To some extent, a discrete counterpart to the theorem above is stated below for implicit Euler and Mil'shtein methods. For the construction and convergence of implicit Mil'shtein methods see in [27, 28]. Let $P_n^{\rho,M}$ denote the matrix of second moments $\mathbb{E}\, Y_n^i Y_n^j$ of the implicit Mil'shtein method for (45) at time t_n, and $P_n^{\rho,E}$ that of the corresponding implicit Euler method (42) using the same step size Δ and implicitness $\rho \in [0,1]$. It is not hard to verify that from the stability of matrix sequences P_n^{ρ} it follows the stability in mean square sense, as required in (49), and vice versa. Thereby one can draw further attention to the behaviour of the sequence $(P_n^{\rho})_{n=0,1,2,\ldots}$ for the both implicit methods. The following result is stated for autonomous systems, i.e. for systems with time–independent drift and diffusion parts.

Theorem B.2. *Assume that the both discrete systems start with the same positive definite matrix P_0 of second moments for the autonomous form of SDE (45). Then, the mean square evolution of both discrete systems is given by the corresponding operators $\mathcal{L}_\rho^E$ and $\mathcal{L}_\rho^M$ mapping from the space of positive definite $d\,x\,d$ – matrices into itself, and the following relations hold*

(i) : $P_{n+1}^{\rho,E} = \left(\mathcal{L}_\rho^E\right)^n (P_0) \quad \leq_{(+)} \quad \left(\mathcal{L}_\rho^M\right)^n (P_0) = P_{n+1}^{\rho,M}$, *i.e.*
$\forall x \in \mathbb{R}^d \; : \; x^T P_{n+1}^{\rho,E} x \leq x^T P_{n+1}^{\rho,M} x \; : \; n = 0, 1, 2, \ldots ,$

(ii) : $\forall S \in \mathbb{R}^{dxd}$, *S positive definite solution of (48),* $\forall \rho \geq 0.5$:
$\mathcal{L}_\rho^E \, S <_{(+)} S$, *hence the eigenvalues of* $\mathcal{L}_\rho^E$ *are smaller than one,*

(iii) : $\rho = 0.5 \implies$
The implicit Euler method possesses a mean square stable null solution iff the null solution is mean square stable for the continuous time system.

(iv) : $\forall S \in \mathbb{R}^{dxd}$, *S positive definite,* $\forall \rho_1, \rho_2,\, 0 \leq \rho_1 \leq \rho_2$: $\mathcal{L}_{\rho_2} S \leq_{(+)} \mathcal{L}_{\rho_1} S$
where $\mathcal{L}_\rho$ *is the mean square operator of the implicit Euler or Mil'shtein method, resp.*

This formulation assembles the contents of the paper [40] where one also finds the proofs. The occured matrix inequalities are understood in terms of positive definiteness of the corrresponding matrix differences as indicated in $B.2.(i)$.

Remarks: The results on mean square stability of implicit Euler and Mil'shtein methods are very effective. As a conclusion, the half-implicit Euler method ($\rho = 0.5$) can be considered as a stability indicator of the continuous time system to be solved numerically by this method. This method possesses a mean square stable null solution iff the null solution is mean square stable for the corresponding continuous time system (see $B.2.(iii)$). Through the item $B.2.(iv)$ the 'monotonic nesting' of mean square stability domains has been discovered, cf. [40]. Thus, the most mean square stable method (within the class $0 \leq \rho \leq 1$) is the fully drift–implicit method, i.e. the numerical method with implicitness $\rho = 1$. This holds for both the family of implicit Euler methods and the family of implicit Mil'shtein methods.
If one is not interested in indicating stability one could also apply Balanced methods to achieve control on the discrete dynamical behaviour. As an advantage of the Balanced methods, the costly inversion of correction matrices is not really necessary. One can use Balanced methods with scalar corrections, e.g. in case of bounded norms of drift $a(.)$ and diffusion $b^j(.)$ take

$$Y_{n+1} \; = \; Y_n + w_n^{-1} \left(a(Y_n)\Delta_n + \sum_{j=1}^{m} b^j(Y_n)\Delta W_n^j \right) \tag{50}$$

where $w_n = 1 + \alpha_0 ||a(.)||\Delta_n + \sum_{j=1}^{m} \alpha_j ||b^j(.)|| |\Delta W_n^j|$ with nonnegative real parameters $(\alpha_j)_{j=0,1,\ldots,m}$. It is not hard to verify that there exist Balanced methods (50) which provide mean square stable numerical solutions. Moreover, with (50) one can state numerical methods which possess a more mean square stable null solution than any of the implicit Euler or implicit Mil'shtein methods for bilinear systems. The only disadvantage of Balanced methods, they reduce slightly the weak convergence order and could be too stable (not applicable as stability indicator) for general continuous time systems. This is due to their large variety. Apart from these drawbacks the Balanced methods seem to be the richest and most robust class of numerical methods so far. For corresponding results involving them, see Schurz (1994).

Higher order Taylor methods, those methods with higher convergence order and their rise from the stochastic Taylor formula, do not seem to be appropriate to achieve stability. For example, Theorem $B.2.(i)$ shows that the implicit Mil'shtein method is worser than the corresponding implicit Euler method with respect to mean square stability. At least it indicates that one has to introduce and add very carefully further terms in the numerical method to achieve higher convergence order and stable behaviour simultaneously. However, the drawbacks of higher order methods still exist. There is no appropriate stochastic control on their stability behaviour up to now.

We end up in the following interesting question which still has to be answered satisfactorily. What is happening with the mean square stability of higher order methods in general? In the hope that we have not discouraged the reader so far, we leave the answere to the readership and future research.

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