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Moment evolution of the outflow–rate from nonlinear conceptual reservoirs

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ABSTRACT. The temporal evolution of moments of outflow-rate is investigated in a stochastically perturbed nonlinear reservoir due to precipitation. The detailed stochastic behaviour of outflow is obtained from the numerical solution of a nonlinear stochastic differential equation with multiplicative noise. The time-development of first two moments is studied for various choices of parameters. Using Stratonovich interpretation, it turns out that the mean outflow-rate is above that given by the deterministic solution. Based on the set of 9000 simulation runs, 90 % confidence intervals for the mean evolution of outflow-rate are computed. The effect of stochastic perturbations with finite correlation time is also investigated.

CONTENTS

1. Introduction

The time distribution of outflows at the outlet of catchments fed by rainfall is usually studied on the basis of a single or a cascade of conceptual reservoirs, see Nash [16]. The outflows driven by deterministic inputs of rainfall have been extensively studied within the framework of cascade of conceptual reservoirs. Efforts have also been made to incorporate the nonlinear relationship between storage and discharge in Singh [20].
Unny & Karmeshu [24] have extended the Nash cascade of reservoirs to take into account the stochastic nature of input. The objective of such an extension is to provide a principal basis for the generation of stochastic stream flows. Incorporating stochastic input, the storage balance equations for the system of reservoirs turn out to be stochastic differential equations (SDE's). There are already examinations for quite general SDE's describing hydrological systems, e.g. in stochastic streamflow modelling with jump diffusions by Konecny & Nachtnebel [10]. However, herein attention is to be drawn to SDE's driven only by white and coloured noise sources. Recently Karmeshu & Lal [6] have further investigated the stochastic behaviour of storage in conceptual reservoirs based on storage balance equations. The explicit time–development of moments of outflow for a reservoir could be obtained only in the linear case. However, in a nonlinear reservoir the resulting nonlinear SDE leads to a rather intractable hierarchy of moment equations. In a recent paper Fujita, Shinohara, Nakao & Kudo [4] have also considered a stochastic nonlinear reservoir and have analyzed the linearized version with additive noise.
The purpose of this paper is to investigate the evolution of moments of outflow in a nonlinear reservoir arising due to stochastic rainfall/precipitation. The resulting SDE with multiplicative noise can be interpreted in two different ways – Stratonovich and Ito prescriptions, cf. Gardiner [3]. These are related to each other in the sense that one can transform results of one prescription to those of the other. We have adopted Stratonovich prescription as it is preferable for modelling a physical process, due to Wong & Zakai [27]. However, for the purpose of comparison we have presented results for Ito prescription as well.
The evolution of moments of outflow is obtained by numerical integration schemes, in general, based on the stochastic Taylor expansion described in Kloeden & Platen [7]. Unny [25] also discussed the numerical integration of SDE's in the context of catchment modelling. Resorting to numerical analysis has enabled us to take into account the time varying nature of the rainfall intensity.
The paper comprises seven sections. Section 2 deals with the formulation of the stochastic model involving a nonlinear stochastic differential equation. Section 3 and 4 are concerned with the stationary probability density function (pdf) and the evolution of moments of outflow–rate. After discussing briefly numerical methods for the solution of stochastic differential equations in section 5, we carry out the numerical simulation of moments in section 6. Succeeding section 7 is devoted to the study of effects of coloured noise fluctuations on the evolution of moments. Eventually, this paper finalizes with some conclusions and remarks in section 8.

2. Formulation of the model

The continuity equation for the storage $S(t)$ in a reservoir can be written as

$$\frac{dS(t)}{dt} = I(t) - Q(t), \tag{2.1}$$

where $Q(t)$ denotes the outflow–rate and $I(t)$ is the inflow–rate. The storage equation takes into account the nonlinearity of the conceptual reservoir considered, following the consideration of Singh [20]. Among several storage equations proposed in the literature, the simplest nonlinear reservoir is defined as

$$S = kQ^n, \tag{2.2}$$

k and n being positive parameters. The exponent n is generally found to lie between 0.4 and 3.3. Combining equations (2.1) and (2.2), we get the differential equation for the outflow–rate

$$\frac{dQ}{dt} = aQ^{1-n}[I(t) - Q], \tag{2.3}$$

where

$$a = 1/(kn). \tag{2.4}$$

The inflow–rate being usually expressible in terms of precipitation is stochastic in nature. Customarily, the stochastic fluctuations in the precipitation can be represented by a white noise process. Thus we set

$$I(t) = \overline{I}(t)[1 + \sigma\xi(t)] \tag{2.5}$$

where $\overline{I}(t)$ denotes the mean precipitation, $\xi(t)$ represents the stochastic fluctuations and σ measures their intensity.
Substituting equation (2.5) in equation (2.3), we get the SDE for the outflow–rate

$$dQ(t) = aQ^{1-n}(t)[\overline{I}(t) - Q(t)]dt + \sigma a\overline{I}(t)Q^{1-n}(t) \circ dW(t) \tag{2.6}$$

where we replace $\xi(t)dt$ by the differential $dW(t)$ of the Wiener process, and the SDE (2.6) is to be interpreted in the Stratonovich sense.
The Itô SDE corresponding to the Stratonovich SDE (2.6) is

$$\begin{aligned} dQ(t) &= \left[aQ^{1-n}(t)(\overline{I}(t) - Q(t)) + \frac{1}{2}\sigma^2 a^2(1-n)\overline{I}^2(t)Q^{1-2n}(t)\right] dt \\ &\quad + \sigma a\overline{I}(t)Q^{1-n}(t)dW(t) \end{aligned} \tag{2.7}$$

The SDE (2.6) as well as (2.7) completely describes the stochastic evolution of the outflow–rate from a conceptual reservoir. The solution process $Q(t)$ is a diffusion process, and accordingly the SDE is subject to the theory of diffusion processes.

3. Stationary probability density of the outflow–rate

The complete probabilistic description of the model can be obtained in terms of the Fokker–Planck equation (FPE) which yields the probability density for the outflow–rate $Q(t)$. The FPE corresponding to the SDE (2.6) is

$$\frac{\partial p(Q,t|Q_0)}{\partial t} = -\frac{\partial}{\partial Q}\left[\{f(Q,t)+\frac{1}{2}g'(Q,t)g(Q,t)\}p\right]+\frac{1}{2}\frac{\partial^2}{\partial Q^2}\left[g^2(Q,t)p\right] \quad (3.1)$$

where

$$\begin{aligned} f(Q,t) &= aQ^{1-n}(\overline{I}(t)-Q), \\ g(Q,t) &= \sigma a\overline{I}(t)Q^{1-n}. \end{aligned} \quad (3.2)$$

The FPE is to be solved under appropriate boundary conditions, one of them is that at $Q=0$ there is a reflecting barrier. This condition can be expressed by the requirement that the current probability vanishes at $Q=0$, i.e.

$$\left\{f(Q,t)p-\frac{1}{2}\frac{\partial}{\partial Q}[g^2(Q,t)p]\right\}_{Q=0}=0. \quad (3.3)$$

The other boundary condition is provided by imposing the natural boundary condition for $Q\to\infty$. The initial condition can be expressed as

$$\lim_{t\to 0} p(Q,t|Q_0)=\delta(Q-Q_0). \quad (3.4)$$

It may be mentioned that to derive the explicit solution of the FPE seems to be impossible. However, the stationary probability density function $p_s(Q)$ (pdf) can be obtained by setting $\frac{\partial p(Q,t)}{\partial t}=0$. Following Horsthemke & Lefever [5], the stationary pdf $p_s(Q)$ is found to be

$$p_s(Q) = \frac{N(n)}{g(Q)}\exp\left[2\int_0^Q \frac{f(u)}{g^2(u)}du\right], \quad Q>0 \quad (3.5)$$

where $N(n)$ is the corresponding normalization constant. Consequently, the stationary probability density $p_s(Q)$ corresponding to the outflow model (2.6) possesses then the form

$$p_s(Q) = \begin{cases} N(n)\cdot C_0\cdot Q^{n-1}\cdot\exp\left[C_1\cdot Q^n - C_2\cdot Q^{n+1}\right] & : \; if \quad Q>0 \\ 0 & : \; else \end{cases} \quad (3.6)$$

$$\text{where} \quad C_0=\frac{1}{k\,n\,\overline{I}},\; C_1=\frac{2\,C_0}{\sigma^2\,n},\; C_2=\frac{C_1\,n}{\overline{I}\,(n+1)}$$

with $\overline{I}, \sigma^2, k, n > 0$.

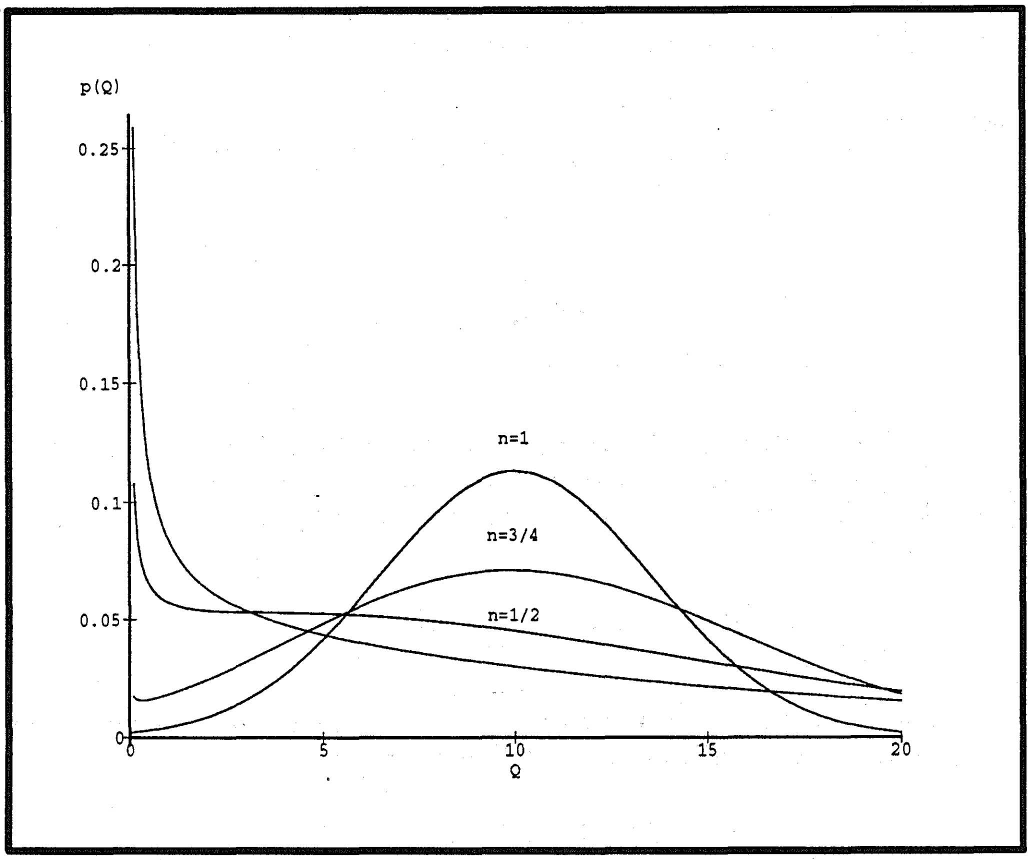

Figure 1. Stationary probability densities of the outflow–rate ($n = 1/3, 1/2, 3/4, 1$).

For the purpose of illustration we present in figure 1 graphs of $p_s(Q)$ for various values of the exponent n. We find that for $n < 1$, the shape of $p_s(Q)$ is highly skewed and tends to be symmetrical as n increases up to 1 and beyond, with more and more concentrating mass at $Q = \overline{I}$. Although it has not been possible to obtain the evolution of pdf $p(Q, t|Q_0)$, nevertheless significant insight could be gained from the evolution of moments pertaining to the pdf.

4. Evolution of moments for the outflow–rate

The differential equations governing the evolution of first two moments of the outflow–rate can be obtained from the SDE (2.6). Using Itô's formula (Arnold [1], Gardiner [3] or Kloeden & Platen [22]), the mentioned differential equations are

$$\frac{d\mathbb{E}[Q(t)]}{dt} = a\overline{I}(t)\mathbb{E}[Q^{1-n}(t)] - a\mathbb{E}[Q^{2-n}(t)] + \frac{1}{2}\sigma^2 a^2(1-n)\overline{I}^2(t)\mathbb{E}[Q^{1-2n}(t)];$$
$$\frac{d\mathbb{E}[Q^2(t)]}{dt} = 2a\overline{I}(t)\mathbb{E}[Q^{2-n}(t)] - 2a\mathbb{E}[Q^{3-n}(t)] + \sigma^2 a^2(2-n)\overline{I}^2(t)\mathbb{E}[Q^{2-2n}(t)]\,. \quad (4.1)$$

Proceeding in a similar manner, differential equations for higher order moments can be derived. It may be noted that except for the linear case ($n = 1$), the equations for the first (and second) moments involve moments of orders other than first (and second). It is easily seen that, in order to study moments, one is confronted with the hierarchy of moment equations (Soong [21]).
To make any progress one could truncate the hierarchy at some stage, but the truncation procedures based on the assumption that the pdf of the process $Q(t)$ is close to a Gaussian process are not suitable for our purpose. The reason is that the model considered here is described by a SDE driven by multiplicative noise which yields a probability distribution being quite different from the Gaussian one. The best one can do under these circumstances is to employ a numerical scheme and to simulate the evolution of the moments for the outflow–rate.

5. Numerical methods for the solution of SDE's

Numerous methods for the numerical treatment of stochastic differential equations can be found in the literature. For the sake of generality we only consider the following type of stochastic differential equations. Given a m–dimensional Wiener process $(W^j(t))_{j=1,\ldots,m}$ which drives the Itô differential equation

$$dQ(t) = a(Q(t))\,dt + \sum_{j=1}^{m} b^j(Q(t))\,dW^j(t) \quad (5.1)$$

starting at $Q(0) = Q_0 \in \mathbb{R}^d$ on the time interval $[0, T]$. As already mentioned the Itô and Stratonovich versions of (5.1) can be transformed each other in a natural way. Solutions $\{Q(t) : t \geq 0\}$ of (5.1) exists and are unique under the assumptions of Lipschitz continuity and of 'appropriate' polynomial boundedness of the functions $a(.)$ and $b(.)$. The simplest method to generate numerically such solutions is the Euler–Maruyama method constructed by the scheme

$$Y_{n+1} = Y_n + a(Y_n)\,\triangle_n + \sum_{j=1}^{m} b^j(Y_n)\,\triangle W_n^j \quad (n = 0, 1, 2, \ldots). \quad (5.2)$$

Here Y_{n+1} means the value of the approximate solution using the step size $\triangle_n = t_{n+1} - t_n$ at the time point t_{n+1}. With $\triangle W_n^j = W^j(t_{n+1}) - W^j(t_n)$ we denote the current j–th increment of the Wiener process $W^j(t)$ which can be generated as a standard Gaussian random variable multiplied by $\sqrt{\triangle_n}$. At least for 'small enough' step sizes $\triangle_n$, by corresponding convergence theorems the application of the method (5.2) to the equation (5.1) is justified to obtain an approximate solution depending on the practical purpose one is going to follow. In case of pathwise approximation (strong) one requires that it exists a positive constant $K = K(T)$ (T terminal time) such that

$$\forall t_n : \quad \mathbb{E}\,||\,Q(t_n) - Y(t_n)\,|| \quad \leq \quad K(T) \cdot \Delta^{\gamma} \tag{5.3}$$

where $\Delta = \sup_n \Delta_n < +\infty$. In contrast to that, for momentwise approximation (weak) it is sufficient to demand only that

$$\forall t_n : \quad ||\,\mathbb{E}\,(g(Q(t_n)) - g(Y(t_n)))\,|| \quad \leq \quad K(T,g) \cdot \Delta^{\beta} \tag{5.4}$$

with respect to a class of 'sufficiently smooth' functions g (often $g \in C_p^{\infty}$). The weak convergence has more practical usage because one is mostly interested in the calculation of moments only. In those cases one may even simplify the generation of the random variables ΔW_n^j in (5.2). For equidistant approximations, it turns out to take any independent random variables $\xi_{j,n}$ instead of ΔW_n^j which satisfy the moment relation

$$|\mathbb{E}\,\xi| + |\mathbb{E}\,\xi^3| + |\mathbb{E}\,\xi^5| + |\mathbb{E}\,\xi^2 - \Delta| \quad \leq \quad C \cdot \Delta^2$$

for a constant $C > 0$. Thus we keep the weak convergence order $\beta = 1.0$ of the simplified Euler method (5.2) with validity of (5.4). For example, this is true for the two–point distributed random variable $\hat{\xi}$ or the three–point distributed $\hat{\hat{\xi}}$ with

$$\mathbb{P}(\hat{\xi} = \pm\sqrt{\Delta}) = \frac{1}{2} \quad \text{or} \quad \mathbb{P}(\hat{\hat{\xi}} = \pm\sqrt{3\Delta}) = \frac{1}{6} \quad \text{and} \quad \mathbb{P}(\hat{\hat{\xi}} = 0) = \frac{2}{3},$$

respectively. This simplification saves time and computational effort, but the same procedure cannot be applied to the scheme (5.2) approximating pathwisely the solution of (5.1) via the requirement (5.3). The method (5.2) possesses the strong convergence order $\gamma = 0.5$ and weak convergence order $\beta = 1.0$. Mil'shtein has done one of the first trials of systematic construction of numerical methods and proved the convergence of the well–known Mil'shtein schemes (with $\gamma = 1.0$ and $\beta = 1.0$). In general, corresponding higher order methods are derived from the stochastic Taylor expansion, which is due to the iterative application of Itô's lemma, by appropriate truncation. This approach has been firstly suggested in Wagner & Platen [26] and is described in Mil'shtein [13] and Kloeden & Platen [7]. For further details, see Clark & Cameron [2], Kushner & Dupuis [11], Newton [17], Pardoux & Talay [18], Talay [22, 23] or Kloeden, Platen & Schurz [8]. In our experiments we used the scheme form (5.2) and obtained reasonable results for the outflow–rates. Note that higher order methods would not be always applicable to our models because of explosions in their numerical solutions close to zero. Furthermore, in Mil'shtein, Platen & Schurz [14] one finds first attempts to achieve control in stochastically stiff situations, such systems where one observes slowly and rapidly varying stochastic components influencing decisively the dynamical behaviour. In some applications (in particular in long–term simulations for stiff systems and in simulations for cascades of water reservoirs) one needs stability of numerical intergration methods, in addition to convergence requirements on finite time intervals. For this purpose implicit methods are introduced, cf. Kloeden et al. [7, 8, 9] or Mil'shtein [13, 14]. Special interest for moment stability, particularly for mean square stability, rises during the simulation experiments, as in this paper. A corresponding mean square stability analysis for linear numerical methods with lower convergence order has been recently worked out by Schurz [19]. Using contribution [19] one can achieve control on the numerical behaviour (here moment evolution of the outflow–rate) of the linearized models up to their second moments, as long this is true for the corresponding continuous time systems. However, such analysis and implicit methods are not generally needed for our hydrological model.

6. Numerical simulation of moments

As discussed in section 4, the numerical solution of SDE's is an effective means of obtaining detailed information about the stochastic behaviour of the outflow–rate. To this end we have simulated the system governed by SDE's for the following situations:

A. Constant mean precipitation,
B. Constant mean precipitation followed by exponential decay.

A. Constant mean precipitation
We have numerically solved the considered SDE for a few choices of the parameter σ when the mean precipitation is assumed to remain constant over the entire simulation run. In the following runs the time step size of the Euler approximation is $\Delta = 0.01$(hr). The initial time $t_0 = 0$ and the simulation is for 4 hrs so that $T = 4$. The other parameters of the model are fixed and their magnitudes are :
Mean precipitation $\overline{I} = 10, k = 1.0, n = 0.75$, Initial outflow–rate $Q(0) = 1.0$.
In figure 2 the deterministic evolution of $Q(t)$ is presented. This corresponds to the case $\sigma = 0$, when there are no fluctuations. While modelling consideration requires the Stratonovich interpretation, for comparison we have also computed the sample paths of $Q(t)$ when the SDE is interpreted in Itô sense as well. For small values of $\sigma = 0.1$, the two sample paths are very close to each other. Divergence between paths increases with σ caused by the growing up difference between the corresponding drift functions. We observe in figure 3 that the sample path under Itô interpretation gives a lower estimate of the outflow–rate as compared with the Stratonovich prescription. In figure 4 the temporal deterministic outflow–rate as well as the mean outflow–rate interpreted in Stratonovich and Itô calculus are given. One observes that Stratonovich prescription gives larger estimates of the mean outflow–rate than the deterministic values which are above the Itô estimates. The time–development of the second moment $\mathbb{E}[Q^2(t)]$ is viewed in figure 5, and we notice that, as time advances, it tends to settle down to a constant value. Figure 6 displays the 90% confidence intervals for the mean evolution in Stratonovich calculus compared with the deterministic values. The estimates are based on the mean of a set of 15 batches of trajectories repeatedly observed 600 times.

B. Constant mean precipitation followed by an exponential decay
In this situation we have assumed that the precipitation remains constant for $T = 2$ hrs and then it falls exponentially. This can be expressed as

$$I(t) = \begin{cases} \overline{I}_1 \,(1 + \sigma\xi(t)) \;, & 0 \leq t \leq 2 \\ \overline{I}_1 \exp(-\alpha(t-2))\,(1+\sigma\xi(t)) \;, & t > 2 \end{cases} \tag{6.1}$$

where α (> 0) is a precipitation decay parameter. The model described by the equation 6.1 besides being more realistic is mathematically interesting. The process evolves till time $T = 2$ governed by (2.6) with given initial condition. Then, terminal points $Q(T)$ of the sample paths serve as random initial conditions for the outflow–rate which has to be incorporated in (2.6). Figure 7 shows the mean evolution $(\alpha = 2)$ in both Stratonovich and Itô calculus compared with the deterministic values firstly increasing up to $T = 2$ and then starts declining.

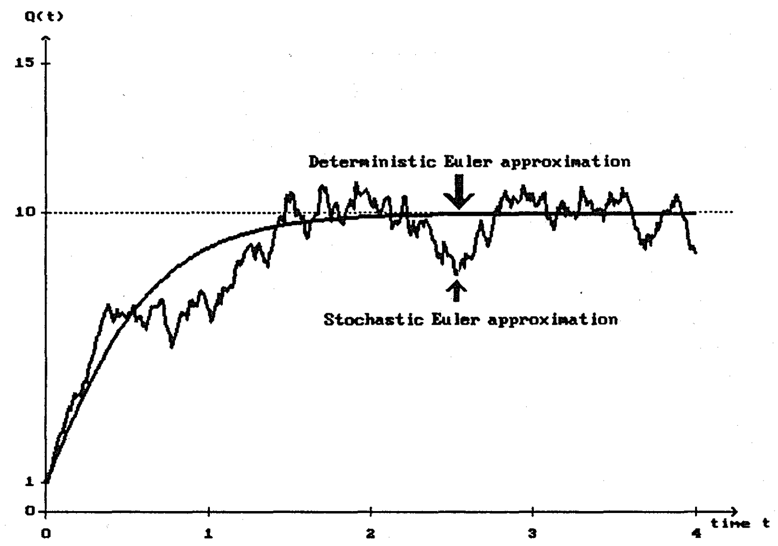

Figure 2. Sample path of the outflow–rate with $\sigma = 0.1$.

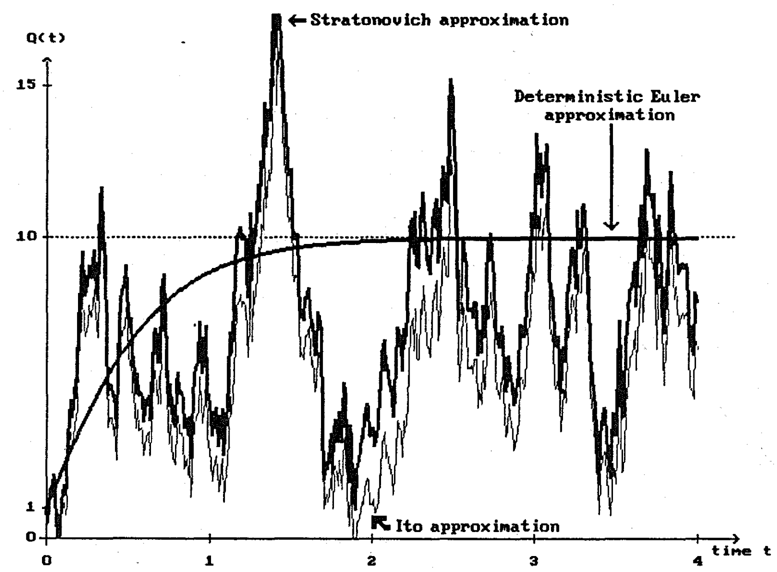

Figure 3. Sample path of the outflow–rate with $\sigma = 0.5$.

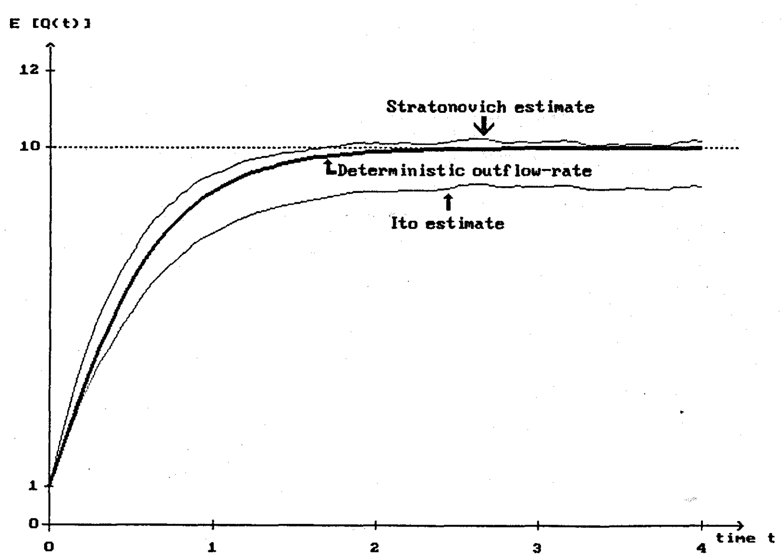

Figure 4. Estimate for the first mean evolution.

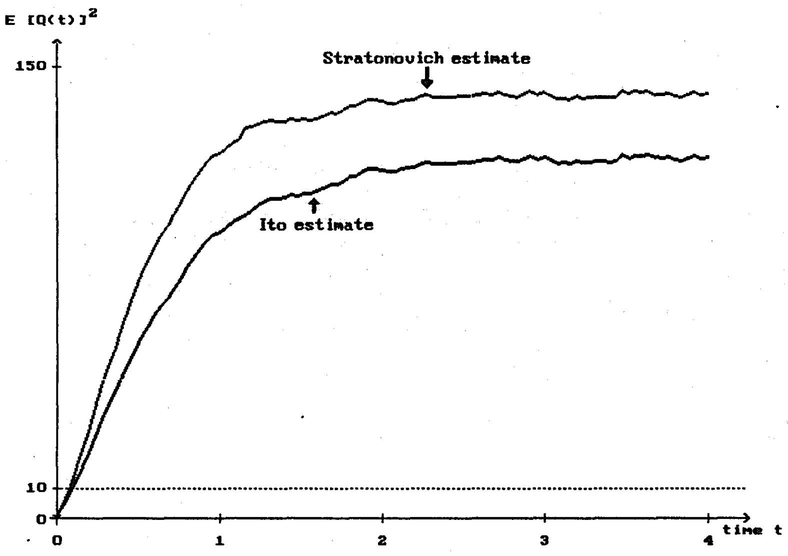

Figure 5. Estimate for the mean square evolution.

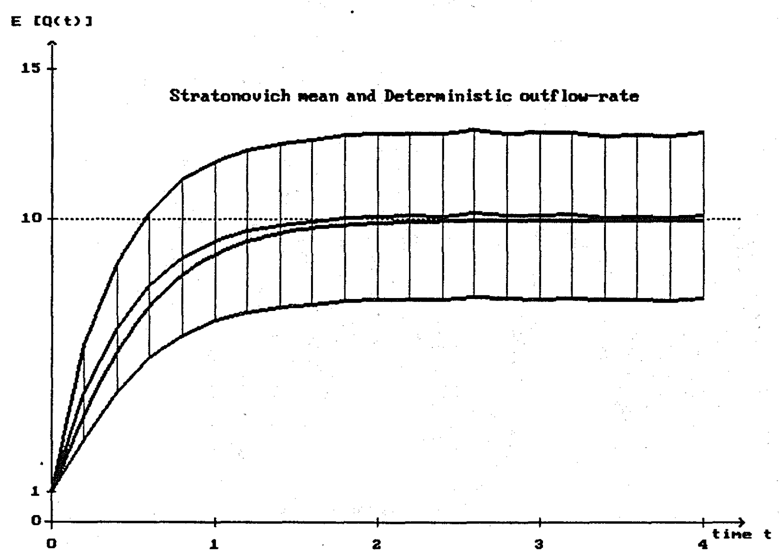

Figure 6. Confidence intervals (90%) for the mean evolution.

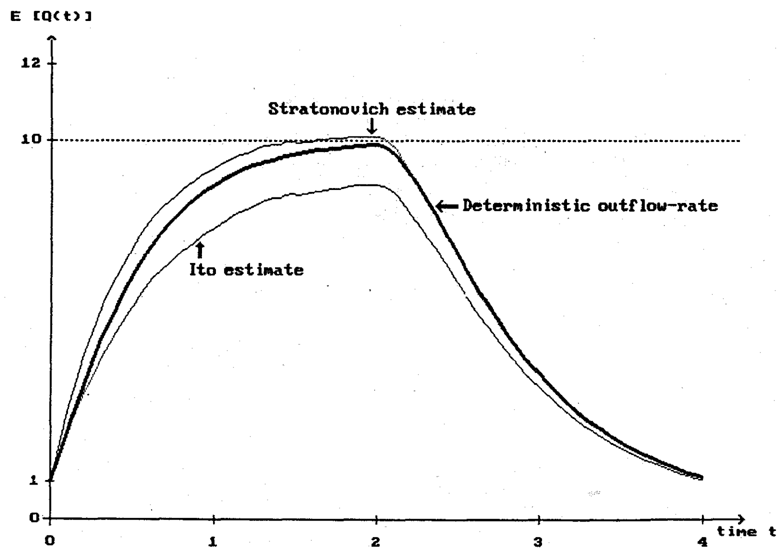

Figure 7. Estimated mean evolution with exponentially decaying precipitation.

7. Effect of coloured noise fluctuations

In the previous section we carried out the analysis to study the effect of white noise fluctuations in the precipitation. The assumption of white noise implies that the time scale of fluctuations is negligibly small compared with the macroscopic time scale of the system. This assumption is in general rather too restrictive, and in a more realistic situation the time scale of fluctuations may not be negligibly small. Accordingly, one has to assume that the stochastic perturbations in the precipitation are represented by coloured noise with a finite correlation time. A realistic version of noise with finite correlation time is the well-known Ornstein-Uhlenbeck process [3]. The auto-correlation of $\xi(t)$ is given by

$$\mathbb{E}[\xi(t)\xi(t')] = \frac{\nu}{2}\exp(-\nu|t-t'|)\,, \quad t > t' \tag{7.1}$$

which tends to the delta–correlated process as $\nu \to \infty$. Equations 7.1 can be used to define the correlation time as $1/\nu$ ($\nu > 0$). Now we rewrite the SDE (2.6) to

$$dQ(t) = aQ^{1-n}(t)(\overline{I}(t) - Q(t))\,dt + \sigma a\overline{I}(t)Q^{1-n}(t)\,\xi(t)\,dt \tag{7.2}$$

where $\xi(t)$ is a stationary Ornstein-Uhlenbeck process determined by

$$d\xi(t) = -\nu\xi(t)\,dt + \nu\,dW(t)\,. \tag{7.3}$$

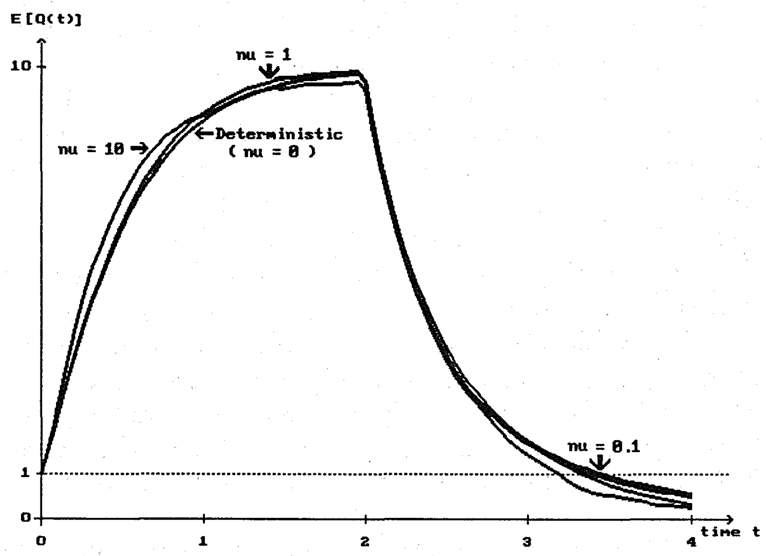

Figure 8. Estimated mean outflow–rate with exponentially decaying precipitation and various correlation parameters ν.

For various values of the parameter ν, the coupled SDE's 7.2 and 7.3 are solved numerically. Figure 8 gives the mean outflow–rate for the same values of parameters as considered in section 6.A. For large values of ν we find that $\mathbb{E}[Q(t)]$ tends to those viewed in the figure 4. However, for smaller values of ν the deviation is significant. In passing, we note that under the peculiarity of coloured noised systems several special effects in their numerical simulation occur. For example, numerical solutions achieve a higher order of convergence to the exact one, cf. Mil'shtein & Tretjakov [15].

8. Summary, Conclusions and Remarks

The described hydrological models can be considered as natural generalization of the corresponding deterministic models. It is worth to introduce and handle with stochasticity influencing outflow–rates. The probabilistic behaviour of the outflow–rate is given by a Fokker–Planck equation which is not explicitly solvable in general. Nevertheless, the stationary behaviour of the outflow–rate could be completely described in terms of the stationary probability density (as normalized stationary solution of its FPE). Because a complete analysis of the FPE was not possible, as often met in nonlinear situations, the main attention has been drawn to the moment evolution of the outflow–rate. Using numerical techniques from [7, 8, 9] arising from stochastic analysis we obtained estimates for the first and second moments, including confidence intervals for these statistics. Thus, we deliberately avoided the application of closure procedures for approximating the unwieldy equations for the moments. Depending on the stochastic calculus – Itô or Stratonovich – one observes an under– and overestimating of the mean evolution compared with the deterministic evolution, respectively. However, Stratonovich interpretation should be prefered for modelling purposes, cf. Wong & Zakai [27]. Instead of white noise sources (uncorrelated increments), we recommend to model with coloured noise. At least in the mean sense, the coloured noised model leads to an estimate which is closer to the deterministic values than those of the Itô and Stratonovich estimates using white noise.
A plenty of generalizations of the presented models for the outflow–rate could now follow. The framework proposed here can be easily extended to a cascade of stochastic nonlinear reservoirs. Besides, the incorporating of more general diffusion processes in the models would be very useful. For example, jump–diffusions reflecting a series of jumps in the rainfall. As we already mentioned, such types of SDE's in Hydrology has been used by Konecny & Nachtnebel [10] in the context of daily streamflow modelling. They discribe the daily discharge series by a SDE with jumps based on the mass balance of a linear reservoir, and also provide some simulations.
Anyway, much work incorporating some stochasticity in modelling has been and is being done. These and other papers demonstrate that even new effects can occur due to the consideration of several forms of stochasticity. Often, stochasticity appropriately reflects the 'erratic behaviour of nature'. In this respect we close this paper with the hope of some encouragement of the readership to work on stochastic modelling.

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